

MOTZKIN INTERVALS AND VALID HOOK CONFIGURATIONS

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ABSTRACT. We define a new natural partial order on Motzkin paths that serves as an intermediate step between two previously-studied partial orders. We provide a bijection between valid hook configurations of 312-avoiding permutations and intervals in these new posets. We also show that valid hook configurations of permutations avoiding 132 (or equivalently, 231) are counted by the same numbers that count intervals in the Motzkin-Tamari posets that Fang recently introduced, and we give an asymptotic formula for these numbers. We then proceed to enumerate valid hook configurations of permutations avoiding other collections of patterns. We also provide enumerative conjectures, one of which links valid hook configurations of 312-avoiding permutations, intervals in the new posets we have defined, and certain closed lattice walks with small steps that are confined to a quarter plane.

1. INTRODUCTION

1.1. Partial Orders on Motzkin Paths. A *Motzkin path* is a lattice path consisting of $(1, 1)$ steps (called *up steps*), $(1, -1)$ steps (called *down steps*), and $(1, 0)$ steps (called *east steps*) that starts at the origin, ends on the horizontal axis, and never passes below the horizontal axis. Let U, D, E represent up, down, and east steps, respectively. We can think of a Motzkin path Λ of length n as a word $\Lambda_1 \cdots \Lambda_n$ of length n over the alphabet $\{U, D, E\}$ that has as many U 's as it has D 's and also has the property that each of its prefixes has at least as many U 's as D 's. The number of Motzkin paths of length n is the n^{th} Motzkin number M_n (OEIS sequence A001006). Let \mathbf{M}_n be the set of Motzkin paths of length n . A *Dyck path* is a Motzkin path that has no east steps. Let \mathbf{D}_k be the set of Dyck paths of length $2k$.

There is a natural partial order \leq_S on \mathbf{M}_n that we obtain by declaring that $\Lambda \leq_S \Lambda'$ if Λ lies below or is equal to Λ' . Alternatively, we have $\Lambda_1 \cdots \Lambda_n \leq_S \Lambda'_1 \cdots \Lambda'_n$ if and only if the number of U 's in $\Lambda_1 \cdots \Lambda_i$ is at most the number of U 's in $\Lambda'_1 \cdots \Lambda'_i$ for every $i \in [n]$. When $n = 2k$, this order relation induces a poset on the subset $\mathbf{D}_k \subseteq \mathbf{M}_n$. Among other results, Ferrari and Pinzani [41] proved that the posets $\mathcal{L}_k^S := (\mathbf{D}_k, \leq_S)$ and $\mathcal{M}_n^S := (\mathbf{M}_n, \leq_S)$ are lattices. Bernardi and Bonichon [2] called \mathcal{L}_k^S the k^{th} *Stanley lattice*. By analogy, we call \mathcal{M}_n^S the n^{th} *Motzkin–Stanley lattice*.

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The k^{th} Tamari lattice, which we denote by \mathcal{L}_k^T , is an extremely important sublattice of the k^{th} Stanley lattice. Tamari lattices have seen a huge amount of attention from researchers in combinatorics, group theory, theoretical computer science, algebraic geometry, and algebraic topology [2, 17, 19, 21, 38, 40, 43, 45, 49, 52, 54, 56]. Recently, Fang introduced new posets defined on the sets \mathbf{M}_n that are analogous to the Tamari lattices. He investigated the structural and enumerative aspects of the components and intervals of these posets. We denote these posets, which we define formally in Section 2, by \mathcal{M}_n^T .

In Section 2, we define new posets \mathcal{M}_n^C that are natural intermediate steps between the Motzkin-Stanley lattices \mathcal{M}_n^S and the Motzkin-Tamari posets \mathcal{M}_n^T . More precisely, \mathcal{M}_n^T is a subposet of \mathcal{M}_n^C , which in turn is a subposet of \mathcal{M}_n^S . The intervals in the lattices \mathcal{L}_k^S and \mathcal{L}_k^T have been the subject of recent investigations [2, 19, 22, 23]. One of our main goals in this paper is to link the intervals in the posets \mathcal{M}_n^C and \mathcal{M}_n^T with recently-introduced combinatorial objects called “valid hook configurations.” We define these objects formally in Section 3, but roughly speaking, they are configurations of L-shaped “hooks” drawn on permutations that satisfy particular constraints. They have been used to understand West’s stack-sorting map, but they also possess their own rich combinatorial structure.

1.2. West’s Stack-Sorting Map. In his book *The Art of Computer Programming*, Knuth [48] introduced a certain “stack-sorting algorithm.” His analysis of this algorithm led to several important advances in combinatorics and computer science, such as the notion of pattern avoidance in permutations [3, 47, 51] and the “kernel method” [1]. In his Ph.D. dissertation, West [58] defined a deterministic variant of Knuth’s algorithm that has now received a considerable amount of attention [3–7, 12–16, 18, 20, 23–34, 36, 37, 39, 44, 57–59]. This variant is a function that sends permutations to permutations; we denote it by s and call it the *stack-sorting map*.

One of the central definitions in the study of the stack-sorting map is that of the *fertility* of a permutation π ; this is simply $|s^{-1}(\pi)|$, the number of preimages of π under s . Computing fertilities of permutations is, a priori, a difficult task. Indeed, West went through a great deal of effort to compute the fertilities of the specific permutations of the forms

$$23 \cdots k1(k+1) \cdots n, \quad 12 \cdots (k-2)k(k-1)(k+1) \cdots n, \quad \text{and} \quad k12 \cdots (k-1)(k+1) \cdots n.$$

Bousquet-Mélou [13] defined a permutation to be *sorted* if its fertility is positive. She gave an algorithm for determining whether or not a given permutation is sorted and stated that it would be interesting to find a general method for computing the fertility of any given permutation. This was accomplished in even greater generality in [30–32]; the method relies on valid hook configurations. The current author and his coauthors have recently developed a theory of valid hook configurations that provides a unified framework for understanding the stack-sorting map. Indeed, the articles [23–25, 27, 28, 30–34] apply these new objects in order to reprove and generalize several old results and formulate and prove new results concerning the stack-sorting map.

The article [33], which was the first to examine valid hook configurations as combinatorial objects in their own right, produced a bijection between valid hook configurations and certain weighted set partitions that Josuat-Vergès [46] had studied in the context of free probability theory. As a consequence, it showed that the total number of valid hook configurations of all permutations in S_n is $-k_{n+1}(-1)$, where $k_{n+1}(\lambda)$ is the $(n+1)^{\text{st}}$ cumulant of the free Poisson law with rate λ .¹ This is also equal to a sum of specific values of Tutte polynomials of crossing graphs of certain set partitions.

¹More precisely, there is a formula for this cumulant when $\lambda > 0$, and substituting $\lambda = -1$ into that formula yields the number of valid hook configurations.

A permutation is called *uniquely sorted* if its fertility is 1. The authors of [33] proved that a permutation in S_n is uniquely sorted if and only if it is sorted and has exactly $\frac{n-1}{2}$ descents. In particular, there are no uniquely sorted permutations of even length. By restricting the aforementioned bijection between valid hook configurations and weighted set partitions, they then proved that uniquely sorted permutations of odd length are counted by ‘‘Lassalle’s sequence,’’ a fascinating new sequence that was introduced in [50]. It turns out that counting uniquely sorted permutations in S_n is equivalent to counting valid hook configurations of permutations in S_n with $\frac{n-1}{2}$ hooks. Thus, results that enumerate certain uniquely sorted permutations can be interpreted as results that enumerate certain valid hook configurations.

The paper [23] produced bijections between uniquely sorted permutations that avoid certain patterns and intervals in posets defined on Dyck paths. Thus, the papers [23] and [33] have counted valid hook configurations of unrestricted permutations in S_n , valid hook configurations of permutations in S_n with $\frac{n-1}{2}$ hooks, and valid hook configurations of pattern-avoiding permutations in S_n with $\frac{n-1}{2}$ hooks. What is missing, which is the focus of the current article, is the investigation of valid hook configurations of pattern-avoiding permutations in S_n with no restriction on the number of hooks.

1.3. Notation and Terminology. A *permutation* is an ordering of a set of positive integers, which we write as a word. Let S_n be the set of permutations of the set $[n] = \{1, \dots, n\}$. If π is a permutation of length n , then the *normalization* of π is the permutation in S_n obtained by replacing the i^{th} -smallest entry in π with i for all $i \in [n]$. Given $\tau \in S_m$, we say a permutation $\sigma = \sigma_1 \cdots \sigma_n$ *contains the pattern* τ if there exist indices $i_1 < \cdots < i_m$ in $[n]$ such that the normalization of $\sigma_{i_1} \cdots \sigma_{i_m}$ is τ . We say σ *avoids* τ if it does not contain τ . Let $\text{Av}_n(\tau^{(1)}, \dots, \tau^{(r)})$ denote the set of permutations in S_n that avoid the patterns $\tau^{(1)}, \dots, \tau^{(r)}$. Let $\text{Av}(\tau^{(1)}, \dots, \tau^{(r)}) = \bigcup_{n \geq 0} \text{Av}_n(\tau^{(1)}, \dots, \tau^{(r)})$.

We let $\text{VHC}(\pi)$ denote the set of valid hook configurations of a permutation π (defined in Section 3). Given a set A of permutations, let $\text{VHC}(A) = \bigcup_{\pi \in A} \text{VHC}(\pi)$. Define the *tail length* of a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$, denoted $\text{tl}(\pi)$, to be the smallest nonnegative integer ℓ such that $\pi_{n-\ell} \neq n - \ell$. We make the convention that $\text{tl}(123 \cdots n) = n$. The tail length is a new permutation statistic that was introduced in [33] and has played a crucial role in [24, 26]; it will also be crucial for our proofs. If $\text{tl}(\pi) = \ell$, then the *tail* of π is the list of points $(n - \ell + 1, n - \ell + 1), \dots, (n, n)$.

An *interval* of a poset P is an ordered pair (x, y) of elements of P such that $x \leq y$. Let $\text{Int}(P)$ denote the set of intervals in the poset P . We let \mathcal{M}_n^S , \mathcal{M}_n^C , and \mathcal{M}_n^T be the posets defined on \mathbf{M}_n in Section 2. We also let \mathcal{M}_n^A denote the antichain on \mathbf{M}_n . Note that $|\text{Int}(\mathcal{M}_n^A)| = |\mathbf{M}_n| = M_n$.

1.4. Summary of Main Results. In Section 4, we produce, for each positive integer n , a bijection

$$\widehat{\mathbf{M}}_n : \text{VHC}(\text{Av}_n(312)) \rightarrow \text{Int}(\mathcal{M}_{n-1}^C).$$

This is an extension of a bijection between 312-avoiding uniquely sorted permutations (equivalently, valid hook configurations of 312-avoiding permutations in S_n with $\frac{n-1}{2}$ descents) and intervals in Stanley lattices that was established in [23]. We also give a recurrence that specifies the numbers $|\text{VHC}(\text{Av}_n(312))|$. We make a conjecture that links these numbers with certain lattice walks in the first quadrant that were studied in [8] and [10]. In Section 5, we prove that

$$\sum_{n \geq 1} |\text{VHC}(\text{Av}_n(132))| x^n = \sum_{n \geq 1} |\text{VHC}(\text{Av}_n(231))| x^n = \sum_{n \geq 1} |\text{Int}(\mathcal{M}_{n-1}^T)| x^n.$$

We will see that this generating function is algebraic of degree 5, and we will derive an asymptotic formula for its coefficients. In Section 6, we prove that

$$|\text{VHC}(\text{Av}_n(132, 231))| = |\text{VHC}(\text{Av}_n(132, 312))| = |\text{VHC}(\text{Av}_n(231, 312))| = M_{n-1}$$

for every positive integer n . We can interpret this last result as the statement that certain valid hook configurations are in bijection with intervals of the antichain \mathcal{M}_{n-1}^A . In Sections 7–10, we prove the following enumerative results:

$$\begin{aligned} \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(132, 321))| x^n &= \frac{1 - 3x + 3x^2}{(1 - x)^4}, \\ \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231, 321))| x^n &= \frac{1 - 2x + 2x^2 - \sqrt{1 - 4x + 4x^2 - 4x^3 + 4x^4}}{2x^2}, \\ |\text{VHC}(\text{Av}_n(312, 321))| &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2k+1} \binom{n-k-1}{k} \binom{n}{2k} \quad \text{for every } n \geq 1, \\ \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231, 1243))| x^n &= 1 + \frac{2x^2}{3x - 1 + \sqrt{1 - 2x - 3x^2}}. \end{aligned}$$

2. MOTZKIN POSETS

Every Motzkin path $\Lambda = \Lambda_1 \cdots \Lambda_n \in \mathbf{M}_n$ can be written uniquely in the form $X_1 D^{\gamma_1} X_2 D^{\gamma_2} \cdots X_m D^{\gamma_m}$ for some $X_1, \dots, X_m \in \{U, E\}$. Note that $n - m$ is the number of D 's appearing in the word, and this is also the number of U 's. For $j \in \{1, \dots, m\}$, we define $\text{long}_j(\Lambda)$ as follows. If $X_j = E$, then $\text{long}_j(\Lambda) = -1$. If $X_j = U$, let $\text{long}_j(\Lambda)$ be the smallest nonnegative integer t such that $\Lambda_{j+1} \cdots \Lambda_{j+t+1}$ contains more D 's than U 's. We call $(\text{long}_1(\Lambda), \dots, \text{long}_m(\Lambda))$ the *longevity sequence* of Λ . The *class* of Λ , denoted $\text{cl}(\Lambda)$, is the set of indices $j \in \{1, \dots, m\}$ such that $X_j = E$. For example, if $\Lambda = UEDUUEDUDED$ is the Motzkin path in Figure 1, then the longevity sequence of Λ is $(1, -1, 6, 1, -1, 0, -1)$, and the class of Λ is $\text{cl}(\Lambda) = \{2, 5, 7\}$.



FIGURE 1. The Motzkin path $UEDUUEDUDED$.

Although we will not actually use the definition of the Motzkin-Tamari poset \mathcal{M}_n^T , we state it for the sake of completeness. This definition is not identical to the one given in [40], but the results in that paper can be used to prove the equivalence of the different definitions.

Definition 2.1. Given Motzkin paths $\Lambda, \Lambda' \in \mathbf{M}_n$, we write $\Lambda \leq_T \Lambda'$ if $\text{cl}(\Lambda) = \text{cl}(\Lambda')$ and $\text{long}_j(\Lambda) \leq \text{long}_j(\Lambda')$ for every positive integer j for which $\text{long}_j(\Lambda)$ and $\text{long}_j(\Lambda')$ are defined. Let \mathcal{M}_n^T be the poset (\mathbf{M}_n, \leq_T) .

The Hasse diagram of \mathcal{M}_n^T has multiple connected components; two Motzkin paths are in the same component if and only if they have the same class. Fang [40] proved that each of these components is isomorphic to an interval in a classical Tamari lattice. It is straightforward to show that $\Lambda \leq_S \Lambda'$ whenever $\Lambda \leq_T \Lambda'$. In other words, \mathcal{M}_n^T is a subposet of \mathcal{M}_n^S . We now define a new partial order on \mathbf{M}_n that serves as a natural intermediate step between \leq_T and \leq_S .

Definition 2.2. Given Motzkin paths $\Lambda, \Lambda' \in \mathbf{M}_n$, we write $\Lambda \leq_C \Lambda'$ if $\text{cl}(\Lambda) = \text{cl}(\Lambda')$ and $\Lambda \leq_S \Lambda'$. Let \mathcal{M}_n^C be the poset (\mathbf{M}_n, \leq_C) .

3. VALID HOOK CONFIGURATIONS

The *plot* of a permutation $\pi = \pi_1 \cdots \pi_n$ is obtained by plotting the points (i, π_i) for all $i \in [n]$. A *descent* of π is an index $i \in [n - 1]$ such that $\pi_i > \pi_{i+1}$. If i is a descent of π , then we call the point (i, π_i) a *descent top of the plot of π* .

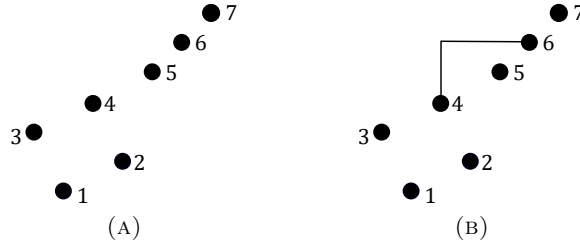


FIGURE 2. The left image is the plot of 3142567. The right image shows this plot along with a single hook.

A *hook* of π is drawn by starting at a point (i, π_i) in the plot of π , drawing a line segment vertically upward, and then drawing a line segment horizontally to the right until reaching another point (j, π_j) . This only makes sense if $i < j$ and $\pi_i < \pi_j$. The point (i, π_i) is called the *southwest endpoint* of the hook, while (j, π_j) is called the *northeast endpoint*. The right image in Figure 2 shows the plot of 3142567 along with a single hook whose southwest endpoint is $(3, 4)$ and whose northeast endpoint is $(6, 6)$.

Definition 3.1. Let π be a permutation of length n with k descents, say $d_1 < \cdots < d_k$. A *valid hook configuration* of π is a tuple (H_1, \dots, H_k) of hooks of π that satisfies the following constraints:

1. For every $i \in [n]$, the southwest endpoint of the hook H_i is the descent top (d_i, π_{d_i}) .
2. A point in the plot of π cannot lie directly above a hook.
3. Hooks cannot intersect or overlap each other except in the case that the northeast endpoint of one hook is the southwest endpoint of the other.

Let $\text{VHC}(\pi)$ denote the set of valid hook configurations of π . We make the convention that a valid hook configuration includes its underlying permutation as part of its definition. In other words, $\text{VHC}(\pi)$ and $\text{VHC}(\pi')$ are disjoint whenever π and π' are distinct. Furthermore, we agree that every increasing permutation (including the empty permutation) has exactly one valid hook configuration (which has no hooks).

Figure 3 shows four arrangements of hooks that cannot appear in a valid hook configuration. Figure 4 shows all of the valid hook configurations of 3142567.

Suppose H is a hook of a permutation π with southwest endpoint (i, π_i) and northeast endpoint (j, π_j) . Let $\text{VHC}^H(\pi)$ be the set of all valid hook configurations of π that include the hook H . Assume that j is larger than every descent of π . The hook H separates π into two parts. One part, which we call the *H -unsheltered subpermutation of π* and denote by π_U^H , is $\pi_1 \cdots \pi_i \pi_{j+1} \cdots \pi_n$. The other part, which we call the *H -sheltered subpermutation of π* and denote by π_S^H , is $\pi_{i+1} \cdots \pi_{j-1}$.

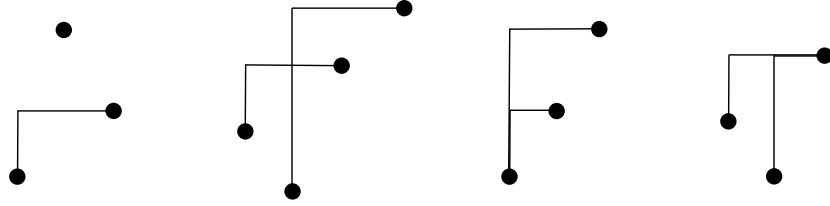


FIGURE 3. Four placements of hooks that are forbidden in a valid hook configuration.

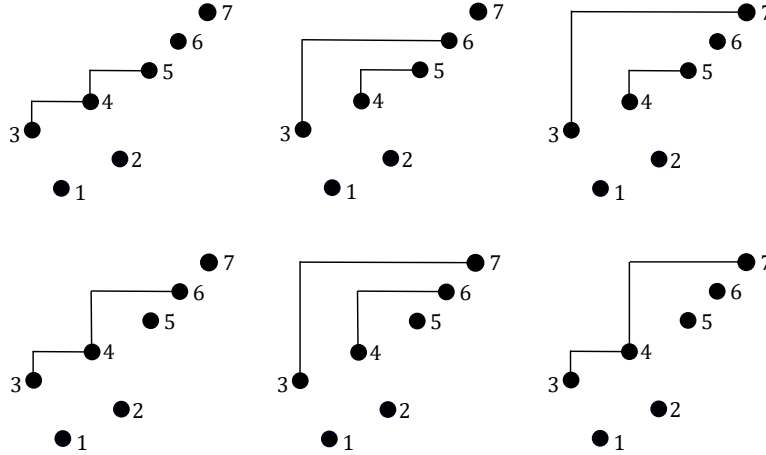
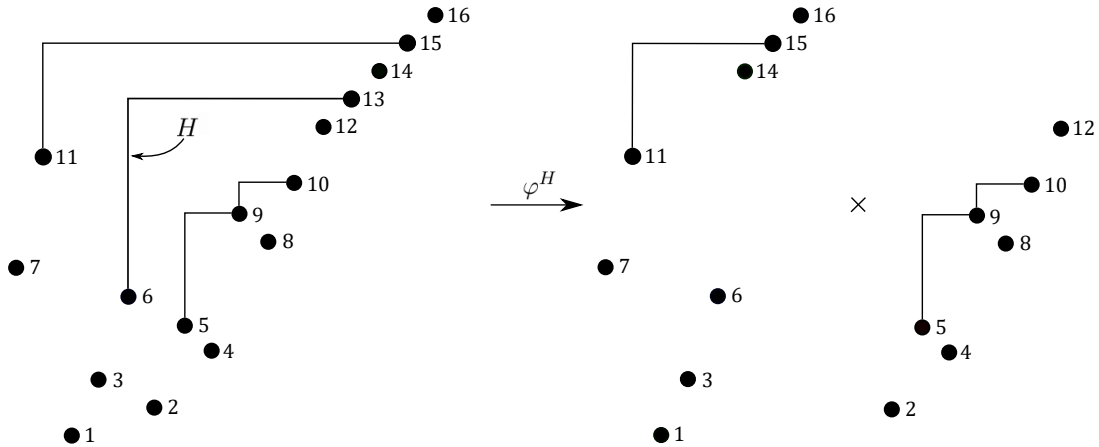


FIGURE 4. The permutation 3142567 has 6 valid hook configurations.

Note that the entry π_j does not appear in either of these two parts. This decomposition of π into the H -unsheltered and H -sheltered subpermutations provides a useful decomposition of valid hook configurations in $\text{VHC}^H(\pi)$, which we state in the following lemma. We use Figure 5 as a substitute for the proof of this lemma.

Lemma 3.1. *Let π be a permutation with descents $d_1 < \dots < d_k$. If H is a hook of π with northeast endpoint (j, π_j) and $j > d_k$, then there exists a bijection*

$$\varphi^H : \text{VHC}^H(\pi) \rightarrow \text{VHC}(\pi_U^H) \times \text{VHC}(\pi_S^H).$$

FIGURE 5. The bijection φ^H from Lemma 3.1

Recall from the introduction that the tail of a permutation $\pi \in S_n$ is the list of points $(n - \ell + 1, n - \ell + 1), \dots, (n, n)$, where $\ell = \text{tl}(\pi)$ is the tail length of π . Let $\text{SW}_i(\pi)$ be the set of hooks of a permutation π with southwest endpoint (i, π_i) . We say a descent d of π is *tail-bound* if every hook in $\text{SW}_d(\pi)$ has its northeast endpoint in the tail of π . The following corollary follows immediately from Lemma 3.1. It is very closely related to the ‘‘Decomposition Lemma’’ used to compute fertilities of permutations in [24] and [26].

Corollary 3.1. *If d is a tail-bound descent of a permutation $\pi \in S_n$, then*

$$|\text{VHC}(\pi)| = \sum_{H \in \text{SW}_d(\pi)} |\text{VHC}(\pi_U^H)| \cdot |\text{VHC}(\pi_S^H)|.$$

4. $\text{VHC}(\text{Av}(312))$

This section begins our exploration with an analysis of valid hook configurations of 312-avoiding permutations. We start by describing a correspondence between 312-avoiding permutations and certain matrices. It will be convenient to first establish one quick piece of terminology. Given an $\ell \times \ell$ matrix $M = (m_{ij})$ and indices $r, r', c, c' \in \{1, \dots, \ell\}$, consider the matrix obtained by deleting all rows of M except rows r and r' and deleting all columns of M except columns c and c' . We say this new matrix is a *lower 2×2 submatrix* of M if $\ell + 1 - c \leq r < r'$ and $c < c'$.

Fix $n \geq 1$ and a permutation $\pi = \pi_1 \cdots \pi_n \in \text{Av}_n(312)$ such that $\text{VHC}(\pi)$ is nonempty. We must have $\pi_n = n$. A *left-to-right maximum of the plot of π* is a point in the plot of π that is higher up than every point to its left. Let $\mathfrak{R}_0, \dots, \mathfrak{R}_\ell$ be these left-to-right maxima listed in order from right to left (so $\mathfrak{R}_0 = (n, n)$ and $\mathfrak{R}_\ell = (1, \pi_1)$). It will be convenient to let $\mathfrak{R}_{\ell+1} = (0, 0)$, although this is not a point in the plot of π . Let $M(\pi) = (m_{ij})$ be the $\ell \times \ell$ matrix in which m_{ij} is the number of points in the plot of π lying strictly vertically between \mathfrak{R}_i and \mathfrak{R}_{i+1} and strictly horizontally between $\mathfrak{R}_{\ell-j}$ and $\mathfrak{R}_{\ell-j+1}$. See Figure 6 for an example.

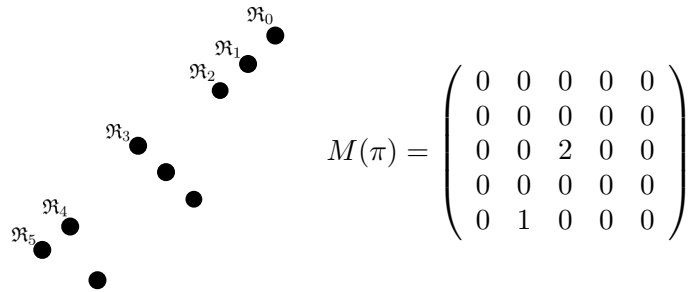


FIGURE 6. The matrix $M(\pi)$, where $\pi = 231654789$.

Remark 4.1. Note that $m_{ij} = 0$ whenever $j \leq \ell - i$. Furthermore, in every lower 2×2 submatrix of $M(\pi)$, either the bottom left entry or the top right entry is 0. Indeed, this follows from the fact that π avoids 312. We can easily reconstruct the plot of the permutation π from the matrix $M(\pi)$ by noting that the points lying horizontally between two consecutive left-to-right maxima must be decreasing in height from left to right and that the same must be true of points lying vertically between two left-to-right maxima. \diamond

Let us now choose a valid hook configuration $\mathcal{H} \in \text{VHC}(\pi)$. Note that every northeast endpoint of a hook in \mathcal{H} is a left-to-right maximum of the plot of π . Indeed, this follows from Condition 2 in

Definition 3.1 and the fact that π avoids 312. Let γ_i be the sum of the entries in column $\ell - i + 1$ of $M(\pi)$, and let γ'_i be the sum of the entries in row i of $M(\pi)$. Because $m_{ij} = 0$ whenever $j \leq \ell - i$, we must have

$$(1) \quad \gamma_1 + \cdots + \gamma_p \geq \gamma'_1 + \cdots + \gamma'_p \quad \text{for every } p \in \{1, \dots, \ell\}.$$

If \mathfrak{R}_{j-1} is the northeast endpoint of a hook in \mathcal{H} , let $X_j = U$; otherwise, let $X_j = E$. Let $\Lambda = X_1 D^{\gamma_1} X_2 D^{\gamma_2} \cdots X_\ell D^{\gamma_\ell}$ and $\Lambda' = X_1 D^{\gamma'_1} X_2 D^{\gamma'_2} \cdots X_\ell D^{\gamma'_\ell}$. Finally, let $\widehat{\mathcal{M}}(\mathcal{H}) = (\Lambda, \Lambda')$.

We claim that $(\Lambda, \Lambda') \in \text{Int}(\mathcal{M}_{n-1}^C)$. To see this, let $\Lambda = \Lambda_1 \cdots \Lambda_{n-1}$. Recall that the southwest endpoints of the hooks of \mathcal{H} are precisely the descent tops of the plot of π . Fix $p \in \{1, \dots, \ell\}$, and let Y_p be the set of points in the plot of π lying to the right of \mathfrak{R}_p . One can easily check that $\gamma_1 + \cdots + \gamma_p$ is the number of southwest endpoints of hooks in \mathcal{H} that are in $Y_p \cup \{R_p\}$. Similarly, the number of U 's in $X_1 \cdots X_p$ is the number of northeast endpoints of hooks in \mathcal{H} that are in Y_p . Since each southwest endpoint in $Y_p \cup \{\mathfrak{R}_p\}$ belongs to a hook whose northeast endpoint is in Y_p , we see that $\gamma_1 + \cdots + \gamma_p$ is at most the number of U 's in $X_1 \cdots X_p$. This is true for every $p \in \{1, \dots, p\}$, so Λ is a Motzkin path. It now follows from (1) that Λ' is also a Motzkin path and that $\Lambda \leq_S \Lambda'$. Finally, an index i appears in the class of Λ if and only if $X_i = E$. This occurs if and only if i is in the class of Λ' . It follows that $\text{cl}(\Lambda) = \text{cl}(\Lambda')$, so $\Lambda \leq_C \Lambda'$. We have now defined a map $\widehat{\mathcal{M}}_n : \text{VHC}(\text{Av}_n(312)) \rightarrow \text{Int}(\mathcal{M}_{n-1}^C)$.

Theorem 4.1. *The each positive integer n , the map*

$$\widehat{\mathcal{M}}_n : \text{VHC}(\text{Av}_n(312)) \rightarrow \text{Int}(\mathcal{M}_{n-1}^C)$$

is a bijection.

Proof. The proof relies on the following fact, which appears as a lemma in [23].

Fact: Let $a_1, \dots, a_\ell, b_1, \dots, b_\ell$ be nonnegative integers such that $a_1 + \cdots + a_\ell = b_1 + \cdots + b_\ell$ and $a_{\ell-i+1} + \cdots + a_\ell \leq b_{\ell-i+1} + \cdots + b_\ell$ for all $i \in \{1, \dots, \ell\}$. There exists an $\ell \times \ell$ matrix $M = (m_{ij})$ with nonnegative integer entries such that

- (i) $m_{ij} = 0$ whenever $j \leq \ell - i$;
- (ii) the sum of the entries in column i of M is b_i for every $i \in \{1, \dots, \ell\}$;
- (iii) the sum of the entries in row i of M is $a_{\ell-i+1}$ for every $i \in \{1, \dots, \ell\}$;
- (iv) in every lower 2×2 submatrix of M , either the bottom left entry or the top right entry is 0.

To prove that $\widehat{\mathcal{M}}_n$ is surjective, fix $(\Lambda, \Lambda') \in \text{Int}(\mathcal{M}_{n-1}^C)$. Because $\text{cl}(\Lambda) = \text{cl}(\Lambda')$, we can write $\Lambda = X_1 D^{\gamma_1} X_2 D^{\gamma_2} \cdots X_\ell D^{\gamma_\ell}$ and $\Lambda' = X_1 D^{\gamma'_1} X_2 D^{\gamma'_2} \cdots X_\ell D^{\gamma'_\ell}$ for some $X_1, \dots, X_\ell \in \{U, E\}$. Let $a_i = \gamma'_{\ell-i+1}$ and $b_i = \gamma_{\ell-i+1}$. Because Λ and Λ' are Motzkin paths, we have $a_1 + \cdots + a_\ell = b_1 + \cdots + b_\ell$. The fact that $\Lambda \leq_S \Lambda'$ tells us that $a_{\ell-i+1} + \cdots + a_\ell \leq b_{\ell-i+1} + \cdots + b_\ell$ for all $i \in \{1, \dots, \ell\}$. The above fact guarantees that there is a matrix $M = (m_{ij})$ satisfying the properties (i)–(iv). According to Remark 4.1, we can use such a matrix to obtain a permutation $\pi \in \text{Av}_n(312)$ such that $\text{VHC}(\pi) \neq \emptyset$ and $M(\pi) = M$. We claim that there is a unique valid hook configuration $\mathcal{H} \in \text{VHC}(\pi)$ such that $\widehat{\mathcal{M}}_n(\mathcal{H}) = (\Lambda, \Lambda')$. Let $\mathfrak{R}_0, \dots, \mathfrak{R}_\ell$ be the left-to-right maxima of the plot of π listed in order from right to left, and let $A = \{\mathfrak{R}_i : X_{i+1} = U\}$. The northeast endpoints of the hooks in \mathcal{H} are precisely the points in A . The specific choices of the hooks themselves are now determined by the conditions in Definition 3.1. We need to make sure that each descent top of the plot of π can actually find a corresponding northeast endpoint for its hook. In other words, we need to know that for each $i \in \{1, \dots, n-1\}$, the number of descent tops in the set $\{(n-i, \pi_{n-i}), \dots, (n, \pi_n)\}$ is at most the

number of elements of A in $\{(n-i+1, \pi_{n-i+1}), \dots, (n, \pi_n)\}$. This follows immediately from the fact that Λ is a Motzkin path.

To prove injectivity, let us assume there exist $\pi, \pi' \in \text{Av}_n(312)$, $\mathcal{H} \in \text{VHC}(\pi)$, and $\mathcal{H}' \in \text{VHC}(\pi')$ with $\widehat{\mathbb{M}}_n(\mathcal{H}) = \widehat{\mathbb{M}}_n(\mathcal{H}') = (\Lambda, \Lambda')$, where (Λ, Λ') is as above. Assume by way of contradiction that $\mathcal{H} \neq \mathcal{H}'$. We saw in the preceding paragraph that \mathcal{H} is uniquely determined by π and (Λ, Λ') . This means we must have $\pi \neq \pi'$. According to Remark 4.1, the matrices $M(\pi) = (m_{ij})$ and $M(\pi') = (m'_{ij})$ uniquely determine π and π' , respectively. Therefore, these matrices are distinct. However, both of these matrices satisfy properties (i)–(iv) from the above fact, where $a_i = \gamma'_{k-i+1}$ and $b_i = \gamma_{k-i+1}$. Because they are distinct, we can find a pair (i_0, j_0) with $m_{i_0 j_0} \neq m'_{i_0 j_0}$. We may assume that j_0 was chosen maximally, which means $m_{ij} = m'_{ij}$ whenever $j > j_0$. We may assume that i_0 was chosen maximally after j_0 was chosen, meaning $m_{ij_0} = m'_{ij_0}$ whenever $i > i_0$. We may assume without loss of generality that $m_{i_0 j_0} > m'_{i_0 j_0}$. Because $M(\pi)$ and $M(\pi')$ satisfy property (ii), their j_0^{th} columns have the same sum. This means that there exists $i_1 \neq i_0$ with $m_{i_1 j_0} < m'_{i_1 j_0}$. In particular, $m'_{i_1 j_0}$ is positive. The maximality of i_0 guarantees that $i_1 < i_0$. Because $M(\pi)$ and $M(\pi')$ satisfy property (iii), their i_1^{th} rows have the same sum. This means that there exists $j_1 \neq j_0$ with $m_{i_1 j_1} > m'_{i_1 j_1}$. The maximality of j_0 guarantees that $j_1 < j_0$. Since $M(\pi)$ satisfies property (i) and $m_{i_1 j_1} > 0$, we must have $\ell + 1 - j_1 \leq i_1$. Now, the j_1^{th} columns of $M(\pi)$ and $M(\pi')$ have the same sum, so there exists $i_2 \neq i_1$ such that $m_{i_2 j_1} < m'_{i_2 j_1}$. If $i_2 > i_1$, then $m'_{i_2 j_1}$ and $m'_{i_1 j_0}$ are positive numbers that form the bottom left and top right entries in a lower 2×2 submatrix of $M(\pi)$. This is impossible since $M(\pi)$ satisfies property (iv), so we must have $i_2 < i_1$. Continuing in this fashion, we find decreasing sequences of positive integers $i_0 > i_1 > i_2 > \dots$ and $j_0 > j_1 > j_2 > \dots$. This is our desired contradiction. \square

The preceding theorem tells us that $|\text{VHC}(\text{Av}_n(312))| = |\text{Int}(\mathcal{M}_{n-1}^C)|$. We now turn our attention toward obtaining a recurrence for these numbers. If $\lambda = \lambda_1 \dots \lambda_\ell \in S_\ell$ and $\mu = \mu_1 \dots \mu_m \in S_m$, then the *sum* of λ and μ , denoted $\lambda \oplus \mu$, is the permutation in $S_{\ell+m}$ obtained by “placing μ above and to the right of λ .” More formally, the i^{th} entry of $\lambda \oplus \mu$ is

$$(\lambda \oplus \mu)_i = \begin{cases} \lambda_i & \text{if } 1 \leq i \leq \ell; \\ \mu_{i-\ell} + \ell & \text{if } \ell + 1 \leq i \leq \ell + m. \end{cases}$$

A permutation is called *sum indecomposable* if it cannot be written as a sum of two shorter permutations. Every normalized permutation π can be written uniquely in the form $\pi = \lambda^{(1)} \oplus \dots \oplus \lambda^{(r)}$ for some sum indecomposable permutations $\lambda^{(1)}, \dots, \lambda^{(r)}$, which are called the *components* of π . Let $\text{comp}(\pi)$ denote the number of components of π . Recall the notation from Corollary 3.1.

Let

$$\mathcal{D}_{\ell,c}(n) = \{\pi \in \text{Av}_{n+\ell}(312) : \text{tl}(\pi) = \ell, \text{comp}(\pi) = c\}$$

and

$$\mathcal{D}_{\geq \ell, \geq c}(n) = \{\pi \in \text{Av}_{n+\ell}(312) : \text{tl}(\pi) \geq \ell, \text{comp}(\pi) \geq c\}.$$

Let $B_{\ell,c}(n) = |\text{VHC}(\mathcal{D}_{\ell,c}(n))|$ and $B_{\geq \ell, \geq c}(n) = |\text{VHC}(\mathcal{D}_{\geq \ell}(n))|$.

Suppose $\pi \in \mathcal{D}_{\ell,c}(n+1)$ for some $n \geq 0$ and $c \geq \ell + 1$. Note that π is not an identity permutation because it has length $\ell + n + 1$ and has tail length ℓ . Because π avoids 312, one can easily check that n is a tail-bound descent of π . Choose $j \in \{1, \dots, \ell\}$, and let H be the hook of π with southwest endpoint (n, π_n) and northeast endpoint $(n+1+j, n+1+j)$. We have $\pi_U^H = \pi_1 \dots \pi_{n-1}(n+2+j) \dots (n+\ell+1)$ and $\pi_S^H = \pi_{n+1}(n+2) \dots (n+j)$. The increasing permutation π_S^H has a unique valid hook configuration (with no hooks). The permutation π_U^H has

the same number of valid hook configurations as its normalization. This normalization is an element of $\mathcal{D}_{\geq \ell-j, \geq c-j}(n)$. Any element of $\mathcal{D}_{\geq \ell-j, \geq c-j}(n)$ can be the normalization of π_U^H . Combining these facts with Corollary 3.1, we find that

$$(2) \quad B_{\ell,c}(n+1) = \sum_{j=1}^{\ell} B_{\geq \ell-j, \geq c-j}(n).$$

Figure 7 illustrates this recurrence.

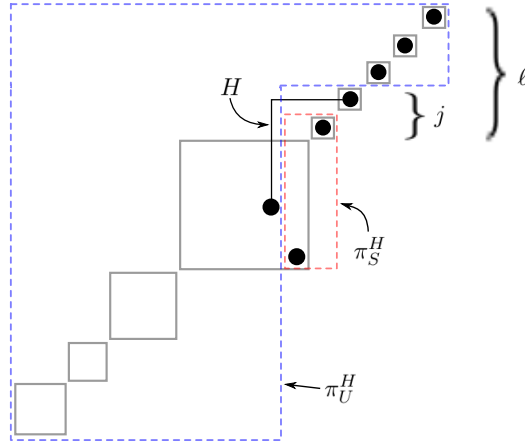


FIGURE 7. An illustration of the recurrence in equation (2). This is an example with $c = 9$, where the grey boxes represent the components of the permutation π .

If $c \geq \ell + 1$, then we also have $B_{\ell,c}(0) = 0$. If $c \leq \ell - 1$, then $B_{\ell,c}(n) = 0$. Finally, $B_{\ell,\ell}(0) = 1$ and $B_{\ell,\ell}(n) = 0$ for $n \neq 0$. These initial conditions and the recurrence in (7) allow us to efficiently compute the values of $B_{\ell,c}(n)$. Hence, we can efficiently compute the numbers $|\text{VHC}(\text{Av}_n(312))| = B_{\geq 0, \geq 0}(n)$. The first few values, starting at $n = 1$, are

1, 1, 2, 5, 14, 44, 148, 528, 1972, 7647, 30605, 125801, 529131, 2270481, 9914870, 43973755, 197744417.

We can also use this recurrence to derive the generating function equation in the following proposition; we omit the details.

Proposition 4.1. *We have*

$$\sum_{n \geq 0} |\text{VHC}(\text{Av}_n(312))| x^n = 1 + \sum_{n \geq 1} |\text{Int}(\mathcal{M}_{n-1}^C)| x^n = Q(x, 0, 0),$$

where $Q(x, y, z)$ is the trivariate power series satisfying

$$\begin{aligned} \frac{y}{1-y} (Q(x, y, z) - Q(x, y, 0)) &= \frac{Q(x, y, z) - 1/(1-y)}{x} - \frac{Q(x, y, z) - Q(x, 0, z)}{y} (z-1) \\ &\quad - \frac{Q(x, y, 0) - Q(x, 0, 0)}{y} - \frac{Q(x, y, z) - Q(x, y, 0)}{xz}. \end{aligned}$$

We have formulated the following conjecture that links valid hook configurations in $\text{VHC}(\text{Av}_n(312))$, intervals in the posets \mathcal{M}_{n-1}^C , and certain lattice walks. More precisely, let $\mathfrak{w}(k)$ be the number of lattice walks of length k that start and end at the origin, always stay in the first quadrant, and use the steps $(-1, 0)$, $(-1, 1)$, $(0, -1)$, $(0, 1)$, $(1, -1)$. These walks were studied in [8] and [10]. The first few values of $\mathfrak{w}(k)$ are given in the OEIS sequence A151347.

Conjecture 4.1. *For every positive integer n , we have*

$$|\text{VHC}(\text{Av}_n(312))| = |\text{Int}(\mathcal{M}_{n-1}^C)| = \sum_{k=0}^{n-1} \binom{n-1}{k} \mathfrak{w}(k).$$

5. $\text{VHC}(\text{Av}(132))$ AND $\text{VHC}(\text{Av}(231))$

We now consider the pattern 132 and the pattern 231. Our goal in this section is to prove the following theorem.

Theorem 5.1. *Let $\rho \approx 4.658905$ be the unique real root of $256x^3 - 645x^2 - 2112x - 2048$, and let $\beta \approx 0.805810$ be the unique positive real root of $41472x^6 - 34749x^4 + 5472x^2 - 256$. We have*

$$\sum_{n \geq 0} |\text{VHC}(\text{Av}_n(132))| x^n = \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231))| x^n = 1 + \sum_{n \geq 1} |\text{Int}(\mathcal{M}_{n-1}^T)| x^n,$$

and this generating function is algebraic of degree 5. Furthermore,

$$|\text{VHC}(\text{Av}_n(132))| = |\text{VHC}(\text{Av}_n(231))| \sim \frac{\beta}{\sqrt{\pi}} \cdot \frac{\rho^n}{n^{5/2}}.$$

The proof of this theorem is similar to the proof of the formula for $|s^{-1}(\text{Av}_n(231))|$ (which is the number of 2-stack-sortable permutations in S_n) given in [24]. In fact, the recurrence relations obtained in that proof and in the following argument are identical except in their initial conditions. The proof in [24] uses the sequence of Catalan numbers for initial values, whereas we use the simpler sequence $1, 1, 1, \dots$ here. What is interesting is that the generating function for $|\text{VHC}(\text{Av}_n(231))|$ is actually more complicated than the generating function for $|s^{-1}(\text{Av}_n(231))|$. Both are algebraic, but their degrees are 5 and 3, respectively. Moreover, the radius of convergence of the former is a cubic irrational while the radius of convergence of the latter is simply $4/27$.

Proof. In [28], the author found a bijection $\text{VHC}(\text{Av}(132)) \rightarrow \text{VHC}(\text{Av}(231))$ that preserves the lengths of the underlying permutations (and much more). Therefore, $\sum_{n \geq 0} |\text{VHC}(\text{Av}_n(132))| x^n = \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231))| x^n$. Let

$$\mathcal{D}_\ell(n) = \{\pi \in \text{Av}_{n+\ell}(231) : \text{tl}(\pi) = \ell\} \quad \text{and} \quad \mathcal{D}_{\geq \ell}(n) = \{\pi \in \text{Av}_{n+\ell}(231) : \text{tl}(\pi) \geq \ell\}.$$

Let $B_\ell(n) = |\text{VHC}(\mathcal{D}_\ell(n))|$ and $B_{\geq \ell}(n) = |\text{VHC}(\mathcal{D}_{\geq \ell}(n))|$.

Suppose $\pi \in \mathcal{D}_\ell(n+1)$ is such that $\pi_{n+1-i} = n+1$ (where $n \geq 0$). Then $n+1-i$ is a tail-bound descent of π . According to Corollary 3.1, $|\text{VHC}(\pi)|$ is equal to the number of triples $(H, \mathcal{H}_U, \mathcal{H}_S)$, where $H \in \text{SW}_{n+1-i}(\pi)$, $\mathcal{H}_U \in \text{VHC}(\pi_U^H)$, and $\mathcal{H}_S \in \text{VHC}(\pi_S^H)$. Choosing H amounts to choosing the number $j \in \{1, \dots, \ell\}$ such that the northeast endpoint of H is $(n+1+j, n+1+j)$. The permutation π and the choice of H determine the permutations π_U^H and π_S^H . On the other hand, the choices of H and the permutations π_U^H and π_S^H uniquely determine π . It follows that $B_\ell(n+1)$, which is the number of ways to choose an element of $\text{VHC}(\mathcal{D}_\ell(n+1))$, is also the number of ways to choose j , the permutations π_U^H and π_S^H , and the valid hook configurations \mathcal{H}_U and \mathcal{H}_S . Let us fix a choice of j .

Because π avoids 231, π_U^H must be a permutation of the set $\{1, \dots, n-i\} \cup \{n+1\} \cup \{n+2+j, \dots, n+\ell+1\}$, while π_S^H must be a permutation of $\{n-i+1, \dots, n+j\} \setminus \{n+1\}$. Therefore, choosing π_U^H and π_S^H is equivalent to choosing their normalizations. The normalization of π_U^H is in $\mathcal{D}_{\geq \ell-j+1}(n-i)$, while the normalization of π_S^H is in $\mathcal{D}_{\geq j-1}(i)$. Any element of

$\mathcal{D}_{\geq \ell-j+1}(n-i)$ can be chosen as the normalization of π_U^H , and any element of $\mathcal{D}_{\geq j-1}(i)$ can be chosen as the normalization of π_S^H . Also, every permutation has the same number of valid hook configurations as its normalization. Combining these facts, we find that the number of choices for π_U^H and \mathcal{H}_U is $|\text{VHC}(\mathcal{D}_{\geq \ell-j+1}(n-i))| = B_{\geq \ell-j+1}(n-i)$. Similarly, the number of choices for π_S^H and \mathcal{H}_S is $B_{\geq j-1}(i)$. Consequently,

$$(3) \quad B_\ell(n+1) = \sum_{i=1}^n \sum_{j=1}^{\ell} B_{\geq \ell-j+1}(n-i) B_{\geq j-1}(i).$$

Let

$$G_\ell(x) = \sum_{n \geq 0} B_{\geq \ell}(n) x^n \quad \text{and} \quad I(x, z) = \sum_{\ell \geq 0} G_\ell(x) z^\ell.$$

Note that

$$G_\ell(0) = B_{\geq \ell}(0) = |\text{VHC}(\mathcal{D}_{\geq \ell}(0))| = |\text{VHC}(123 \cdots \ell)| = 1.$$

Because $B_{\geq 0}(n) = |\text{VHC}(\text{Av}_n(231))|$, we wish to understand the generating function

$$I(x, 0) = G_0(x) = \sum_{n \geq 0} B_{\geq 0}(n) x^n = \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231))| x^n.$$

By (3), we have

$$(4) \quad \begin{aligned} \sum_{\ell \geq 0} \sum_{n \geq 0} B_\ell(n+1) x^n z^\ell &= \sum_{\ell \geq 0} \sum_{j=1}^{\ell} \sum_{n \geq 0} \sum_{i=1}^n B_{\geq \ell-j+1}(n-i) B_{\geq j-1}(i) x^n z^\ell \\ &= \sum_{\ell \geq 0} \sum_{j=1}^{\ell} G_{\ell-j+1}(x) (G_{j-1}(x) - G_{j-1}(0)) z^\ell = \sum_{\ell \geq 0} \sum_{j=1}^{\ell} G_{\ell-j+1}(x) (G_{j-1}(x) - 1) z^\ell \\ &= \left(\sum_{r \geq 0} G_{r+1}(x) z^r \right) \left(\sum_{j \geq 1} (G_{j-1}(x) - 1) z^j \right) = (I(x, z) - I(x, 0))(I(x, z) - 1/(1-z)). \end{aligned}$$

On the other hand,

$$B_\ell(n+1) = B_{\geq \ell}(n+1) - B_{\geq \ell+1}(n),$$

so

$$(5) \quad \begin{aligned} \sum_{\ell \geq 0} \sum_{n \geq 0} B_\ell(n+1) x^n z^\ell &= \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell}(n+1) x^n z^\ell - \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell+1}(n) x^n z^\ell \\ &= \frac{1}{x} \sum_{\ell \geq 0} (G_\ell(x) - 1) z^\ell - \frac{1}{z} \sum_{\ell \geq 0} G_{\ell+1}(x) z^{\ell+1} = \frac{I(x, z) - 1/(1-z)}{x} - \frac{I(x, z) - I(x, 0)}{z}. \end{aligned}$$

By (4) and (5), we have

$$(6) \quad xz(I(x, z) - I(x, 0))(I(x, z)(1-z) - 1) - z(I(x, z)(1-z) - 1) + x(1-z)(I(x, z) - I(x, 0)) = 0.$$

At this point, we use the ‘‘quadratic method,’’ which is discussed and substantially generalized in [9]. We can rewrite (6) as

$$(7) \quad R(I(x, z), I(x, 0), x, z)^2 = \Delta(I(x, 0), x, z),$$

where

$$R(u, v, x, z) = 2xz(1-z)u + x - z - 2xz + z^2 - xz(1-z)v$$

and

$$\Delta(v, x, z) = (x - (2 + v)xz + (-1 + z)z + vxz^2)^2 - 4xz(1 - z)(z + vx(2z - 1)).$$

There is a unique power series $Z = Z(x)$ such that $Z(x) = x + O(x^2)$ and $R(I(x, Z), I(x, 0), x, Z) = 0$. According to (7), $z = Z(x)$ is a repeated root of $\Delta(I(x, 0), x, z)$. This means that the discriminant of $\Delta(I(x, 0), x, z)$ with respect to z vanishes. Computing this discriminant, we find that $Q(I(x, 0), x) = 0$, where

$$(8) \quad Q(v, x) = (-1 + 6x + 15x^2 + 8x^3) + (1 - 11x + 28x^3 + 16x^4)v + (4x - 19x^2 - 14x^3)v^2 + (6x^2 - 9x^3 + 8x^4)v^3 + 4x^3v^4 + x^4v^5.$$

This polynomial is irreducible, so the generating function $I(x, 0) = \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231))|x^n = \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(132))|x^n$ is algebraic of degree 5 over the field $\mathbb{R}(x)$. Fang [40] found an algebraic equation (written a bit differently) that the generating function $1 + \sum_{n \geq 1} |\text{Int}(\mathcal{M}_{n-1}^T)|x^n$ satisfies, and one can show (after some straightforward manipulations that we omit) that it matches the equation we have just found for $I(x, 0)$.

We are left to prove the desired asymptotic formula for the coefficients of $I(x, 0)$. To do this, we invoke the techniques of singularity analysis outlined in Chapters VI and VII of [42]. We refer the reader to that book for the relevant definitions and details. The singularities of $I(x, 0)$ are contained in the set of roots of $x^{19}(1+x)^2(2048x^3 + 2112x^2 + 645x - 256)^3$, which is the discriminant of $Q(v, x)$ with respect to v . Pringsheim's theorem guarantees that the radius of convergence of $I(x, 0)$ is one of these singularities. It follows that the radius of convergence must be $1/\rho$, where ρ is as in the statement of the theorem. It is now routine to identify the branch of $Q(v, x)$ near $v = 1/\rho$ that corresponds to the combinatorially-defined generating function $I(x, 0)$ and expand it as a Puiseux series. We find that

$$I(x, 0) = \alpha_1 + \alpha_2(1/\rho - x) + \alpha_3(1/\rho - x)^{3/2} + o(1/\rho - x)^{3/2}$$

for some algebraic numbers $\alpha_1, \alpha_2, \alpha_3$. Moreover, $\alpha_3 = 4\beta\rho^{3/2}/3$, where β is as in the statement of the theorem. This translates to the asymptotic formula

$$|\text{VHC}(\text{Av}_n(231))| \sim \alpha_3\rho^{n-3/2} \frac{\Gamma(n-3/2)}{\Gamma(-3/2)\Gamma(n+1)} \sim \frac{\beta}{\sqrt{\pi}} \cdot \frac{\rho^n}{n^{5/2}}. \quad \square$$

Remark 5.1. We have shown that $|\text{VHC}(\text{Av}_n(132))| = |\text{Int}(\mathcal{M}_{n-1}^T)|$ for each $n \geq 1$. It would be very interesting to have a combinatorial proof of this fact. For example, one could hope for an injection $\psi_n : \text{VHC}(\text{Av}_n(132)) \rightarrow \text{VHC}(\text{Av}_n(312))$ such that $\widehat{\mathbb{M}}_n(\psi_n(\text{VHC}(\text{Av}_n(132)))) = \text{Int}(\mathcal{M}_{n-1}^T)$, where $\widehat{\mathbb{M}}_n$ is as in Section 4.

6. $\text{VHC}(\text{Av}(132, 231))$, $\text{VHC}(\text{Av}(132, 312))$, AND $\text{VHC}(\text{Av}(231, 312))$

This brief section is dedicated to proving the following theorem. Recall that \mathcal{M}_n^A is the antichain on the set \mathbf{M}_n and that $M_n = |\mathbf{M}_n| = |\text{Int}(\mathcal{M}_n^A)|$ is the n^{th} Motzkin number.

Theorem 6.1. *For every positive integer n , the bijection $\widehat{\mathbb{M}}_n : \text{VHC}(\text{Av}_n(312)) \rightarrow \text{Int}(\mathcal{M}_{n-1}^C)$ restricts to a bijection $\text{VHC}(\text{Av}_n(231, 312)) \rightarrow \text{Int}(\mathcal{M}_{n-1}^A)$. Furthermore,*

$$|\text{VHC}(\text{Av}_n(132, 231))| = |\text{VHC}(\text{Av}_n(132, 312))| = |\text{VHC}(\text{Av}_n(231, 312))| = M_{n-1}.$$

Proof. Recall the definition of the components of a permutation from Section 4. A permutation is called *layered* if its components are decreasing permutations. For example, the permutation $2143765 = (21) \oplus (21) \oplus (321)$ is layered because its components are 21, 21, and 321. It is well known

that the set of layered permutations in S_n is precisely $\text{Av}_n(231, 312)$. If $\pi \in \text{Av}_n(312)$, $\mathcal{H} \in \text{VHC}(\pi)$, and $\widehat{\mathbb{M}}_n(\mathcal{H}) = (\Lambda, \Lambda')$, then it is straightforward to check that $\Lambda = \Lambda'$ if and only if π is layered. In other words, $\mathbb{M}_n(\mathcal{H}) \in \text{Int}(\mathcal{M}_{n-1}^A)$ if and only if $\mathcal{H} \in \text{VHC}(\text{Av}_n(231, 312))$. In [28], the author found bijections $\text{VHC}(\text{Av}(231, 312)) \rightarrow \text{VHC}(\text{Av}(132, 312))$ and $\text{VHC}(\text{Av}(132, 231)) \rightarrow \text{VHC}(\text{Av}(231, 312))$ that preserve the lengths of the underlying permutations. This proves the last statement of the theorem. \square

7. $\text{VHC}(\text{Av}(132, 321))$

Theorem 7.1. *We have*

$$\sum_{n \geq 0} |\text{VHC}(\text{Av}_n(132, 321))| x^n = \frac{1 - 3x + 3x^2}{(1 - x)^4}.$$

Proof. Choose $n \geq 1$ and $\ell \in \{1, \dots, n-1\}$. By concatenating the increasing permutations $(i+1) \cdots (n-\ell)$, $1 \cdots i$, and $(n-\ell+1) \cdots n$, we obtain the permutation $\zeta_i = (i+1) \cdots (n-\ell) 1 \cdots i (n-\ell+1) \cdots n$. One can easily check that $\zeta_1, \dots, \zeta_{n-\ell-1}$ are precisely the permutations in $\text{Av}_n(132, 321)$ with tail length ℓ and that each of these permutations has exactly ℓ valid hook configurations. It follows that

$$|\text{VHC}_n(132, 321)| = 1 + \sum_{\ell=1}^{n-1} (n-\ell-1)\ell.$$

The remainder of the proof is routine. \square

8. $\text{VHC}(231, 321)$

Theorem 8.1. *We have*

$$\sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231, 321))| x^n = \frac{1 - 2x + 2x^2 - \sqrt{1 - 4x + 4x^2 - 4x^3 + 4x^4}}{2x^2}.$$

Proof. Let

$$\mathcal{D}_\ell(n) = \{\pi \in \text{Av}_{n+\ell}(231, 321) : \text{tl}(\pi) = \ell\} \quad \text{and} \quad \mathcal{D}_{\geq \ell}(n) = \{\pi \in \text{Av}_{n+\ell}(231, 321) : \text{tl}(\pi) \geq \ell\}.$$

Let $B_\ell(n) = |\text{VHC}(\mathcal{D}_\ell(n))|$ and $B_{\geq \ell}(n) = |\text{VHC}(\mathcal{D}_{\geq \ell}(n))|$.

Suppose $\pi \in \mathcal{D}_\ell(n+1)$ is such that $\pi_{n+1-i} = n+1$ (where $n \geq 0$). Then $n+1-i$ is a tail-bound descent of π . Corollary 3.1 tells us that $|\text{VHC}(\pi)|$ is equal to the number of triples $(H, \mathcal{H}_U, \mathcal{H}_S)$, where $H \in \text{SW}_{n+1-i}(\pi)$, $\mathcal{H}_U \in \text{VHC}(\pi_U^H)$, and $\mathcal{H}_S \in \text{VHC}(\pi_S^H)$. Choosing H amounts to choosing the number $j \in \{1, \dots, \ell\}$ such that the northeast endpoint of H is $(n+1+j, n+1+j)$. As in the proof of Theorem 5.1, we find that $B_\ell(n+1)$ is the number of ways to choose j , the permutations π_U^H and π_S^H , and the valid hook configurations \mathcal{H}_U and \mathcal{H}_S . Fix a choice of j .

Because π avoids 231, π_U^H must be a permutation of the set $\{1, \dots, n-i\} \cup \{n+1\} \cup \{n+2+j, \dots, n+\ell+1\}$, while π_S^H must be a permutation of $\{n-i+1, \dots, n+j\} \setminus \{n+1\}$. Because π avoids 321, π_S^H is the increasing permutation on the set $\{n-i+1, \dots, n+j\} \setminus \{n+1\}$. There is one choice for π_S^H and \mathcal{H}_S . Choosing π_U^H is equivalent to choosing its normalization, which is in $\mathcal{D}_{\geq \ell-j+1}(n-i)$. Any element of $\mathcal{D}_{\geq \ell-j+1}(n-i)$ can be chosen as the normalization

of π_U^H . Furthermore, every permutation has the same number of valid hook configurations as its normalization. Combining these facts, we find that the number of choices for π_U^H and \mathcal{H}_U is $|\text{VHC}(\mathcal{D}_{\geq \ell-j+1}(n-i))| = B_{\geq \ell-j+1}(n-i)$. Thus,

$$(9) \quad B_\ell(n+1) = \sum_{i=1}^n \sum_{j=1}^{\ell} B_{\geq \ell-j+1}(n-i).$$

Let

$$G_\ell(x) = \sum_{n \geq 0} B_{\geq \ell}(n)x^n \quad \text{and} \quad I(x, z) = \sum_{\ell \geq 0} G_\ell(x)z^\ell.$$

Note that $G_\ell(0) = B_{\geq \ell}(0) = |\text{VHC}(\mathcal{D}_{\geq \ell}(0))| = |\text{VHC}(123 \cdots \ell)| = 1$. We wish to understand the generating function

$$I(x, 0) = G_0(x) = \sum_{n \geq 0} B_{\geq 0}(n)x^n = \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231, 321))|x^n.$$

By (9), we have

$$(10) \quad \begin{aligned} \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell}(n+1)x^n z^\ell &= \sum_{\ell \geq 0} \sum_{n \geq 0} \sum_{i=1}^n \sum_{j=1}^{\ell} B_{\geq \ell-j+1}(n-i)x^n z^\ell = \sum_{\ell \geq 0} \sum_{j=1}^{\ell} \frac{x}{1-x} G_{\ell-j+1}(x)z^\ell \\ &= \frac{x(I(x, z) - I(x, 0))}{(1-x)(1-z)}. \end{aligned}$$

The same argument used to deduce (5) in the proof of Theorem 5.1 shows that

$$(11) \quad \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell}(n+1)x^n z^\ell = \frac{I(x, z) - 1/(1-z)}{x} - \frac{I(x, z) - I(x, 0)}{z}.$$

We can combine (10) and (11) and rearrange terms to obtain the equation

$$(12) \quad I(x, z)((1-x)z^2 + (-1+2x^2)z + x - x^2) = (-1+x)z + I(x, 0)(x - x^2 + (-x+2x^2)z).$$

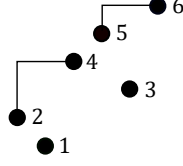
We now use the *kernel method* (see [1, 9, 11, 55] for details about this method). Let $Z(x) = \frac{1 - 2x^2 - \sqrt{1 - 4x + 4x^2 - 4x^3 + 4x^4}}{2(1-x)}$ so that $(1-x)Z(x)^2 + (-1+2x^2)Z(x) + x - x^2 = 0$. We can substitute $z = Z(x)$ in (12) to find that $(-1+x)Z(x) + I(x, 0)(x - x^2 + (-x+2x^2)Z(x)) = 0$. Thus,

$$\begin{aligned} I(x, 0) &= \frac{(1-x)Z(x)}{x - x^2 + (-x+2x^2)Z(x)} = \frac{(1-x)Z(x)}{-((1-x)Z(x)^2 + (-1+2x^2)Z(x)) + (-x+2x^2)Z(x)} \\ &= \frac{1}{1-Z(x)} = \frac{1 - 2x + 2x^2 - \sqrt{1 - 4x + 4x^2 - 4x^3 + 4x^4}}{2x^2}. \quad \square \end{aligned}$$

9. $\text{VHC}(\text{Av}(312, 321))$

In the proof of the following theorem, it will be helpful to consider a new statistic defined on valid hook configurations. Suppose \mathcal{H} is a valid hook configuration of a permutation π . Recall that a left-to-right maximum of the plot of π is a point in the plot of π that is higher than every point to its left. An *open site* of \mathcal{H} is a left-to-right maximum of the plot of π that is not a northeast

endpoint of a hook in \mathcal{H} . Define the *abundancy* of \mathcal{H} , denoted $\text{ab}(\mathcal{H})$, to be the number of open sites of \mathcal{H} . For example, the abundancy of the valid hook configuration



is 2 because the open sites are (1, 2) and (4, 5).

Theorem 9.1. *For each positive integer n , we have*

$$|\text{VHC}(\text{Av}_n(312, 321))| = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2k+1} \binom{n-k-1}{k} \binom{n}{2k}.$$

Proof. Let

$$\mathcal{E}_a(n) = \{\mathcal{H} \in \text{VHC}(\text{Av}_{n+a}(312, 321)) : \text{ab}(\mathcal{H}) = a\}$$

and

$$\mathcal{E}_{\geq a}(n) = \{\mathcal{H} \in \text{VHC}(\text{Av}_{n+a}(312, 321)) : \text{ab}(\mathcal{H}) \geq a\}.$$

Let $A_a(n) = |\mathcal{E}_a(n)|$ and $A_{\geq a}(n) = |\mathcal{E}_{\geq a}(n)|$. Let $\tilde{\mathcal{E}}_a(n)$ be the set of valid hook configurations $\mathcal{H} \in \mathcal{E}_a(n)$ such that the first entry of the underlying permutation of \mathcal{H} is not 1. Removing the leftmost point (which is also the point with height 1) from each valid hook configuration in $\mathcal{E}_a(n+1) \setminus \tilde{\mathcal{E}}_a(n+1)$ yields a bijection from $\mathcal{E}_a(n+1) \setminus \tilde{\mathcal{E}}_a(n+1)$ to $\mathcal{E}_{a-1}(n+1)$, so $A_a(n+1) = A_{a-1}(n+1) + |\tilde{\mathcal{E}}_a(n+1)|$.

Now suppose $\mathcal{H} \in \tilde{\mathcal{E}}_a(n+1)$, and let $\pi = \pi_1 \cdots \pi_{n+a+1}$ be the underlying permutation of \mathcal{H} . Let $r \geq 1$ be such that $\pi_{r+1} = 1$. Because π avoids 312 and 321, we must have $\pi_1 \cdots \pi_r = 23 \cdots (r+1)$. The points $(i, i+1)$ for $i \in \{1, \dots, r\}$ are all open sites of \mathcal{H} , so $r \leq a$. There must be a hook of \mathcal{H} with southwest endpoint $(r, r+1)$. If we remove this hook along with all of the points (i, π_i) for $i \in \{1, \dots, r+1\}$ and then “normalize” the remaining points and hooks, we obtain a valid hook configuration $\mathcal{H}' \in \mathcal{E}_{\geq a-r+1}(n-1)$. On the other hand, it is easy to recover \mathcal{H} if we are just given \mathcal{H}' and the values of n, a, r . We depict this in Figure 8. It follows that

$$|\tilde{\mathcal{E}}_a(n+1)| = \sum_{r=1}^a |\mathcal{E}_{\geq a-r+1}(n-1)| = \sum_{r=1}^a A_{\geq a-r+1}(n-1).$$

Consequently,

$$(13) \quad A_a(n+1) = A_{a-1}(n+1) + \sum_{r=1}^a A_{\geq a-r+1}(n-1).$$

Now let $J(x, z) = \sum_{a \geq 0} \sum_{n \geq 0} A_{\geq a}(n) x^n z^a$. Note that we are primarily interested in the generating function

$$J(x, 0) = \sum_{n \geq 0} A_{\geq 0}(n) x^n = \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(312, 321))| x^n.$$

We have

$$(14) \quad \begin{aligned} \sum_{a \geq 0} \sum_{n \geq 0} A_a(n+1) x^n z^a &= \sum_{a \geq 0} \sum_{n \geq 0} A_{\geq a}(n+1) x^n z^a - \sum_{a \geq 0} \sum_{n \geq 0} A_{\geq a+1}(n) x^n z^a \\ &= \frac{J(x, z) - 1/(1-z)}{x} - \frac{J(x, z) - J(x, 0)}{z} \end{aligned}$$

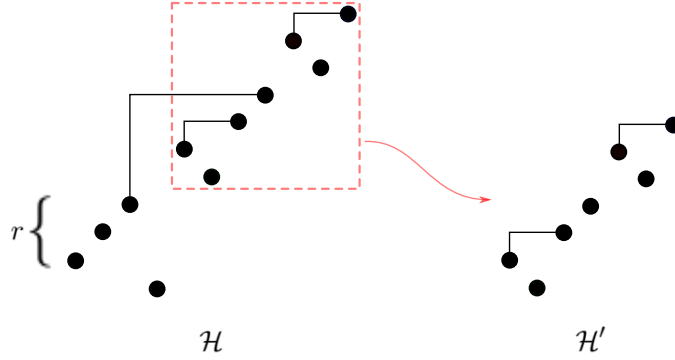


FIGURE 8. An illustration of the proof of Theorem 9.1. In this example, $n = 6$, $a = 4$, and $r = 3$.

and

$$\begin{aligned}
 \sum_{a \geq 0} \sum_{n \geq 0} A_{a-1}(n+1)x^n z^a &= \sum_{a \geq 1} \sum_{n \geq 0} A_{a-1}(n+1)x^n z^a = \sum_{a \geq 1} \sum_{n \geq 0} A_{\geq a-1}(n+1)x^n z^a - \sum_{a \geq 1} \sum_{n \geq 0} A_{\geq a}(n)x^n z^a \\
 (15) \qquad \qquad \qquad &= \frac{z}{x}(J(x, z) - 1/(1-z)) - (J(x, z) - J(x, 0)).
 \end{aligned}$$

Combining (13) and (15) gives

$$\begin{aligned}
 \sum_{a \geq 0} \sum_{n \geq 0} A_a(n+1)x^n z^a &= \sum_{a \geq 0} \sum_{n \geq 0} A_{a-1}(n+1)x^n z^a + \sum_{a \geq 0} \sum_{n \geq 0} \sum_{r=1}^a A_{\geq a-r+1}(n-1)x^n z^a \\
 &= \frac{z}{x}(J(x, z) - 1/(1-z)) - (J(x, z) - J(x, 0)) + x \sum_{a \geq 0} \sum_{r=1}^a \sum_{n \geq 0} A_{\geq a-r+1}(n)x^n z^a \\
 &= \frac{z}{x}(J(x, z) - 1/(1-z)) - (J(x, z) - J(x, 0)) + \frac{x}{1-z}(J(x, z) - J(x, 0)) \\
 &= J(x, z) \left(\frac{z}{x} + \frac{x}{1-z} - 1 \right) - \frac{z}{x(1-z)} + J(x, 0) \left(1 - \frac{x}{1-z} \right).
 \end{aligned}$$

We now combine this with (14) to obtain the equation

$$\frac{J(x, z) - 1/(1-z)}{x} - \frac{J(x, z) - J(x, 0)}{z} = J(x, z) \left(\frac{z}{x} + \frac{x}{1-z} - 1 \right) - \frac{z}{x(1-z)} + J(x, 0) \left(1 - \frac{x}{1-z} \right).$$

Rearranging this equation yields

$$(16) \qquad \frac{1}{z(1-z)^2}(J(x, z) - J(x, 0)) (x(1-z)^2 + x^2z - z(1-z)^2) = J(x, 0) - \frac{1}{1-z}.$$

We now employ the kernel method. There is a unique power series $Z = Z(x)$ such that

$$(17) \qquad x(1-Z)^2 + x^2Z - Z(1-Z)^2 = 0 \quad \text{and} \quad \frac{1}{1-Z(x)} = 1 + x + x^2 + O(x^3).$$

If we substitute $z = Z(x)$ in (16), we find that $J(x, 0) = \frac{1}{1-Z(x)}$. Now,

$$J(x, 0) - 1 - \frac{xJ(x, 0)}{1 - x^2J(x, 0)^2} = \frac{1}{1-Z} - 1 - \frac{x \frac{1}{1-Z}}{1 - x^2 \frac{1}{(1-Z)^2}} = \frac{Z}{1-Z} - \frac{x(1-Z)}{(1-Z)^2 - x^2} = 0,$$

where the last equality follows from the first equation in (17). This means that

$$J(x, 0) = 1 + \frac{xJ(x, 0)}{1 - x^2J(x, 0)^2}.$$

The theorem now follows from the Lagrange Inversion formula. \square

Remark 9.1. One can show that $\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{1}{2k+1} \binom{n-k-1}{k} \binom{n}{2k}$ is also the number of Dyck paths of length $2n$ in which every string of consecutive down steps has odd length (see [53]). It would be interesting to have a bijection between the set of such Dyck paths and the set $\text{VHC}(\text{Av}_n(312, 321))$.

10. $\text{VHC}(\text{Av}(231, 1243))$

In [23], the current author considered uniquely sorted permutations avoiding one length-3 pattern and one length-4 pattern. He found connections between these uniquely sorted permutations and intervals in lattices of Dyck paths. He also gave several additional enumerative conjectures concerning uniquely sorted permutations avoiding a length-3 pattern and a length-4 pattern. In a similar vein, it seems promising to enumerate valid hook configurations of permutations that avoid a length-3 pattern and a length-4 pattern. The purpose of this section is to initiate this investigation by enumerating valid hook configurations of permutations that avoid 231 and 1243. To do this, we rely on a lemma that makes use of our connection between valid hook configurations and Motzkin paths. In what follows, let

$$A_\ell(n) = |\text{VHC}(\{\pi \in \text{Av}_{n+\ell}(132, 231) : \text{tl}(\pi) = \ell\})|,$$

$$A_{\geq \ell}(n) = |\text{VHC}(\{\pi \in \text{Av}_{n+\ell}(132, 231) : \text{tl}(\pi) \geq \ell\})|,$$

and

$$J(x, z) = \sum_{\ell \geq 0} \sum_{n \geq 0} A_{\geq \ell}(n) x^n z^\ell.$$

Lemma 10.1. *We have*

$$J(x, z) = \frac{(-1 + 2z)(1 - \sqrt{1 - 2x - 3x^2}) - x}{(1 - z)(x(-2 + z) + (1 - \sqrt{1 - 2x - 3x^2})z)}$$

Proof. Let $M(x) = \sum_{n \geq 0} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}$ be the generating function of the sequence of Motzkin numbers, and consider the generating function $F(x, z) = \frac{xz^2(1 + xM(x))}{1 - xz(1 + xM(x))}$. For $n \geq 1$, let $\mathbf{a}(n, \ell)$ be the number of Motzkin paths of length n in which ℓ endpoints of steps touch the horizontal axis. Also, let $\mathbf{b}(n, \ell)$ be the number of Motzkin paths of length n in which the first down step is the ℓ^{th} step (with the convention that $\mathbf{b}(n, n+1) = 1$). In [35], Deutsch described a simple involution on Dyck paths. By extending this involution in an obvious way to Motzkin paths, one can show that $\mathbf{a}(n, \ell) = \mathbf{b}(n, \ell)$. Furthermore, it is known [53] that $\mathbf{a}(n, \ell)$ is the coefficient of $x^n z^\ell$ in $F(x, z)$. Hence, $\mathbf{b}(n, \ell)$ is the coefficient of $x^n z^\ell$ in $F(x, z)$.

We now refer back to the proof of Theorem 6.1. If Λ is a Motzkin path of length $n - 1$ and $\ell \in \{0, \dots, n - 2\}$, then the first down step in Λ is the $(\ell + 1)^{\text{st}}$ step if and only if the tail length of the underlying permutation of $\widehat{\mathbb{M}}_n^{-1}(\Lambda, \Lambda)$ is ℓ . Moreover, the bijection $\text{VHC}(\text{Av}(132, 231)) \rightarrow \text{VHC}(\text{Av}(231, 312))$ from [28] preserves lengths and tail lengths of the underlying permutations of valid hook configurations. It follows that $A_\ell(n - \ell) = \mathbf{b}(n - 1, \ell + 1)$ whenever $n \geq 3$ and

$\ell \in \{1, \dots, n-2\}$. We also have $A_\ell(0) = 1$ and $A_\ell(1) = 0$ for all $\ell \geq 0$. After putting this all together, we find that

$$(18) \quad \sum_{\ell \geq 0} \sum_{n \geq 0} A_\ell(n-\ell) x^n z^\ell = 1 + xz + x^2 \left(\frac{1}{xz} F(x, z) - \frac{z(1-z)}{1-xz} \right).$$

Let $\tilde{F}(x, z)$ be the generating function in (18). Straightforward manipulations allow us to find that

$$(19) \quad J(x, z) = \frac{(z/x)\tilde{F}(x, z/x) - J(x, 0)}{z/x - 1}.$$

It follows from Theorem 6.1 that $J(x, 0) = 1 + xM(x)$. The remainder of the proof now amounts to substituting the relevant expressions into (19) and simplifying. \square

Theorem 10.1. *We have*

$$\sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231, 1243))| x^n = 1 + \frac{2x^2}{3x - 1 + \sqrt{1 - 2x - 3x^2}}.$$

Proof. Let

$$\mathcal{D}_\ell(n) = \{\pi \in \text{Av}_{n+\ell}(231, 1243) : \text{tl}(\pi) = \ell\} \quad \text{and} \quad \mathcal{D}_{\geq \ell}(n) = \{\pi \in \text{Av}_{n+\ell}(231, 1243) : \text{tl}(\pi) \geq \ell\}.$$

Let $B_\ell(n) = |\text{VHC}(\mathcal{D}_\ell(n))|$ and $B_{\geq \ell}(n) = |\text{VHC}(\mathcal{D}_{\geq \ell}(n))|$.

Suppose $\pi \in \mathcal{D}_\ell(n+1)$ is such that $\pi_{n+1-i} = n+1$ (where $n \geq 0$). Then $n+1-i$ is a tail-bound descent of π . We can use Corollary 3.1 to see that $|\text{VHC}(\pi)|$ is equal to the number of triples $(H, \mathcal{H}_U, \mathcal{H}_S)$, where $H \in \text{SW}_{n+1-i}(\pi)$, $\mathcal{H}_U \in \text{VHC}(\pi_U^H)$, and $\mathcal{H}_S \in \text{VHC}(\pi_S^H)$. Choosing H amounts to choosing the number $j \in \{1, \dots, \ell\}$ such that the northeast endpoint of H is $(n+1+j, n+1+j)$. The permutation π and the choice of H determine the permutations π_U^H and π_S^H . On the other hand, the choices of H and the permutations π_U^H and π_S^H uniquely determine π . It follows that $B_\ell(n+1)$, which is the number of ways to choose an element of $\text{VHC}(\mathcal{D}_\ell(n+1))$, is also the number of ways to choose j , the permutations π_U^H and π_S^H , and the valid hook configurations \mathcal{H}_U and \mathcal{H}_S . Let us fix a choice of j .

First, suppose $i \leq n-1$. Because π avoids 231 and 1243, one can show that

$$\pi_U^H = (n-i)(n-i-1) \cdots 1(n+1)(n+2+j)(n+3+j) \cdots (n+\ell+1),$$

while π_S^H must be a permutation of $\{n-i+1, \dots, n+j\} \setminus \{n+1\}$ that avoids 132 and 231. The valid hook configuration \mathcal{H}_U has $n-i-1$ hooks; choosing this valid hook configuration amounts to choosing the northeast endpoints of these hooks from the top $\ell-j+1$ points in π_U^H . Thus, the number of choices for \mathcal{H}_U is $\binom{\ell-j+1}{n-i-1}$. The normalization of π_S^H is in $\text{Av}_{i+j-1}(132, 231)$ and has tail length at least $j-1$. Any permutation in $\text{Av}_{i+j-1}(132, 231)$ and has tail length at least $j-1$ can be chosen as the normalization of π_S^H . Also, every permutation has the same number of valid hook configurations as its normalization. Consequently, the number of choices for π_S^H and \mathcal{H}_S is $A_{\geq j-1}(i)$, where we have preserved the notation immediately preceding Lemma 10.1.

If $i = n$, then we repeat the same argument, except that there is only one choice for π_U^H and \mathcal{H}_U and that the number of choices for π_S^H and \mathcal{H}_S is $B_{\geq j-1}(n)$. We now obtain the recurrence

$$B_\ell(n+1) = \sum_{j=1}^{\ell} \sum_{i=1}^{n-1} \binom{\ell-j+1}{n-i-1} A_{\geq j-1}(i) + \sum_{j=1}^{\ell} B_{\geq j-1}(n).$$

After multiplying this equation by $x^n z^\ell$, summing over $\ell \geq 0$ and $n \geq 0$, and simplifying, we find that

$$(20) \quad \sum_{\ell \geq 0} \sum_{n \geq 0} B_\ell(n+1)x^n z^\ell = \frac{z}{1-z} \left(I(x, z) - \frac{1}{1-z} \right) + \frac{xz(1+x)}{1-(1+x)z} \left(J(x, z) - \frac{1}{1-z} \right),$$

where $I(x, z) = \sum_{\ell \geq 0} \sum_{n \geq 0} B_{\geq \ell}(n)x^n z^\ell$ and $J(x, z)$ is as in Lemma 10.1. The same argument used to derive (5) in the proof of Theorem 5.1 shows that

$$(21) \quad \sum_{\ell \geq 0} \sum_{n \geq 0} B_\ell(n+1)x^n z^\ell = \frac{I(x, z) - 1/(1-z)}{x} - \frac{I(x, z) - I(x, 0)}{z}.$$

We now put (20) and (21) together and rearrange terms to find that

$$(22) \quad I(x, z) \left(1 - \frac{z}{x} + \frac{z^2}{1-z} \right) = I(x, 0) - T(x, z),$$

where

$$T(x, z) = \frac{z}{x(1-z)} - \frac{z^2}{(1-z)^2} + \frac{xz^2(1+x)(J(x, z) - 1/(1-z))}{1-(1+x)z}.$$

As in previous proofs, we now use the kernel method. Let $Z(x) = \frac{x(1+xM(x))}{1+x}$, where $M(x) = \sum_{n \geq 0} M_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$ is the Motzkin generating function. We have $1 - \frac{Z(x)}{x} + \frac{Z(x)^2}{1-Z(x)} = 0$, so substituting $z = Z(x)$ into (22) yields $I(x, 0) = T(x, Z(x))$. The expression $T(x, Z(x))$ simplifies to $1 + \frac{2x^2}{3x-1+\sqrt{1-2x-3x^2}}$. Finally,

$$I(x, 0) = \sum_{n \geq 0} B_{\geq 0}(n)x^n = \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231, 1243))| x^n. \quad \square$$

11. CONCLUDING REMARKS AND FUTURE DIRECTIONS

Most of the present article has concerned sets of the form $\text{VHC}(\text{Av}(\tau^{(1)}, \dots, \tau^{(r)}))$, where $\tau^{(1)}, \dots, \tau^{(r)} \in S_3$. These sets are completely uninteresting when one of the permutations $\tau^{(1)}, \dots, \tau^{(r)}$ is 123 or 213, so we can restrict our attention to the cases in which $\tau^{(1)}, \dots, \tau^{(r)} \in \{132, 231, 312, 321\}$. We have said nothing about this problem when $r \geq 3$, but that is because the enumerative results are not terribly difficult in these cases. For completeness, we state these results (without proof) in the following proposition. Let F_n denote the n^{th} Fibonacci number (with $F_1 = F_2 = 1$).

Proposition 11.1. *We have*

$$\sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231, 312, 321))| x^n = \frac{1-x+x^2-\sqrt{1-2x-x^2-2x^3+x^4}}{2x^2}.$$

For every $n \geq 1$, we have

$$|\text{VHC}(\text{Av}_n(132, 231, 321))| = |\text{VHC}(\text{Av}_n(132, 312, 321))| = 1 + \binom{n-1}{2},$$

$$|\text{VHC}(\text{Av}_n(132, 231, 312))| = F_n,$$

and

$$|\text{VHC}(\text{Av}_n(132, 231, 312, 321))| = n-1.$$

We have said nothing about the numbers $|\text{VHC}(\text{Av}_n(321))|$; it would be interesting to have non-trivial results concerning these numbers or their generating function. We wish to recall Conjecture 4.1, which connects valid hook configurations of 312-avoiding permutations, intervals in the posets \mathcal{M}_{n-1}^C , and lattices walks in the first quadrant that were studied in [8] and [10]. We also wish to remind the reader of the combinatorial proofs requested in Remarks 5.1 and 9.1.

Finally, we believe it could be interesting to enumerate valid hook configurations of permutations avoiding collections of length-4 patterns. One could also enumerate valid hook configurations of permutations avoiding one length-3 pattern and one length-4 pattern. We initiated this direction in Section 10 when we enumerated $\text{VHC}(\text{Av}(231, 1243))$. We also have the following conjecture.

Conjecture 11.1. *We have*

$$\sum_{n \geq 0} |\text{VHC}(\text{Av}_n(132, 3241))| x^n = \sum_{n \geq 0} |\text{VHC}(\text{Av}_n(231, 2143))| x^n = \frac{1 + x^2 - \sqrt{1 - 4x + 2x^2 + x^4}}{2x}.$$

Let us remark that the first equality in Conjecture 11.1 follows from the results in [28]. Thus, the actual content of the conjecture lies in the explicit form of the generating function.

12. ACKNOWLEDGMENTS

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