An Asymptotic Form of the Generating Function $\prod_{k=1}^{\infty}\left(1+\frac{x^k}{k}\right)$

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Abstract

It is shown that the sequence of rational numbers $r(k)$ generated by the ordinary generating function $\prod_{k=1}^{\infty}\left(1+\frac{x^k}{k}\right)$ converges to a limit $C > 0$. C can be expressed as $C = \exp\left(-\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\,\zeta(k)\right)$ where $\zeta()$ denotes the Riemann zeta function.

The ordinary generating function (OGF)

$$R(x) \ := \ \prod_{k=1}^{\infty}\left(1+\frac{x^k}{k}\right) \ = \ \sum_{k=0}^{\infty} r(k)\ x^k \tag{1a}$$

is closely related to the well known OGF

$$Q(x) \ := \ \prod_{k=1}^{\infty}\left(1+x^k\right) \ = \ \sum_{k=0}^{\infty} q(k)\ x^k \tag{1b}$$

$Q(x)$ generates the sequence of counters for the number of integer partitions with distinct parts [Wil]. $q(k)$ is equal to the number of partitions of k into distinct parts for each $k \geq 0$ [Int].

A partition with distinct parts of integer k can be regarded as a finite set S of (distinct) positive integers $i \geq 1$ whose sum equals k. Let $\mathcal{P}(k)$ denote the set of all such partitions of k and let $S \in \mathcal{P}(k)$. We then have

$$\sum_{i\in S} i = k \tag{2}$$

With each partition $S \in \mathcal{P}(k)$ we can associate the inverse of the product of its (distinct) elements

$$\text{IP}(S) := \frac{1}{\prod\limits_{i \in S} i} \tag{3}$$

With this in mind $r(k)$ can be written as

$$r(k) = 1 \qquad : \ r = 0 \tag{4a}$$

$$r(k) = \sum_{S \in \mathcal{P}(k)} \text{IP}(S) = \sum_{S \in \mathcal{P}(k)} \frac{1}{\prod\limits_{i \in S} i} \qquad : \ r \geq 1 \tag{4b}$$

In other words, $r(k)$ is equal to the sum over all partitions $S \in \mathcal{P}(k)$ of the reciprocal of the product of the elements of S .

How does the sequence $r(k)$ given in (4a) and (4b) behave? Does it converge to some limit $C > 0$? Taking the logarithm of (1a), applying the Mercator series expansion [Wol], and summing up columns first gives

$$\begin{aligned}
\ln R(x) = \sum_{k \geq 1} \ln\left(1 + \tfrac{x^k}{k}\right) &= x \qquad - \tfrac{1}{2}\, x^2 \qquad + \tfrac{1}{3}\, x^3 \qquad - \cdots \\
&+ \tfrac{x^2}{2} \qquad - \tfrac{1}{2}\left[\tfrac{x^2}{2}\right]^2 \quad + \tfrac{1}{3}\left[\tfrac{x^2}{2}\right]^3 \quad - \cdots \\
&+ \tfrac{x^3}{3} \qquad - \tfrac{1}{2}\left[\tfrac{x^3}{3}\right]^2 \quad + \tfrac{1}{3}\left[\tfrac{x^3}{3}\right]^3 \quad - \cdots \\
&\ \vdots \\
&= \mathrm{Li}_1(x) - \tfrac{1}{2}\, \mathrm{Li}_2(x^2) + \tfrac{1}{3}\, \mathrm{Li}_3(x^3) - \cdots
\end{aligned} \tag{5}$$

Here $\mathrm{Li}_s(x)$ denotes the so-called polylogarithm [Wikc], a Dirichlet type series [Wika].

We are looking for an asymptotic relation of the form

$$R(x) \xrightarrow{x \to 1-} \frac{C}{1 - x} \tag{6}$$

for some constant $C > 0$. This is equivalent to the existence of the limit

$$C \ = \lim_{x \to 1-} (1 - x) R(x) \tag{7}$$

Taking the logarithm of (7) gives

$$\ln C \ = \lim_{x \to 1-} \big(\ln(1 - x) + \ln R(x)\big) \tag{8}$$

If we insert (5), observe the identity [Wikd]

$$\mathrm{Li}_1(x) = -\ln(1-x) \tag{9}$$

and finally set $x = 1$, we arrive at the condition

$$\begin{aligned} \ln C &= -\tfrac{1}{2}\,\mathrm{Li}_2(1) + \tfrac{1}{3}\,\mathrm{Li}_3(1) - \tfrac{1}{4}\,\mathrm{Li}_4(1) + \cdots \\ &= -\tfrac{1}{2}\,\zeta(2) + \tfrac{1}{3}\,\zeta(3) - \tfrac{1}{4}\,\zeta(4) + \cdots \end{aligned} \tag{10}$$

where $\zeta(s)$ denotes the Riemann Zeta function [Wike]. We therefore have

$$C = \exp\left(-\sum_{k=2}^{\infty} \frac{(-1)^k}{k}\,\zeta(k) \right) \tag{11}$$

We observe that $\zeta(k)$ converges rapidly towards 1 [Wikb]:

k	$\zeta(k)-1$	
2	$\frac{\pi^2}{6}-1$	0.644934
3	-	0.202057
4	$\frac{\pi^4}{90}-1$	0.082323
5	-	0.036928
6	$\frac{\pi^6}{945}-1$	0.017343
7	-	0.008349
8	$\frac{\pi^8}{9450}-1$	0.004077
9	-	0.002008
10	$\frac{\pi^{10}}{93555}-1$	0.000995
11	-	0.000494

$$\zeta(k) \xrightarrow{k\to\infty} 1$$

This motivates the decomposition of (10)

$$\begin{aligned} \ln C = & -\tfrac{1}{2}\,\zeta(2) + \tfrac{1}{3}\,\zeta(3) - \tfrac{1}{4}\,\zeta(4) + \cdots \\ = & -\tfrac{1}{2}\Big[\zeta(2)-1\Big] + \tfrac{1}{3}\Big[\zeta(3)-1\Big] - \tfrac{1}{4}\Big[\zeta(4)-1\Big] + \cdots \\ & -\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots \\ = & -\Delta + \ln 2 - 1 \end{aligned} \tag{12}$$

where Δ is defined as

$$\Delta := +\ \tfrac{1}{2}\left[\zeta(2)-1\right]\ -\ \tfrac{1}{3}\left[\zeta(3)-1\right]\ +\ \tfrac{1}{4}\left[\zeta(4)-1\right] -\ \cdots \tag{13}$$

We therefore have from (12)

$$C = \frac{2}{e^{1+\Delta}} \tag{14}$$

From (13) we derive the sequence of corrections Δ_m as follows

$$\Delta_m = \begin{cases} 0 & : \ m = 1 \\ \sum\limits_{k=2}^{m} \frac{(-1)^k}{k}\big(\zeta(k)-1\big) & : \ m \geq 2 \end{cases} \tag{15}$$

This creates the sequence

$$C_m = \frac{2}{\exp\left(1+\Delta_m\right)} \qquad : m \geq 1 \tag{16}$$

of approximations of C whose first elements are listed in table 1.

m	Δ_m	$\frac{2}{\exp(1+\Delta_m)}$
1	0.0	0.7357589
2	0.3224670	0.5329542
3	0.2551147	0.5700863
4	0.2756955	0.5584734
5	0.2683100	0.5626133
6	0.2712005	0.5609894
7	0.2700078	0.5616589
8	0.2705174	0.5613727
9	0.2702943	0.5614980
10	0.2703937	0.5614421
11	0.2703488	0.5614674
12	0.2703693	0.5614559
13	0.2703599	0.5614612

Table 1: Approximation of C

Useful recurrence relations for computation

For $n > 0$ we define the finite products

$$R_n(x) \;:=\; \prod_{k=1}^{n} \left(1 + \frac{x^k}{k}\right) \;=\; \sum_{k=0}^{\infty} r_n(k)\; x^k \tag{17a}$$

$$Q_n(x) \;:=\; \prod_{k=1}^{n} (1 + x^k) \;=\; \sum_{k=0}^{\infty} q_n(k)\; x^k \tag{17b}$$

The integer numbers $q_n(k)$ in (17b) count the number of partitions of k with distinct parts *where no part exceeds* n. The coefficients $q_n(k)$ clearly have 3 basic properties:

$$q_n(k) \;=\; q(k) \qquad \text{if} \quad k \;<=\; n \tag{18a}$$

$$q_n(k) \;=\; 0 \qquad \text{if} \quad k \;>\; \tfrac{n(n+1)}{2} \tag{18b}$$

$$\sum_{k\geq 0} q_n(k) \;=\; 2^n \tag{18c}$$

where (18c) follows from evaluation of $Q_n(1)$. The $q_n(k)$ obey the recurrence relations

$$q_0(k) = \begin{cases} 1 \;:\; k = 0 \\ 0 \;:\; k \geq 1 \end{cases} \tag{19a}$$

$$q_n(k) = \; q_{n-1}(k) \quad : \; 0 \leq k < n \tag{19b}$$

$$q_n(k) = \; q_{n-1}(k-n) \;+\; q_{n-2}(k-n+1) \;+\; q_{n-3}(k-n+2) \;+\; \cdots$$
$$+\; q_1(k-2) \;+\; q_0(k-1) \quad : \; k \geq n > 0 \tag{19c}$$

Initial values are prescribed in row $n = 0$ (19a). The values in any subsequent row $n \geq 1$ are determined by values in previous rows $m < n$.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	1	2	1	1	1	0	0	0	0	0	0	0	0	0	0
4	1	1	1	2	2	2	2	2	1	1	1	0	0	0	0	0	0
5	1	1	1	2	2	3	3	3	3	3	3	2	2	1	1	1	0

Table 2: Upper left section of the $q_n(k)$ field [$0 \leq n \leq 5$, $0 \leq k \leq 16$]

Analogous properties and relations hold for the *rational* numbers $r_n(k)$ in (17a):

$$r_n(k) \;=\; r(k) \qquad \text{if} \quad k \;<=\; n \tag{20a}$$

$$r_n(k) \;=\; 0 \qquad \text{if} \quad k \;>\; \tfrac{n(n+1)}{2} \tag{20b}$$

$$\sum_{k\geq 0} r_n(k) \;=\; n+1 \tag{20c}$$

$$r_0(k) = \begin{cases} 1 \;:\; k=0 \\ 0 \;:\; k \geq 1 \end{cases} \tag{21a}$$

$$r_n(k) = \; r_{n-1}(k) \quad : \; 0 \leq k < n \tag{21b}$$

$$\begin{aligned} r_n(k) = \; & \tfrac{1}{n}\, r_{n-1}(k-n) \;+\; \tfrac{1}{n-1}\, r_{n-2}(k-n+1) \;+\; \tfrac{1}{n-2}\, r_{n-3}(k-n+2) \;+\; \cdots \\ & + \; \tfrac{1}{2}\, r_1(k-2) \;+\; \tfrac{1}{1}\, r_0(k-1) \quad : \; k \geq n > 0 \end{aligned} \tag{21c}$$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0	0	0	0	0	0	0	0
4	1	1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{5}{12}$	$\frac{7}{24}$	$\frac{5}{24}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{24}$	0	0	0	0	0	0
5	1	1	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{7}{12}$	$\frac{37}{60}$	$\frac{59}{120}$	$\frac{37}{120}$	$\frac{1}{4}$	$\frac{19}{120}$	$\frac{1}{8}$	$\frac{7}{120}$	$\frac{1}{24}$	$\frac{1}{60}$	$\frac{1}{120}$	$\frac{1}{120}$	0

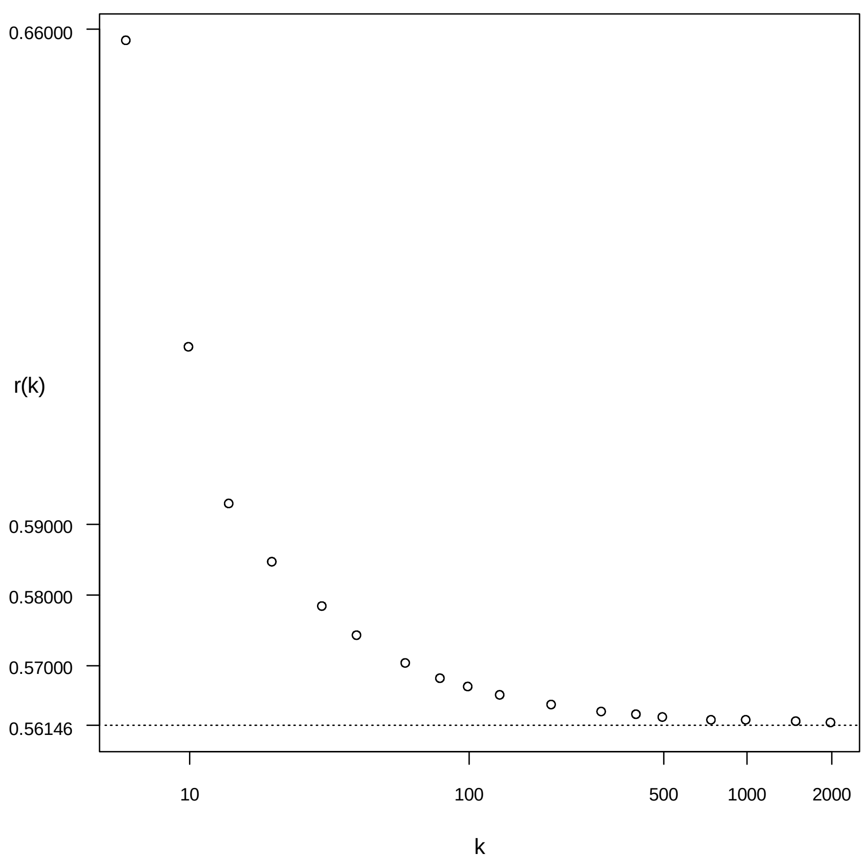
Table 3: Upper left section of the $r_n(k)$ field [$0 \leq n \leq 5$, $0 \leq k \leq 16$]

Figure 1 assembles some instances of $r(k)$ which have been computed on the R platform for statistical computing [RPr] using recurrence relations (21a), (21b), and (21c). The plot shows that the $r(k)$ approach the asymptotic value

$$C = 0.56146\ldots$$

from above as k increases. The constant C is determined by (11) and (14) and is marked by a dashed horizontal line.

Figure 1: Some computed instances of $r(k)$



Conclusion

It has been shown that the function

$$f(x) = \frac{C}{1 - x}$$

is an asymptotic form of the generating function (1a) in the sense that the sequence of rational numbers $r(k)$ generated by (1a) converges towards $C > 0$ which is determined by (11) and (14).

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