Enumeration of Flats of the Extended Catalan and Shi Arrangements with Species

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The number of flats of a hyperplane arrangement is considered as a generalization of the Bell number and the Stirling number of the second kind. In 1998, Robert Gill gave the exponential generating function of the number of flats of the extended Catalan arrangements, using species. In this article, we introduce the species of flats of the extended Catalan and Shi arrangements and they are given by iterated substitution of species of sets and lists. Moreover, we enumerate the flats of these arrangements in terms of infinite matrices.

Keywords: hyperplane arrangement, Shi arrangement, Catalan arrangement, intersection poset, species, gain graph, set partition, Bell number, Stirling number, Lah number

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Contents

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1. Introduction

A **hyperplane arrangement** is a finite collection of affine subspaces of codimension 1 in an affine space over an arbitrary field $\mathbb{K}$. In spite of its simple definition, arrangements are investigated in a variety of ways, such as topological, algebrogeometric, and combinatorial aspects. A standard reference for hyperplane arrangements is [15].

Given an arrangement $\mathcal{A}$, let $L(\mathcal{A})$ denote the set of nonempty intersections of hyperplanes in $\mathcal{A}$. Note that the ambient space is a member of $L(\mathcal{A})$ since it is regarded as the intersection over the empty set. We call an element of $L(\mathcal{A})$ a **flat**. Define a partial order on $L(\mathcal{A})$ by the reverse inclusion, that is, $X \leq Y \Leftrightarrow X \supseteq Y$ for $X, Y \in L(\mathcal{A})$. We call $L(\mathcal{A})$ the **intersection poset** of $\mathcal{A}$. This poset plays an important role in the theory of hyperplane arrangements. For each nonnegative integer k, let

$$L_k(\mathcal{A}) \coloneqq \{\, X \in L(\mathcal{A}) \mid \dim X = k \,\}.$$

When $\mathcal{A}$ is **central**, that is, the intersection of all hyperplanes in $\mathcal{A}$ is nonempty, the poset $L(\mathcal{A})$ is a geometric lattice.

A **set partition** of a finite set V is a collection $\pi = \{B_1, \dots, B_k\}$ of nonempty subsets $B_i \subseteq V$ such that $B_i \cap B_j = \varnothing$ for $i \neq j$ and $\bigcup_{i=1}^{k} B_i = V$. Each B_i is called a **block** of π. Let π, π' be set partitions of V. Define a partial order $\pi \leq \pi'$ if each block of π is a subset of some block of π'. We also say that π **refines** π' if $\pi \leq \pi'$. Then the collection of the set partitions of V forms a lattice.

For a positive integer n, let $[n]$ be the set $\{1, \dots, n\}$ and $[0]$ the empty set. The number of set partitions of $[n]$ is called the **Bell number**, denoted by $B(n)$, where $B(0) = 1$. The number of set partitions of $[n]$ into k blocks is called the **Stirling number of the second kind**, denoted by $S(n, k)$, where $S(0, 0) = 1$.

Let $x_1, \dots, x_n$ denote coordinates of $\mathbb{R}^n$ and $\mathcal{B}_n$ the n-dimensional **braid arrangement** (also known as the **Weyl arrangement of type** A_{n-1}), which consists of hyperplanes $\{x_i - x_j = 0\}$ in $\mathbb{R}^n$ for $1 \leq i < j \leq n$. It is well known that there exists an isomorphism from $L(\mathcal{B}_n)$ to the lattice of set partitions of $[n]$ which sends a k-dimensional flat to a set partition into k blocks. In other words, $|L(\mathcal{B}_n)| = B(n)$ and $|L_k(\mathcal{B}_n)| = S(n, k)$. The numbers of flats and k-dimensional flats of an arrangement is considered to be generalizations of the Bell numbers and the Stirling numbers.

Define the **extended Catalan arrangement** $\mathcal{C}_n^m$ and the **extended Shi arrangement** $\mathcal{S}_n^m$ in $\mathbb{R}^n$ as follows.

$$\begin{aligned}\mathcal{C}_n^m &\coloneqq \{\, \{x_i - x_j = a\} \mid 1 \le i < j \le n, -m \le a \le m \,\}, \quad (m \ge 0),\\ \mathcal{S}_n^m &\coloneqq \{\, \{x_i - x_j = a\} \mid 1 \le i < j \le n, 1-m \le a \le m \,\}, \quad (m \ge 1).\end{aligned}$$

Note that $\mathcal{C}_n^0 = \mathcal{B}_n$ for every nonnegative integer n and $L_0(\mathcal{C}_n^m) = L_0(\mathcal{S}_n^m) = \varnothing$ unless $n = 0$. Gill [10] investigated the intersection posets of the extended Catalan arrangements. First Gill determined the number of maximal elements of the poset $L(\mathcal{C}_n^m)$ as follows.

Theorem 1.1 (Gill [10, Theorem 1])**.** *Let m be a nonnegative integer. Then*

$$\sum_{n=1}^{\infty} |L_1(\mathcal{C}_n^m)| \frac{x^n}{n!} = \frac{e^x - 1}{1 - m(e^x - 1)}.$$

In Gill's works in [10], the use of species, which were initiated by Joyal [13], is a noteworthy point. Standard references for species are [3, 4]. Let $\mathbb{B}$ denote the category of finite sets and bijections and $\mathbb{E}$ the category of finite sets and maps. A **species, or $\mathbb{B}$-species,** is a functor $\mathsf{F}\colon \mathbb{B} \to \mathbb{E}$. The value of a species F at a finite set V is denoted by $\mathsf{F}[V]$. Moreover, we write simply $\mathsf{F}[n]$ instead of $\mathsf{F}[[n]]$. The symbol $\mathsf{F}(x)$ is used for the exponential generating function

$$\mathsf{F}(x) \coloneqq \sum_{n=0}^{\infty} |\mathsf{F}[n]| \frac{x^n}{n!}.$$

Example 1.2. Let E denote the **species of sets**. Namely $\mathsf{E}[V] \coloneqq \{V\}$. Then we have $|\mathsf{E}[n]| = 1$ for every n and thus $\mathsf{E}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

Example 1.3. Let L denote the **species of lists**. Namely

$$\mathsf{L}[V] \coloneqq \{\, (v_1, \dots, v_n) \mid n = |V| \text{ and } V = \{v_1, \dots, v_n\} \,\}.$$

We have $|\mathsf{L}[n]| = n!$ for every n and hence $\mathsf{L}(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

Making use of species enables us to calculate generating functions systematically. Namely we can define operations for species such as sum, product, and so on. These operations are compatible with the corresponding operations for the generating functions. Hence we may say that species is a refinement of the exponential generating function.

Definition 1.4. For species F and G, we define the **sum** $\mathsf{F} + \mathsf{G}$ by $(\mathsf{F} + \mathsf{G})[V] \coloneqq \mathsf{F}[V] \sqcup \mathsf{G}[V]$, where $\sqcup$ means the disjoint union.

Then we have $(\mathsf{F} + \mathsf{G})(x) = \mathsf{F}(x) + \mathsf{G}(x)$.

Definition 1.5. Let

$$\mathsf{F}_k[V] := \begin{cases} \mathsf{F}[V] & \text{if } |V| = k, \\ \varnothing & \text{otherwise.} \end{cases}$$

for every species F. The species F has a canonical decomposition $\mathsf{F} = \sum_{k=0}^{\infty} \mathsf{F}_k$. Furthermore, we define a species $\mathsf{F}_+ := \sum_{k=1}^{\infty} \mathsf{F}_k$.

Then $\mathsf{F}_+[\varnothing] = \varnothing$ and $\mathsf{F}_+(x) = \mathsf{F}(x) - \mathsf{F}(0)$.

Example 1.6.

$$\mathsf{E}_+(x) = e^x - 1, \quad \mathsf{E}_k(x) = \frac{x^k}{k!}, \quad \text{and} \quad \mathsf{L}_+(x) = \frac{1}{1-x} - 1 = \frac{x}{1-x}.$$

In this article, we frequently use the operation $\mathsf{F} \circ \mathsf{G}$ of species, called **substitution** (or **composition**).

Definition 1.7. Let F and G be species with $\mathsf{G}[\varnothing] = \varnothing$. Define

$$(\mathsf{F} \circ \mathsf{G})[V] := \bigsqcup_{\pi \in \Pi[V]} \left(\mathsf{F}[\pi] \times \prod_{B \in \pi} \mathsf{G}[B] \right),$$

where Π denotes the **species of set partitions**.

Substitution of species corresponds to substitution of generating functions, that is, $(\mathsf{F} \circ \mathsf{G})(x) = \mathsf{F}(\mathsf{G}(x))$. It is a functional tool for computing the generating function and explains why the exponential function e^x sometimes appears in the generating function.

Example 1.8. By definition, the exponential generating function of the Bell numbers is given by $\Pi(x)$. The species of set partitions Π coincides with the species $\mathsf{E} \circ \mathsf{E}_+$ (See Example 2.1 for details). Hence we have the following.

$$\Pi(x) = (\mathsf{E} \circ \mathsf{E}_+)(x) = \exp\left(e^x - 1\right).$$

Example 1.9. Given a species F, let $\mathsf{F}_+^{\circ m}$ be the m-times iterated self-substitution of F_+. For the species L of lists, we have

$$\mathsf{L}_+^{\circ 2}(x) = (\mathsf{L}_+ \circ \mathsf{L}_+)(x) = \frac{\frac{x}{1-x}}{1 - \frac{x}{1-x}} = \frac{x}{1-2x}.$$

More generally, one can show easily that

$$\mathsf{L}_+^{\circ m}(x) = \frac{x}{1-mx}.$$

The construction of the extended Catalan arrangement $\mathcal{C}_n^m$ is functorial in n. Namely, for every finite set V, we may construct the corresponding Catalan arrangement in the vector space $\mathbb{R}^V = \mathrm{Map}(V, \mathbb{R})$ and hence there exist species $L\mathcal{C}^m$ and $L_k\mathcal{C}^m$ such that $L\mathcal{C}^m[n] = L(\mathcal{C}_n^m)$ and $L_k\mathcal{C}^m[n] = L_k(\mathcal{C}_n^m)$. Gill's second theorem is as follows.

Theorem 1.10 (Gill [10, Theorem 2])**.** *For every $m \geq 0$, the equality $L\mathcal{C}^m = \mathsf{E} \circ L_1\mathcal{C}^m$ holds. Moreover, the bivariate generating function of $|L_k(\mathcal{C}_n^m)|$ is given by*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} |L_k(\mathcal{C}_n^m)| \, t^k \frac{x^n}{n!} = \exp\left(t \frac{e^x - 1}{1 - m(e^x - 1)} \right).$$

We write $\mathsf{F} = \mathsf{G}$ if there exists a natural isomorphism between species F and G. In this paper, we will improve Gill's results in terms of species as follows.

Theorem 1.11. *Let m and k be nonnegative integers. Then*

$$L_k\mathcal{C}^m = \mathsf{E}_k \circ \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+ \quad \text{and} \quad L\mathcal{C}^m = \mathsf{E} \circ \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+.$$

This theorem together with Example 1.6, Example 1.8, and Example 1.9 leads to the following corollary. Note that one can deduce Theorem 1.1 and Theorem 1.10 from this corollary.

Corollary 1.12.

$$L_k\mathcal{C}^m(x) = \sum_{n=0}^{\infty} |L_k(\mathcal{C}_n^m)| \frac{x^n}{n!} = \frac{1}{k!} \left(\frac{e^x - 1}{1 - m(e^x - 1)} \right)^k,$$
$$L\mathcal{C}^m(x) = \sum_{n=0}^{\infty} |L(\mathcal{C}_n^m)| \frac{x^n}{n!} = \exp\left(\frac{e^x - 1}{1 - m(e^x - 1)} \right).$$

We will also give an analog for the extended Shi arrangement $\mathcal{S}_n^m$. However, the construction of $\mathcal{S}_n^m$ cannot be regarded as a functor on $\mathbb{B}$ since the construction requires the linear order on the set $[n]$. For this reason we consider $\mathbb{L}$-species, that is, a functor $\mathsf{F} \colon \mathbb{L} \to \mathbb{E}$, where $\mathbb{L}$ denotes the category of linearly ordered finite sets and order-preserving bijections. Note that every $\mathbb{B}$-species can be considered as an $\mathbb{L}$-species via the forgetful functor from $\mathbb{L}$ to $\mathbb{B}$.

Let $L\mathcal{S}^m$ and $L_k\mathcal{S}^m$ denote the $\mathbb{L}$-species such that $L\mathcal{S}^m[n] = L(\mathcal{S}_n^m)$ and $L_k\mathcal{S}^m[n] = L_k(\mathcal{S}_n^m)$. The following is another main result of this article.

Theorem 1.13. *Let m and k be nonnegative integers. Then*

$$L_k\mathcal{S}^m = \mathsf{E}_k \circ \mathsf{L}_+^{\circ m} \quad \text{and} \quad L\mathcal{S}^m = \mathsf{E} \circ \mathsf{L}_+^{\circ m}.$$

Corollary 1.14.

$$L_k\mathcal{S}^m(x) = \sum_{n=0}^{\infty} |L_k(\mathcal{S}_n^m)| \frac{x^n}{n!} = \frac{1}{k!} \left(\frac{x}{1 - mx} \right)^k,$$
$$L\mathcal{S}^m(x) = \sum_{n=0}^{\infty} |L(\mathcal{S}_n^m)| \frac{x^n}{n!} = \exp\left(\frac{x}{1 - mx} \right).$$

Moreover, we will give explicit formulas for the numbers $|L_k(\mathcal{S}_n^m)|$ and $|L_k(\mathcal{C}_n^m)|$ of k-dimensional flats of n-dimensional extended Catalan and extended Shi arrangements with infinite matrices. Let $[a_{ij}]$ denote the infinite matrix whose entry in the i-th row and the j-th column is a_{ij}, where i and j run over all positive integers. Let

$$c := [c(j,i)] \quad \text{and} \quad S := [S(j,i)]\,,$$

where $c(j,i)$ denote the **unsigned Stirling number of the first kind**, that is, the number of ways to partition a j-element set into i cycles. Note that most of tables, including Table 5, consisting of such numbers are lower triangular. However, our infinite matrices c and S are transposed and hence upper triangular. Namely,

$$c = \begin{bmatrix} 1 & 1 & 2 & 6 & 24 & \cdots \\ 0 & 1 & 3 & 11 & 50 & \cdots \\ 0 & 0 & 1 & 6 & 35 & \cdots \\ 0 & 0 & 0 & 1 & 10 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 3 & 7 & 15 & \cdots \\ 0 & 0 & 1 & 6 & 25 & \cdots \\ 0 & 0 & 0 & 1 & 10 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Theorem 1.15.

$$\Big[\big|L_i(\mathcal{C}_j^m)\big|\Big] = (Sc)^m S \quad \textit{and} \quad \Big[\big|L_i(\mathcal{S}_j^m)\big|\Big] = (Sc)^m.$$

From Theorem 1.15, we may also calculate the matrices recursively as follows.

$$\begin{array}{ccccccccccc} \mathcal{C}^0 & \xrightarrow{c} & \mathcal{S}^1 & \xrightarrow{S} & \mathcal{C}^1 & \xrightarrow{c} & \mathcal{S}^2 & \xrightarrow{S} & \mathcal{C}^2 & \xrightarrow{c} & \cdots \\ S & & Sc & & ScS & & ScSc & & ScScS & & \end{array}$$

The **Lah number** is the number of ways to partition an n-element set into k nonempty lists, which is equal to the cardinality of $(\mathsf{E}_k \circ \mathsf{L}_+)[n]$ (See Example 2.1 for details). It is well known that the Lah number is given by the following formula.

Proposition 1.16 (See [16, p.44] for example)**.** *Let k and n be nonnegative integers. Then*

$$\big|(\mathsf{E}_k \circ \mathsf{L}_+)[n]\big| = \frac{n!(n-1)!}{k!(k-1)!(n-k)!}.$$

Using the Lah numbers, we give an explicit formula for the number of flats of the extended Shi arrangement.

Theorem 1.17.

$$\big|L_k(\mathcal{S}_n^m)\big| = m^{n-k}\frac{n!(n-1)!}{k!(k-1)!(n-k)!}.$$

The organization of this paper is as follows.

In §2, we give some examples of substitutions of species. Also, we introduce tree notation for the species $\mathsf{L}_+^{\circ m}$. Although actually this notation is not required for the proofs of our main theorems, it helps us to recognize elements of $\mathsf{L}_+^{\circ m}$.

In §3, we review the theory of gain graphs developed by Zaslavsky. Since the extended Catalan and Shi arrangements are expressed by using gain graphs, the intersection posets of these arrangements can be represented by a kind of partitions of the vertices of the corresponding gain graphs. This guarantees that it suffices to know the 1-dimensional flats in order to know all flats, that is, $L\mathcal{C}^m = \mathsf{E} \circ L_1\mathcal{C}^m$ and $L\mathcal{S}^m = \mathsf{E} \circ L_1\mathcal{S}^m$.

In §4, we give proofs of Theorem 1.11, Theorem 1.13, Theorem 1.15, and Theorem 1.17.

In §5, we state a relation between species of the form $\mathsf{E}_k \circ \mathsf{F}$ and the partial Bell polynomials.

In §A, as an appendix we give numerical tables for the number of flats with ID numbers in the On-Line Encyclopedia of Integer Sequences (OEIS) [12].

Note that we will construct several natural transformations. However, we will omit proofs of commutativity of diagrams for the natural transformations since they are obvious.

2. Substitution of species and tree notation

2.1. Examples

The substitution $\mathsf{F} \circ \mathsf{G}$ has the external and internal structure. Namely, for each set partition π, we take an element of $\mathsf{F}[\pi]$, which is the external structure. For every block B of π, we choose an element of $\mathsf{G}[B]$, which is the internal structure. In the usual case, a constituent of the external structure is labeled with blocks of a partition and if we "substitute" the labels with internal structures, then we obtain an element of $(\mathsf{F} \circ \mathsf{G})[V]$.

Example 2.1. Let G be a species such that if $g \in \mathsf{G}[B], g' \in \mathsf{G}[B']$, and $g = g'$, then $B = B'$. We will see that the species $\mathsf{E} \circ \mathsf{G}$ may be considered as a species of set partitions consisting of G-structures. By definition,

$$(\mathsf{E} \circ \mathsf{G})[V] = \bigsqcup_{\pi \in \Pi[V]} \left(\mathsf{E}[\pi] \times \prod_{B \in \pi} \mathsf{G}[B] \right) = \bigsqcup_{\pi \in \Pi[V]} \left(\{\pi\} \times \prod_{B \in \pi} \mathsf{G}[B] \right).$$

For any set partition $\pi = \{B_1, \dots, B_k\}$, every element $(\pi, (g_{B_1}, \dots, g_{B_k})) \in \{\pi\} \times \prod_{B\in\pi} \mathsf{G}[B]$ can be identified with the set $\{g_{B_1}, \dots, g_{B_k}\}$. Then the set $(\mathsf{E} \circ \mathsf{G})[V]$ is identified with

$$\{\, \{g_{B_1}, \dots, g_{B_k}\} \mid \{B_1, \dots, B_k\} \in \Pi[V],\ g_{B_i} \in \mathsf{G}[B_i] \,\}.$$

Especially $\mathsf{E} \circ \mathsf{E}_+$ gives set partitions consisting of sets, that is, ordinary set partitions. Namely $\mathsf{E} \circ \mathsf{E}_+ = \Pi$. Therefore $\left|(\mathsf{E}_k \circ \mathsf{E}_+)[n]\right|$ is the Stirling number of the second kind. For the same reason $\left|(\mathsf{E}_k \circ \mathsf{L}_+)[n]\right|$ yields the Lah number.

Figure 1: (123) and (2431)

We sometimes omit commas in sets, lists, and so on. For example, we write $\{123\}$ instead of $\{1,2,3\}$.

Example 2.2. Let G be as above. By the similar discussion, we may say that the species $\mathsf{L}\circ\mathsf{G}$ may be considered as a species of lists consisting of G-structures. Namely the set $(\mathsf{L}\circ\mathsf{G})[V]$ can be identified naturally with

$$\{\,(g_{B_1},\cdots,g_{B_k})\mid\{B_1,\dots,B_k\}\in\Pi[V],\ g_{B_i}\in\mathsf{G}[B_i]\,\}\,.$$

The species $\mathsf{L}\circ\mathsf{E}_+$ is known as a species of **set compositions** (or **ordered set partitions**). The cardinality $|(\mathsf{L}\circ\mathsf{E}_+)[n]|$ is called the **ordered Bell number** (or **Fubini number**). For instance $|(\mathsf{L}\circ\mathsf{E}_+)[3]|=13$ since it consists of the following set compositions.

$$(\{123\}),(\{12\}\{3\}),(\{13\}\{2\}),(\{23\}\{1\}),(\{3\}\{12\}),(\{2\}\{13\}),(\{1\}\{23\}),$$
$$(\{1\}\{2\}\{3\}),(\{1\}\{3\}\{2\}),(\{2\}\{1\}\{3\}),(\{2\}\{3\}\{1\}),(\{3\}\{1\}\{2\}),(\{3\}\{2\}\{1\}).$$

Example 2.3. The species E_1 of singletons behaves as the identity element with respect to substitution. Namely, $\mathsf{E}_1\circ\mathsf{G}=\mathsf{G}$ and $\mathsf{F}\circ\mathsf{E}_1=\mathsf{F}$.

A lot of researchers have been studied iterated substitutions of species of sets and lists. For example, Motzkin [14] investigated several structures including, "sets of sets" $\mathsf{E}\circ\mathsf{E}_+$, "sets of lists" $\mathsf{E}\circ\mathsf{L}_+$, "lists of sets" $\mathsf{L}\circ\mathsf{E}_+$, and "lists of lists" $\mathsf{L}\circ\mathsf{L}_+$. Sloane and Wieder [18] call an element of $(\mathsf{E}\circ\mathsf{L}_+\circ\mathsf{E}_+)[n]$ a **hierarchical ordering** (or **society**). Callan [5, Section 2] gave a bijection between lists of noncrossing sets and sets of lists $\mathsf{E}\circ\mathsf{L}_+$. Hedmark [11, Subsection 5.2] introduced an α-colored partition lattice for a positive integer α and stated that it can be regarded as $\mathsf{L}_+^{\circ\alpha}\circ\mathsf{E}_+$.

2.2. Tree notation for $\mathsf{L}_+^{\circ m}$

The species $\mathsf{L}_+^{\circ m}$ can be considered as the species of "m-dimensional lists". For example $(((49)(5))((3)(71)(6))((82)))$ is an element of $\mathsf{L}_+^{\circ 3}[9]$. However, it is difficult to understand the structure of this at first glance. Hence we introduce tree notation for $\mathsf{L}_+^{\circ m}$.

The idea is very simple. We just regard a list as an ordered rooted tree of height one with labeled leaves, where an **ordered rooted tree** means a rooted tree whose sibling sets are linearly ordered as lists. For example the lists (123) and (2413) are expressed as in Figure 1.

Then the species $\mathsf{L}_+^{\circ m}$ can be regarded as a rooted tree of height m. For example, $(((49)(5))((3)(71)(6))((82)))\in\mathsf{L}_+^{\circ 3}[9]$ is expressed as the left in Figure 2. We also can express elements of $\mathsf{L}_+^{\circ m}\circ\mathsf{E}_+$ by taking labels consisting of sets. For example $((\{57\}\{3\})(\{149\}\{26\}\{8\}))\in(\mathsf{L}_+^{\circ 2}\circ\mathsf{E}_+)[9]$ is as the right in Figure 2.

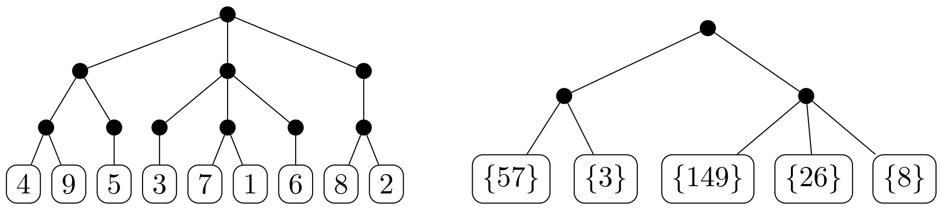


Figure 2: Elements in $\mathsf{L}_+^{\circ 3}[9]$ and $(\mathsf{L}_+^{\circ 2} \circ \mathsf{E}_+)[9]$

3. Gain graphs and the associated posets

3.1. Review of graphic arrangements

First we recall graphic arrangements and their intersection lattices. Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a simple graph on vertex set $V_\Gamma = [n]$. We can associate Γ with a hyperplane arrangement $\mathcal{A}_\Gamma$ in $\mathbb{K}^n$, called the **graphic arrangement**, consisting of hyperplanes defined by $\{x_i - x_j = 0\}$ with $\{i, j\} \in E_\Gamma$, where $x_1, \dots, x_\ell$ denote coordinates of $\mathbb{K}^\ell$.

Also, we can construct a matroid M_Γ on E_Γ, called the **graphic matroid**, from the linear independence of $\mathcal{A}_\Gamma$. Hence the **lattice of flats** $L(M_\Gamma)$ is naturally isomorphic to the intersection lattice $L(\mathcal{A}_\Gamma)$. Rota [17] first introduced $L(M_\Gamma)$ and called it the **bond lattice** of the graph Γ.

It is well known that the lattices $L(\mathcal{A}_\Gamma)$ and $L(M_\Gamma)$ also can be represented by using set partitions as explained below. A **connected partition** of Γ is a set partition of V_G whose every block induces a connected subgraph of Γ. Let $L(\Gamma)$ be the set of all connected partitions of Γ with the partial order defined by refinement. Namely, $\pi \leq \pi'$ if each block of π' is the union of some blocks of π. We call $L(\Gamma)$ the **lattice of connected partitions** (or the **lattice of contractions**), which is naturally isomorphic to $L(\mathcal{A}_\Gamma)$ and $L(M_\Gamma)$. Thus we can associate a simple graph Γ with three isomorphic lattices $L(\Gamma), L(\mathcal{A}_\Gamma)$ and $L(M_\Gamma)$.

Note that the braid arrangement $\mathcal{B}_n$ can be regarded as a graphic arrangement with the complete graph K_n. Since $L(K_n)$ consists of all set partitions of $[n]$, the number of flats of the braid arrangement is associated with the Bell number and the Stirling number of the second kind.

3.2. Gain graphs and affinographic arrangements

The extended Catalan and Shi arrangements are represented by using gain graphs. Gain graphs were introduced by Zaslavsky [21] for abstraction of linear independence of two kind of hyperplanes of the form

$$\{x_i - x_j = a\} \quad \text{and} \quad \{x_i - ax_j = 0\} \quad (a \neq 0).$$

Roughly speaking, a gain graph is a graph with labeled edges (i, j, a), which corresponds to the hyperplanes above. However, since

$$\{x_i - x_j = a\} = \{x_j - x_i = -a\} \quad \text{and} \quad \{x_i - ax_j = 0\} = \{x_j - a^{-1}x_i = 0\},$$

we must identify (i, j, a) with $(j, i, -a)$ for the former case and (i, j, a) with (j, i, a^{-1}) for the latter case. Although these two cases can be treated uniformly to a certain extent, we are interested in the former case in this article since extended Catalan and Shi arrangements consists of these hyperplanes. For this reason, we use the additive notation for labels of gain graphs. Here we give a formal definition of gain graphs.

Definition 3.1. A **simple gain graph** is a triple $\Gamma = (V_\Gamma, E_\Gamma, G_\Gamma)$ satisfying the following conditions.

(i) V_Γ is a finite set.

(ii) G_Γ is a group (not necessarily finite).

(iii) E_Γ is a finite subset of $\{ (u, v, a) \in V_\Gamma \times V_\Gamma \times G_\Gamma \mid u \neq v \}$ divided by the equivalence relation $\sim$ generated by $(u, v, a) \sim (v, u, -a)$.

Let $\{u, v\}_a$ denote the equivalence class containing (u, v, a). Then $\{u, v\}_a = \{v, u\}_{-a}$. Elements in V_Γ and E_Γ are called vertices and edges of the gain graph Γ. The group G_Γ is called the **gain group** of Γ.

Note that we quite simplify the notion of gain graphs for our purpose. Especially we do not consider multiple edges with the same elements in the gain group, loops, or half edges. See Zaslavsky [21] for a general treatment. For our main results, one may assume that the gain group is always commutative.

Definition 3.2. Suppose that Γ is a gain graph on $[n]$ and $G_\Gamma = \mathbb{K}^+$, the additive group of a field $\mathbb{K}$. Define an affine arrangement $\mathcal{A}_\Gamma$ in $\mathbb{K}^n$ by

$$\mathcal{A}_\Gamma \coloneqq \{ \{x_i - x_j = a\} \mid \{i, j\}_a \in E_\Gamma \} .$$

We call $\mathcal{A}_\Gamma$ the **affinographic arrangement** of Γ.

Note that every simple graph is regarded as a gain graph such that all elements in the gain group are 0. So the graphic arrangement is the affinographic arrangement. Hence there is no confusion to use the same symbol $\mathcal{A}_\Gamma$ for the graphic and affinographic arrangements.

Definition 3.3. Let G be a group and $A \subseteq G$ a finite subset. For every positive integer n, define K_n^A as a gain graph on $[n]$ with gain group G and edges

$$E_{K_n^A} \coloneqq \{ \{i, j\}_a \mid 1 \leq i < j \leq n,\, a \in A \} .$$

We call K_n^A the **complete gain graph with gain** A.

Remark 3.4. The construction of K_n^A requires the linear order on $[n]$ unless $-A = A$.

For integers a, b with $a \leq b$, let $[a, b] := \{a, a + 1, \dots, b\} \subseteq \mathbb{Z}$.

Example 3.5. The extended Catalan arrangements and the extended Shi arrangements are obtained by

$$\mathcal{C}_n^m = \mathcal{A}_{K_n^{[-m,m]}} \quad \text{and} \quad \mathcal{S}_n^m = \mathcal{A}_{K_n^{[1-m,m]}} .$$

Moreover, $\mathcal{A}_{K_n^{[1,m]}}$ is known as the (extended) Linial arrangements.

Let $v_1, v_2, \ldots, v_r$ be distinct vertices of a gain graph Γ. A **path** on the vertices $v_1, \ldots, v_r$ is a set of edges $\{v_1, v_2\}_{a_1}, \{v_2, v_3\}_{a_2}, \ldots, \{v_{r-1}, v_r\}_{a_{r-1}}$. A **cycle** on the vertices $v_1, \ldots, v_r$ is a set of edges $\{v_1, v_2\}_{a_1}, \{v_2, v_3\}_{a_2}, \ldots, \{v_{r-1}, v_r\}_{a_{r-1}}, \{v_r, v_1\}_{a_r}$. The cycle is called **balanced** if $a_1 + a_2 + \cdots + a_r = 0$. This definition is independent of choice of the initial vertex and the direction of the cycle. A subset $S \subseteq E_\Gamma$ is called **balanced** if any cycle contained in S is balanced. A balanced subset $S \subseteq E_\Gamma$ is called **balanced-closed** if

$$S = \{\, e \in E_\Gamma \mid e \text{ forms a balanced cycle together with a subset of } S \,\}.$$

Definition 3.6. Given a gain graph Γ, define a poset $L(M_\Gamma)$ by

$$L(M_\Gamma) \coloneqq \{\, F \subseteq E_\Gamma \mid F \text{ is balanced and balanced-closed} \,\}$$

and its order is given by inclusion. The poset $L(M_\Gamma)$ is called the **poset of balanced flats** of Γ.

Remark 3.7. The poset $L(M_\Gamma)$ is denoted by the symbol $\mathrm{Lat}^{\mathrm{b}}$ in Zaslavsky's papers and is actually the poset of flats of the **semimatroid** M_Γ associated with a gain graph Γ. Ardila [1] introduced the notion of semimatroids for combinatorial abstraction of linear independence of affine arrangements and described a relation between semimatroids and geometric semilattices introduced by Wachs and Walker [19], which is a generalization of a well-known relation between matroids and geometric lattices.

In this paper we omit the definition of semimatroids and state only the outline. Roughly speaking, a semimatroid is defined by the ground sets, the collection of central sets, and the rank function. An equivalent definition can be found in [9, Subsection 4.1]. Given an affine arrangement $\mathcal{A}$, we can define the corresponding semimatroid $M_\mathcal{A}$ if we define the ground set by $\mathcal{A}$, the collection of central sets by central subarrangements of $\mathcal{A}$, and the rank function by the rank of arrangements. Moreover, keep in mind that we can define flats and the poset of flats $L(M)$ of each semimatroid M and the intersection poset $L(\mathcal{A})$ of an affine arrangement $\mathcal{A}$ is naturally isomorphic to $L(M_\mathcal{A})$.

When $G_\Gamma = \mathbb{K}^+$, the linear independence of the affinographic arrangement $\mathcal{A}_\Gamma$ yields a semimatroid on E_Γ. Zaslavsky [21, 22, 23, 24, 25] gave an abstraction of this semimatroid, that is, defined the semimatroid M_Γ on E_Γ for a gain graph Γ with any gain group. The semimatroid M_Γ is called the **balanced semimatroid** [25] or the **semimatroid of graph balance** [9].

Theorem 3.8 (Zaslavsky [24, Corollary 4.5(a)])**.** *Let Γ be a gain graph on $[n]$ and the gain group is $\mathbb{K}^+$. Then the map*

$$\begin{array}{ccc} L(M_\Gamma) & \longrightarrow & L(\mathcal{A}_\Gamma) \\ F & \longmapsto & \displaystyle\bigcap_{\{i,j\}_a \in F} \{x_i - x_j = a\}. \end{array}$$

is an isomorphism of posets.

3.3. Poset of connected partitions

Let G be a group. A **G-labeled set** is a pair (V, θ) of a set V and a map $\theta\colon V \to G$. Two G-labeled sets (V, θ) and (V, η) are said to be **equivalent** if there exists an element $g \in G$ satisfying $\theta(v) = g + \eta(v)$ for any $v \in V$. Let $[V, \theta]$ denote the equivalent class containing (V, θ).

Let Γ be a gain graph. Consider an equivalence class of G_Γ-labeled sets $[B, \theta]$ with $B \subseteq V_\Gamma$. Define $\Gamma[B, \theta]$ as a gain graph on B with gain group G_Γ and edges

$$\begin{aligned} E_{\Gamma[B,\theta]} &\coloneqq \{\, \{u, v\}_a \in E_\Gamma \mid u, v \in B,\ \theta(u) + a = \theta(v) \,\} \\ &= \left\{\, \{u, v\}_{-\theta(u)+\theta(v)} \in E_\Gamma \;\middle|\; u, v \in B \,\right\}. \end{aligned}$$

One can show that the gain graph $\Gamma[B, \theta]$ is independent of choice of a representative (B, θ).

A **connected partition** of Γ is a collection $\pi = \{[B_1, \theta_1], \ldots, [B_k, \theta_k]\}$ such that $\{B_1, \ldots, B_k\}$ is a set partition of V_Γ and each $\Gamma[B_i, \theta_i]$ is connected, that is, there exists a path joining any two distinct vertices.

Let π and π' be connected partitions. We say that π **refines** π', denoted by $\pi \leq \pi'$ if for any $[B, \theta] \in \pi$ there exist $[B', \theta'] \in \pi'$ and $g \in G_\Gamma$ such that $B \subseteq B'$ and $\theta(v) = g + \theta'(v)$ for any $v \in B$. Let $L(\Gamma)$ denote the set of all connected partitions, which forms a poset together with the refinement. Call $L(\Gamma)$ the **poset of connected partitions** of Γ.

Note that when a simple graph Γ is viewed as a gain graph, connected partitions coincide with the usual connected partitions of the simple graph Γ. Therefore the poset of connected partitions of a gain graph is a generalization of the lattice of connected partitions of a simple graph.

Remark 3.9. For any finite group G Dowling [8] introduced G-labeled sets and G-partitions, which is equivalent to K_n^G-partitions, and showed that the collection consisting of G-partitions of the subsets of $[n]$ forms a geometric lattice. This lattice is known as the **Dowling lattice**.

Theorem 3.10 (Zaslavsky [20, Lemma 3.1A and 3.1B])**.** *Given a gain graph Γ, the following map is an isomorphism of posets.*

$$\begin{array}{rcl} L(\Gamma) & \longrightarrow & L(M_\Gamma) \\ \pi & \longmapsto & \displaystyle\bigcup_{[B,\theta]\in\pi} E_{\Gamma[B,\theta]}. \end{array}$$

Corollary 3.11. *Let Γ be a gain graph on $[n]$ and the gain group is $\mathbb{K}^+$.*

(1) If $[B, \theta]$ is a block of a connected partition of Γ with $B = \{i_1, \ldots, i_r\}$, then

$$X_{[B,\theta]} \coloneqq \bigcap_{\{i,j\}_a \in E_{\Gamma[B,\theta]}} \{x_i - x_j = a\} = \{x_{i_1} + \theta(i_1) = \cdots = x_{i_r} + \theta(i_r)\}.$$

(2) The following map is an isomorphism of posets.

$$\begin{array}{rcl} L(\Gamma) & \longrightarrow & L(\mathcal{A}_\Gamma) \\ \pi & \longmapsto & \displaystyle\bigcap_{[B,\theta]\in\pi} X_{[B,\theta]}. \end{array}$$

(3) For any nonnegative integer k, the isomorphism in (2) induces a bijection between $L_k(\Gamma)$ and $L_k(\mathcal{A}_\Gamma)$, where

$$L_k(\Gamma) := \{\, \pi \in L(\Gamma) \mid |\pi| = k \,\}.$$

Proof. (1) Recall $E_{\Gamma[B,\theta]} = \left\{ \{i,j\}_{-\theta(i)+\theta(j)} \in E_\Gamma \;\middle|\; i,j \in B \right\}$. The hyperplane corresponding to the edge $\{i,j\}_{-\theta(i)+\theta(j)}$ is

$$\{x_i - x_j = -\theta(i) + \theta(j)\} = \{x_i + \theta(i) = x_j + \theta(j)\}.$$

Since $\Gamma[B,\theta]$ is connected, the intersection coincides with the right hand side.

(2) This follows from Theorem 3.8 and 3.10.

(3) Obvious from (2). □

3.4. Partitional decompositions

Let G be a group and $A \subseteq G$ a finite subset. The construction of the complete gain graph with gain A is considered to be an $\mathbb{L}$-species. Namely, for each finite linearly ordered set V, we may construct the complete gain graph K_V^A on V with gain A and its poset of connected partitions $L(K_V^A)$ in an obvious way. In other words, there exist $\mathbb{L}$-species LK^A and L_kK^A such that $LK^A[n] = L(K_n^A)$ and $L_kK^A[n] = L_k(K_n^A)$.

Proposition 3.12. *Let G be a group and $A \subseteq G$ a finite subset. Then*

$$L_kK^A = \mathsf{E}_k \circ L_1K^A \quad \text{and} \quad LK^A = \mathsf{E} \circ L_1K^A$$

Proof. Let $[B,\theta]$ be a block of a connected partition of K_n^A. Then the block $[B,\theta]$ may be regarded as an element of $L_1K^A[B]$. This identification leads to the desired natural isomorphisms of the species. □

3.5. Integral gain graphs and height functions

We call a gain graph with the gain group $\mathbb{Z}$ an **integral gain graph**. The notion of height function on an integral gain graph was introduced in [7]. Let $\mathbb{S}$ be the category of sets and maps.

Definition 3.13. A **height function** on a finite set V is a function $h\colon V \to \mathbb{Z}$ such that $\min(h(V)) = 0$. Let $\mathsf{H}[V]$ be the set of height functions on V and H a functor from $\mathbb{B}$ to $\mathbb{S}$.

Let $\mathbb{Z}_+$ denote the set of positive integers. Given a nonnegative integer ℓ, a tuple $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_+^k$ is called an **integer composition** of length k. Note that there exists a unique integer composition $\varnothing$ of length 0. When α is a tuple, including a list and an integer composition, let $\ell(\alpha)$ denote the **length** of α.

Definition 3.14. Define a functor $\mathsf{I} : \mathbb{B} \to \mathbb{S}$ by

$$\mathsf{I}[V] := \{\, (\sigma, \alpha) \mid \sigma \in (\mathsf{L}_+ \circ \mathsf{E}_+)[V],\ \ell(\alpha) = \ell(\sigma) - 1 \,\}.$$

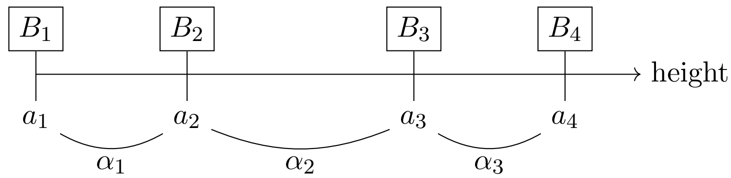


Figure 3: A height function and the corresponding set and integer compositions

Let h be a height function on V and we write $h(V) = \{a_1, \ldots, a_k\}$ with $0 = a_1 < a_2 < \cdots < a_k$. We define the set composition σ_h and the integer composition α_h by

$$\sigma_h \coloneqq (h^{-1}(a_1), \ldots, h^{-1}(a_k)),$$
$$\alpha_h \coloneqq (a_2 - a_1, \ldots, a_k - a_{k-1}).$$

Now we prove that a height function is identified with a pair of set and integer compositions (See Figure 3).

Lemma 3.15. $\mathsf{H} = \mathsf{I}$.

Proof. Define a natural transformation $\eta\colon \mathsf{H} \to \mathsf{I}$ by $\eta_V(h) \coloneqq (\sigma_h, \alpha_h)$. Let $(\sigma, \alpha) \in \mathsf{I}[V]$ and write $\sigma = (B_1, \ldots, B_k)$ and $\alpha = (\alpha_1, \ldots, \alpha_{k-1})$. Define a natural transformation $\zeta\colon \mathsf{I} \to \mathsf{H}$ by

$$\zeta_V(\sigma, \alpha)(v) \coloneqq \sum_{j=1}^{i-1} \alpha_i \text{ if } v \in B_i.$$

It is easy to show that η and ζ are inverse to each other. Hence we conclude that $\mathsf{H} = \mathsf{I}$. □

4. Proofs

4.1. Proof of Theorem 1.11

In order to describe $L\mathcal{C}^m$, we need to determine the species $L_1K^{[-m,m]}$ by Corollary 3.11(2) and Proposition 3.12.

For an integer composition $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_+^k$, define $\max(\alpha)$ by

$$\max(\alpha) \coloneqq \begin{cases} \max\{\, \alpha_i \mid 1 \leq i \leq k \,\} & \text{if } \alpha \neq \varnothing, \\ 0 & \text{if } \alpha = \varnothing. \end{cases}$$

Definition 4.1. Let m be a nonnegative integer. We define a species P^m by

$$\mathsf{P}^m[V] \coloneqq \{\, (\sigma, \alpha) \in \mathsf{I}[V] \mid \max(\alpha) \leq m \,\},$$

where I is the functor defined in Definition 3.14.

Note that $\mathsf{P}^0 = \mathsf{E}_+ = L_1 K^{\{0\}}$ since $\mathsf{P}^0[V]$ consists of a single element $((V), \varnothing)$. Furthermore, the following lemma holds.

Lemma 4.2. *Let m be a nonnegative integer. Then $L_1 K^{[-m,m]} = \mathsf{P}^m$.*

Proof. First note that every element in $L_1 K^{[-m,m]}[V] = L_1(K_V^{[-m,m]})$ is identified with the height function $h\colon V \to \mathbb{Z}$ such that the gain graph $K_V^{[-m,m]}[V,h]$ is connected. Hence, by Lemma 3.15, it is satisfied to show that for every height function $h\colon V \to \mathbb{Z}$, the gain graph $K_V^{[-m,m]}[V,h]$ is connected if and only if $\max(\alpha_h) \le m$.

Write $\sigma_h = (B_1, \dots, B_k)$ and $\alpha_h = (\alpha_1, \dots, \alpha_{k-1})$. Suppose that $\max(\alpha_h) \le m$. For each $i \in \{1, \dots, k-1\}$, since $\alpha_i \le m$, there exists an edge $\{u,v\}_{\alpha_i}$ such that $u \in B_i$ and $v \in B_{i+1}$. Therefore $K_V^{[-m,m]}[V,h]$ is connected.

Next suppose that $K_V^{[-m,m]}[V,h]$ is connected. For every $p \in \{1, \dots, k-1\}$, there exists an edge $\{u,v\}_a$ such that $u \in B_i$ $(1 \le i \le p)$ and $v \in B_j$ $(p+1 \le j \le k)$. Then

$$m \ge a = h(v) - h(u) = \alpha_{j-1} + \alpha_{j-2} + \cdots + \alpha_p + \cdots + \alpha_i \ge \alpha_p$$

and we have $\alpha_p \le m$. Therefore $\max(\alpha_h) \le m$ and hence the assertion holds. □

Let $\alpha = (\alpha_1, \dots, \alpha_k)$ and $\alpha' = (\alpha'_1, \dots, \alpha'_\ell)$ be tuples. The **concatenation** $\alpha * \alpha'$ is defined by

$$\alpha * \alpha' \coloneqq (\alpha_1, \dots, \alpha_k, \alpha'_1, \dots, \alpha'_\ell).$$

Lemma 4.3. *Let m be a nonnegative integer. Then $\mathsf{P}^m = \mathsf{L}_+^m \circ \mathsf{E}_+$.*

Proof. We prove the assertion by induction on m. Since $\mathsf{P}^0 = \mathsf{E}_+ = \mathsf{E}_1 \circ \mathsf{E}_+ = \mathsf{L}_+^{\circ 0} \circ \mathsf{E}_+$, the assertion holds when $m = 0$. Suppose that $m > 0$. In order to prove $\mathsf{P}^m = \mathsf{L}_+^m \circ \mathsf{E}_+$, we construct a natural transformation $\eta\colon \mathsf{L}_+ \circ \mathsf{P}^{m-1} \to \mathsf{P}^m$ and prove that η is a natural isomorphism. Suppose that $((\tau_1, \beta_1), \dots, (\tau_k, \beta_k)) \in (\mathsf{L}_+ \circ \mathsf{P}^{m-1})[V]$. We define σ and α by

$$\begin{aligned}
\sigma &\coloneqq \tau_1 * \cdots * \tau_k, \\
\alpha &\coloneqq \beta_1 * (m) * \beta_2 * (m) * \cdots * (m) * \beta_k.
\end{aligned}$$

Then we have

$$\ell(\alpha) = \sum_{i=1}^{k} \ell(\beta_i) + k - 1 = \sum_{i=1}^{k} (\ell(\tau_i) - 1) + k - 1 = \sum_{i=1}^{k} \ell(\tau) - 1 = \ell(\sigma) - 1$$

and $\max(\alpha) \le m$ since $\max(\beta_i) \le m-1$ for each $i \in \{1, \dots, k\}$. Moreover we have $\sigma \in (\mathsf{L}_+ \circ \mathsf{E}_+)[V]$ since $\tau_i \in (\mathsf{L}_+ \circ \mathsf{E}_+)[B_i]$ for some $\{B_1, \dots, B_k\} \in \Pi[V]$. We define a natural transformation η by

$$\eta_V(((\tau_1, \beta_1), \dots, (\tau_k, \beta_k))) \coloneqq (\sigma, \alpha) \in \mathsf{P}^m[V].$$

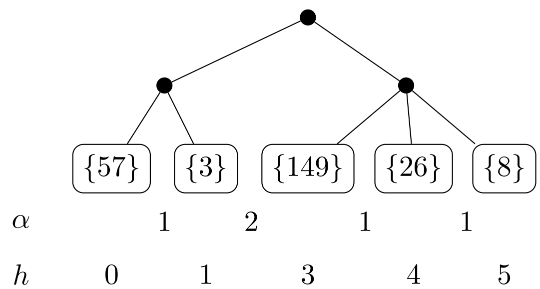


Figure 4: An example of the correspondence for the extended Catalan arrangement

We construct another natural transformation $\zeta\colon \mathsf{P}^m \to \mathsf{L}_+ \circ \mathsf{P}^{m-1}$. Let $(\sigma, \alpha) \in \mathsf{P}^m[V]$ and suppose that the number of m's appearing in α is $k-1$. Decompose α as $\alpha = \beta_1 * (m) * \beta_2 * (m) * \cdots * (m) * \beta_k$. Then each β_i satisfies $\max(\beta_i) \leq m-1$ since $\max(\alpha) \leq m$. Moreover, decompose σ as $\sigma = \tau_1 * \cdots * \tau_k$ such that $\ell(\tau_i) = \ell(\beta_i)+1$. Then there exists a set partition $\{B_1, \ldots, B_k\} \in \Pi[V]$ such that $\sigma_i \in (\mathsf{L}_+ \circ \mathsf{E}_+)[B_i]$ for each $i \in \{1, \ldots, k\}$. We define a natural transformation ζ by

$$\zeta_V((\sigma, \alpha)) \coloneqq ((\tau_1, \beta_1), \ldots, (\tau_k, \beta_k)) \in (\mathsf{L}_+ \circ \mathsf{P}^{m-1})[V].$$

It is obvious that η and ζ are inverse to each other. Thus $\mathsf{P}^m = \mathsf{L}_+ \circ \mathsf{P}^{m-1}$. □

Proof of Theorem 1.11. By Corollary 3.11(3), Lemma 4.2, and Lemma 4.3

$$L_1 \mathcal{C}^m = L_1 K^{[-m,m]} = \mathsf{P}^m = \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+.$$

Moreover by Corollary 3.11(2) and Proposition 3.12,

$$L\mathcal{C}^m = LK^{[-m,m]} = \mathsf{E} \circ L_1 K^{[-m,m]} = \mathsf{E} \circ \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+.$$

Use Corollary 3.11(3) to obtain $L_k \mathcal{C}^m = \mathsf{E}_k \circ \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+$. □

Example 4.4. Consider $((B_1B_2)(B_3B_4B_5)) = ((\{57\}\{3\})(\{149\}\{26\}\{8\})) \in (\mathsf{L}_+^{\circ 2} \circ \mathsf{E}_+)[9]$ (See Figure 4). We construct the corresponding flat of the extended Catalan arrangement $\mathcal{C}_9^2$. First, for each i, let α_i denote the height of the minimal tree containing leaves B_i and B_{i+1}. In this case we have the integer composition $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 2, 1, 1)$. By taking the partial sum $\sum_{j=1}^{i-1} \alpha_j$ for each i, we obtain the sequence of heights $(0, 1, 3, 4, 5)$. The height function $h : [9] \to \mathbb{Z}$ is obtained by the following table.

v	5	7	3	1	4	9	2	6	8
$h(v)$	0	0	1	3	3	3	4	4	5

The corresponding flat in $\mathcal{C}_9^2$ is

$$\{x_5 = x_7 = x_3 + 1 = x_1 + 3 = x_4 + 3 = x_9 + 3 = x_2 + 4 = x_6 + 4 = x_8 + 5\}.$$

4.2. Proof of Theorem 1.13

We assume that all species in this subsection are $\mathbb{L}$-species and V denotes a finite linearly ordered set. The symbols $\min(B)$ and $\max(B)$ stand for the minimum and the maximum elements of a subset $B \subseteq V$.

Definition 4.5. Let m be a positive integer. Define an $\mathbb{L}$-species Q^m by

$$\mathsf{Q}^m[V] \coloneqq \{ (\sigma, \alpha) \in \mathsf{P}^m[V] \mid \alpha_i = m \Rightarrow \min(B_i) < \max(B_{i+1}) \},$$

where P^m is the species defined in Definition 4.1 and α_i and B_i denote the i-th entries of α and σ, respectively.

Lemma 4.6. *Let m be a positive integer. Then $L_1 K^{[1-m,m]} = \mathsf{Q}^m$.*

Proof. Let h be a height function on a finite linearly ordered set V and we write $\sigma_h = (B_1, \dots, B_k)$ and $\alpha_h = (\alpha_1, \dots, \alpha_{k-1})$. Similarly to the proof of Lemma 4.2, it is satisfied to show that $K_V^{[1-m,m]}[V, h]$ is connected if and only if $\max(\alpha_h) \leq m$ and $\alpha_i = m$ implies $\min(B_i) < \min(B_{i+1})$.

First we assume that the latter conditions and prove that $K_V^{[1-m,m]}[V, h]$ is connected. If $\alpha_i \leq m-1$, then there exists an edge connecting a vertex in B_i and a vertex in B_{i+1}. Suppose that $\alpha_i = m$. Then the condition $\min(B_i) < \min(B_{i+1})$ guarantees the existence of the edge $\{u, v\}_m$ such that $u = \min(B_i)$ and $v = \max(B_{i+1})$. Thus $K_V^{[1-m,m]}[V, h]$ is connected.

Second we assume that $K_V^{[1-m,m]}[V, h]$ is connected. If $\alpha_p < m$ for every $p \in \{1, \dots, k-1\}$, then obviously the later conditions hold. Hence we may assume that there exists $p \in \{1, \dots, k-1\}$ such that $\alpha_p \geq m$. From the connectedness, there exists an edge $\{u, v\}_a$ such that $u \in B_i$ $(1 \leq i \leq p)$, $v \in B_j$ $(p+1 \leq j \leq k)$. If $j - i \geq 2$, then

$$m \geq a = h(v) - h(u) = \alpha_{j-1} + \alpha_{j-2} + \cdots + \alpha_i > \alpha_p,$$

which contradicts to the assumption $\alpha_p \geq m$. Hence $j - i = 1$ and we have $i = p$, $j = p+1$, and $a = \alpha_p$. If $a = \alpha_p > m$, then $\{u, v\}_a$ is not an edge, which is a contradiction. Therefore $\alpha_p = m$. This implies $\min(B_p) \leq u < v \leq \max(B_{p+1})$ and the assertion holds. ∎

Lemma 4.7. *Let m be a positive integer. Then $\mathsf{Q}^m = \mathsf{L}_+^{\circ m}$.*

Proof. We proceed by induction on m. Suppose that $m = 1$. We may omit integer compositions from the notation of Q^1 since only integer compositions consisting of 1 are allowed. Namely we may consider Q^1 as

$$\mathsf{Q}^1[V] = \{ (B_1, \dots, B_k) \in (\mathsf{L}_+ \circ \mathsf{E}_+)[V] \mid \min B_i < \max B_{i+1} \ (1 \leq i \leq k-1) \}.$$

Now we construct a natural transformation $\eta \colon \mathsf{L}_+ \to \mathsf{Q}^1$. Let $(v_1, \dots, v_n) \in \mathsf{L}_+[V]$ be a list of a finite linearly ordered set V. Suppose that $(v_{i_1}, \dots, v_{i_{k-1}})$ is the sublist such that $\{i_1, \dots, i_{k-1}\} =$

$\{ i \in [n] \mid v_i < v_{i+1} \}$. For each $j \in \{1, \dots, k\}$, put $B_j \coloneqq \{ v_i \in V \mid i_{j-1} < i \le i_j \}$, where we consider $i_0 \coloneqq 0$ and $i_k \coloneqq n$. Then $\min B_j = v_{i_j} < v_{i_j+1} = \max B_{j+1}$ for each $j \in \{1, \dots, k-1\}$. We define η by

$$\eta_V((v_1, \dots, v_n)) \coloneqq (B_1, \dots, B_k) \in \mathsf{Q}^1.$$

Next we construct a natural transformation $\zeta \colon \mathsf{Q}^1 \to \mathsf{L}_+$. Given an element $(B_1, \dots, B_k) \in \mathsf{Q}^1[V]$, let β_i be the list of B_i with decreasing order for each $i \in \{1, \dots, k\}$. Define ζ by

$$\zeta_V((B_1, \dots, B_k)) \coloneqq \beta_1 * \cdots * \beta_k \in \mathsf{L}_+[V].$$

One can show that η and ζ are inverse to each other and hence $\mathsf{Q}^1 = \mathsf{L}_+$.

Suppose that $m \ge 2$. We will prove that $\mathsf{Q}^m = \mathsf{L}_+ \circ \mathsf{Q}^{m-1}$. First we construct a natural transformation $\eta \colon \mathsf{L}_+ \circ \mathsf{Q} \to \mathsf{Q}^m$. Take an element $((\tau_1, \beta_1), \dots, (\tau_k, \beta_k)) \in (\mathsf{L}_+ \circ \mathsf{Q}^{m-1})[V]$. For each $i \in \{1, \dots, k\}$, let $T(\tau_i)$ and $I(\tau_i)$ denote the terminal and initial entries of τ_i and define an integer μ_i by

$$\mu_i \coloneqq \begin{cases} m & \text{if } \min T(\tau_i) < \max I(\tau_{i+1}), \\ m-1 & \text{if } \min T(\tau_i) > \max I(\tau_{i+1}). \end{cases}$$

Now we define σ and α by

$$\begin{aligned} \sigma &\coloneqq \tau_1 * \cdots * \tau_k, \\ \alpha &\coloneqq \beta_1 * (\mu_1) * \beta_2 * (\mu_2) * \cdots * (\mu_{k-1}) * \beta_k. \end{aligned}$$

Then $\sigma \in (\mathsf{L}_+ \circ \mathsf{E}_+)[V]$, $\ell(\alpha) = \ell(\sigma) - 1$, and $\max(\alpha) \le m$. Moreover we write $\sigma = (B_1, \dots, B_{k'})$ and $\alpha = (\alpha_1, \dots, \alpha_{k'-1})$. Then $\alpha_i = m$ implies $\max B_i < \max B_{i+1}$. This means $(\sigma, \alpha) \in \mathsf{Q}^m[V]$. We define η by

$$\eta_V(((\tau_1, \beta_1), \dots, (\tau_k, \beta_k))) \coloneqq (\sigma, \alpha) \in \mathsf{Q}^m[V].$$

Next we construct another natural transformation $\zeta \colon \mathsf{Q}^m \to \mathsf{L}_+ \circ \mathsf{Q}^{m-1}$. Suppose that $(\sigma, \alpha) \in \mathsf{Q}^m[V]$ and write $\sigma = (B_1, \dots, B_\ell)$ and $\alpha = (\alpha_1, \dots, \alpha_{\ell-1})$. Let

$$S \coloneqq \{ i \in [\ell - 1] \mid \alpha_i = m \} \cup \{ i \in [\ell - 1] \mid \alpha_i = m - 1 \text{ and } \min B_i > \max B_{i+1} \}.$$

We write $S = \{i_1, \dots, i_{k-1}\}$ and decompose α as

$$\alpha = \beta_1 * (\alpha_{i_1}) * \beta_2 * (\alpha_{i_2}) * \cdots * (\alpha_{i_{k-1}}) * \beta_k.$$

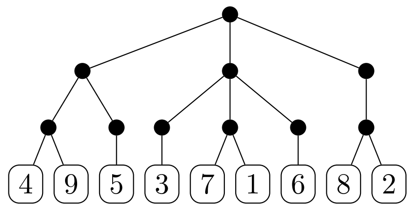
Moreover decompose σ as $\sigma = \tau_1 * \cdots * \tau_k$ with $\ell(\tau_i) = \ell(\beta_i) + 1$ for each $i \in \{1, \dots, k\}$. Then $\max(\beta_i) \le m-1$ and note that if β_i contains an entry α_j such that $\alpha_j = m-1$, then $\min B_j > \max B_{j+1}$ by the definition of S. Hence we may define ζ by

$$\zeta_V((\sigma, \alpha)) \coloneqq ((\tau_1, \beta_1), \dots, (\tau_k, \beta_k)) \in (\mathsf{L}_+ \circ \mathsf{Q}^{m-1})[V].$$

It is easy to show that η and ζ are inverse to each other and hence $\mathsf{Q}^m = \mathsf{L}_+ \circ \mathsf{Q}^{m-1}$. Finally by the induction hypothesis we have

$$\mathsf{Q}^m = \mathsf{L}_+ \circ \mathsf{Q}^{m-1} = \mathsf{L}_+ \circ \mathsf{L}_+^{\circ(m-1)} = \mathsf{L}_+^{\circ m},$$

which is the desired result. □



α		1	2	3	2	1	2	3	1
α'		1	1	2	2	0	2	3	0
h	0	1	2	4	6	6	8	11	11

Figure 5: An example of the correspondence for the extended Shi arrangement

Proof of Theorem 1.13. By Corollary 3.11(3), Lemma 4.6, and Lemma 4.7

$$L_1\mathcal{S}^m = L_1K^{[1-m,m]} = \mathsf{Q}^m = \mathsf{L}_+^{\circ m}.$$

Moreover by Corollary 3.11(2) and Proposition 3.12,

$$L\mathcal{S}^m = LK^{[1-m,m]} = \mathsf{E} \circ L_1K^{[1-m,m]} = \mathsf{E} \circ \mathsf{L}_+^{\circ m}.$$

Similarly by Corollary 3.11(3), we obtain $L_k\mathcal{S}^m = \mathsf{E}_k \circ \mathsf{L}_+^{\circ m}$. □

Example 4.8. Consider $(((v_1v_2)(v_3))((v_4)(v_5v_6)(v_7))((v_8v_9))) = (((49)(5))((3)(71)(6))((82))) \in \mathsf{L}_+^{\circ 3}[9]$ (See Figure 5). We construct the corresponding flat of the extended Shi arrangement $\mathcal{S}_9^3$. First let α be the integer composition obtained in a similar way in Example 4.4. In this case $\alpha = (1,2,3,2,1,2,3,1)$. Next define the integer composition α' by

$$\alpha'_i := \begin{cases} \alpha_i & \text{if } v_i < v_{i+1}, \\ \alpha_i - 1 & \text{if } v_i > v_{i+1}. \end{cases}$$

In this case $\alpha' = (1,1,2,2,0,2,3,0)$. By taking the partial sum $\sum_{j=1}^{i-1} \alpha_j$ for each i, we obtain the sequence of heights $(0,1,2,4,6,6,8,11,11)$. The height function h is obtained by the following table.

v	4	9	5	3	7	1	6	8	2
$h(v)$	0	1	2	4	6	6	8	11	11

The corresponding flat is

$$\{x_4 = x_9 + 1 = x_5 + 2 = x_3 + 4 = x_7 + 6 = x_1 + 6 = x_6 + 8 = x_8 + 11 = x_2 + 11\}.$$

4.3. Proof of Theorem 1.15

Let F be a species with $\mathsf{F}[\varnothing] = \varnothing$. We consider the infinite matrix $\left[\left|(\mathsf{E}_i \circ \mathsf{F})[j]\right|\right]$. Note that almost all entries of each column of the matrix are 0 since

$$\sum_{i=1}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F})[j]\right| = \left|(\mathsf{E} \circ \mathsf{F})[j]\right| < \infty.$$

We show that substitution of species is compatible with product of the infinite matrices.

Proposition 4.9. *Let F and G be species with $\mathsf{F}[\varnothing] = \mathsf{G}[\varnothing] = \varnothing$. Then*

$$\left[\left|(\mathsf{E}_i \circ \mathsf{F})[j]\right|\right]\left[\left|(\mathsf{E}_i \circ \mathsf{G})[j]\right|\right] = \left[\left|(\mathsf{E}_i \circ \mathsf{F} \circ \mathsf{G})[j]\right|\right].$$

Proof. Fix a positive integer i. By definition,

$$(\mathsf{E}_i \circ \mathsf{F} \circ \mathsf{G})(x) = \sum_{j=0}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F} \circ \mathsf{G})[j]\right| \frac{x^j}{j!}.$$

We give another calculation of the series as follows.

$$\begin{aligned}
(\mathsf{E}_i \circ \mathsf{F} \circ \mathsf{G})(x) &= (\mathsf{E}_i \circ \mathsf{F})(\mathsf{G}(x)) \\
&= \sum_{k=0}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F})[k]\right| \frac{\mathsf{G}(x)^k}{k!} \\
&= \sum_{k=0}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F})[k]\right| (\mathsf{E}_k \circ \mathsf{G})(x) \\
&= \sum_{k=0}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F})[k]\right| \sum_{j=0}^{\infty} \left|(\mathsf{E}_k \circ \mathsf{G})[j]\right| \frac{x^j}{j!} \\
&= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \left|(\mathsf{E}_i \circ \mathsf{F})[k]\right| \left|\mathsf{E}_k \circ \mathsf{G}[j]\right| \right) \frac{x^j}{j!}.
\end{aligned}$$

Therefore we have

$$\left|\mathsf{E}_i \circ \mathsf{F} \circ \mathsf{G}[j]\right| = \sum_{k=0}^{\infty} \left|\mathsf{E}_i \circ \mathsf{F}[k]\right| \left|\mathsf{E}_k \circ \mathsf{G}[j]\right|$$

for any positive integers i and j. Hence the assertion holds. □

Example 4.10. Let C denote the **species of cyclic permutations**. Then the substitution $\mathsf{E} \circ \mathsf{C}_+$ coincides with the **species of permutations**. As mentioned in [3, p. 346], we have that $\mathsf{L} = \mathsf{E} \circ \mathsf{C}_+$ as $\mathbb{L}$-species. Indeed let each permutation $\sigma \in (\mathsf{E} \circ \mathsf{C}_+)[n]$ correspond to the list $(\sigma(1), \sigma(2), \cdots, \sigma(n)) \in \mathsf{L}[n]$.

It is easy to see that this correspondence is bijective. Note that L and $\mathsf{E} \circ \mathsf{C}_+$ are not isomorphic as $\mathbb{B}$-species. Recall that c, S, and $\left[\left|(\mathsf{E}_i \circ \mathsf{L}_+)[j]\right|\right]$ is the infinite upper triangular matrix consisting of Stirling numbers of the first and second kind, and Lah numbers. We can recover the following well-known equality.

$$\left[\left|(\mathsf{E}_i \circ \mathsf{L}_+)[j]\right|\right] = \left[\left|(\mathsf{E}_i \circ \mathsf{E}_+ \circ \mathsf{C}_+)[j]\right|\right] = \left[\left|(\mathsf{E}_i \circ \mathsf{E}_+)[j]\right|\right]\left[\left|(\mathsf{E}_i \circ \mathsf{C}_+)[j]\right|\right] = Sc.$$

Proof of Theorem 1.15. From Theorem 1.11, Proposition 4.9, and Example 4.10

$$\begin{aligned}\left[\,\left|L_i(\mathcal{C}_j^m)\right|\,\right] &= \left[\,\left|L_i\mathcal{C}^m[j]\right|\,\right] = \left[\,\left|(\mathsf{E}_i \circ \mathsf{L}_+^{\circ m} \circ \mathsf{E}_+)[j]\right|\,\right] \\ &= \left[\left|(\mathsf{E}_i \circ \mathsf{L}_+)[j]\right|\right]^m \left[\left|(\mathsf{E}_i \circ \mathsf{E}_+)[j]\right|\right] = (Sc)^m S.\end{aligned}$$

Similarly from Theorem 1.13, Proposition 4.9, and Example 4.10

$$\begin{aligned}\left[\,\left|L_i(\mathcal{S}_j^m)\right|\,\right] &= \left[\,\left|L_i\mathcal{S}^m[j]\right|\,\right] = \left[\,\left|(\mathsf{E}_i \circ \mathsf{L}_+^{\circ m})[j]\right|\,\right] \\ &= \left[\left|(\mathsf{E}_i \circ \mathsf{L}_+)[j]\right|\right]^m = (Sc)^m.\end{aligned}$$

□

4.4. Proof of Theorem 1.17

Let a_{ij} denote the Lah number $\left|(\mathsf{E}_i \circ \mathsf{L}_+)[j]\right|$. By Proposition 1.16, $a_{ij} = \frac{j!(j-1)!}{i!(i-1)!(j-i)!}$. Note that if $i > j$, then $a_{ij} = 0$.

Lemma 4.11. *For every positive integer* m, $[a_{ij}]^m = [m^{j-i}a_{ij}]$.

Proof. We proceed by induction on m. If $m = 1$, then it is trivial. Assume that $m \geq 2$. By the induction hypothesis, the (i, j)-entry of the matrix $[a_{ij}]^m = [a_{ij}][a_{ij}]^{m-1}$ is

$$\begin{aligned}\sum_{k=i}^{j} a_{ik}(m-1)^{j-k}a_{kj} &= \sum_{k=i}^{j} \frac{k!(k-1)!}{i!(i-1)!(k-i)!}(m-1)^{j-k}\frac{j!(j-1)!}{k!(k-1)!(j-k)!} \\ &= \frac{j!(j-1)!}{i!(i-1)!(j-i)!}\sum_{k=i}^{j}\frac{(j-i)!}{(j-k)!(k-i)!}(m-1)^{j-k} \\ &= a_{ij}\sum_{k=i}^{j}\binom{j-i}{k-i}(m-1)^{j-k} = a_{ij}\sum_{k=0}^{j-i}\binom{j-i}{k}(m-1)^{j-i-k} = m^{j-i}a_{ij}.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.17. The assertion holds immediately from Proposition 1.16, Theorem 1.15 and Lemma 4.11. □

5. Relation with the partial Bell polynomials

Let n and k be nonnegative integers. Bell [2] introduced the **partial Bell polynomial** $B_{n,k} = B_{n,k}(z_1, z_2, \ldots, z_{n-k+1})$, which is characterized by the following formal series expansion (See [6, p. 133, [3a']] for example).

$$\frac{1}{k!}\left(\sum_{n=1}^{\infty} z_n \frac{x^n}{n!}\right)^k = \sum_{n=k}^{\infty} B_{n,k}(z_1, \ldots, z_{n-k+1})\frac{x^n}{n!}.$$

See also [6, p. 134, Theorem A] for an explicit formula of the partial Bell polynomials.

Given a sequence $(a_n)_{n=1}^{\infty}$, let $T(n,k) := B_{n,k}(a_1, \ldots, a_{n-k+1})$. The transformation from a sequence $(a_n)_{n=1}^{\infty}$ to the triangular array consisting of the numbers $T(n,k)$ is often called the **Bell transform**. For example, the Bell transform of the sequence $(1)_{n=1}^{\infty}$ is the Stirling number of the second kind. Namely, $S(n,k) = B_{n,k}(1, \ldots, 1)$. We can say that $\left|(\mathsf{E}_k \circ \mathsf{E}_+)[n]\right| = S(n,k)$ is the Bell transform of the sequence $(\left|E_+[n]\right|)_{n=1}^{\infty} = (1)_{n=1}^{\infty}$. More generally, the following proposition holds.

Proposition 5.1. *Let* F *be a species such that* $\mathsf{F}[\varnothing] = \varnothing$. *Then the Bell transform of the sequence* $(\mathsf{F}[n])_{n=1}^{\infty}$ *is the triangular array consisting of the numbers* $\left|(\mathsf{E}_k \circ \mathsf{F})[n]\right|$. *Namely,* $\left|(\mathsf{E}_k \circ \mathsf{F})[n]\right| = B_{n,k}\left(\left|\mathsf{F}[1]\right|, \left|\mathsf{F}[2]\right|, \ldots, \left|\mathsf{F}[n-k+1]\right|\right)$.

Proof.

$$\begin{aligned}(\mathsf{E}_k \circ \mathsf{F})(x) = \mathsf{E}_k(\mathsf{F}(x)) = \frac{1}{k!}\mathsf{F}(x)^k &= \frac{1}{k!}\left(\sum_{n=1}^{\infty} \left|\mathsf{F}[n]\right| \frac{x^n}{n!}\right)^k \\ &= \sum_{n=k}^{\infty} B_{n,k}(\left|\mathsf{F}[1]\right|, \left|\mathsf{F}[2]\right|, \ldots, \left|\mathsf{F}[n-k+1]\right|)\frac{x^n}{n!}.\end{aligned}$$

Thus $\left|(\mathsf{E}_k \circ \mathsf{F})[n]\right| = B_{n,k}\left(\left|\mathsf{F}[1]\right|, \left|\mathsf{F}[2]\right|, \ldots, \left|\mathsf{F}[n-k+1]\right|\right)$. □

Example 5.2. Since $\left|\mathsf{L}[n]\right| = n!$ for $n \geq 1$, we can recover the well-known formula for the Lah numbers:

$$\left|(\mathsf{E}_k \circ \mathsf{L}_+)[n]\right| = B_{n,k}(1!, 2!, \ldots, (n-k+1)!).$$

Moreover, since $\left|\mathsf{C}[n]\right| = (n-1)!$ for $n \geq 1$, we can also recover the well-known equality for the unsigned Stirling numbers of the first kind:

$$c(n,k) = \left|(\mathsf{E}_k \circ \mathsf{C}_+)[n]\right| = B_{n,k}(0!, 1!, \ldots, (n-k)!).$$

The sum of the partial Bell polynomials

$$B_n(z_1, \ldots, z_n) := \sum_{k=1}^{n} B_{n,k}(z_1, \ldots, z_{n-k+1})$$

is called the **complete Bell polynomial**. We can deduce easily the following corollary to Proposition 5.1.

Corollary 5.3. *Let* F *be a species such that* $\mathsf{F}[\varnothing] = \varnothing$. *Then* $\left|(\mathsf{E} \circ \mathsf{F})[n]\right| = B_n(\left|\mathsf{F}[1]\right|, \left|\mathsf{F}[2]\right|, \ldots, \left|\mathsf{F}[n]\right|)$.

A. Numerical tables

n	1	2	3	4	5	6	7	OEIS
$L(\mathcal{B}_n)$	1	2	5	15	52	203	877	A000110
$L(\mathcal{C}_n^1)$	1	4	23	173	1602	17575	222497	A075729
$L(\mathcal{C}_n^2)$	1	6	53	619	8972	155067	3109269	A109092
$L(\mathcal{C}_n^3)$	1	8	95	1497	29362	688439	18766393	None
$L(\mathcal{C}_n^4)$	1	10	149	2951	72852	2152651	74031869	None

Table 1: The numbers of flats of the extended Catalan arrangements

n	1	2	3	4	5	6	7	OEIS
$L(\mathcal{S}_n^1)$	1	3	13	73	501	4051	37633	A000262
$L(\mathcal{S}_n^2)$	1	5	37	361	4361	62701	1044205	A025168
$L(\mathcal{S}_n^3)$	1	7	73	1009	17341	355951	8488117	A321837
$L(\mathcal{S}_n^4)$	1	9	121	2161	48081	1279801	39631369	A321847
$L(\mathcal{S}_n^5)$	1	11	181	3961	108101	3532651	134415961	A321848

Table 2: The numbers of flats of the extended Shi arrangements

n	1	2	3	4	5	6	7	OEIS
$L_1(\mathcal{B}_n)$	1	1	1	1	1	1	1	A000012
$L_1(\mathcal{C}_n^1)$	1	3	13	75	541	4683	47293	A000670
$L_1(\mathcal{C}_n^2)$	1	5	37	365	4501	66605	1149877	A050351
$L_1(\mathcal{C}_n^3)$	1	7	73	1015	17641	367927	8952553	A050352
$L_1(\mathcal{C}_n^4)$	1	9	121	2169	48601	1306809	40994521	A050353

Table 3: The numbers of 1-dimensional flats of the extended Catalan arrangements

n	1	2	3	4	5	6	7	OEIS
$L_1(\mathcal{S}_n^1)$	1	2	6	24	120	720	5040	A000142
$L_1(\mathcal{S}_n^2)$	1	4	24	192	1920	23040	322560	A002866
$L_1(\mathcal{S}_n^3)$	1	6	54	648	9720	174960	3674160	A034001
$L_1(\mathcal{S}_n^4)$	1	8	96	1536	30720	737280	20643840	A034177
$L_1(\mathcal{S}_n^5)$	1	10	150	3000	75000	2250000	78750000	A034325

Table 4: The numbers of 1-dimensional flats of the extended Shi arrangements

$L_k(\mathcal{B}_n)$					A008277
$n\backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

$L_k(\mathcal{S}_n^1)$					A105278
$n\backslash k$	1	2	3	4	5
1	1				
2	2	1			
3	6	6	1		
4	24	36	12	1	
5	120	240	120	20	1

$L_k(\mathcal{C}_n^1)$					A088729
$n\backslash k$	1	2	3	4	5
1	1				
2	3	1			
3	13	9	1		
4	75	79	18	1	
5	541	765	265	30	1

$L_k(\mathcal{S}_n^2)$					A079621
$n\backslash k$	1	2	3	4	5
1	1				
2	4	1			
3	24	12	1		
4	192	144	24	1	
5	1920	1920	480	40	1

$L_k(\mathcal{C}_n^2)$					NONE
$n\backslash k$	1	2	3	4	5
1	1				
2	5	1			
3	37	15	1		
4	365	223	30	1	
5	4501	3675	745	50	1

$L_k(\mathcal{S}_n^3)$					NONE
$n\backslash k$	1	2	3	4	5
1	1				
2	6	1			
3	54	18	1		
4	648	324	36	1	
5	9720	6480	1080	60	1

$L_k(\mathcal{C}_n^3)$					NONE
$n\backslash k$	1	2	3	4	5
1	1				
2	7	1			
3	73	21	1		
4	1015	439	42	1	
5	17641	10185	1465	70	1

$L_k(\mathcal{S}_n^4)$					A048786
$n\backslash k$	1	2	3	4	5
1	1				
2	8	1			
3	96	24	1		
4	1536	576	48	1	
5	30720	15360	1920	80	1

$L_k(\mathcal{C}_n^4)$					NONE
$n\backslash k$	1	2	3	4	5
1	1				
2	9	1			
3	121	27	1		
4	2169	727	54	1	
5	48601	21735	2425	90	1

$L_k(\mathcal{S}_n^5)$					NONE
$n\backslash k$	1	2	3	4	5
1	1				
2	10	1			
3	150	30	1		
4	3000	900	60	1	
5	75000	30000	3000	100	1

Table 5: Triangles of the number of k-dimensional flats

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