Electrostatic T-matrix for a torus on bases of toroidal and spherical harmonics

Matt Majic[*]

The MacDiarmid Institute for Advanced Materials and Nanotechnology, School of Chemical and Physical Sciences,
Victoria University of Wellington, PO Box 600, Wellington 6140, New Zealand

Semi-analytic expressions for the static limit of the T-matrix for electromagnetic scattering are derived for a circular torus, expressed in both a basis of toroidal harmonics and spherical harmonics. The scattering problem for an arbitrary static excitation is solved using toroidal harmonics, and these are then compared to the extended boundary condition method to obtain analytic expressions for auxiliary Q and P-matrices, from which $\mathbf{T} = \mathbf{P}\mathbf{Q}^{-1}$ (in a toroidal basis). By applying the basis transformations between toroidal and spherical harmonics, the quasi-static limit of the T-matrix block $\mathbf{T}^{22}$ for electric-electric multipole coupling is obtained. For the toroidal geometry there are two similar T-matrices on a spherical basis, for computing the scattered field both near the origin and in the far field. Static limits of the optical cross-sections are computed, and analytic expressions for the limit of a thin ring are derived.

I. Introduction

The T-matrix method is a semi-analytical tool for calculating the properties of electromagnetic or acoustic scattering by macroscopic particles [1, 2]. The incident and scattered electromagnetic fields are expanded on bases of orthogonal functions, e.g. spherical wavefunctions, and the T-matrix essentially outputs the series coefficients of the scattered field in terms of the known coefficients for the incident field. Typically the T-matrix is calculated via the extended boundary condition method which involves numerically evaluating surface integrals, however this approach is unstable for particles with highly non-spherical shape, and much work has been done on determining conditions for the T-matrix to be applicable [3]. The Rayleigh hypothesis is central to the T-matrix formalism and is the assumption that the series of spherical functions converge on the entire surface of the particle, although it is known that this is not necessary and weaker criteria have been suggested [3].

In particular, the EBCM has not been applied successfully to a torus. Aside from the T-matrix method, scattering by a torus has been considered analytically in the long wavelength limit using toroidal harmonics, the partially separable solutions to Laplace's equation in toroidal coordinates. Toroidal harmonics are a relatively new tool in computational physics due to their complexity. Also, being only partially separable solutions makes toroidal harmonics difficult to apply to problems even involving the torus. Toroidal harmonics have been applied to specific problems including a conducting torus illuminated by a low frequency plane wave [4] or point dipole [5], and a dielectric torus in a uniform field [6]. A detailed computational analysis of many aspects of quasistatic scattering by single and multiple layered tori is conducted in [7]. For conducting tori, fairly simple series solutions can be obtained, but for dielectric tori, the series coefficients are not explicit. The coefficients are calculated using a three step recurrence relation, with initial values given by a continued fraction.

Here we use an alternative approach to this recurrence scheme by computing the T-matrix on a basis of toroidal harmonics. The series coefficients are then given by a matrix inversion.

Spherical harmonics are far easier to compute, better known and more applicable than toroidal harmonics, so it is of interest to re-express this T-matrix on a basis of spherical harmonics - in particular, this allows us to compute the long wavelength limit of the T-matrix for electromagnetic or acoustic scattering, which is usually expressed on a basis of spherical wavefunctions. The series relationships between spherical and toroidal harmonics are largely unknown, exept for the low degrees: toroidal harmonics of degree zero, corresponding to the potential of rings of sinusoidal charge distributions, are known as series of spherical harmonics, and spherical harmonics corresponding to point charges and dipoles are known as series of toroidal harmonics. In fact, relationships for all degrees and orders have been derived in a Russian paper from 1983 [8], although this paper does not appear to be well known, except that the relationships were subsequently used to study the electrostatic interaction of a torus and a sphere [9]. Here we re-derive these expansions and express the expansion coefficients in a simpler form.

The document is organised as follows. Section II defines toroidal coordinates and harmonics, and investigates the charge distributions which generate these harmonics, some of which are unknown. Then in section III the scattering problem for a torus is formulated where the incident field is assumed to be expanded as a series of toroidal harmonics, and using the electrostatic null field equations [10], a toroidal T-matrix is derived which gives the scattered field also on a basis of toroidal harmonics. Section IV presents relationships between spherical harmonics and a new simple recurrence for their expansion coefficients. Section V then uses these relationships to re-expresses the T-matrix on a spherical harmonic basis. Section VI uses these results to compute physical quantities including capacitance, polarizability, plasmon resonances optical cross-sections, and obtain analytic asymptotic expressions for the T-matrix elements in the thin ring limit.

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[*] mattmajic@gmail.com

II. Toroidal coordinates and harmonics

From spherical coordinates (r, θ, ϕ), and cylindrical coordinates (ρ, z, ϕ), toroidal coordinates (ξ, η, ϕ) with focal ring radius a are defined as

$$\eta = \arcsin \frac{2az}{\sqrt{(r^2 + a^2)^2 - 4\rho^2 a^2}}, \tag{1}$$

$$\xi = \frac{1}{2} \log \frac{(\rho + a)^2 + z^2}{(\rho - a)^2 + z^2}, \tag{2}$$

$$\beta = \cosh \xi, \tag{3}$$

with $\eta \in [-\pi, \pi]$, $\xi \in [0, \infty)$, $\beta \in [1, \infty)$. β corresponds to the torus size ($\beta = 1$ is a tight torus covering all space and $\beta = \infty$ is the focal ring) and η corresponds to the angle subtending to the nearest point on the focal ring. Laplace's equation is partially separable in toroidal coordinates, so that the solutions have an additional prefactor. The toroidal harmonics are defined for integer n, m as

$$\psi_n^{mc} = \Delta Q_{n-1/2}^m(\beta) \cos n\eta \; e^{im\phi}, \tag{4}$$

$$\psi_n^{ms} = \Delta Q_{n-1/2}^m(\beta) \sin n\eta \; e^{im\phi}, \tag{5}$$

$$\Psi_n^{mc} = \Delta P_{n-1/2}^m(\beta) \cos n\eta \; e^{im\phi}, \tag{6}$$

$$\Psi_n^{ms} = \Delta P_{n-1/2}^m(\beta) \sin n\eta \; e^{im\phi}, \tag{7}$$

with $\Delta = \sqrt{2(\beta - \cos \eta)}$, \tag{8}

where $P_{n-1/2}^m$ and $Q_{n-1/2}^m$ are the Legendre functions of the first and second kinds. The superscript ^{v} will be used to denote ^{c} or ^{s}. We will call Ψ_n^{mv} the "ring toroidal harmonics" and ψ_n^{mv} the "axial toroidal harmonics" since they are singular on the focal ring and z-axis respectively. There are essentially twice as many physically applicable toroidal harmonics as spherical harmonics - for spherical harmonics the angular solutions $Q_n(\cos \theta)$ are discarded due to their singularity at the poles, while for toroidal harmonics, all Ψ_n^{mv} and ψ_n^{mv} (for integer n, m) are smooth on the torus. Now for insight we investigate the charge distributions that create toroidal harmonics.

A. Charge distributions of ring toroidal harmonics Ψ_n^{mv}

Ψ_n^{mv} are finite everywhere except $\beta = \infty$, on the focal ring. We want to express the functions as integrals of charge distributions on this focal ring, weighted by the inverse distance between a point on the focal ring $\mathbf{r}'$ and an arbitrary point in space $\mathbf{r}$, which is $|\mathbf{r}' - \mathbf{r}| = \rho^2 + a^2 - 2\rho a \cos(\phi - \phi') + z^2$. We start with $n = 0$, looking at $\Psi_0^{mc} = \Delta P_{-1/2}^m(\beta) e^{im\phi}$ (note that $\Psi_0^{ms} = 0$). The Legendre functions approaching the ring behave as

$$P_{-1/2}^m(\beta \to \infty) \to \frac{(2m-1)!!}{(-2)^m \pi} \sqrt{\frac{2}{\beta}} \log \beta \tag{9}$$

while $\Delta \to \sqrt{2\beta}$. Approaching the focal ring, $1/\beta$ becomes equal to the distance d from the ring. Compare this to approaching infinitely close to a line source with arbitrary charge distribution, where the potential goes as $2 \log d$ if the charge has unit density at the point of closest approach. And the line charge distribution must be proportional to $e^{im\phi}$, so we deduce that

$$\Psi_0^{mc} = \frac{(2m-1)!!}{(-2)^m \pi} \int_0^{2\pi} \frac{e^{im\phi'} a \, d\phi'}{\sqrt{r^2 + a^2 - 2\rho a \cos(\phi - \phi')}}. \tag{10}$$

We should also check the limit as $r \to \infty$. Here $\beta \to 1$, $P_{n-1/2} \to 1$, $P_{n-1/2}^{m>0} \to 0$, and $\Delta \to 2a/r$, so we can rule out the possibility of sources at $r = \infty$.

For $n = 1$, the charge distributions are double rings - rings with a dipole moment either pointing outward from the origin or along the z-axis. The harmonics for $n = 1$ may be generated by application of the operators $\partial_z = \partial/\partial z$ and $r\partial_r$ (see appendix B 1 for details), so we apply these to the integral expression (10):

$$\Psi_1^{mc} = -\frac{2r\partial_r + 1}{m - 1/2} \Psi_0^{mc}$$
$$= \frac{(2m-3)!!}{(-2)^m \pi} \int_0^{2\pi} \frac{a(a^2 - r^2)e^{im\phi'} \, d\phi'}{(r^2 + a^2 - 2\rho a \cos(\phi - \phi'))^{3/2}} \tag{11}$$

The charge distribution is two oppositely charged rings on the xy-plane, one with an infinitesimally greater radius than the other. This produces an infinitesimal charge imbalance - a net monopole moment.

And for Ψ_1^{ms}:

$$\Psi_1^{ms} = \frac{-a\partial_z}{m - 1/2} \Psi_0^{mc}$$
$$= \frac{(2m-3)!!}{(-2)^m \pi} \int_0^{2\pi} \frac{-2a^2 z e^{im\phi'} \, d\phi'}{(r^2 + a^2 - 2\rho a \cos(\phi - \phi'))^{3/2}} \tag{12}$$

This is the potential of two oppositely charged rings with an infinitesimal separation in the z-direction. These charge distributions are represented in figure 1.

For higher n the harmonics can be generated by repeated application of the operator $r\partial_r$. We leave the details for appendix B 1, but state that the toroidal harmonics follow a recurrence relation (B14) involving $r\partial_r$, and this recurrence is also satisfied by coefficients c_{nk}^m, s_{nk}^m relating toroidal and spherical harmonics (see (54)). We can then deduce the differential operators that generate the nth order harmonic from the 0th or 1st order harmonics:

$$\Psi_n^{mc} = \frac{(-)^n}{2} c_n^m(r\partial_r) \Psi_0^{mc} \tag{13}$$

$$\Psi_n^{ms} = (-)^{n+1} \frac{s_n^m(r\partial_r)}{s_0^m(r\partial_r)} \Psi_1^{ms} \tag{14}$$

where $c_n^m(r\partial_r) = c_{nk}^m$ with $k \to r\partial_r$, and is a polynomial degree n in $r\partial_r$.

The relationship above for the $\sin n\eta$ dependent harmonics starts from $n = 1$ because $\Psi_0^{ms} = 0$. Also note that $s_n^m(r\partial_r)$ always contains a factor $s_0^m(r\partial_r)$ so the denominator in (14) simply cancels this.

The corresponding integral expressions for Ψ_n^{mv} are given in operator form by applying (13) and (14) to the
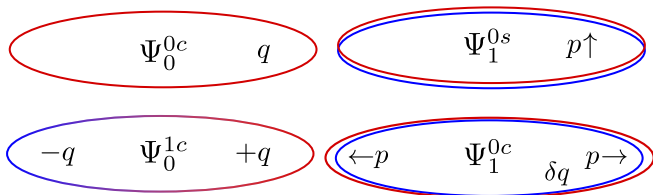
FIG. 1. Representation of the source charge configurations for the low order ring toroidal harmonics. Red and blue represent positive and negative charge q, and dipole moments are denoted p. For Ψ_1^{0c} the slightly larger radius of the outer ring creates an infinitesimal charge δq.

integral expression for Ψ_0^{mc} (10) and Ψ_1^{ms} (12). However, explicit evaluations of these derivatives do not appear to reveal any simple patterns when explicitly evaluating the derivatives.

These integral expressions have been checked numerically up to $n = 3$.

B. Charge distributions of axial toroidal harmonics ψ_n^{mv}

We will use heuristic arguments to derive the line source distributions for ψ_n^{mv}, which are singular on the z axis. First consider $m = 0$. As in the previous section, we use the fact that for an arbitrary line charge distribution, close enough to the line, the potential will only depend on the magnitude of the charge distribution at the point of approach. So we can determine the line charge distribution by matching it to the limit of the potential as it approaches the line. The exact proportionality is obtained by considering that the potential near a line source with unit magnitude goes as $2\log(\rho)$. The behaviour of the components of the toroidal harmonics as $\rho \to 0$ are, using $v = z/a$:

$$\lim_{\rho \to 0} \Delta = \frac{2}{\sqrt{v^2+1}} \tag{15}$$

$$\lim_{\rho \to 0} \eta = \text{sign}(v)\text{acos}\frac{v^2-1}{v^2+1} \equiv \eta' \tag{16}$$

$$\lim_{\beta \to 1} Q_{n-1/2}(\beta) = \log\frac{1}{\sqrt{\beta-1}} \tag{17}$$

$$\lim_{\rho \to 0} \frac{1}{\sqrt{\beta-1}} = \frac{a}{\rho}(v^2+1). \tag{18}$$

Using the notation $e^{in\eta}$ to deal with ψ_n^c and ψ_n^s simultaneously, we see that the line source distribution of $\psi_n^c + i\psi_n^s$ must be $ae^{in\eta'}/\sqrt{v^2+1}$:

$$\psi_n^c + i\psi_n^s = \int_{-\infty}^{\infty} \frac{a\, e^{in\eta'} \mathrm{d}v}{\sqrt{v^2+1}\sqrt{\rho^2+(z-av)^2}}. \tag{19}$$

And for $\rho \to \infty$, we have $\psi_n^v \to 0$ and there is no contribution from sources at $\rho \to \infty$. This is unlike the spherical harmonics of the second kind $r^n Q_n(\cos\theta)$, whose charge distributions are the difference between a line source that produces an infinite potential and a

sum of multipoles at infinity of infinite strength (work currently in progress).

Note that the function $\text{sign}(v)$ makes the charge distribution continuous, and that it can also be expressed in terms of Chebyshev polynomials of the first and second kinds T_n and U_n:

$$e^{in\eta'} = T_n\left(\frac{v^2-1}{v^2+1}\right) + \frac{2iv}{v^2+1}U_{n-1}\left(\frac{v^2-1}{v^2+1}\right). \tag{20}$$

For $m > 0$, the charge distributions are multi-line, which can be deduced from the $e^{im\phi}$ dependence. For example ψ_n^{1v} has a charge distribution of two line sources infinitely close together but of opposite charge. The behaviour of the Legendre functions $Q_{n-1/2}^m(\beta)$ is:

$$\lim_{\rho \to 0} Q_{n-1/2}^m(\beta) = \frac{(-)^m}{2}(m-1)!\left(\frac{a(v^2+1)}{\rho}\right)^m. \tag{21}$$

So for $m > 0$ we have, $\psi_n^{mv} \propto \rho^{-m}$ as $\rho = 0$.
And in the limit $\rho \to \infty$, we have $\psi_n^{mv} \propto \rho^{m-1}$. Despite the divergence for $m > 1$, there is no contribution from sources at $\rho = \infty$, which can be explained as follows. If sources at $\rho = \infty$ existed, then we can split ψ_n^{mv} into its contributions from charges on the z-axis and at $\rho = \infty$. The potential due to charges at $\rho = \infty$, being finite at the origin, can then be expressed as a series of regular spherical harmonics $\hat{S}_n^m = r^n P_n^m(\cos\theta)e^{im\phi}$ of the same m as ψ_n^{mv}. However, $\hat{S}_n^m(\rho \to \infty) \propto \rho^k$ where $k \geq m$, so it is impossible to express the ρ^{m-1} dependence as a series of $\hat{S}_n^m$. Therefore there cannot exist charges at $\rho = \infty$, and counter-intuitively, this ρ^{m-1} dependence must be entirely due to the line source on the z-axis.

We can compare this to the potential near an m-fold line charge distribution which similarly goes as ρ^{-m} as $\rho \to 0$, with integral kernel $|\mathbf{r} - \mathbf{r}'|^{-2m-1}$. (21) also shows non-constant z-dependence $(v^2+1)^m$, of which we multiplied the charge distribution by. Putting this together gives

$$\psi_n^{mc} + i\psi_n^{ms} = (2m-1)!!a\left(\frac{-a\rho}{2}\right)^m e^{im\phi}$$

$$\times \int_{-\infty}^{\infty} \frac{(v^2+1)^{m-1/2}e^{in\eta'}}{(\rho^2+(z-av)^2)^{m+1/2}}\mathrm{d}v \tag{22}$$

Andrews ([11]- eq. 27) proved (22) via direct integration for all m, and $n = 0$ (in his notation $n \leftrightarrow m$).

III. T-matrix for a torus on a toroidal harmonic basis

Consider a circular torus with major radius R_0, minor radius r_0 and relative permittivity to the surroundings ϵ as shown in figure 2. The surface is defined by the aspect ratio $\beta = \beta_0 = R_0/r_0$, and the focal ring radius is $a = \sqrt{R_0^2 - r_0^2}$. The torus is excited by an arbitrary external electric potential V_e, and we want to determine the scattered potential V_s (where $V_o = V_e + V_s$), and internal potential V_i. The boundary conditions are:

$$V_i = V_o, \qquad \epsilon\frac{\partial V_i}{\partial \beta} = \frac{\partial V_o}{\partial \beta}, \qquad \text{at } \beta = \beta_0. \tag{23}$$
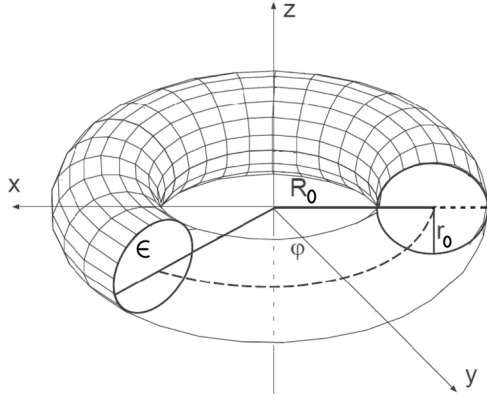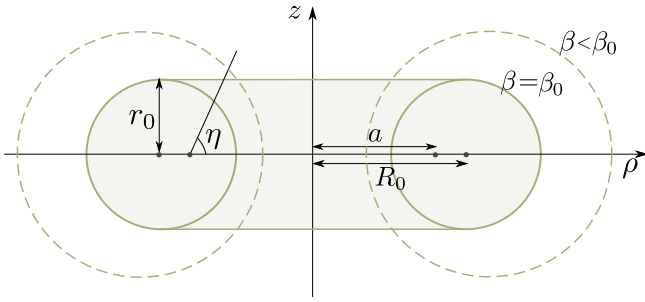
FIG. 2. Above: Parameters defining the torus including relative permittivity to the surroundings ϵ. Below: relationships between toroidal geometry and the separable toroidal coordinate system.

The fields are assumed to be expanded in terms of toroidal harmonics as

$$V_e = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{v=c,s} a_n^{mv} \psi_n^{mv}, \tag{24}$$

$$V_i = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{v=c,s} b_n^{mv} \psi_n^{mv}, \tag{25}$$

$$V_s = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{v=c,s} c_n^{mv} \Psi_n^{mv}. \tag{26}$$

In this section we derive the matrix relations between the incident, scattered, and internal expansion coefficients.
In general the boundary conditions are difficult to solve, and may lead to an inhomogeneous three term recurrence relation with initial conditions given by an infinite continued fraction [6]. Instead we will use the boundary integral equation formulation to find an analogue of the T-matrix in toroidal coordinates. The boundary integral equations are derived from Green's second identity on the torus surface S [12]:

$$\frac{\epsilon - 1}{4\pi} \int_S \frac{\partial V_i(\mathbf{r}')}{\partial n'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \mathrm{d}S' = \begin{cases} V_e(\mathbf{r}) - V_i(\mathbf{r}) & \mathbf{r} \in V \\ -V_s(\mathbf{r}) & \mathbf{r} \notin V. \end{cases} \tag{27}$$

where $\partial/\partial n'$ is the derivative with respect to the surface normal and $\mathrm{d}S$ is the infinitesimal surface element. In the T-matrix approach, Green's function $1/|\mathbf{r} - \mathbf{r}'|$ is expanded as a series of separable harmonics; here we use

toroidal harmonics:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{2\pi a} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{v=c,s} \varepsilon_n (-)^m$$

$$\times \begin{cases} \psi_n^{mv}(\mathbf{r}) \Psi_n^{-m,v}(\mathbf{r}') & \mathbf{r} \in V \\ \Psi_n^{mv}(\mathbf{r}) \psi_n^{-m,v}(\mathbf{r}') & \mathbf{r} \notin V \end{cases} \tag{28}$$

$$\text{with } \varepsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n > 0 \end{cases}.$$

We now insert this and the expansions of the potentials into the integral equation(27), and equate the coefficients of the toroidal harmonics. For example, equating the coefficients of $\psi_n^{mc}(\mathbf{r})$ for $\mathbf{r}$ inside the torus leads to (dropping the integration primes):

$$\varepsilon_n (-)^m \frac{\epsilon - 1}{8\pi^2 a} \sum_{k=0}^{\infty} \int_S \frac{\partial}{\partial n} \{\Delta Q_{k-1/2}^m(\beta_0)[b_k^{mc}\cos(k\eta) +$$

$$b_k^{ms}\sin(k\eta)]\} \Delta P_{n-1/2}^{-m}(\beta_0) \cos(n\eta) \mathrm{d}S = a_n^{mc} - b_n^{mc}. \tag{29}$$

The sum over m has been omitted because the torus is rotationally symmetric and only terms with the same m in the expansions of the V_i and G survive the integration. This decouples the problem for each m. Furthermore, the integration is over an even interval of η so $\sin(k\eta)\cos(n\eta)$ integrates to zero, decoupling the sine and cosine expansion coefficients. The relationships between the expansion coefficients can be formulated with infinite dimensional matrices:

$$\mathbf{a}^{mv} = \bar{\mathbf{Q}}^{mv} \mathbf{b}^{mv}, \quad \mathbf{c}^{mv} = \bar{\mathbf{P}}^{mv} \mathbf{b}^{mv}, \quad \mathbf{c}^{mv} = \bar{\mathbf{T}}^{mv} \mathbf{a}^{mv},$$
$$\text{for } m \in \mathbb{Z}, \quad v = c, s, \tag{30}$$

which are analogous to the P, Q, T-matrices in the T-matrix method for scattering of waves. $\mathbf{a}^{mv}$ are $1 \times N$ (N being the numerical truncation order) column vectors containing the elements a_n^{mv} for fixed m, v. When convenient the superscripts m and v will be omitted.
For a conducting torus, $\epsilon \to \infty$ and the T-matrix $\bar{\mathbf{T}} = \bar{\mathbf{T}}^{\infty}$ is diagonal:

$$[\bar{T}^{\infty}]_{nk}^{mv} = -\frac{Q_{n-1/2}^m(\beta_0)}{P_{n-1/2}^m(\beta_0)} \delta_{nk} \tag{31}$$

which is equivalent to well known solutions for conducting tori, see for example [13].
For general ϵ, the integral equations 27 allow us to calculate $\bar{\mathbf{P}}$ and $\bar{\mathbf{Q}}$, while $\bar{\mathbf{T}}$ is obtained from

$$\bar{\mathbf{T}} = \bar{\mathbf{P}} \bar{\mathbf{Q}}^{-1}. \tag{32}$$

We now look for analytic expressions for the matrix elements, we have

$$\frac{\partial}{\partial n} = \frac{-\Delta^2 \sinh\xi}{2a} \frac{\partial}{\partial \beta}, \qquad \mathrm{d}S = \frac{4a^2 \sinh\xi}{\Delta^4} \mathrm{d}\eta \mathrm{d}\phi. \tag{33}$$

Then $\bar{\mathbf{Q}}^c$ becomes (using (29)):

$$\bar{Q}_{nk}^{mc} = \delta_{nk} - \frac{\epsilon-1}{2\pi}P_{n-1/2}^{-m}(\beta_0)\varepsilon_n$$
$$\times \left[\frac{\partial Q_{k-1/2}^m(\beta_0)}{\partial\beta_0}\int_{-\pi}^{\pi}\cos(n\eta)\cos(k\eta)\mathrm{d}\eta\right.$$
$$\left. + \frac{1}{2}Q_{k-1/2}^m(\beta_0)\int_{-\pi}^{\pi}\frac{\cos(n\eta)\cos(k\eta)}{\beta_0-\cos\eta}\mathrm{d}\eta\right]. \qquad (34)$$

The first integral is simple:

$$\int_{-\pi}^{\pi}\cos(n\eta)\cos(k\eta)\mathrm{d}\eta = 2\pi\frac{\delta_{nk}}{\varepsilon_n}, \qquad (35)$$

and the second integral evaluates to

$$\int_{-\pi}^{\pi}\frac{\cos(n\eta)\cos(k\eta)}{\cosh\xi_0-\cos\eta}\mathrm{d}\eta = \pi\frac{e^{-|n-k|\xi_0}+e^{-(n+k)\xi_0}}{\sinh\xi_0}. \qquad (36)$$

See appendix A for proof. For $\bar{\mathbf{Q}}^s$ the integrals are

$$\int_{-\pi}^{\pi}\sin(n\eta)\sin(k\eta)\mathrm{d}\eta = \pi\delta_{nk}(1-\delta_{n0}), \qquad (37)$$

and

$$\int_{-\pi}^{\pi}\frac{\sin(n\eta)\sin(k\eta)}{\cosh\xi_0-\cos\eta}\mathrm{d}\eta = -\pi\frac{e^{-|n-k|\xi_0}-e^{-(n+k)\xi_0}}{\sinh\xi_0}. \qquad (38)$$

The matrix elements are then

$$\bar{Q}_{nk}^{mc} = \delta_{nk} - (\epsilon-1)\sinh^2\xi_0 P_{n-1/2}^{-m}(\beta_0)\left[\partial_{\beta_0}Q_{n-1/2}^m(\beta_0)\delta_{nk}\right.$$
$$\left. + \frac{\varepsilon_n}{2}Q_{k-1/2}^m(\beta_0)\frac{e^{-|n-k|\xi_0}+e^{-(n+k)\xi_0}}{2\sinh\xi_0}\right], \qquad (39)$$

$$\bar{Q}_{nk}^{ms} = \delta_{nk} - (\epsilon-1)\sinh^2\xi_0 P_{n-1/2}^{-m}(\beta_0)\left[\partial_{\beta_0}Q_{n-1/2}^m(\beta_0)\delta_{nk}\right.$$
$$\left. \times(1-\delta_{n0}) - Q_{k-1/2}^m(\beta_0)\frac{e^{-|n-k|\xi_0}-e^{-(n+k)\xi_0}}{2\sinh\xi_0}\right]. \qquad (40)$$

A similar derivation shows that $\bar{\mathbf{P}}$ is related to $\bar{\mathbf{Q}}$ by

$$\bar{\mathbf{P}} = \bar{\mathbf{T}}^\infty(\bar{\mathbf{Q}}-\mathbf{I}) \qquad (41)$$

for all m,v. Hence the T-matrix can also be expressed as

$$\bar{\mathbf{T}} = \bar{\mathbf{T}}^\infty(\mathbf{I}-\bar{\mathbf{Q}}^{-1}). \qquad (42)$$

We suspect this form applies to the T-matrix for any particle whos geometry is a coordinate of a coordinate system with partially-separable solutions to Laplace's equation. in particular for bispherical coordinates (currently a work in progress). $\bar{\mathbf{T}}$ may be calculated via (32) or (42); both give the same numerical accuracy. To avoid possible numerical instability in inverting $\bar{\mathbf{Q}}^{m=0,c}$, the matrix should be transposed, inverted then transposed back.

The toroidal T-matrix has the following symmetry property:

$$\bar{T}_{kn}^{mv} = \frac{\varepsilon_k\Gamma(k-m+\frac{1}{2})\Gamma(n+m+\frac{1}{2})}{\varepsilon_n\Gamma(k+m+\frac{1}{2})\Gamma(n-m+\frac{1}{2})}\bar{T}_{nk}^{mv}. \qquad (43)$$

This can be derived from the analysis in [10], which proves the symmetry of the T-matrix from deriving two equivalent boundary integral equations, and applying Green's theorem. The T-matrix here is not exactly symmetric due to using unnormalised basis functions.

A. Comparison to recurrence approach

The usual way to solve electrostatic problems for the dielectric torus is to apply the boundary conditions in differential form (23) directly to the series expansions for the potentials (24-26), to obtain a recurrence relation for the coefficients. This has been done for a uniform electric field [6], and point charge [14]. For arbitrary excitation, we get

$$c_{n+1}^{mv}\Lambda_{n+1}^m - c_n^{mv}\left(2\beta_0\Lambda_n^m + (\epsilon-1)P_{n-1/2}^m\right) + c_{n-1}^{mv}\Lambda_{n-1}^m$$
$$= (\epsilon-1)\left[-a_{n+1}^{mv}Q_{n+1/2}^{m\prime} + a_n^{mv}(Q_{n-1/2}^m+2\beta_0 Q_{n-1/2}^{m\prime})\right.$$
$$\left. - a_{n-1}^{mv}Q_{n-3/2}^{m\prime}\right] \qquad (44)$$

where $\quad \Lambda_n^m = \epsilon Q_{n-1/2}^{m\prime}(\beta_0)\frac{P_{n-1/2}^m(\beta_0)}{Q_{n-1/2}^m(\beta_0)} - P_{n-1/2}^{m\prime}(\beta_0).$

$$\qquad (45)$$

The difficulty in computing the coefficients using this scheme is finding the initial values; these are expressed as a series of products of continued fractions. Nevertheless, we can compare this to the T-matrix approach. Combining this recurrence with the definition of the T-matrix, (30) , we find

$$\sum_{k=0}^{\infty}\left[\bar{T}_{n+1k}^{mv}\Lambda_{n+1}^m - \bar{T}_{nk}^{mv}\left(2\beta_0\Lambda_n^m + (\epsilon-1)P_{n-1/2}^m\right)\right.$$
$$\left. + \bar{T}_{n-1k}^{mv}\Lambda_{n-1}^m\right]a_k^{mv}$$
$$= (\epsilon-1)\left[-a_{n+1}^{mv}Q_{n+1/2}^{m\prime} + a_n^{mv}(Q_{n-1/2}^m+2\beta_0 Q_{n-1/2}^{m\prime})\right.$$
$$\left. - a_{n-1}^{mv}Q_{n-3/2}^{m\prime}\right] \qquad (46)$$

which must hold for any set of coefficients a_n^{mv}. In particular if we choose an excitation with $a_n^{mv} = \delta_{np}$, a recurrence for the elements of the toroidal T-matrix can be found:

$$\bar{T}_{n+1p}^{mv}\Lambda_{n+1}^m - \bar{T}_{np}^{mv}\left(2\beta_0\Lambda_n^m + (\epsilon-1)P_{n-1/2}^m\right) + \bar{T}_{n-1p}^{mv}\Lambda_{n-1}^m$$
$$= (\epsilon-1)\left[\delta_{np}(Q_{p-1/2}^m+2\beta_0 Q_{p-1/2}^{m\prime})\right.$$
$$\left. - (\delta_{n+1p}+\delta_{n-1p})Q_{p-1/2}^{m\prime}\right]. \qquad (47)$$

Which has been used as a check for the T-matrix.

IV. Relationships between spherical and toroidal harmonics

In order to express the T-matrix on a basis of spherical harmonics, we need series relationships between spherical and toroidal harmonics. We define the regular and irregular solid spherical harmonics as:

$$\hat{S}_n^m = \left(\frac{r}{a}\right)^n P_n^m(\cos\theta)e^{im\phi}, \tag{48}$$

$$S_n^m = \left(\frac{a}{r}\right)^{n+1} P_n^m(\cos\theta)e^{im\phi}. \tag{49}$$

In the appendix we derive the following linear relationships between spherical and toroidal harmonics:

$$\Psi_n^{mc} = \sum_{k=m}^{\infty} c_{nk}^m P_k^{-m}(0) \begin{cases} (-)^n \hat{S}_k^m & r < a \\ S_k^m & r > a \end{cases} \tag{50}$$

$$\Psi_n^{ms} = \sum_{k=m}^{\infty} s_{nk}^m P_{k+1}^{-m}(0) \begin{cases} (-)^n \hat{S}_k^m & r < a \\ -S_k^m & r > a \end{cases} \tag{51}$$

$$\hat{S}_n^m = \frac{1}{\pi} \begin{cases} P_n^m(0) \displaystyle\sum_{k=0}^{\infty} \frac{\varepsilon_k}{2} c_{kn}^{-m} \psi_k^{mc} & n+m \text{ even} \\ -P_{n+1}^m(0) \displaystyle\sum_{k=1}^{\infty} s_{kn}^{-m} \psi_k^{ms} & n+m \text{ odd.} \end{cases} \tag{52}$$

$$S_n^m = \frac{1}{\pi} \begin{cases} P_n^m(0) \displaystyle\sum_{k=0}^{\infty} \frac{\varepsilon_k}{2}(-)^k c_{kn}^{-m} \psi_k^{mc} & n+m \text{ even} \\ P_{n+1}^m(0) \displaystyle\sum_{k=1}^{\infty} (-)^k s_{kn}^{-m} \psi_k^{ms} & n+m \text{ odd} \end{cases} \tag{53}$$

c_{nk}^m, s_{nk}^m are rational numbers defined by recurrence:

$$\left(n-m+\frac{1}{2}\right)c_{n+1,k}^m = (2k+1)c_{nk}^m + \left(n+m-\frac{1}{2}\right)c_{n-1,k}^m \tag{54}$$

and the same for s_{nk}^m. The initial values are

$$c_{0k}^m = \frac{(2m-1)!!}{2^{m-1}}, \qquad c_{1k}^m = -\frac{(2m-3)!!}{2^{m-1}}(2k+1),$$

$$s_{0k}^m = 0, \qquad s_{1k}^m = -\frac{(2m-3)!!}{2^m}(k+m+1). \tag{55}$$

(the double factorial can be extended to odd negative integers via recurrence, or equivalently through the gamma function). Note for $m < 0$:

$$c_{nk}^{-m} = \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})}c_{nk}^m,$$

$$s_{nk}^{-m} = \frac{\Gamma(n-m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})}\frac{k-m+1}{k+m+1}s_{nk}^m. \tag{56}$$

The Legendre functions at 0 are, for $m \in \mathbb{Z}$

$$P_n^m(0) = \begin{cases} (-)^{(n-m)/2}\dfrac{(n+m-1)!!}{(n-m)!!} & n+m \text{ even} \\ 0 & n+m \text{ odd.} \end{cases} \tag{57}$$

It is interesting that for $m = 0$, the series (50) starts from $k = 0$ regardless of n; these toroidal harmonics all have non-zero monopole moment.

Existence of expansions

Toroidal harmonics do not follow the same notion of internal and external as do spherical or say spheroidal harmonics. We have shown that the ring toroidal harmonics Ψ_m^{mv} can be written as a series of either internal or external spherical harmonics. This is due to the fact that they are finite at both the origin and at $r = \infty$. However the axial toroidal harmonics ψ_n^{mv} are singular at the origin and infinity, so cannot be expanded as a series of spherical harmonics at all [1]. Also, neither internal and external spherical harmonics can be expressed as a series of ring harmonics Ψ_n^{mv}. This is because a series of ring harmonics can only converge outside some toroidal boundary, and this boundary must enclose the singularities of the function being expanded - the external spherical harmonics S_n^m are singular at the origin, so this toroidal boundary must cover the origin and thus extend to all space, while the internal spherical harmonics $\hat{S}_n^m$ cannot be expanded for a similar reason - the torus must extend to all space to cover the "singularity" at $r = \infty$. Contrarily, both internal and external spherical harmonics can be expanded with axial toroidal harmonics ψ_n^{mv}, because a series of axial toroidal harmonics will converge *inside* some torus - for the internal spherical harmonics, this toroidal boundary may extend up to infinity since the functions are continuous in all space. For external spherical harmonics the toroidal boundary may extend to the origin. To check this we can determine the boundary of convergence of expansions (52) and (53) from the behaviour of the kth term in the series as $k \to \infty$. The Legendre functions grow as [15] pg 191 ((58) is presented for completeness):

$$\lim_{k\to\infty} P_{k-1/2}^m(\cosh\xi) = \frac{k^m e^{k\xi}}{\sqrt{(2k-1)\sinh\xi}} \tag{58}$$

$$\lim_{k\to\infty} Q_{k-1/2}^m(\cosh\xi) = \frac{\sqrt{\pi}(-k)^m e^{-k\xi}}{\sqrt{(2k-1)\sinh\xi}} \tag{59}$$

while c_{kn}^{-m} and s_{kn}^{-m} are bounded by the sequence $e_{k+1} = \frac{2n+1}{k}e_k + e_{k-1}$ (with the same initial values), which itself grows slower than k^{2n+1}. So the series coefficients decay geometrically as $e^{-k\xi}$, and $\xi > 0$ everywhere except the z-axis and $r = \infty$, so the series converge everywhere, although converge slowly near the z axis and also for large r. In these cases the series terms grow significantly in magnitude before converging, which sacrifices accuracy because the series can only be accurate to the last digit of the largest term in the series. This causes problems in dealing with extremely tight tori.

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[1] They can neither be expressed as a series of spherical harmonics of the second kind, $r^n Q_n(\cos\theta)$ or $r^{-n-1}Q_n(\cos\theta)$ which are also singular on the z axis. This is due to differences in parity about z

V. T-matrix on a spherical harmonic basis

We now apply the linear basis transformations between toroidal and spherical harmonics to express the T-matrix on a basis of spherical harmonics.

Because of the hole in the torus, the incident and scattered fields may both be expandable on interior or exterior spherical harmonics (see sec. V A). First we look at the standard case where the external field is expanded on regular spherical harmonics and the scattered field will be expanded on irregular harmonics. The potentials are now assumed to have the following expansions:

$$V_e = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} A_n^m \hat{S}^m, \tag{60}$$

$$V_i = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} B_n^m \hat{S}^m, \tag{61}$$

$$V_s = \sum_{m=-\infty}^{\infty} \sum_{n=|m|}^{\infty} C_n^m S_n^m, \tag{62}$$

the sum over n can be written in matrix notation so that

$$V_e = \sum_m \mathbf{A}^T \hat{\mathbf{S}}, \tag{63}$$

$$V_i = \sum_m \mathbf{B}^T \hat{\mathbf{S}}, \qquad \mathbf{B} = \mathbf{Q}^{-1} \mathbf{A}, \tag{64}$$

$$V_s = \sum_m \mathbf{C}^T \mathbf{S}, \qquad \mathbf{C} = \mathbf{T} \mathbf{A}. \tag{65}$$

where for example $\mathbf{A}, \hat{\mathbf{S}}$ are $1 \times N$ (N being the numerical truncation order) column vectors containing the elements A_n^m or $\hat{S}_n^m$ for all $n \geq 0$, for a fixed m, and $\mathbf{Q}, \mathbf{T}$ are the $N \times N$ Q-and T-matrices. The vectors and matrices start their index from $n = 0$ despite that the spherical harmonics are zero for $n < m$; this is to have the same dimensionality as the toroidal-basis matrices which start from $n = 0$.

We can then apply the linear relationships between spherical and toroidal harmonics, expressed in matrix notation:

$$\hat{\mathbf{S}} = \mathbf{M}_r^c \boldsymbol{\psi}^c + \mathbf{M}_r^s \boldsymbol{\psi}^s, \tag{66}$$

$$\boldsymbol{\Psi}^c = \mathbf{N}_i^c \mathbf{S}, \qquad \boldsymbol{\Psi}^s = \mathbf{N}_i^s \mathbf{S}, \tag{67}$$

where the elements of matrices $\mathbf{M}$ and $\mathbf{N}$ can be determined from (50,51,52) to be

$$[M_r^c]_{nk}^m = \frac{\varepsilon_k}{2\pi} P_n^m(0) c_{kn}^{-m},$$

$$[M_r^s]_{nk}^m = \frac{-1}{\pi} P_{n+1}^m(0) s_{kn}^{-m},$$

$$[N_i^c]_{nk}^m = P_k^{-m}(0) c_{nk}^m,$$

$$[N_i^s]_{nk}^m = -P_{k+1}^{-m}(0) s_{nk}^m. \tag{68}$$

The subscripts r, i stand for regular or irregular (referring to the type of spherical harmonic), and the superscripts

c, s refer to cosine or sine of η.

Applying (66) to V_e:

$$V_e = \sum_m \mathbf{A}^T [\mathbf{M}_r^c \boldsymbol{\psi}^c + \mathbf{M}_r^s \boldsymbol{\psi}^s] \tag{69}$$

and obtaining V_s via the toroidal-basis T-matrix solution (30):

$$V_s = \sum_m \mathbf{A}^T [\mathbf{M}_r^c (\bar{\mathbf{T}}^c)^T \boldsymbol{\Psi}^c + \mathbf{M}_r^s (\bar{\mathbf{T}}^s)^T \boldsymbol{\Psi}^s]. \tag{70}$$

Then expanding the toroidal functions back into spherical harmonics with (67):

$$V_s = \sum_m \mathbf{A}^T [\mathbf{M}_r^c (\bar{\mathbf{T}}^c)^T \mathbf{N}_i^c + \mathbf{M}_r^s (\bar{\mathbf{T}}^s)^T \mathbf{N}_i^s] \mathbf{S}. \tag{71}$$

Comparing this to (65), and noting that $\mathbf{C}^T = \mathbf{A}^T \mathbf{T}^T$, the T-matrix is found to be

$$\mathbf{T} = (\mathbf{N}_i^c)^T \bar{\mathbf{T}}^c (\mathbf{M}_r^c)^T + (\mathbf{N}_i^s)^T \bar{\mathbf{T}}^s (\mathbf{M}_r^s)^T. \tag{72}$$

$T_{nk}^m = 0$ for $n + k$ odd as expected for particles with reflection symmetry about z. In the paper [9] that analysed the electrostatic interaction between a conducting torus and a partial spherical shell, they use what is effectively the T-matrix for the conducting torus (eq. (32)), expressed on a spherical basis as part of a kernel of an integral equation.

Here we have not used normalised spherical basis functions, and as a consequence $\mathbf{T}$ is not symmetric for $m > 0$. The quasistatic limit of the conventional symmetric electromagnetic T-matrix in [2] can be obtained from:

$$\lim_{k_1 \to 0} T_{nk}^{22,m} = \frac{-i(k_1 a)^{n+k+1}}{(2n-1)!!(2k-1)!!}$$

$$\times \sqrt{\frac{(n+1)(k+1)}{nk(2n+1)(2k+1)} \frac{(n+m)!(k-m)!}{(n-m)!(k+m)!}} T_{nk}^m \tag{73}$$

where k_1 is the wavenumber in the surrounding medium.

A. Interior/exterior T-matrices

In the above derivation we assumed that the external potential was created by a source that could be expanded as a series of *regular* spherical harmonics, but because the torus excludes the origin, the external field may come from near the origin (for example a point source at the origin) and *irregular* harmonics must be used instead. Similarly, to compute the scattered field near the origin, a regular basis must be used. The T-matrix derived above applies only if V_e is expanded on $\hat{S}_m^m$ and V_s is expanded on S_n^m, and we denote this matrix $\mathbf{T}(r \to i)$. Following similar derivations, expressions for the other matrices can be found, and all four variations of these T-matrices are

$$\mathbf{T}(r \to i) = (\mathbf{N}_i^c)^T \bar{\mathbf{T}}^c (\mathbf{M}_r^c)^T + (\mathbf{N}_i^s)^T \bar{\mathbf{T}}^s (\mathbf{M}_r^s)^T$$

$$\mathbf{T}(r \to r) = (\mathbf{N}_r^c)^T \bar{\mathbf{T}}^c (\mathbf{M}_r^c)^T + (\mathbf{N}_r^s)^T \bar{\mathbf{T}}^s (\mathbf{M}_r^s)^T$$

$$\mathbf{T}(i \to r) = (\mathbf{N}_r^c)^T \bar{\mathbf{T}}^c (\mathbf{M}_i^c)^T + (\mathbf{N}_r^s)^T \bar{\mathbf{T}}^s (\mathbf{M}_i^s)^T$$

$$\mathbf{T}(i \to i) = (\mathbf{N}_i^c)^T \bar{\mathbf{T}}^c (\mathbf{M}_i^c)^T + (\mathbf{N}_i^s)^T \bar{\mathbf{T}}^s (\mathbf{M}_i^s)^T \tag{74}$$

The transformation matrices not already defined in (68) differ only in alternating signs, and are

$$[M_i^c]_{nk}^m = \frac{\varepsilon_k(-)^k}{2\pi} P_n^m(0) c_{kn}^{-m},$$

$$[M_i^s]_{nk}^m = \frac{(-)^k}{\pi} P_{n+1}^m(0) s_{kn}^{-m},$$

$$[N_r^c]_{nk}^m = (-)^n P_k^{-m}(0) c_{nk}^m,$$

$$[N_r^s]_{nk}^m = (-)^n P_{k+1}^{-m}(0) s_{nk}^m. \tag{75}$$

For example, $\mathbf{T}(r \to r)$ applies if the source is outside the torus' circumscribing sphere, and if the scattered field is to be evaluated near the hole.

For a conducting torus two pairs of these T-matrices in (74) are identical. Mathematically it is straightforward to show that $\mathbf{T}(i \to r) = \mathbf{T}(r \to i)$. Physically this is because the problem for a conductor does not distinguish the scattered and external/incident fields - the problem of determining the incident field from the scattered is the same as determining the scattered from the incident. And $\mathbf{T}(r \to r) = \mathbf{T}(i \to i)$, which can be shown by applying radial inversion about the sphere of radius a centred at the origin. The geometry transforms as $r \to a^2/r$ which preserves the torus, while potentials transform as $V \to (a/r)V(r \to a^2/r)$, so $\hat{S}_n^m \leftrightarrow S_n^m$. This argument only applies to the conducting torus since permittivity also transforms as $\epsilon \to (a^2/r^2)\epsilon$.

From the static T-matrix we can make deductions of the applicability of the full-wave T-matrix. Since the high order limit of the spherical wave functions tends towards the solid spherical harmonics, the boundaries of convergence of the full wave solution are identical to that for the quasistatic solution. It is not possible to compute the scattered field in a spherical annulus around the mid section of the torus $r \sim a$, due to the singularity of the scattered field. Even in the case of a charged conducting torus (constant external potential), it can be shown that both the interior and exterior spherical harmonic solutions diverge in the annulus $a < r < R_0$. This is analogous to spheroids with high aspect ratio - there is a region near the spheroid where the fields cannot be expressed using spherical basis functions.

VI. Derived physical quantities

We now use the T-matrix formulation to derive physical quantities.

A. Capacitance and dipolar response of conducting torus

The capacity C of a conducting torus held at a uniform potential V_0 can be deduced from the induced charge Q, and is related to T_{00}^0:

$$C = 4\pi\varepsilon_0 a T_{00}^0, \tag{76}$$

$$T_{00}^0 = -\frac{2}{\pi} \sum_{q=0}^{\infty} \varepsilon_q \frac{Q_{q-1/2}(\beta_0)}{P_{q-1/2}(\beta_0)}. \tag{77}$$
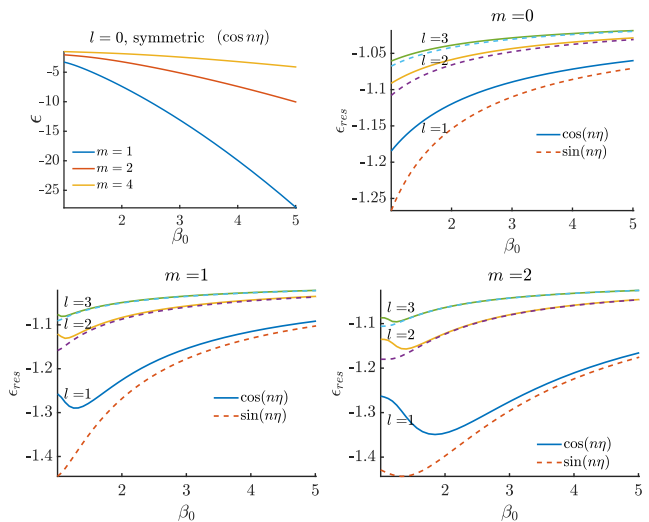

FIG. 3. Relative resonant permittivities of tori of aspect ratio $\beta_0 = R_0/r_0$. The dashed (solid) curves correspond to the resonances of $\bar{\mathbf{T}}^c$ ($\bar{\mathbf{T}}^s$), which are symmetric (antisymmetric) about the torus plane. Resonances are labelled with mode number l in order of increasing ϵ. The antisymmetric s modes start at $l = 1$. The top left plot shows the strong resonances that only occur for $\bar{\mathbf{T}}^c$. The other plots show the next three strongest resonances for both $v = c, s$; these resonances tend towards $\epsilon = -1$, for all values of m, v, β_0. The matrix size used in calculations ranged from $N = 250$ for $\beta_0 = 1.01$ to $N = 12$ for $\beta_0 = 5$.

And for the dipole polarizability α we consider a uniform field along a cartesian axis, exciting dipole moment $\mathbf{p}$ in a conducting torus. The polarizability is related to T_{11}^0, T_{11}^1:

$$p_w = \alpha_{ww} E_w, \qquad w = x, y, z, \tag{78}$$

$$\alpha_{zz} = 4\pi\epsilon_0 a^3 T_{11}^0, \tag{79}$$

$$\alpha_{xx} = \alpha_{yy} = 4\pi\epsilon_0 a^3 T_{11}^1, \tag{80}$$

where

$$T_{11}^0 = -\frac{16}{\pi} \sum_{q=1}^{\infty} q^2 \frac{Q_{q-1/2}(\beta_0)}{P_{q-1/2}(\beta_0)}, \tag{81}$$

$$T_{11}^1 = \frac{1}{\pi} \sum_{q=0}^{\infty} \varepsilon_q (4q^2 - 1) \frac{Q_{q-1/2}^1(\beta_0)}{P_{q-1/2}^1(\beta_0)}. \tag{82}$$

These results agree with [16].

VII. Plasmon resonances

From the explicit expressions for the toroidal matrices in (40), we can calculate the values of ϵ that produce a plasmon resonance, where the potential is infinite. These are $\epsilon = \epsilon_l^{mv}$ for $l = 0, 1, 2...$, $m = 0, 1, 2...$, $v = c, s$, where we have replaced the index n with l to highlight that the modes do not excite toroidal harmonics of a single degree n. The index l instead orders these in terms of magnitude. These can be found from the condition $\det(\bar{\mathbf{Q}}) = 0$, and it can be shown that $\epsilon_l^{mv} = 1 - 1/\lambda_l^{mv}$ where λ_l^{mv} are the

eigenvalues of $\hat{\mathbf{Q}}^{mv}$ where $\bar{\mathbf{Q}}^{mv} = \mathbf{I} - (\epsilon - 1)\hat{\mathbf{Q}}^{mv}$. Note that $\hat{\mathbf{Q}}$ is independent of ϵ.

In figure 3 the conditions for resonances are plotted as a function of aspect ratio β_0, and agree with the continued fraction approach [17], [18], [7]. Also, these results confirm the discussion of [19] claiming that the $m = 0$ resonances exist, even though they were not obtained in [18] or [17].

A. Optical cross sections

The cross sections are obtained from the elements of the full-wave T-matrix [2]. In the small particle limit, only the dominant terms $T_{11}^{22,m}$ for $m = 0, 1$ contribute. For example we look at the orientation averaged cross-sections:

$$\lim_{k_1 \to 0} \langle C_{ext} \rangle = -\frac{2\pi}{k_1^2} \mathrm{Re}\{T_{11}^{22,0} + 2T_{11}^{22,1}\}, \tag{83}$$

$$\lim_{k_1 \to 0} \langle C_{sca} \rangle = \frac{2\pi}{k_1^2} \left(|T_{11}^{22,0}|^2 + 2|T_{11}^{22,1}|^2 \right), \tag{84}$$

with

$$\lim_{k_1 \to 0} T_{11}^{22,0} = -i(k_1 a)^3 \frac{32}{3\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} nk \, \bar{T}_{nk}^{m=0,s}, \tag{85}$$

$$\lim_{k_1 \to 0} T_{11}^{22,1} = i(k_1 a)^3 \frac{2}{3\pi} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \varepsilon_k (4n^2 - 1) \bar{T}_{nk}^{m=1,c}. \tag{86}$$

The extinction cross-section is plotted in figure 4 for various nano-tori and appear consistent with results from [20–22]. The dominant response comes from $T_{11}^{22,1}$ - excitation along the xy- plane - this is the $l = 0, m = 1$ plasmon resonance in figure 3. We cannot see the resonances for $l \geq 1$ that occur for small negative epsilon due to the considerable imaginary part of the dielectric function of gold. On the other hand, $T_{11}^{22,0}$ is almost negligible and has no resonance.

VIII. Thin ring limit

We can find analytic results for the matrix elements in the thin ring limit. The limits of the Legendre functions are [15]

$$P_{-1/2}^{m}(\beta_0) \to \frac{\sqrt{2/(\pi\beta_0)}}{\Gamma(-m + \frac{1}{2})} \left[\log(8\beta_0) - 2\sum_{p=1}^{m} \frac{1}{2n-1} \right] \quad m \geq 0 \tag{87}$$

$$P_{n-1/2}^{m}(\beta_0) \to \frac{(n-1)!(2\beta_0)^{n-1/2}}{\sqrt{\pi}\Gamma(n-m+\frac{1}{2})}, \quad n > 0 \tag{88}$$

$$Q_{n-1/2}^{m}(\beta_0) \to \frac{(-)^m \sqrt{\pi}\Gamma(n+m+\frac{1}{2})}{n!(2\beta_0)^{n+1/2}}, \tag{89}$$
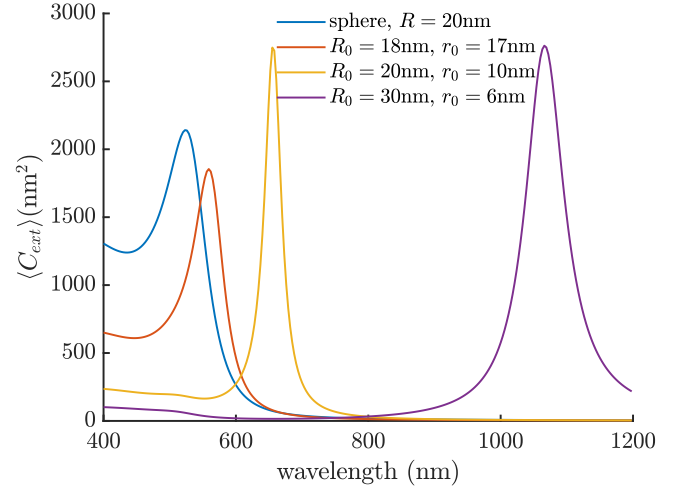

FIG. 4. Extinction cross sections for a sphere and gold nano-tori in water. The dielectric function of gold was taken from eq. (E.2) of [23], and $\epsilon_{\text{water}} = 1.77$. The radiative correction has been applied.

and we will need $Q_{-1/2}^{m}(\beta_0)$ to second order:

$$Q_{-1/2}^{m}(\beta_0) \to \frac{\pi(-)^m(2m-1)!!}{2^m \sqrt{2\beta_0}} \left[1 + \frac{4m^2 + 3}{16\beta_0^2} \right]. \tag{90}$$

which can be obtained from its hypergeometric function definition. (87) agrees with [13] as $\beta \to \infty$ for $m = 0, 1, 2, ...$, but is more accurate. It can be shown that the toroidal Q-matrices in this limit are lower triangular - $\bar{Q}_{nk}^{mc} = \bar{Q}_{nk}^{ms} = 0$ for $n < k$, with

$$\bar{Q}_{nk}^{m_s^c} \to \delta_{nk} \frac{\epsilon+1}{2} + (1 - \delta_{nk})(1 \pm \delta_{k0})$$
$$\times \frac{\epsilon-1}{4} \frac{\Gamma(k+m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})} \frac{(n-1)!}{k!} \quad n \geq k, n \neq 0$$

$$\bar{Q}_{00}^{mc} \to 1 + \frac{\epsilon-1}{\beta_0^2} \frac{4m^2+1}{8} \left[\log(8\beta) - 2\sum_{p=1}^{|m|} \frac{1}{2p-1} \right]$$

$$\bar{Q}_{00}^{ms} = 1 \quad \forall \beta_0. \tag{91}$$

In the goal of finding results for the T-matrix, we introduce the R-matrix $\mathbf{R} = \mathbf{Q}^{-1}$, which has the following limit:

$$\bar{R}_{nk}^{m,c} \to \frac{2\delta_{nk}}{\epsilon+1} + (1 - \delta_{nk}) \left(\frac{1 + \epsilon\delta_{k0}}{1 - \delta_{k0}} \right) \frac{\epsilon-1}{(\epsilon+1)^{n-k+1}}$$
$$\times \frac{\Gamma(k+m+\frac{1}{2})}{\Gamma(n+m+\frac{1}{2})} \prod_{p=k+1}^{n-1} \left(\frac{\epsilon-1}{2} + p(\epsilon+1) \right),$$
$$n \geq k, n \neq 0 \tag{92}$$

This expression for $\bar{\mathbf{R}}$ is the inverse of $\bar{\mathbf{Q}}$ in (91), i.e. $\sum_{p=0}^{\infty} \bar{Q}_{np} \bar{R}_{pk} = \delta_{nk}$, which can be proven with the help of Mathematica. To obtain a result for the T-matrix entry T_{00}^{m}, it is also necessary to include the second order for

the element R_{00}^{mc}:

$$\bar{R}_{00}^{mc} \to 1 - \frac{\epsilon - 1}{\epsilon + 1} \frac{2m^2(\epsilon+1)+1}{4\beta_0^2} \left[\log(8\beta) - 2\sum_{p=1}^{|m|} \frac{1}{2n-1} \right]$$

$$\bar{R}_{00}^{ms} = 1 \quad \forall \beta_0. \tag{93}$$

The second order for $\bar{R}_{00}^{mc}$ is obtained by including the 2nd order element Q_{01}^{mc} (above the diagonal) and inverting the matrix consisting of just the top left 2×2 block of $\bar{\mathbf{Q}}^{mc}$. Then the T-matrix (94) is obtained through (42):

$$\bar{T}_{nk}^{mv} \to (-)^m \frac{\pi \Gamma(n+m+\frac{1}{2})\Gamma(n-m+\frac{1}{2})}{n!(n-1)!(2\beta_0)^{2n}} (\bar{R}_{nk}^{mv} - \delta_{nk})$$
$$n \geq k, n \neq 0$$

$$\bar{T}_{00}^{mc} \to -\frac{\pi^2}{8} \frac{\epsilon - 1}{\epsilon + 1} \frac{2m^2(\epsilon+1)+1}{\beta_0^2}$$

$$\bar{T}_{n0}^{ms} = \bar{T}_{0k}^{ms} = 0. \tag{94}$$

The upper diagonal entries of $\bar{\mathbf{T}}^{mv}$ with $n \leq k$ can be obtained from the symmetry property of the toroidal T-matrix (43). In this limit there is just one resonance at $\epsilon = -1$, consistent with the trends seen in figure 3 for large β_0. All elements have been checked numerically against the exact matrix elements in section III.

An important feature is that the top left 2×2 square of $\bar{\mathbf{T}}^{mc}$ are all order β_0^{-2}, while all other entries decay quickly as $\beta_0 \to \infty$, meaning that toroidal modes of degrees 0 and 1 dominate both the excitation and response. For $\bar{\mathbf{T}}^{ms}$, there is only one dominant element, T_{11}^{ms}.

We can now evaluate the static dipolar polarizabilities per unit volume, to $\mathcal{O}(\beta_0^{-2})$, using (85,86) with (94):

$$\alpha_{zz}^{\text{ring}} \to 2\frac{\epsilon - 1}{\epsilon + 1}, \tag{95}$$

$$\alpha_{xx}^{\text{ring}} \to \frac{\epsilon + 3}{2} \frac{\epsilon - 1}{\epsilon + 1}. \tag{96}$$

These produce reasonable accuracy ($\lesssim 10\%$) for aspect ratios $\beta_0 \gtrsim 5$, but fail when $|\epsilon| >> \beta_0$. It is interesting to compare these to the polarizabilities of a thin prolate spheroid, or needle:

$$\alpha_{zz}^{\text{needle}} \to \epsilon - 1, \tag{97}$$

$$\alpha_{xx}^{\text{needle}} \to 2\frac{\epsilon - 1}{\epsilon + 1}. \tag{98}$$

$\alpha_{zz}^{\text{ring}}$, $\alpha_{xx}^{\text{needle}}$ apply when the long dimension of the wire is perpendicular to the applied electric field, and in fact both are of the form $\alpha_\perp = 2(\epsilon-1)/(\epsilon+1)$. On the other hand $\alpha_{xx}^{\text{ring}}$ and $\alpha_{zz}^{\text{needle}}$ both diverge as $\epsilon \to \infty$, where the long dimension of the wire is aligned with the electric field. $\alpha_{zz}^{\text{needle}}$ is of the form $\alpha_{||} = \epsilon - 1$. Intuitively, for $\alpha_{xx}^{\text{ring}}$ the ring is half aligned perpendicular and half parallel to the incident field, so that $\alpha_{xx}^{\text{ring}} = (\alpha_\perp + \alpha_{||})/2$.

Numerical tests show that for very large ϵ and very thin rings, the approximate expressions for $\bar{\mathbf{R}}^{m,c}$ break down, particularly for $m = 0$. For analysis of the thin ring limit for conducting tori, see [13].

IX. Conclusion

The problem of a dielectric torus in an arbitrary electrostatic field has been solved semi-analytically, and expressions are found for the T-matrix on both a toroidal and spherical harmonic basis. For this, new forms of the series relationships between spherical and toroidal harmonics were derived. Resonant permittivities are calculated from the eigenvalues of a matrix and agree with results from solving a continued fraction equation. Fully analytic assymptotic expressions are given in the conducting and thin ring limits. The T-matrix has then been transformed on to a basis of solid spherical harmonics so that the incident and reflected fields are also expressed as sums of spherical harmonics. This matrix has then been converted to a basis of spherical wavefunctions to coincide with the limit of the time-harmonic T-matrix governing electric multipole interactions, $\mathbf{T}^{22}$. These results prove the existence of the T-matrix for such a complex shape, atleast in the small size limit. For any bounded scatterer, the scattered field is expandable on a basis of outgoing spherical wave functions that converge outside the circumscribing sphere, and by linearity of the Helmholtz equation, the expansion coefficients are related to the coefficients of the incident field. Therefore the T-matrix method should be applicable for any bounded scatterer. It is also found that for a scatterer that excludes the origin, similar T-matrices can be defined depending on whether the source is near or far from the torus hole and whether the scattered field is to be evaluated near or far from the origin.

A. Evaluation of the surface integral (36)

First we express the integral using the product formula:

$$\int_{-\pi}^{\pi} \frac{\cos(n\eta)\cos(k\eta)}{\cosh\xi - \cos\eta} d\eta = \frac{1}{2} \int_{-\pi}^{\pi} \frac{\cos((n+k)\eta)}{\cosh\xi - \cos\eta} d\eta$$
$$+ \frac{1}{2} \int_{-\pi}^{\pi} \frac{\cos((n-k)\eta)}{\cosh\xi - \cos\eta} d\eta \tag{A1}$$

We make the substitution $z = e^{i\eta}$. Following the method of [24] (lemma 8.3) with minor changes we can rewrite

one of these integrals as

$$\int_{-\pi}^{\pi} \frac{\cos(p\eta)}{\cosh\xi - \cos\eta} d\eta$$
$$= \int_{|z|=1} 2i \frac{z^p + z^{-p}}{\sinh\xi} \left(\frac{1}{z - e^\xi} - \frac{1}{z - e^{-\xi}}\right) dz \quad (A2)$$

There are two poles inside the integration path, a simple pole at $z = e^{-\xi}$ and a pole of order p at $z = 0$. Applying the residue theorem we find:

$$\int_{-\pi}^{\pi} \frac{\cos(p\eta)}{\cosh\xi - \cos\eta} d\eta = \frac{2\pi e^{-p\xi}}{\sinh\xi}. \quad (A3)$$

Then combine this with (A1) to obtain (36). The proof of (38) is similar.

B. Derivations of the relationships between spherical and toroidal harmonics

1. Expansions of toroidal harmonics

We can first obtain the expansions of toroidal harmonics Ψ_0^{mc} in terms of spherical harmonics $S_n^m, \hat{S}_n^m$, by equating two expansions of Green's function. For points $\mathbf{r}_1$ and $\mathbf{r}_2$ with $r_1 < r_2$, the spherical harmonic expansion of Green's function is [25]

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{m=-\infty}^{\infty} (-)^m \sum_{n=|m|}^{\infty} \frac{r_1^n}{r_2^{n+1}} P_n^m(u_1) P_n^{-m}(u_2)$$
$$\times \cos m(\phi_1 - \phi_2), \quad (B1)$$

note $P_n^{-m} = (-)^m \frac{(n-m)!}{(n+m)!} P_n^m$. In this document $P_k^m(u)$ are defined without the phase $(-)^m$.

Also we have the 'cylindrical' expansion which converges in all space[26]:

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{\pi\sqrt{\rho_1\rho_2}} \sum_{m=-\infty}^{\infty} Q_{m-1/2}(\chi_{12}) \cos m(\phi_1 - \phi_2),$$
$$\chi_{12} = \frac{\rho_1^2 + \rho_2^2 + (z_1 - z_2)^2}{2\rho_1\rho_2} \quad (B2)$$

which converges for all $\mathbf{r}_1 \neq \mathbf{r}_2$. Note $Q_{-m-1/2}^n = Q_{m-1/2}^n$.

Evaluating both (B1) and (B2) at $\rho_2 = a, z_2 = 0$ ($\Rightarrow u_2 = 0$, $r_2 = a$, $\chi_{12} = \chi = \beta/\sqrt{\beta^2 - 1}$), and equating the mth term gives for $r < a$:

$$\sqrt{\frac{a}{\rho}} Q_{m-1/2}(\chi) = \pi \sum_{k=|m|}^{\infty} (-)^m P_k^{-m}(0) \left(\frac{r}{a}\right)^k P_k^m(u)$$
$$(B3)$$

Similarly the expansion for $r > a$ can be found by setting $\rho_1 = a, z_1 = 0$ in (B1) and (B2):

$$\sqrt{\frac{a}{\rho}} Q_{m-1/2}(\chi) = \pi \sum_{k=|m|}^{\infty} (-)^m P_k^{-m}(0) \left(\frac{a}{r}\right)^{k+1} P_k^m(u)$$
$$(B4)$$

Despite the appearance, the left hand side is actually proportional to a toroidal harmonic Ψ_0^{mc}. This can be seen by application of the Whipple formulae, which expressed in toroidal coordinates read as

$$\Delta P_{n-1/2}^m(\beta) = \frac{(-)^n 2/\sqrt{\pi}}{\Gamma(n - m + \frac{1}{2})} \sqrt{\frac{a}{\rho}} Q_{m-1/2}^n(\chi) \quad (B5)$$

$$\Delta Q_{n-1/2}^m(\beta) = \frac{(-)^n \pi\sqrt{\pi}}{\Gamma(n - m + \frac{1}{2})} \sqrt{\frac{a}{\rho}} P_{m-1/2}^n(\chi). \quad (B6)$$

And for toroidal harmonics of negative order:

$$P_{n-1/2}^{-m} = \frac{\Gamma(n - m + \frac{1}{2})}{\Gamma(n + m + \frac{1}{2})} P_{n-1/2}^m$$

$$Q_{n-1/2}^{-m} = \frac{\Gamma(n - m + \frac{1}{2})}{\Gamma(n + m + \frac{1}{2})} Q_{n-1/2}^m \quad (B7)$$

We may then rewrite the expansions (B3,B4) as

$$\Delta P_{-1/2}^m(\beta) = \frac{2\sqrt{\pi}(-)^m}{\Gamma(-m + 1/2)} \sum_{k=|m|}^{\infty} P_k^{-m}(0)$$
$$\times \begin{cases} \left(\frac{r}{a}\right)^k P_k^m(u) & r < a \\ \left(\frac{a}{r}\right)^{k+1} P_k^m(u) & r > a \end{cases} \quad (B8)$$

which is (50) for $n = 0$. (B8) has been generalized to the Helmholtz equation (harmonic time dependence), as a spherical wave function expansion of a circular ring with current distribution expressed as a Fourier series [27].

The $n = 1$ toroidal harmonics Ψ_1^{ms} are the potential of a ring of dipoles pointing in the z-direction. This can be obtained by applying the operator ∂_z which transforms a charged ring into a double ring with dipole moment in the z-direction. Explicitly:

$$a\frac{\partial}{\partial z} \Psi_0^{mc} = \left(m - \frac{1}{2}\right) \Psi_1^{ms}. \quad (B9)$$

∂_z is also a ladder operator for the spherical harmonics:

$$a\partial_z \hat{S}_n^m = (n + m)\hat{S}_{n-1}^m \quad (B10)$$
$$a\partial_z S_n^m = -(n - m + 1)S_{n+1}^m \quad (B11)$$

Applying $a\partial_z$ to the toroidal harmonic expansion (B8), and noting an identity for the derivative of the Legendre functions, leads directly to the expansion for Ψ_1^{ms}.

Ψ_1^{mc} is the potential of a ring of dipoles oriented outwards from the origin. To generate this we apply $r\partial_r$ which preserves harmonicity as does ∂_z, but turns a ring of charge on the xy plane into a ring of dipoles pointing inward, plus, as it turns out, a net charge on the ring, as noted in section II A.

Applying $r\partial_r$ to the $n = 0$ toroidal harmonic expansion (B8) and rearranging gives the expansion for ψ_1^{mc}. We can now derive the expansions for general n by repeated

application of $r\partial_r$. For the spherical harmonics, we have

$$r\frac{\partial}{\partial r}S_n^m = -(n+1)S_n^m \tag{B12}$$

$$r\frac{\partial}{\partial r}\hat{S}_n^m = n\hat{S}_n^m \tag{B13}$$

And applying $r\partial_r$ to Ψ_n^{mv} and rearranging, making use of the product to sum formulae for trigonometric functions and the recurrence relation for the Legendre functions $P_{n-1/2}^m$ for increasing n, we obtain:

$$2r\frac{\partial}{\partial r}\Psi_n^{mv} = \left(n+m-\frac{1}{2}\right)\Psi_{n-1}^{mv} - \Psi_n^{mv}$$
$$- \left(n-m+\frac{1}{2}\right)\Psi_{n+1}^{mv} \tag{B14}$$

By assuming some expansion coefficients c_{nk}^m, s_{nk}^m for the expansions of the toroidal harmonics as in (50,51), we can deduce that they satisfy the recurrence relation (54), which is sufficient to calculate the coefficients for all n, k, m without numerical problems.
The first few orders for $m = 0$ are:

$$c_{0,k}^0 = 2$$
$$c_{1,k}^0 = 2(2k+1)$$
$$c_{2,k}^0 = \frac{16}{3}(k^2+k+\frac{3}{8})$$
$$c_{3,k}^0 = \frac{32}{15}(2k+1)(k^2+k+\frac{15}{16})$$
$$c_{4,k}^0 = \frac{64}{105}(4k^4+8k^3+15k^2+11k+\frac{105}{32})$$
$$c_{5,k}^0 = \frac{128}{945}(2k+1)(4k^4+8k^3+\frac{109}{4}k^2+\frac{93}{4}k+\frac{945}{64})$$

$$s_{0,k}^0 = 0$$
$$s_{1,k}^0 = 4(k+1)$$
$$s_{2,k}^0 = \frac{8}{3}(k+1)(2k+1)$$
$$s_{3,k}^0 = \frac{64}{15}(k+1)(k^2+k+\frac{13}{16})$$
$$s_{4,k}^0 = \frac{128}{105}(k+1)(2k+1)(k^2+k+\frac{76}{32})$$
$$s_{5,k}^0 = \frac{256}{945}(k+1)(4k^4+8k^3+\frac{107}{4}k^2+\frac{91}{4}k+\frac{789}{64})$$

Although these coefficients may be computed for all integer n, k, m, c_{nk}^m are only relevant for $k+m$ even, while s_{nk}^m only for $k+m$ odd.

2. Expansions of spherical harmonics

We start by applying a trigonometric identity to rearrange the toroidal Green's function expansion [25] as

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{\Delta_1\Delta_2}{2\pi a}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}P_{k-1/2}^m(\beta_1)Q_{k-1/2}^{-m}(\beta_2)$$

$$\times [\cos(k\eta_1)\cos(k\eta_2) + \sin(k\eta_1)\sin(k\eta_2)]\cos m(\phi_1-\phi_2),$$
$$\beta_1 < \beta_2. \tag{B15}$$

We substitute the spherical harmonic expansions for point $\mathbf{r}_2$ (50,51),

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{\Delta_2}{2\pi a}\sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}Q_{k-1/2}^{-m}(\beta_2)(-)^k$$

$$\times \sum_{n=|m|}^{\infty}\left[\cos(k\eta_2)c_{kn}^m P_n^{-m}(0) + \sin(k\eta_2)s_{kn}^m P_{n+1}^{-m}(0)\right]$$

$$\times \left(\frac{r_1}{a}\right)^n P_n^m(u_1)\cos m(\phi_1-\phi_2), \tag{B16}$$

and compare this to the spherical expansion of Green's function (B1), equating the coefficients of $P_n^m(u_1)$ for all n, m. This gives the expansion of spherical harmonics $S_n^m, \hat{S}_n^m$ in terms of toroidal harmonics ψ_n^{mv}, as presented in (52,53).

Looking at the low orders, for $n = 0$, (52) is the expansion of a constant onto the basis of toroidal harmonics, and is known as Heine's expansion. And $n = 0$ for (53) is simply Green's function expansion (B15) for $\mathbf{r}_1 = 0$. Also, (52),(53) have been derived for $n = 1$, $m = 0, 1$, for analysis of low frequency plane wave scattering [4], and point dipole scattering [5]. Some low orders of the coefficients are

$$c_{k0}^0 = 2, \qquad s_{k1}^0 = 8k, \tag{B17}$$

$$c_{k2}^0 = 8k^2 + 2, \qquad s_{k3}^0 = \frac{16}{9}(4k^3+5k), \tag{B18}$$

$$c_{k1}^1 = -4k^2 + 1, \qquad c_{k1}^{-1} = -4. \tag{B19}$$

These can be proven by substitution into the recurrence formula (54).

We were unable to find a simple closed form for c_{nk}^m or s_{nk}^m, or find any links with known sequences on the OEIS. However, in [8] these coefficients were expressed as a sum (they used coefficients h_{nk}^m which dealt with Ψ_n^{mc} and Ψ_n^{ms} simultaneously):

$$h_{nk}^m = \frac{(-)^{k+m}}{a^k}[c_{nk}^m P_k^{-m}(0) + is_{nk}^m P_{k+1}^{-m}(0)]$$
$$= \frac{2^{2m}m!}{(2m)!} + \sum_{p=0}^{n}\frac{(-)^p 2^{3p+2m}(p+m)!(n+p-1)!}{(2p)!(2p+2m)!(n-p)!}$$
$$\times \left[nP_{k+p}^{m+p}(0) + i(k-m+1)pP_{k+p}^{m+p-1}(0)\right] \tag{B20}$$

which was found to have a triangular recurrence over n and k (by application of ∂_z instead of $r\partial_r$):

$$2i(k-m)h_{n,k-1}^m = \left(n-m+\frac{1}{2}\right)h_{n+1,k}^m - 2nh_{nk}^m$$
$$+ \left(n+m-\frac{1}{2}\right)h_{n-1,k}^m. \tag{B21}$$

h_{nk}^m was actually generalised to consider the centre of the spherical coordinates being translated up the z-axis - this

is obtained essentially by replacing the 0 in $P_k^m(0)$ with some angle. These generalised coefficients could be used to obtain the T-matrix for the offset torus, allowing an analytic study of stacked tori, as considered in [21].

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