A MODIFICATION OF WYTHOFF'S NIM

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Abstract. We modify Wythoff's game by allowing an additional move. The P-positions in our game can be derived from the table of letter positions in the Tribonacci word. This is related to the recent solution of the Greedy Queens in a spiral problem. Our analysis involves the table of letter positions of arbitrary k-bonacci words. We find a mex-rule that generates the Quadribonacci table, extending work by Duchêne and Rigo on the Tribonacci table.

1. Splythoff's Nim

The following take-away game is known as Wythoff's Nim and was described in 1907 in the Nieuw Archief voor Wiskunde [18]. The game is played by two persons. Two piles of counters are placed on a table. The two players alternately either take an arbitrary number of counters from a single pile or an equal arbitrary number from both piles. The player who takes the last counter, or counters, wins.

We modify Wythoff's Nim by allowing the additional option of a *split*. If a player takes an equal number of counters from both piles and *only one pile remains*, then he can split the remaining pile into two. For instance, suppose there are 4 counters on the first pile and 7 on the second. In this position a player is allowed to take 4 counters from both piles and split the remainder into piles of 1 and 2. A split is only allowed after taking counters from both piles. A player is not allowed to take all counters from a single pile and then split. We call this Splythoff's Nim. We refer to the three possible moves as *single*, *double*, and *split*.

A P-position is a combination of two numbers a and b that is losing for the player that moves next. In other words, it is winning for the previous player. Otherwise, it is called an N-position. It is obvious that:

(1) There are no moves between P-positions.
(2) Every N-position has a move to a P-position.
(3) $(0, 0)$ is a P-position.

We reserve a for the number of counters on the smaller pile and b for the larger pile. The set of all positions (a, b) form the vertices of a direct graph, and the moves form the edges. The graph is connected since all games end

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at $(0,0)$. It is acyclic since moves remove counters: we have a partial order. More specifically, the position p is larger than the position q iff there is a sequence of moves from p to q.

The standard algorithm to compute the P-positions is as follows. Let N_0 be the set of all (a,b) that are neighbors of $(0,0)$. Delete all vertices in $N_0 \cup \{0,0\}$ from the graph, along with their edges. Let P_1 be the set of minimal elements of the remaining digraph. Let N_1 be the set of neighbors of P_1. Remove $N_1 \cup P_1$ from the graph. Etc. The set of P-positions is equal to the union of all P_i.

Recall that for a proper subset $S \subset \mathbb{N}$ the minimal excluded value $\mathrm{mex}(S)$ is the minimum of $\mathbb{N} \setminus S$. For Wythoff's Nim, the P_i are singletons $\{(a_i, b_i)\}$ that can be computed with the mex operator. Let $A_i = \{a_1, \ldots, a_i\}$ and let $B_i = \{b_1, \ldots, b_i\}$. Then

$$\begin{aligned} a_{i+1} &= \mathrm{mex}\,(A_i \cup B_i) \\ b_{i+1} &= a_{i+1} + i + 1 \end{aligned}$$

One can also generate the sequence (a_i, b_i) from the Fibonacci substitution

$$\begin{aligned} 0 &\mapsto 01 \\ 1 &\mapsto 0 \end{aligned}$$

which has fixed point $010010100100101\cdots$ known as the *Fibonacci word*. The numbers a_i correspond to the positions of the zeroes in this sequence. The b_i correspond to the ones. We will see that for Splythoff's Nim the P-positions can be derived from the Tribonacci substitution

$$\begin{aligned} 0 &\mapsto 01 \\ 1 &\mapsto 02 \\ 2 &\mapsto 0 \end{aligned}$$

which has fixed point $\omega^3 = 0102010010201010201001\cdots$ known as the *Tribonacci word*.

Theorem 1. *Let x_i, y_i, and z_i be the locations of the i-th $0, 1$, and 2 in the Tribonacci word. Then the i-th P-position in Splythoff's Nim (a_i, b_i) is given by $a_i = y_i - x_i$ and $b_i = z_i - y_i$.*

We shall prove this theorem first and then extend it to the *k-bonacci substitution* [16]. This extension concerns integer sequences and does not seem to have an impartial game that comes with it.

Our paper is strongly related to previous work of Duchêne and Rigo [7] on a three-pile take-away game. The game introduced by Duchêne and Rido has P-positions in the 'positions table' (defined below) of the Tribonacci word ω^3. The P-positions of Splythoff's Nim are in the 'difference table' of ω^3.

The literature on modifications of Wythoff's Nim is huge. We briefly discuss it in the final section of our paper. An extensive bibliography on Wythoff's game can be found in [5].

2. The P-positions

We generate a table with two rows A and B and infinitely many columns. Each natural number appears exactly once in the table. We add a row $B-A$ on top of the table, and also a row $A+B$ below. We let Δ and Σ respectively denote these two rows.

Δ	1	2	4	5	6	7	9	10	11	13	$\cdots$
A	1	3	4	6	7	9	10	12	14	15	$\cdots$
B	2	5	8	11	13	16	19	22	25	28	$\cdots$
Σ	3	8	12	17	20	25	29	34	39	43	$\cdots$

Table 1. The table of A and B alongside their sum and difference

For any set $S \subset \mathbb{N}$ let S_i be the subset of its least i elements. The columns of the table are generated by

$$\begin{aligned} \delta_{i+1} = & \quad \operatorname{mex}(\Delta_i \cup \Sigma_i) \\ a_{i+1} = & \quad \operatorname{mex}(A_i \cup B_i) \\ b_{i+1} = & \quad a_{i+1} + \delta_{i+1} \\ \sigma_{i+1} = & \quad a_{i+1} + b_{i+1} \end{aligned}$$

It follows from the definitions that all four sequences are strictly increasing. Since A and Δ are defined by a mex relation, both Δ, Σ and A, B are partitions of $\mathbb{N}$. A is sequence A140100 and B is sequence A140101 in Sloane's On-Line Encyclopedia of Integer Sequences. These two sequences arise from the Greedy Queens in a spiral problem, which has recently been solved by Dekking, Shallit, and Sloane [4]. The table also appears in the study of a take-away game on three piles by Duchêne and Rigo [7].

We write $\delta(a, b) = b - a$ and $\sigma(a, b) = a + b$. A single preserves either a or b. A double preserves $\delta(a, b)$. A split preserves $\sigma(a, b)$. The following result is immediate.

Lemma 1. *If there is a move from (a, b) to (a', b') then either $\{a, b\} \cap \{a', b'\}$ or $\{\delta(a, b), \sigma(a, b)\} \cap \{\delta(a', b'), \sigma(a', b')\}$ is non-empty.*

Observe that $\{a, b\}$ is a multiset if $a = b$ and that $\{\delta(a, b), \sigma(a, b)\}$ is a multiset if $a = 0$. These are exactly the positions with a move to $(0, 0)$. In other words, these are the positions in N_0. The complement of $P_0 \cup N_0$ consists of all positions (m, n) such that $m > 0$ and $m \neq n$. Each of these can be moved to $(1, 2)$ within two moves. This proves that $P_1 = \{(1, 2)\}$. The folllowing lemma generalizes this observation.

Lemma 2. $P_i = \{(a_i, b_i)\}$.

Proof. Assuming that the statement is true for all P_i up to P_k for a fixed $k \geq 1$. We prove that it is true for P_{k+1}. Let M_k be the set of all (m, n)

with a move to P_i for some $i \leq k$. CLAIM: these are exactly the positions such that one of the following holds:

(i) $\{m, n\}$ intersects $A_k \cup B_k$.
(ii) $\{\delta(m,n), \sigma(m,n)\}$ intersects $\Delta_k \cup \Sigma_k$.

If (i) holds, then m or n occurs as a coordinate in (a_i, b_i) for some $i \leq k$. If $m = b_i$ then $a_i < n$ and reduce n to a_i. If $m = a_i$ and $n > b_i$ then reduce n to a_i. If $m = a_i$ and $b < n_i$ then $\delta(m,n) < \delta_i$. In this case, $n - m$ is equal to δ_j or σ_j for some $j < i$. In the first case, there is a double from (m,n) to P_j. In the second case, there is a split from (m,n) to P_j. If $n = b_i$ and $m > a_i$ then reduce m to a_i. If $n = b_i$ and $m < a_i$ then m is equal to a_j or b_j for some $j < i$. In this case, there is a single from (m,n) to P_j. If $n = a_i$ then $m = a_j$ or $m = b_j$ for some $j < i$ and there exists a single to P_j. This takes care of case (i).

We now consider case (ii) in which $\{\delta(m,n), \sigma(m,n)\}$ intersects some $\{\delta_j, \sigma_j\}$ for $j \leq k$. If $m \leq a_k$ then we are in case (i), which we already settled. We may therefore assume that $m > a_k$. If $\delta(m,n) = \delta_j$ then there is a double to P_j. If $\delta(m,n) = \sigma_j$ then there is a split to P_j. If $\sigma(m,n) = \delta_j$ then $m \leq a_j$ which contradicts our assumption that $m > a_k$. Finally, if $\sigma(m,n) = \sigma_j$, then $n < b_j$ since $m > a_j$, It follows that $\delta(m,n) < \delta_j$ and we have a double to P_i for some $i < j$. All positions that satisfy (i) or (ii) have a move to a P_j for some $j \leq k$. Conversely, Lemma 1 implies that each position that admits such a move satisfies either (i) or (ii). We conclude that the claim holds.

We need to show that P_{k+1} is the unique minimum of the complement of M_k. By the definition of Table 1, P_{k+1} does not satisfy (i) or (ii). Therefore, it is in the complement of M_k. Let (m,n) be any position that in the complement of M_k. We need to show that there exists a sequence of moves from (m,n) to P_{k+1}. We have that $m \geq a_{k+1}$ since $m \notin A_k \cup B_k$. We have that $\delta(m,n) \geq \delta_{k+1}$ since $\delta(m,n) \notin \Delta_k \cup \Sigma_k$. Apply a single to n to reduce $\delta(m,n)$ to δ_{j+1}. Now apply a double to reduce m to a_{k+1}. We have reached P_{k+1}. It is the unique minimum. $\square$

Proof of Theorem 1. The theorem is a direct consequence of Lemma 2 and the fact that the rows A and B can be obtained as difference sequences from the Tribonacci word. This is the main result in [4] and it is also related to Corollary 3.6 in [7]. We will generalize these results in the next sections, in particular see Corollary 1 below.

3. The positions table of the k-bonacci substitution

The k-bonacci substitution θ_k on the alphabet $\{0, \ldots, k-1\}$ is given by

$$\theta_k \colon \begin{cases} \quad j & \mapsto \quad 0\,(j+1), \quad \text{if } j < k-1 \\ k-1 & \mapsto \quad 0 \end{cases}.$$

The k-bonacci word ω^k is the unique fixed point of this substitution. In our analysis, $k > 2$ is fixed and we will often simply write θ and ω instead of θ_k

and ω^k. We start by recalling some well-known properties of the k-bonacci word, before turning to the tables.

Lemma 3. *The letters $j > 0$ are isolated in ω, i.e, each is preceded and followed by a* 0.

Proof. It follows from the definition of θ that each $i > 0$ is preceded by a 0. It has to be succeeded by 0 as well, for the same reason. □

In fact, we can say much more.

Lemma 4. *Starting from the empty word $w_{-1} = \epsilon$, inductively define the palindromes*

$$w_j = w_{j-1} j w_{j-1}.$$

Each $j \in \omega$ is preceded and succeeded by w_{j-1}. In particular, each j occurs in a w_j at location 2^j.

Proof. Each $j + 1$ is created from a j by a subsitution. By our induction, each j occurs in $w_j j w_j$. Therefore, each $j+1$ occurs in $\theta(w_j)0(j+1)\theta(w_j)$. Observe that the final letter of $\theta(w_j)$ is a 1 and therefore it is followed by a 0. We need to prove that $\theta(w_j)0 = w_{j+1}$. This follows from:

$$\begin{aligned} \theta(w_j)0 = \theta(w_{j-1} j w_{j-1})0 &= \theta(w_{j-1})0(j+1)\theta(w_{j-1})0 \\ &= w_j(j+1)w_j = w_{j+1}. \end{aligned} \tag{1}$$

We also obtain by induction that the length of w_{j-1} is equal to $2^j - 1$, which is why j occurs at location 2^j in w_j. □

Expanding equation 1 we find that

$$w_j = \theta^j(0)\theta^{j-1}(0)\cdots\theta(0)0$$

Since w_{j-1} has length $2^j - 1$, $\theta^j(0)$ has length 2^j, and $\theta^j(0)$ ends with a j, we see that

$$w_{j-1} j = \theta^j(0) \tag{2}$$

Lemma 5. *Each w_j is a prefix or a suffix of a w_{j+1}.*

Proof. By induction. Assume that the statement is true for $j-1$. Since w_{j-1} is followed by 0, we may even assume that $w_{j-1}0$ is a prefix or a suffix of $w_j 0$. By the previous lemma, each w_j is created from $w_{j-1}0$ by substitution. By our assumption, $w_{j-1}0$ is a prefix or a suffix of $w_j 0$. Therefore, w_j is a prefix or a suffix of $\theta(w_j 0) = w_{j+1}$. □

Note that it can happen that w_j is both a prefix and a suffix. Since $w_0 w_0 = 00$ occurs in ω, we have that 0 occurs as a prefix and a suffix of w_1 in $\theta(0)\theta(0)0 = 01\mathbf{0}10$. Applying the substitution once more, we find that w_1 is a prefix and a suffix in $\theta^2(0)\theta^2(0)\theta(0)0 = 0102\mathbf{010}2010$, etc.

Lemma 6. *For all $i \in \{0, \ldots, k-1\}$ the letter $j > 0$ occurs exactly once in $\theta^{j+1}(i)$ at location 2^j.*

Proof. Thie follows from the fact that

$$\theta^{j+1}(i) = \theta^j(0)\theta^j(i+1)$$

where we agree that $i+1$ is the empty word if $i = k-1$. Equation 2 implies that j occurs at location 2^j. We only need to prove that it does not occur in $\theta^j(i+1)$. By induction we may assume that $j-1$ does not occur in $\theta^{j-1}(i+1)$. This immediately implies that j does not occur in $\theta^j(i+1)$. □

Lemma 6 implies that the n-th occurrence of j in $\omega = \omega_1\omega_2\omega_3\ldots$ is in the n-th word of $\theta^{j+1}(\omega_1)\theta^{j+1}(\omega_2)\theta^{j+1}(\omega_3)\ldots$. For a letter j we will say that the distance between consecutive positions of j are *steps*. The steps of j can be computed from the k-bonacci word by applying θ^{j+1}.

The positions table for the Quadribonacci word is given below. Each row X^j contains the locations of j in ω. The steps between the letter j

ω^4	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	0
X^0	1	3	5	7	9	11	13	15	16	18	20	22	24	26	28	30
X^1	2	6	10	14	17	21	25	29	31	35	39	43	46	50	54	58
X^2	4	12	19	27	33	41	48	56	60	68	75	83	89	97	104	112
X^3	8	23	37	52	64	79	93	108	116	131	145	160	172	187	201	216

TABLE 2. The positions table for the Quadribonacci word

are the differences between consecutive entries in X^j. The steps between columns are vectors and one quickly verifies that each such step is one of the four vectors $(2,4,8,15), (2,4,7,14), (2,3,6,12), (1,2,4,8)$, depending on the corresponding letter in the k-bonacci word. For instance, the difference between the fourth and the fifth column (and the twelfth and the thirteenth) is $(2,3,6,12)$ because 2 is the fourth letter (and twelfth) in ω^4. For the positions table of the general k-bonacci word, the result is as follows.

Lemma 7. *Let $\ell^j(i)$ be the length of $\theta^{j+1}(i)$. The positions table can be generated from the vector valued substitution*

$$\nu\colon i \mapsto \begin{bmatrix} \ell^0(i) \\ \ell^1(i) \\ \vdots \\ \ell^{k-1}(i) \end{bmatrix} \tag{3}$$

The initial column of the table contains increasing powers of 2, starting from 2^0, and the $n+1$-th column is generated from the n-th column by adding $\nu(\omega_n^k)$.

Proof. By Lemma 6 the letter j first occurs in $\theta^{j+1}(\omega_0^k)$ at position 2^j. That is why the first column of the table contains the powers of two. Since j occurs at the same location in $\theta^{j+1}(i)$, independent of i, each row X^j has step sizes $\ell^j(i)$. □

The length $\ell^j(i)$ is minimal if $i = k-1$, in which case it is equal to 2^j. If $i \neq k-1$ then $\ell(i) > 2^j$. This means that 2^j is the minimal step between consecutive elements in X^j. The minimal step occurs at columns that are headed by the letter $k-1$ in ω. For instance, in the Quadrobonacci table, the step at the eighth column is minimal because the eighth letter in ω^4 is a 3.

Lemma 8. *Both $X^{j+1} - 2^j$ and $X^{j+1} + 2^j$ are subsets of X^j and their union is equal to X^j. An element of X^j can be written as $m + 2^j$ and $n - 2^j$ exactly if m is taken from a $k-1$-column in the positions table, and n is its successor.*

Proof. Suppose that $j+1$ occurs at location $m \in X^{j+1}$. By Lemma 4 we have that both $m - 2^j$ and $m + 2^j$ are elements of X^j. Hence $X^{j+1} - 2^j$ and $X^{j+1} + 2^j$ are subsets of X^j.

Say $h \in X^j$, i.e., j occurs at location h in ω. Each j occurs in a w_j, which is either a prefix or a suffix of a w_{j+1}. Within w_j, j occurs in the middle location 2^j. Since w_j is a palindrome, the distance between j and $j+1$ in $w_{j+1} = w_j j + 1 w_j$ is the same for the prefix and the suffix. That distance is equal to 2^j. Therefore, $h \in X^{j+1} \pm 2^j$.

A number in X^j can be written as a sum and a difference if it occurs in a step of size 2^{j+1} in X^{j+1}. These steps occur at the $k-1$-columns. $\square$

4. The difference table

The rows in the positions table form a partition of $\mathbb{N}$ by definition. Remarkably, this also holds for the rows of the *difference table*, which has rows $\Delta^j = X^{j+1} - X^j$ that we put alongside the k-bonacci word, as illustrated below. That is, these rows Δ^j again form a partition of $\mathbb{N}$.

ω^4	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	0
Δ^0	1	3	5	7	8	10	12	14	15	17	19	21	22	24	26	28
Δ^1	2	6	9	13	16	20	23	27	29	33	36	40	43	47	50	54
Δ^2	4	11	18	25	31	38	45	52	56	63	70	77	83	90	97	104

TABLE 3. The difference table for the Quadribonacci word

It follows from Lemma 7 and Equation 3 that the difference table can be derived from ω. The step between columns is

$$\begin{bmatrix} \ell^1(i) - \ell^0(i) \\ \ell^2(i) - \ell^1(i) \\ \vdots \\ \ell^{k-1}(i) - \ell^{k-2}(i) \end{bmatrix} \tag{4}$$

Lemma 9. *For each $1 \leq h \leq k-1$ and each letter i*

$$\ell^h(i) = 2\ell^{h-1}(i) - \delta^{k-1}_{h+i},$$

where δ^i_j is Kronecker's delta.

Proof. We defined $\ell^h(i)$ as the word length after $h+1$ substitutions starting from i. Each letter is doubled by our substitution unless it is the final letter. Therefore, the equation says that $\theta^h(i)$ contains no final letter unless $k-1 = h+i$, in which case it contains one final letter. The substitution raises each letter by one, adding a zero. If we start from i, the only letter that reaches the final letter is produced by $i \to i+1 \to \cdots$. It reaches the final letter after $k-1-i$ substitutions. Hence, we have one non-doubling letter if $h = k-1-i$. $\square$

We already observed that $\ell^h(i) > \ell^h(k-1)$ if $i < k-1$. Therefore $\ell^h(i) - \ell^{h-1}(i) \geq \ell^{h-1}(k-1) = 2^{h-1}$. It follows that the minimal step in Δ^{h-1} is equal to 2^{h-1}. Again, the minimal step occurs at the columns marked by $k-1$. However, the minimal steps do not exclusively occur in these columns. For instance, in the Quadribonacci table the minimal step in the first row is 1. It occurs in columns marked by 2 and 3.

Lemma 10. *$\Delta^{j+1} - 2^j$ and $\Delta^{j+1} + 2^j$ are subsets of Δ^j and their union is equal to Δ^j. An element of Δ^j can be written as $m + 2^j$ and $n - 2^j$ exactly if m is taken from a $k-1$-column in the positions table, and n is its successor.*

Proof. By Lemma 5 we have

$$\Delta^j = X^{j+1} - X^j = \left(X^{j+2} - 2^{j+1} \cup X^{j+2} + 2^{j+1}\right) - \left(X^{j+1} - 2^j \cup X^{j+1} + 2^j\right)$$

By the alternating property, this is equal to

$$\left(X^{j+2} - X^j - 2^{j+1} + 2^j\right) \cup \left(X^{j+2} - X^{j+1} + 2^{j+1} - 2^j\right)$$

which is $\Delta^{j+1} - 2^j \cup \Delta^{j+1} + 2^j$ $\square$

Lemma 11. *The rows in the difference table are disjoint as sets. That is,*

$$\Delta^i \cap \Delta^j = \emptyset \text{ if } i \neq j.$$

Proof. Suppose $j > i$. By iterating Lemma 10 we find that each element of Δ^i is in some $\Delta^j \pm 2^{j-1} \pm \ldots \pm 2^i$. The minimal step size in Δ^j is 2^j and $2^{j-1} + \ldots + 1 < 2^j$. Therefore, Δ^i and Δ^j are disjoint. $\square$

Theorem 2. *The rows in the difference table form a partition of $\mathbb{N}$.*

Proof. We only need to prove that the rows cover $\mathbb{N}$. By iterating Lemma 10 each Δ^i is the union of all $\Delta^{k-2} \pm 2^{k-2} \pm \ldots \pm 2^i$, with Δ^{k-2} the bottom row of our table. The union of the rows is equal to the union of $\Delta^{k-2} + n$ for all integers n that can be written as $\pm 2^{k-2} \pm \ldots \pm 2^i$ for some i and some choice of the signs. Here we take $n = 0$ if $i = k-2$. It is not hard to verify that each $n \in \{-2^{k-2}+1, \ldots, 2^{k-2}-1\}$ admits such an expansion. Hence, it suffices to prove that the maximal step in Δ^{k-2} is $2^{k-1} - 1$.

By Lemma 9 and Equation 4 the steps in Δ^{k-2} are given by

$$\ell^{k-1}(i) - \ell^{k-2}(i) = \ell^{k-2}(i) - \delta^i_0.$$

This is maximal if $i = 0$ or $i = 1$, when it is indeed equal to $2^{k-1} - 1$. $\square$

5. The double-difference table

We continue by differencing the differences

$$d\Delta^i = \Delta^{i+1} - \Delta^i$$

and collect these as rows in a *double-difference table*. For good measure, we also add a bottom row containing the sum

$$\Sigma = \Delta^0 + \ldots + \Delta^{k-2}.$$

We call this the *sum row* and we call the other rows the *difference rows*. We add the k-bonacci word ω as a headline, since the increments of the columns are ruled by this word. If $k = 3$, i.e., Tribonacci, we have only one difference row and one sum row. It turns out that these are the Δ and Σ of Table 1.

ω^4	0	1	0	2	0	1	0	3	0	1	0	2	0	1	0	0
$d\Delta^0$	1	3	4	6	8	10	11	13	14	16	17	19	21	23	24	26
$d\Delta^1$	2	5	9	12	15	18	22	25	27	30	34	37	40	43	47	50
Σ	7	20	32	45	55	68	80	93	100	113	125	138	148	161	173	193

Table 4. The double-difference table for the Quadribonacci word

Lemma 9 implies that the step between columns is

$$\tag{5} \begin{bmatrix} \ell^2(i) - 2\ell^1(i) + \ell^0(i) \\ \vdots \\ \ell^{k-1}(i) - 2\ell^{k-2}(i) + \ell^{k-3}(i) \\ \ell^{k-1}(i) - \ell^0(i) \end{bmatrix} = \begin{bmatrix} \ell^0(i) & - & \delta^{k-1}_{2+i} \\ & \vdots & \\ \ell^{k-3}(i) & - & \delta^{k-1}_{k-1+i} \\ \ell^{k-1}(i) & - & \ell^0(i) \end{bmatrix}$$

The minimum step occurs at $(k-1)$-columns, where $\ell^h(i) - \delta^{k-1}_{h+2+i} = \ell^h(k-1) = 2^h$ and $\ell^{k-1}(k-1) - \ell^0(k-1) = 2^{k-1} - 1$. The maximum step in the final difference row is $\ell^{k-3}(i) - \delta^{k-1}_{k-1+i}$. If $i \in \{0, 1\}$ then the length of $\theta^h(i)$ is doubled at each step up to $k-2$. If $i > 1$ then at some step it is doubled minus one (see Lemma 9). Therefore $\ell^{k-3}(i)$ has maximal value 2^{k-3} at $i = 0, 1$. Now $\delta^{k-1}_{k-1+i} = 1$ if $i = 0$ and it is 1 at $i = 1$. Therefore, the step $\ell^{k-3}(i) - \delta^{k-1}_{k-1+i}$ is maximal in the columns marked by 1. Let $M \subset d\Delta^{k-3}$ be the subsequence that is taken from these 1-columns.

The steps in M are sums of steps in Δ^{k-3} along the intermediate letters between two 1's. In other words, we sum the steps over the *return word* [9]: the j-th word in ω that starts with a 1 and leads up to, but does not include, the $j+1$-th 1. Recall that the 1's in ω are the second letters in the sequence $\theta^2(\omega_1)\theta^2(\omega_2)\theta^2(\omega_3)\ldots$. If $\omega_j = i < k-2$ then the j-th return word is $10(i+2)0$. If $\omega_j = k-2$ then the return word is 100 and if $\omega_j = k-1$ then it is 10.

Lemma 12. *The j-th step in M has length $\ell^{k-1}(\omega_j) - \ell^0(\omega_j)$*

Proof. Suppose the letters in the j-th return word are $10(i+2)0$ for $i < k-2$. The sum of the steps is

$$\ell^{k-3}(1) + \ell^{k-3}(0) - 1 + \ell^{k-3}(i+1) + \ell^{k-3}(0) - 1$$

If we agree that $\ell(w)$ is the sum of ℓ over the letters in the word, and if we denote the j-th return word by v_j, then we can write this as

$$\ell^{k-3}(v_j) - 2 = \ell^{k-3}(v_j) - \ell^0(i)$$

So far, we have used $i < k-2$ but our notation carries over to $k-2$ and $k-1$. By definition $\ell^{k-3}(w)$ is the length of $\theta^{k-2}(w)$ and $v_j = \theta^2(i) = \theta^2(\omega_j)$. We find that the sum of the steps is $\ell^{k-3}(\theta^2(\omega_j)) - \ell^0(\omega_j)$. $\square$

We now repeat Lemmas 10 and 11 for the double-difference table.

Lemma 13. *$d\Delta^{j+1} - 2^j$ and $d\Delta^{j+1} + 2^j$ are subsets of $d\Delta^j$ and their union is equal to $d\Delta^j$. An element of $d\Delta^j$ can be written as $m + 2^j$ and $n - 2^j$ exactly if m is taken from a $k-1$-column in the positions table.*

Proof. The same as the proof of Lemma 10, with minor editing.

$$d\Delta^j = \Delta^{j+1} - \Delta^j = \left(\Delta^{j+2} - 2^{j+1} \cup \Delta^{j+2} + 2^{j+1}\right) - \left(\Delta^{j+1} - 2^j \cup \Delta^{j+1} + 2^j\right)$$

By the alternating property, this is equal to

$$\left(\Delta^{j+2} - \Delta^j - 2^{j+1} + 2^j\right) \cup \left(\Delta^{j+2} - \Delta^{j+1} + 2^{j+1} - 2^j\right)$$

which is $d\Delta^{j+1} - 2^j \cup d\Delta^{j+1} + 2^j$ $\square$

Lemma 14. *The difference rows in the double-difference table are disjoint as sets*

$$d\Delta^i \cap d\Delta^j = \emptyset \text{ if } i \neq j$$

Proof. Suppose $j > i$. By iterating Lemma 10 we find that each element of Δ^i is in some $d\Delta^j \pm 2^{j-1} \pm \ldots \pm 2^i$. The minimal step size in $d\Delta^j$ is 2^j and $2^{j-1} + \ldots + 1 < 2^j$. Therefore, $d\Delta^i$ and $d\Delta^j$ are disjoint. $\square$

Lemma 15. *Let $S \subset \mathbb{N}$ be the set $S = M + 2^{k-3}$. The difference rows in the difference table form a partition of $\mathbb{N} \setminus S$.*

Proof. We only need to prove that the difference rows cover $\mathbb{N} \setminus S$. By iterating Lemma 13 each $d\Delta^i$ is the union of all $d\Delta^{k-3} \pm 2^{k-4} \pm \ldots \pm 2^i$, with $d\Delta^{k-3}$ the final difference row of our table. The union of the rows is equal to the union of $d\Delta^{k-3} + n$ for all integers n that can be written as $\pm 2^{k-3} \pm \ldots \pm 2^i$ for some i and some choice of the signs. As before we have $n \in \{-2^{k-4}+1, \ldots, 2^{k-4}-1\}$, which is a set of $2^{k-3}-1$ consecutive elements. The maximal step in $d\Delta^{k-3}$ is 2^{k-2}, which occurs in the 1-columns. All other elements are covered. $\square$

Theorem 3. *The rows in the double-difference table form a partition of $\mathbb{N}$.*

Proof. By Lemma 12 the sequences S and Σ have equal steps. We only need to show that they have the same initial element. The initial element of Σ is the sum of the initial elements of the difference sequences. This is $\ell^{k-1}(0)-\ell^0(0)=2^{k-1}-1$. The initial element of M is the second element of $d\Delta^{k-3}$. The first element is 2^{k-3} and the second element is $2^{k-3}+\ell^{k-3}(0)-1=2^{k-3}+2^{k-2}-1$. Finally, the initial element of S is $2^{k+3}+2^{k-3}+2^{k-2}-1=2^{k-1}-1$. Indeed, S and Σ have the same initial element. □

6. Generating the Quadribonacci table from a mex-rule

We defined k-bonacci tables and showed that their rows partition $\mathbb{N}$. We now consider a method for generating these tables. A mex-rule for the positions table of the Tribonacci word appeared as Theorem 3.1 in [7]. We derive a mex-rule for the Quadribonacci word.

We denote the elements of the difference sequence Δ^j by $a_1^j, a_2^j, \ldots$ and the elements of $d\Delta^j$ by $b_1^j, b_2^j, \ldots$. Recall our notation that X_i contains the first i entries of the sequence X.

Lemma 16. $a_{i+1}^0 = \operatorname{mex}\left(\Delta_i^0 \cup \Delta_i^1 \cup \cdots \cup \Delta_i^{k-2}\right)$.

Proof. The rows partition $\mathbb{N}$ and therefore $\operatorname{mex}\left(\Delta_i^0 \cup \Delta_i^1 \cup \cdots \cup \Delta_i^{k-2}\right)$ occurs in one of the rows. The columns are strictly increasing, therefore this element occurs in the first row. It has to be the next element in that row, since rows are increasing. □

By the same argument we find

Lemma 17. $b_{i+1}^0 = \operatorname{mex}\left(d\Delta_i^0 \cup d\Delta_i^1 \cup \cdots \cup d\Delta_i^{k-3} \cup \Sigma\right)$.

We can now complete the proof of Theorem 1.

Corollary 1. *The rows in Table 1 are identical to the difference table of the Tribonacci word. The header and the footer are identical to the double-difference table.*

Proof. If $k=3$ there are only two rows in the difference table. The previous two lemmas suffice to construct these tables. We have $a_i^2 = a_i^1 + b_i^1$ and $b_i^2 = a_i^1 + a_i^2$. These are the mex-rules that generate Table 1. □

For the Tribonacci word, we can thus generate the columns of the tables one step at a time. We will now describe a column generation for the Quadribonacci word, but we do not have such a procedure for k-bonacci words with $k>4$.

Lemma 18. *The bottom row of the k-bonacci positions table can be written as a sum*

$$X^{k-1} = E + X^0 + \ldots + X^{k-1}$$

where E denotes the enumerating sequence $1,2,3,\ldots$.

Proof. This is true for the initial column. Each next column is an increment by the vector in Equation 3. We need to show that

$$\ell^{k-1}(i) = 1 + \ell^0(i) + \ldots + \ell^{j-2}(i)$$

This is a consequence of the following observation. Suppose you start from 1 and double each time, except once, when you double and subtract one. Then the final number is the sum of the other numbers. We leave the verification to the reader. The equality above follows from the observation for $i < k-1$. In that case, we have $\ell^0(i) = 2$ and each next $\ell^j(i)$ is doubled, unless $k-1$ appears in $\theta^j(i)$, which happens once. If $i = k-1$ then $\ell^j(k-1) = 2^j$ and again the equation holds. □

According to this lemma it suffices to generate the first three rows of the Quadribonacci table, and compute the fourth row as a sum. These three rows can be derived from the first two rows of the difference table, which can be derived from the first row of the double-difference table, which follows from the mex rule. That is the idea behind Theorem 4 below, in which we generate all three tables of the Quadribonacci word simultaneously.

We write x_i^j for the elements of the positions table.

Theorem 4. *The following rules generate the three tables for the Quadribonacci word:*

$$\begin{aligned}
a_{i+1}^0 &= \operatorname{mex}\left(\Delta_i^0 \cup \Delta_i^1 \cup \Delta_i^2\right)\\
b_{i+1}^0 &= \operatorname{mex}\left(d\Delta_i^0 \cup d\Delta_i^1 \cup \Sigma_i\right)\\
x_{i+1}^0 &= \operatorname{mex}\left(X_i^0 \cup X_i^1 \cup X_i^2 \cup X_i^3\right)\\
a_{i+1}^1 &= a_{i+1}^0 + b_{i+1}^0\\
x_{i+1}^1 &= x_{i+1}^0 + a_{i+1}^0\\
x_{i+1}^2 &= x_{i+1}^1 + a_{i+1}^1\\
x_{i+1}^3 &= x_{i+1}^0 + x_{i+1}^1 + x_{i+1}^2 + i + 1\\
a_{i+1}^2 &= x_{i+1}^3 - x_{i+1}^2\\
b_{i+1}^2 &= a_{i+1}^1 + a_{i+1}^2 + a_{i+1}^3
\end{aligned}$$

Proof. The first and second equation follow from Lemma 16 and 17. The third equation can be derived in an equivalent manner. The other equations follow from Lemma 17 and the definition of the difference tables. □

Duchêne and Rigo used the difference table to derive a column generating mex-rule for the Tribonacci table. We used the double-difference table to do the same for the Quadribonacci table. Perhaps there is a way to use triple-differences for the Cinquibonacci table, but this remains to be solved.

7. Other games and other tables

Splythoff's Nim is a simple modification of Wythoff's Nim. What if we modify Splythoff's Nim and also allow a split after a single? Without going into a full analysis of this game we take a first look at its P-positions, along with their sums and differences. The difference sequence is no longer

Δ	1	2	4	5	7	6	8	9	11	12	13	15	16	17	18	20	21
A	1	4	5	7	8	11	13	16	18	20	22	24	26	27	30	31	33
B	2	6	9	12	15	17	21	25	29	32	35	39	42	44	48	51	54
Σ	3	10	14	19	23	28	34	41	47	52	57	63	68	71	78	82	87

increasing and an analysis of this game appears to be difficult.

An appealing feature of Splythoff's Nim is the mex-rule describing its P-positions. The mex operation is applied to A, B and Δ, Σ separately, and so we get separate partitions of $\mathbb{N}$. What if we apply the mex operation to all the rows, so that together they form a partition of $\mathbb{N}$? One obvious way to do this is by putting the smallest element in $\delta \in \Delta$, and then selecting the next smallest element $a \in A$ such that $a + \delta \notin B$. As described by the following mex-rule.

$$\begin{aligned}
\delta_{i+1} &= \operatorname{mex}(\Delta_i \cup A_i \cup B_i \cup \Sigma_i) \\
a_{i+1} &= \operatorname{mex}(\Delta_{i+1} \cup A_i \cup B_i \cup \Sigma_i \cup B_i - \delta_{i+1}) \\
b_{i+1} &= a_{i+1} + \delta_{i+1} \\
\sigma_{i+1} &= a_{i+1} + b_{i+1}
\end{aligned}$$

The resulting table appears to be a new object, which is not contained in the OEIS. Is there a take-away game that is associated to this table?

Δ	1	4	7	9	12	14	18	21	24	27	29	32	34	36	40	42
A	2	6	8	11	13	17	19	22	26	28	30	33	35	39	41	44
B	3	10	15	20	25	31	37	43	50	55	59	65	69	75	81	86
Σ	5	16	23	31	38	48	56	65	76	83	89	98	104	114	122	130

8. Epilogue

Wythoff's Nim has been modified in many different ways and some of these modifications are similar to ours. Larsson [14] introduced a modification in which one is allowed to remove counters from both piles in fixed proportions. His analysis of the game involves 'splitting pairs', but this notion of splitting is different from ours. Duchêne and Rigo [8] devised *invariant games*; a wide-ranging generalization of Wythoff's Nim. Larsson, Hegarty, and Fraenkel showed that for every pair of complementary Beatty sequences, there exists an invariant game which has P-positions along these sequences. This work has been extended to non-homogeneous Beatty sequences in [3]. Splythoff's

Nim is not an invariant game and its P-positions are close to, but not equal to, complementary Beatty sequences [4].

The relation between Wythoff's Nim and Fibonacci numeration is well known and has been extended to other Wythoff-like games by Fraenkel [11, 13]. Our game and its tables are related to Tribonacci numeration. Table 2 and methods for generating the entries go back to Carlitz, et al. [2]. Their results were simplified and derived from the Tribonacci substitution by Barucci, et al. [1].

We did not consider the Sprague-Grundy values of Splythoff's Nim. It is much harder to determine these values and even for Wythoff's Nim this remains an object of study [17]. Duchêne et al. [6] recently developed a profound method for studying the complexity of the Sprague-Grundy function for a general class of games. Interestingly, though unrelated to our work, they use merging of positions instead of our splitting.

Finally, let us mention that Willem Wythoff did not only contribute a game but also a symbol to mathematics [10].

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