

# STUDY OF THE $p, q$ -DEFORMED TOUCHARD POLYNOMIALS

ORLI HERSCOVICI

ABSTRACT. A two-parameter deformation of the Touchard polynomials, based on the NEXT  $q$ -exponential function of Tsallis, defines two statistics on set partitions. The generating function of classical Touchard polynomials is a composition of two exponential functions. By applying analysis of a combinatorial structure of the deformed exponential function, we establish explicit formulae for both statistics. Moreover, the explicit formulae for the deformed Touchard polynomials makes possible to evaluate coefficients of Taylor series expansion for wide variety of functions with different values of parameters  $p$  and  $q$ .

KEYWORDS: Touchard polynomials; Stirling numbers; degenerate exponential function; statistics on set partitions; permutations; right-to-left minimum;  
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## 1. INTRODUCTION

The  $p, q$ -deformed Touchard polynomials  $T_n(x; p, q)$  for  $n \in \mathbb{N}$  were defined in [5] by means of the generating function

$$(1.1) \quad \exp_p(x(\exp_q(t) - 1)) = \sum_{n=0}^{\infty} T_n(x; p, q) \frac{t^n}{n!},$$

where  $\exp_q(t)$  is the NEXT  $q$ -exponential function given for  $q \neq 1$  as follows (see [9])

$$(1.2) \quad \exp_q(t) = (1 + (1 - q)t)^{\frac{1}{1-q}}.$$

We extend this definition by assuming that for  $q = 1$  this deformed exponential function equals its limiting value  $\lim_{q \rightarrow 1} \exp_q(t) = e^t$ .

In [5] it was shown that  $T_n(x; p, q)$  is indeed an  $n$ th degree polynomial, and its coefficients were denoted by  $S_{p,q}(n, k)$ ; that is

$$(1.3) \quad T_n(x; p, q) = \sum_{k=0}^n S_{p,q}(n, k)x^k.$$

In the case that  $p = q = 1$ , where one has

$$e^{x(e^t-1)} = \sum_{n=0}^{\infty} T_n(x; 1, 1) \frac{t^n}{n!},$$

the polynomials  $T_n(x; 1, 1)$  are the classical Touchard polynomials, and the coefficients  $S_{1,1}(n, k)$  are the Stirling numbers of the second kind  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ . Recall that the Stirling numbers of the second kind count the number of partitions of  $[n]$  into exactly  $k$  disjoint non-empty sets. We can state this alternatively as follows:  $S_{1,1}(n, k) = \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$  counts the number of unordered partitions of  $[n]$  into exactly  $k$  unordered blocks. In the language of [7], such objects are called “sets of sets.” Thus, letting  $\text{SSP}_{n,k}$  denote the collection of all sets of sets of  $[n]$  into  $k$  blocks, we have  $|\text{SSP}_{n,k}| = S_{1,1}(n, k) = \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ . One could also consider ordered partitions of  $[n]$  into  $k$  non-empty unordered blocks; that is, the order of the blocks is taken into account, but not the order of the elements inside the blocks. Alternatively, one could consider unordered partitions of  $[n]$  into  $k$  non-empty ordered blocks; that is, the order of the elements in each block is taken into account, but not the order of the blocks. And finally, one could consider ordered partitions of  $[n]$  into  $k$  non-empty ordered blocks; that is, both the order of the blocks and the order of the elements within the blocks is taken into account. In the language of [7], these three types of objects are called “lists of sets,” “sets of lists,” and “lists of lists” respectively. We will denote these collections by  $\text{LSP}_{n,k}$ ,  $\text{SLP}_{n,k}$  and  $\text{LLP}_{n,k}$  respectively.

The work of [5] revealed an intimate connection between the above objects and the deformed Touchard polynomials. We now describe that connection. Consider an element  $\pi \in \text{LLP}_{n,k}$ . For each of its blocks, define the “standard opener” of the block to be the smallest number in that block. Label the blocks by these numbers. In

general, if we look at the blocks from left to right, the labels will not be in increasing order. If the labels are in increasing order, we call the set of blocks a set of “standard blocks.” Imagine now that we can switch the order of the blocks so that their labels will be in increasing order and thus the set of blocks will become a set of standard blocks. However, the only way we are allowed to switch the positions of blocks is by moving blocks to the right. Let  $\text{nsb}(\pi)$  (“nonstandard blocks”) be the statistic denoting the number of blocks in  $\pi$  that must be moved to the right to obtain a set of standard blocks. For example, let  $n = 8$  and  $k = 4$ . If  $\pi = 32/681/57/4$ , then  $\text{nsb}(\pi) = 2$  because we need to move to the right the block 32 and the block 57, in order to obtain the standard set of blocks 618/32/4/57. Now consider any particular block of  $\pi$ . In general, the elements in the block are not in increasing order. If they are, we call the set of elements of that block “standard elements.” Imagine now that we can switch the order of the elements in the block so that they will be in increasing order and become a set of standard elements. However, the only way we are allowed to switch the positions of the elements is by moving them to the right. Let  $\text{nse}(\pi)$  (“nonstandard elements”) be the statistic denoting the number of elements from all the blocks of  $\pi$  that must be moved to the right in order that all of its blocks will be sets of standard elements. Thus for  $\pi$  given above, we have  $\text{nse}(\pi) = 3$ , because we need to move to the right the numbers 3, 6 and 8, in order to obtain 23/168/57/4.

From their definitions, it follows that  $0 \leq \text{nsb}(\pi) \leq k - 1$  and  $0 \leq \text{nse}(\pi) \leq n - k$ , for all  $\pi \in \text{LLP}_{n,k}$ . Define a generating function  $S(n, k; u, v)$  for the objects  $\{\text{nsb}(\pi), \text{nse}(\pi); \pi \in \text{LLP}_{n,k}\}$  by

$$(1.4) \quad S(n, k; u, v) = \sum_{\pi \in \text{LLP}_{n,k}} u^{\text{nsb}(\pi)} v^{\text{nse}(\pi)}.$$

Here is the connection between  $\text{LLP}_{n,k}$  and the deformed Touchard polynomials that was shown in [5].

**Theorem 1.1.** (*Theorem 3.4 in [5]*) *For all integers  $n, k$  with  $1 \leq k \leq n$ , we have*

$$(1.5) \quad S_{p,q}(n, k) = S(n, k; p - 1, q - 1).$$

This work continues the study of the  $p, q$ -deformed Touchard polynomials defined by (1.1). In [5], an explicit formulae for  $S(n, k; u, 0)$  was obtained by applying techniques of generating functions. However, no explicit formulae were given for  $S(n, k; 0, v)$  or  $S(n, k; u, v)$ . The reason is a more complicated recurrence relation involving the size  $n$  of the set. In this work we establish those explicit formulae not by techniques of generating functions but rather by applying combinatorial analysis of the deformed exponential functions. These results are described in Section 2. In Section 3, we give an explicit expression for the  $p, q$ -deformed Touchard polynomials. Finally, in Section 4, we describe a natural connection of the statistic  $nse$  with the right-to-left minima statistic on permutations.

## 2. SETS OF LISTS AND LISTS OF LISTS

We begin this note by noting the special role of the coefficients  $S_{p,q}(n, k)$  of the deformed Touchard polynomials  $T_n(x; p, q)$  in the particular cases that  $p, q \in \{1, 2\}$ .

**Corollary 2.1.**

$$(2.1) \quad \begin{aligned} \begin{Bmatrix} n \\ k \end{Bmatrix} &= S_{1,1}(n, k) = |SSP_{n,k}|; & S_{2,1}(n, k) &= |LSP_{n,k}|; \\ S_{1,2}(n, k) &= |SLP_{n,k}|; & S_{2,2}(n, k) &= |LLP_{n,k}|. \end{aligned}$$

*Combinatorial Proof.* The result follows immediately from (1.4) and (1.5). □

**Remark 2.2.** *It is easy to give a combinatorial proof that*

$$|LSP_{n,k}| = k! \begin{Bmatrix} n \\ k \end{Bmatrix}, \quad |LLP_{n,k}| = n! \binom{n-1}{k-1} \quad \text{and} \quad |SLP_{n,k}| = \frac{n!}{k!} \binom{n-1}{k-1}.$$

More generally, if we set  $u = p - 1$  and  $v = q - 1$ , then note that

$$\begin{aligned} \left[ \frac{t^n}{n!} x^k u^i v^j \right] (\exp_{u+1}(x(\exp_{v+1}(t) - 1))) &= [u^i v^j] S(n, k, u, v) = \\ |\{\pi \in LLP_{n,k} : nsb(\pi) = i \text{ and } nse(\pi) = j\}|. \end{aligned}$$

Note that we can also define  $\text{nsb}(\pi)$ , for  $\pi \in \text{LSP}_{n,k}$ , and we can also define  $\text{nse}(\pi)$ , for  $\pi \in \text{SLP}_{n,k}$ , and that we have

$$\begin{aligned} [u^i v^0]S(n, k, u, v) &= [u^i]S(n, k, u, 0) = |\{\pi \in \text{LSP}_{n,k} : \text{nsb}(\pi) = i\}|; \\ [u^0 v^j]S(n, k, u, v) &= [v^j]S(n, k, 0, v) = |\{\pi \in \text{SLP}_{n,k} : \text{nse}(\pi) = j\}|. \end{aligned}$$

It was shown (see Theorem 3.5 in [5]) that

$$(2.2) \quad [u^i v^0]S(n, k; u, v) = [u^i]S(n, k; u, 0) = \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{bmatrix} k \\ k-i \end{bmatrix},$$

where  $\begin{bmatrix} n \\ k \end{bmatrix}$  denotes unsigned Stirling numbers of the first kind. Thus,

$$|\{\pi \in \text{LSP}_{n,k} : \text{nsb}(\pi) = i\}| = \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{bmatrix} k \\ k-i \end{bmatrix}.$$

**Remark 2.3.** *This formula has an additional aspect. Note that  $|\{\pi \in \text{LSP}_{n,n} : \text{nsb}(\pi) = i\}| = \begin{bmatrix} n \\ n-i \end{bmatrix}$ . Now converting  $\pi \in \text{LSP}_{n,n}$  to a standard block is equivalent to converting a single block of  $n$  elements to standard elements. Thus we can translate the statistic  $\text{nse}$ , defined initially on partitions of sets, into a statistic defined on permutations. There are  $\begin{bmatrix} n \\ n-i \end{bmatrix}$  permutations  $\sigma \in S_n$  with exactly  $i$  elements that must be moved to the right in order to get the identity permutation.*

We will now calculate

$$[u^0 v^j]S(n, k, u, v) = [v^j]S(n, k, 0, v) = |\{\pi \in \text{SLP}_{n,k} : \text{nse}(\pi) = j\}|$$

and

$$[u^i v^j]S(n, k, u, v) = |\{\pi \in \text{LLP}_{n,k} : \text{nsb}(\pi) = i \text{ and } \text{nse}(\pi) = j\}|.$$

Let us consider a combinatorial aspects of (1.2). For this purpose we replace  $q-1 = v$  as it was done in [5]. Note that (1.2) has the following Taylor series expansion (see [3])

$$\exp_q(t) = 1 + \sum_{n=1}^{\infty} Q_{n-1}(q) \frac{t^n}{n!},$$

where  $Q_n(q) = 1 \cdot q \cdot (2q - 1) \cdot (3q - 2) \cdot \dots \cdot (nq - (n - 1))$  with initial condition  $Q_0(q) = 1$ . Therefore, by applying our substitution, we obtain

$$(2.3) \quad \exp_q(t) \Big|_{q-1=v} = (1 - vt)^{\frac{1}{-v}} = 1 + \sum_{n=1}^{\infty} Q_{n-1}(v+1) \frac{t^n}{n!}.$$

Let us consider now the coefficients of this exponential generating function for  $n \geq 1$ .

$$(2.4) \quad \left[ \frac{t^n}{n!} \right] \exp_q(t) \Big|_{q-1=v} = Q_{n-1}(v+1)$$

$$(2.5) \quad = (1+v)(1+2v)(1+3v) \cdot \dots \cdot (1+(n-1)v)$$

$$(2.6) \quad = \prod_{m=0}^{n-1} (mv+1) = \sum_{j=0}^{n-1} \left[ \begin{matrix} n \\ n-j \end{matrix} \right] v^j.$$

From comparing (2.6) with (2.2) we can see that counting an nse in permutations of  $[n]$  coincides with counting an nsb in partitions of  $[n]$  with exactly  $n$  blocks. Therefore the coefficients of the exponential generating function (2.3) count the number of elements with respect to statistic nse. To be more specific, let us denote by  $\text{nse}(\sigma_n, j)$  a number of permutations  $\sigma_n$  of a set  $[n]$  with exactly  $j$  elements which should be moved to the right (see Remark 2.3). Then for  $0 \leq j \leq n-1$ , we have  $\text{nse}(\sigma_n, j) = \left[ \begin{matrix} n \\ n-j \end{matrix} \right]$ .

Now we can find the explicit distribution of the statistic nse as a composition of exponential generating functions [1, Theorem 3.3]. Composition of (2.3) and the exponential function given by  $e^{(1-vt)^{\frac{1}{-v}}}$  is another e.g.f. defining a distribution of the number of elements which must be moved to the right (w.r.t. nse) in partitions of a set  $[n]$  into lists. The inner structure of the lists with respect to the statistic nse is described by (2.6). The outer structure – a set partition – is described by the Stirling numbers of the second kind. Therefore we can state the following theorem.

**Theorem 2.4.**

$$(2.7) \quad [u^0 v^j] S(n, k, u, v) = [v^j] (S(n, k, 0, v) = |\{\pi \in SLP_{n,k} : \text{nse}(\pi) = j\}| = \left[ \begin{matrix} n \\ n-j \end{matrix} \right] \left\{ \begin{matrix} n-j \\ k \end{matrix} \right\}.$$

**Corollary 2.5.** *The average number of elements in partitions of  $[n]$  into sets of lists in accordance with the statistic  $nse$  is given by*

$$\frac{\sum_{j=0}^{n-1} j \begin{bmatrix} n \\ n-j \end{bmatrix} B_{n-j}}{\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} B_k}.$$

*Proof.* Let us define a function  $f_n(v) = \sum_{k=1}^n \sum_{j=0}^{n-k} \begin{bmatrix} n \\ n-j \end{bmatrix} \begin{Bmatrix} n-j \\ k \end{Bmatrix} v^j$ . After changing the summation order, we obtain  $f_n(v) = \sum_{j=0}^{n-1} \begin{bmatrix} n \\ n-j \end{bmatrix} B_{n-j} v^j$ . Therefore the average number of elements w.r.t statistics  $nse$  is given by

$$(2.8) \quad \frac{\partial_v f_n(v) \Big|_{v=1}}{|\text{SLP}_n|} = \frac{\sum_{j=0}^{n-1} j \begin{bmatrix} n \\ n-j \end{bmatrix} B_{n-j}}{\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} B_k},$$

and the proof is complete.  $\square$

We have the following corollary concerning Stirling numbers of the second kind and unsigned Stirling numbers of the first kind.

**Corollary 2.6.**

$$(2.9) \quad \sum_{l=k}^n \begin{bmatrix} n \\ l \end{bmatrix} \begin{Bmatrix} l \\ k \end{Bmatrix} = \frac{n!}{k!} \binom{n-1}{k-1}.$$

*Proof.* The corollary follows immediately from the first equation in (2.7) and the formula for  $|\text{SLP}_{n,k}|$  appearing in the remark following Corollary 2.1.  $\square$

**Remark.** A completely combinatorial proof of (2.9) can be found in [6]. By comparison, recall the classical formula concerning Stirling numbers of the second kind

and (signed) Stirling numbers of the first kind  $((-1)^{n-l} \begin{bmatrix} n \\ l \end{bmatrix})$ :

$$\sum_{l=k}^n (-1)^{n-l} \begin{bmatrix} n \\ l \end{bmatrix} \left\{ \begin{matrix} l \\ k \end{matrix} \right\} = \sum_{l=k}^n \left\{ \begin{matrix} n \\ l \end{matrix} \right\} (-1)^{l-k} \begin{bmatrix} l \\ k \end{bmatrix} = \delta_{nk}.$$

Now we can state the results about the lists of lists.

**Theorem 2.7.**

$$(2.10) \quad \begin{aligned} [u^i v^j] S(n, k, u, v) &= |\{\pi \in LLP_{n,k} : \text{nsb}(\pi) = i \text{ and } \text{nse}(\pi) = j\}| \\ &= \begin{bmatrix} n \\ n-j \end{bmatrix} \left\{ \begin{matrix} n-j \\ k \end{matrix} \right\} \begin{bmatrix} k \\ k-i \end{bmatrix}. \end{aligned}$$

*Proof.* The proof can be easily obtained from the result (2.2) and the Theorem 2.7 as following. The number of partitions of  $[n]$  into  $k$  lists of sets w.r.t. the statistic  $\text{nsb}$  is given by  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \begin{bmatrix} k \\ k-i \end{bmatrix}$  (see (2.2)). The statistic  $\text{nse}$  is responsible for the arrangement of elements *inside* the blocks and we already know that there are exactly  $\begin{bmatrix} n \\ n-j \end{bmatrix} \left\{ \begin{matrix} n-j \\ k \end{matrix} \right\}$  partitions of  $[n]$  into  $k$  blocks with  $j$  elements which must be moved to the right (see (2.7)). Therefore, by combining these results we obtain the statement (2.10).  $\square$

### 3. $p, q$ -DEFORMED TOUCHARD POLYNOMIALS

It was shown in [5] that the coefficients of the  $p, q$ -deformed Touchard polynomials are closely related to the distribution of the statistics  $\text{nsb}$  and  $\text{nse}$  on set partitions (1.5). In previous section we obtained the explicit expression (2.10) for coefficients of the terms  $u^i v^j$ . Now we can get an explicit expression for the deformed Touchard polynomials as polynomials in the variables  $x, p$ , and  $q$ .



**Theorem 3.1.** *The NEXT  $p, q$ -deformed Touchard polynomials  $T_n(x; p, q)$  defined by the generating function (1.1) are polynomials in the variables  $x$ ,  $p$ , and  $q$  given by*

(3.1)

$$T_n(x; p, q) = \sum_{k=0}^n \sum_{m=0}^{k-1} \sum_{\ell=0}^{n-k} \sum_{i=m}^{k-1} \sum_{j=\ell}^{n-k} \binom{i}{m} \binom{j}{\ell} (-1)^{m+\ell} s(n, n-j) S(n-j, k) s(k, k-i) p^m q^\ell x^k,$$

where  $s(i, j)$  and  $S(i, j)$  are the Stirling numbers of the first and second kind respectively.

*Proof.* From (1.3) and (1.5) we obtain

$$T_n(x; p, q) = \sum_{k=0}^n S_{p,q}(n, k) x^k = \sum_{k=0}^n S(n, k; p-1, q-1) x^k.$$

By applying (2.10), we get

$$T_n(x; p, q) = \sum_{k=0}^n \sum_{j=0}^{n-k} \sum_{i=0}^{k-1} \begin{bmatrix} n \\ n-j \end{bmatrix} \left\{ \begin{matrix} n-j \\ k \end{matrix} \right\} \begin{bmatrix} k \\ k-i \end{bmatrix} (p-1)^i (q-1)^j x^k.$$

Binomial expansion of  $(p-1)^i$  and  $(q-1)^j$  leads to

$$\begin{aligned} T_n(x; p, q) &= \sum_{k=0}^n \sum_{j=0}^{n-k} \sum_{i=0}^{k-1} \sum_{m=0}^i \sum_{\ell=0}^j \binom{i}{m} \binom{j}{\ell} (-1)^{(i-m)+(j-\ell)} \begin{bmatrix} n \\ n-j \end{bmatrix} \left\{ \begin{matrix} n-j \\ k \end{matrix} \right\} \begin{bmatrix} k \\ k-i \end{bmatrix} p^m q^\ell x^k. \end{aligned}$$

By applying the identity  $s(i, j) = (-1)^{i-j} \begin{bmatrix} i \\ j \end{bmatrix}$  and changing the summation's order, we obtain

$$\begin{aligned} T_n(x; p, q) &= \sum_{k=0}^n \sum_{m=0}^{k-1} \sum_{\ell=0}^{n-k} \sum_{i=m}^{k-1} \sum_{j=\ell}^{n-k} \binom{i}{m} \binom{j}{\ell} (-1)^{m+\ell} s(n, n-j) S(n-j, k) s(k, k-i) p^m q^\ell x^k, \end{aligned}$$

which proves the theorem.  $\square$

**Remark 3.2.** *Note that for all  $n \in \mathbb{N}$  it holds that  $[x^0]T_n(x; p, q) = 0$  and  $T_0(x; p, q) = 1$  for all values of  $p$  and  $q$ .*

**Remark 3.3.** *We already mentioned that the cases when  $p, q \in \{1, 2\}$  are related to combinatorial objects – partitions of a set. However, the result of Theorem 3.1 is significantly wider. It actually gives an explicit expression for the coefficients of expansion into Taylor series for a variety of functions defined for different values of the parameters  $p$ ,  $q$ , and  $x$ .*

#### 4. ANOTHER CONNECTION TO PERMUTATIONS

We already show that statistic  $nse$ , defined originally for set partitions, can be naturally translated into permutations. Now we present a natural connection between this statistic and another statistic on permutations, namely, right-to-left minima. An element  $\sigma_j$  of permutation  $\sigma = \sigma_1\sigma_2 \dots \sigma_n$  is a right-to-left minimum of  $\sigma$  if  $\sigma_j < \sigma_k$  for all  $k > j$ . We can state the following proposition.

**Proposition 4.1.** *Given a permutation  $\sigma \in S_n$ . Let  $RLM(\sigma)$  be a set of the right-to-left minima of  $\sigma$ ,  $NSE(\sigma)$  be a set of the elements that might be moved w.r.t. the statistic  $nse$ , and  $nse(\sigma) = |NSE(\sigma)|$ ,  $rlm(\sigma) = |RLM(\sigma)|$ . It holds that*

$$NSE(\sigma) \cap RLM(\sigma) = \emptyset, \quad nse(\sigma) + rlm(\sigma) = n.$$

*Proof.* Given a permutation  $\sigma = \sigma_1\sigma_2 \dots \sigma_n \in S_n$ , then a statistic  $nse(\sigma) = |\{j : \sigma_j > \sigma_k \text{ for some } k > j\}|$ . Let us sort the elements of  $\sigma$  to two subsets:  $NSE(\sigma)$ , containing all elements of  $\sigma$  that must be moved to the right w.r.t. the statistic  $nse$ , and  $RLM(\sigma)$ , containing all elements that must be left in their relative to each other order. Now we are looking for the minimal element of  $\sigma$ . Let us assume that it is the element  $\sigma_{j_1}$ . Therefore the elements  $\sigma_1, \dots, \sigma_{j_1-1}$  go to the  $NSE(\sigma)$  and the element  $\sigma_{j_1}$  goes to  $RLM(\sigma)$ . Now we consider the rest of the permutation  $\sigma$ , namely, the sequence  $\sigma_{j_1+1} \dots \sigma_n$ . Let us assume that the element  $\sigma_{j_2}$  is its minimum. Therefore the elements  $\sigma_{j_1+1}, \dots, \sigma_{j_2-1}$  go to the  $NSE(\sigma)$  and the element  $\sigma_{j_2}$  goes to  $RLM(\sigma)$ . We continue this process till all the elements of  $\sigma$  are sorted. At the end of sorting each element of  $\sigma$  belongs to only one of two subsets. It follows from the sorting process that the subset  $RLM(\sigma)$  is the set of right-to-left minima of the permutation  $\sigma$ . Obviously,  $|NSE(\sigma)| = nse(\sigma)$ , and, with notation  $rlm(\sigma) = |RLM(\sigma)|$ , the proof is complete.  $\square$

**Remark 4.2.** *There are three other related terms for permutations: left-to-right maxima, right-to-left maxima, and left-to-right minima, whose distributions are connected to each other. It is known that the number of permutations  $\sigma \in S_n$  with exactly  $k$  left-to-right maxima is  $\begin{bmatrix} n \\ k \end{bmatrix}$  (see [2, Corollary 3.42]), which equals to the number of permutations of  $\sigma \in S_n$  such that  $\text{nse}(\sigma) = n - k$ .*

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#### REFERENCES

- [1] Aigner, M., *A course in enumeration*, Graduate Texts in Mathematics, 238. Springer, Berlin, 2007.
- [2] Bóna, M., *Combinatorics of permutations*, Discrete Mathematics and its Applications, CRC Press, Boca Raton, 2012.
- [3] Borges, E.P., On a  $q$ -generalization of circular and hyperbolic functions, *J. Phys. A*, **31** (1998), no. 23, 5281–5288.
- [4] Flajolet, P. and Sedgewick, R., *Analytic combinatorics*, Cambridge University Press, Cambridge, 2009.
- [5] Herscovici, O. and Mansour, T., Tsallis  $p, q$ -deformed Touchard polynomials and Stirling numbers, *Russ. J. Math. Phys.*, **24** (2017), no. 1, 37–50.
- [6] Herscovici, O. and Pinsky, R.G., A combinatorial proof of an identity involving Stirling numbers of the second kind and unsigned Stirling numbers of the first kind, 2018, preprint.
- [7] Motzkin, T.S., Sorting numbers for cylinders and other classification numbers, in *Combinatorics*, Proc. Sympos. Pure Math., **19** (1971), 167–176.
- [8] Sloane, N., The On-Line Encyclopedia of Integer Sequences, <http://oeis.org>.
- [9] Tsallis, C., What are the numbers that experiments provide?, *Quimica Nova*, **17** (1994), no. 6, 468–471.

O. HERSCOVICI, DEPARTMENT OF MATHEMATICS, TECHNION – ISRAEL INSTITUTE OF TECHNOLOGY, 3200003 HAIFA, ISRAEL

*E-mail address:* orli.herscovici@gmail.com