Polynomial overreproduction by Hermite subdivision operators, and p–Cauchy numbers

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Abstract

We study the case of Hermite subdivision operators satisfying a spectral condition of order greater than their size. We show that this can be characterized by operator factorizations involving Taylor operators and difference factorizations of a rank one vector scheme. Giving explicit expressions for the factorization operators, we put into evidence that the factorization only depends on the order of the spectral condition but not on the polynomials that define it. We further show that the derivation of these operators is based on an interplay between Stirling numbers and p–Cauchy numbers (or generalized Gregory coefficients).

Keywords: Hermite subdivision schemes; operator factorization; p–Cauchy numbers

MSC: 65D15; 41A58; 11B73

1 Introduction

A dyadic stationary subdivision operator S_a acts on a sequence $c : \mathbb{Z} \to \mathbb{R}$ by means of the convolution like and hence stationary operation

$$c \mapsto S_a c := \sum_{\alpha \in \mathbb{Z}} a(\cdot - 2\alpha)\, c(\alpha).$$

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Here a, the so-called *mask* of the subdivision operator, is a finitely supported sequence. There are various ways of generalizing subdivision operators. For example, one can consider several variables, dilation factors greater than 2 or even expansive dilation matrices, or vector- or matrix-valued data which requires the mask to be a finitely supported matrix-valued sequence, cf. [4]. A *subdivision scheme* is an iteration of subdivision operators that may even depend on the level of iteration, where the nth iteration is seen as data defined on the grid $2^{-n}\mathbb{Z}$. Since these grids get finer and finer, there is the concept of a *limit function* of subdivision schemes, cf. [4].

Hermite subdivision is a special case of subdivision operators with matrix masks acting on vector data, where the components of these vectors are interpreted as consecutive derivatives. Such schemes have been considered and analyzed first in [11, 17]. The chain rule then enforces a subdivision process of a mildly level-dependent form that consists of a left and right multiplication by dyadic diagonal matrices. Also the notion of *convergence* is special for Hermite subdivision schemes: If the input data is in $\mathbb{R}^{d+1}$, the limit function is vector-valued of size $d+1$ and consists of a C^d function and its derivatives up to order d.

It is well–known in subdivision theory [4, 10] that the regularity of a limit function implies the preservation of certain polynomials by the subdivision scheme. For Hermite subdivision schemes this is usually formulated in terms of the *spectral condition* and has been related to Taylor polynomials in [9]. In [18] it is shown that the spectral condition is essentially equivalent to an operator factorization of the form

$$TS_{\boldsymbol{A}} = S_{\boldsymbol{B}}T \tag{1}$$

where T is the so–called *Taylor operator*. T is a discrete version of the Taylor formula and relates successive entries of vector-valued data in accordance with the assumption that they are consecutive derivatives. Moreover, the contractivity of $S_{\boldsymbol{B}}$ plays an important role in the analysis of convergence, cf. [18].

In [20] it is conjectured that convergence implies a generalized spectral condition of order at least d to be satisfied. This is in accordance with similar results for scalar subdivision schemes, cf. [4]. Therefore, if one is interested in Hermite schemes of regularity $n>d$, that is, limit functions consisting of a C^n function and its first d derivatives, the Hermite scheme should satisfy a spectral condition of order at least n. Schemes of regularity $n>d$ are considered in e.g. [6, 13, 23].

We call this phenomenon *polynomial overreproduction* and it is the main topic of this paper. We describe conditions under which the subdivision operator $S_{\boldsymbol{A}}$ satisfies a spectral condition of degree *higher* than d, providing a generalization of [24]. It turns out that this property fits well into the existing theory: $S_{\boldsymbol{A}}$ has to have a factorization by means of a Taylor operator as in (1) and the *rank one* vector subdivision scheme $S_{\boldsymbol{B}}$ has to be factorizable in the sense defined in [21, 22]. There is, however, a peculiarity: The matrices that appear in the factorizations of rank one schemes are derived from the spectral condition, but do not depend on the concrete choice of $\boldsymbol{A}$.

The paper is organized as follows. We start by introducing notation and give detailed definitions of the above properties in Section 2; factorizations of subdivision operators are revised in Section 3. In Section 4 we introduce Stirling numbers and their connection to p–Cauchy numbers. Based on the technical preliminaries of Section 5, the main result of the paper, namely the factorization with respect to the *augmented Taylor operator*, is given in Section 6 with a rather short proof.

2 Notation and subdivision schemes

Throughout this paper, d denotes an integer, and $d \geq 1$. Vectors in $\mathbb{R}^{d+1}$ are written as $\boldsymbol{c}$, that is, with boldface lowercase letters, while matrices $\boldsymbol{A}$ are written with boldface uppercase letters. The standard basis in $\mathbb{R}^{d+1}$ is denoted by $\boldsymbol{e}_0,\dots,\boldsymbol{e}_d$. The identity matrix of dimension $d+1$ is denoted by $\boldsymbol{I}_{d+1}$. We also use the Matlab-like notation $\boldsymbol{c}_{k:\ell}$ to extract subvectors. Furthermore, for a vector $\boldsymbol{c} \in \mathbb{R}^{d+1}$ we introduce the notation $\hat{\boldsymbol{c}} = (\boldsymbol{c}_0,\dots,\boldsymbol{c}_{d-1},0)^T$ for the canonical embedding of $\boldsymbol{c}$ into $\mathbb{R}^{d+1+k}$, $k \geq 1$.

The space of all polynomials in one variable is written as Π, while Π_n denotes all such polynomials with degree at most n.

By $\ell^{d+1}(\mathbb{Z})$ we denote the space of all sequences $\boldsymbol{c} : \mathbb{Z} \to \mathbb{R}^{d+1}$, while $\ell^{(d+1)\times(d+1)}(\mathbb{Z})$ is the space of matrix-valued sequences $\boldsymbol{A} : \mathbb{Z} \to \mathbb{R}^{(d+1)\times(d+1)}$. We use the same notation for vectors (matrices) and sequences of vectors (matrices); it will be clear from the context what is meant. The notation $\ell_{00}^{d+1}(\mathbb{Z})$ and $\ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ is used to denote sequences with finite support.

To distinguish them from input data for subdivision schemes, we denote sequences of vector valued parameters by $\boldsymbol{c}_n, n \in \mathbb{N}$, in accordance with the notation $\boldsymbol{e}_0,\dots,\boldsymbol{e}_d$ of the unit coordinate vectors. The k–th entry of an element of such a sequence is accessed by $\boldsymbol{c}_{n,k}, k = 0,\dots,d, n \in \mathbb{N}$.

The *forward difference operator* Δ is used both in the context of functions and sequences. If f is a function, then $(\Delta f)(x) = f(x+1) - f(x), x \in \mathbb{R}$. For $\boldsymbol{c} \in \ell^{d+1}(\mathbb{Z})$ we have $(\Delta \boldsymbol{c})(\alpha) = \boldsymbol{c}(\alpha+1) - \boldsymbol{c}(\alpha), \alpha \in \mathbb{Z}$. Higher order forward difference operators are defined by $\Delta^n = \Delta(\Delta^{n-1})$, $n \geq 1$, with $\Delta^0 = \mathrm{id}$.

A *stationary subdivision operator* with *mask* $\boldsymbol{A} \in \ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ is a map $S_{\boldsymbol{A}} : \ell^{d+1}(\mathbb{Z}) \to \ell^{d+1}(\mathbb{Z})$ defined by

$$(S_{\boldsymbol{A}}\boldsymbol{c})(\alpha) = \sum_{\beta\in\mathbb{Z}} \boldsymbol{A}(\alpha - 2\beta)\boldsymbol{c}(\beta), \quad \alpha \in \mathbb{Z}, \quad \boldsymbol{c} \in \ell^{d+1}(\mathbb{Z}).$$

We consider a vector $\boldsymbol{c} \in \mathbb{R}^{d+1}$ as a constant sequence, so that $S_{\boldsymbol{A}}\boldsymbol{c}$ means the application of $S_{\boldsymbol{A}}$ to the constant sequence $\boldsymbol{c}(\alpha) = \boldsymbol{c}, \alpha \in \mathbb{Z}$.

A level-dependent *subdivision scheme* $(S_{\boldsymbol{A}^{[n]}}, n \in \mathbb{N})$ is the procedure of iteratively constructing vector-valued sequences by

$$\boldsymbol{c}^{[n+1]} = S_{\boldsymbol{A}^{[n]}}\boldsymbol{c}^{[n]}, \quad n \in \mathbb{N}, \tag{2}$$

from initial data $\boldsymbol{c}^{[0]} \in \ell^{d+1}(\mathbb{Z})$. In this paper we consider two cases of such subdivision schemes based on stationary subdivision operators: *vector subdivision schemes* which use the same mask in every iteration level, i.e. $\boldsymbol{A}^{[n]} = \boldsymbol{A}, n \in \mathbb{N}$, cf. [22], and *Hermite subdivision schemes* which use the mildly level-dependent masks

$$\boldsymbol{A}^{[n]} = \boldsymbol{D}^{-n-1}\boldsymbol{A}\boldsymbol{D}^{n} \tag{3}$$

where $\boldsymbol{D} = \mathrm{diag}\left(1, 2^{-1}, \ldots, 2^{-d}\right)$ and $\boldsymbol{A} \in \ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ is fixed. In Hermite subdivision, the data $c^{[n]}$ represents function and consecutive derivative values at $2^{-n}\alpha, \alpha \in \mathbb{Z}$, leading to the mask (3) via the chain rule.

For $p \in \Pi$ we define the vector-valued function

$$\boldsymbol{v}(p)(x) := \left(p^{(k)}(x) : k = 0, \ldots, d\right)^T, \quad x \in \mathbb{R}. \tag{4}$$

We also consider $\boldsymbol{v}(p)$ as a sequence in $\ell^{d+1}(\mathbb{Z})$, by evaluating at integers only. The particular meaning of $\boldsymbol{v}(p)$ will be clear from the context.

A Hermite subdivision scheme is said to satisfy the *spectral condition of order* $n \geq d$ if there exist $p_k \in \Pi_k$, normalized as $p_k(x) = \frac{1}{k!}x^k + \cdots$, such that

$$S_{\boldsymbol{A}}\boldsymbol{v}(p_k) = 2^{-k}\boldsymbol{v}(p_k), \qquad k = 0, \ldots, n. \tag{5}$$

The spectral condition for $n = d$ has first been introduced by [9], see also [18]. The case $n > d$ is a higher order spectral condition studied in [6], and we denote it by *polynomial overreproduction*. The recent paper [20] introduces *spectral chains*, which generalize (5). We briefly discuss spectral chains in Section 6.

While the spectral condition of order d is important for factorization of Hermite subdivision operators [18], it has been shown that it is not necessary for convergence [19, 20].

3 Factorization of subdivision operators

The factorization of subdivision operators is a standard method for proving convergence of the associated subdivision schemes and regularity of their limits. In this paper, we are concerned with factorizations of rank 1 vector schemes as derived in [5, 21, 22, 28] and Taylor factorizations of Hermite schemes [7, 18, 20]. We now introduce these concepts.

Following [22], for a subdivision operator $S_{\boldsymbol{B}}$, we define

$$\mathscr{E}_{\boldsymbol{B}} = \left\{ \boldsymbol{c} \in \mathbb{R}^{d+1} : S_{\boldsymbol{B}} \boldsymbol{c} = \boldsymbol{c} \right\}, \tag{6}$$

which is the eigenspace (of constant sequences) of $S_{\boldsymbol{B}}$ with respect to the eigenvalue 1. The dimension $\dim \mathscr{E}_{\boldsymbol{B}}$ is called the *rank* of the subdivision scheme. In this paper we are only concerned with *rank* 1 *schemes*, i.e. operators $S_{\boldsymbol{B}}$ satisfying $\dim \mathscr{E}_{\boldsymbol{B}} = 1$, cf. [21]. We call a matrix $\boldsymbol{V} = (v_0, \ldots, v_d)$ with $v_j \in \mathbb{R}^{d+1}$, $j = 0, \ldots, d$, an $\mathscr{E}_{\boldsymbol{B}}$*-generator* if $\{v_0, \ldots, v_d\}$ is a basis of $\mathbb{R}^{d+1}$ and if v_d spans $\mathscr{E}_{\boldsymbol{B}}$.

With the operator

$$D := \begin{pmatrix} \boldsymbol{I}_d & \\ & \Delta \end{pmatrix} \tag{7}$$

the following result has been shown, cf. [21, 22]:

Lemma 1. *Let $S_{\boldsymbol{B}}$ be a subdivision operator with $\dim \mathscr{E}_{\boldsymbol{B}} = 1$. If $\boldsymbol{V}$ is an $\mathscr{E}_{\boldsymbol{B}}$-generator, then there exists a subdivision operator $S_{\boldsymbol{C}}$ such that*

$$D\boldsymbol{V}^{-1} S_{\boldsymbol{B}} = S_{\boldsymbol{C}} D\boldsymbol{V}^{-1}.$$

Furthermore, $\dim \mathscr{E}_{\boldsymbol{C}} = 1$.

From [18] recall the *(incomplete) Taylor operator*

$$T_d = \begin{pmatrix} \Delta & -1 & -\frac{1}{2} & \cdots & -\frac{1}{d!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \Delta & -1 & -\frac{1}{2!} \\ & & & \Delta & -1 \\ & & & & 1 \end{pmatrix}$$

and the *complete Taylor operator*

$$\widetilde{T}_d = D\,T_d = \begin{pmatrix} \Delta & -1 & -\frac{1}{2} & \dots & -\frac{1}{d!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \Delta & -1 & -\frac{1}{2!} \\ & & & \Delta & -1 \\ & & & & \Delta \end{pmatrix}.$$

We also consider the following operator which has been defined and studied in [9]:

$$T'_d = \begin{pmatrix} \Delta & -1 & \dots & -\frac{1}{(d-1)!} & 0 \\ & \ddots & \ddots & \vdots & \vdots \\ & & \Delta & -1 & 0 \\ & & & \Delta & 0 \\ & & & & 1 \end{pmatrix}.$$

We furthermore define $\widetilde{T}_0 = \Delta$ and $T_0 = T'_0 = \mathrm{id}$. Generalizations of these Taylor operators have been introduced in [20]; we discuss them in Section 6.

It has been shown in [18, Theorem 4] that a subdivision operator $S_{\boldsymbol{A}}$ satisfying the spectral condition of order d (5) can be factorized with respect to the Taylor operator: There exists a subdivision operator $S_{\boldsymbol{B}}$ such that

$$T_d S_{\boldsymbol{A}} = 2^{-d} S_{\boldsymbol{B}} T_d. \tag{8}$$

If $S_{\boldsymbol{A}}$ factorize-s as in (8), but stepwise, i.e. with respect to operators

$$\begin{pmatrix} T_k & \\ & \boldsymbol{I}_{d-k} \end{pmatrix}, \quad k = 0, \dots, d,$$

then this is even a characterization of the spectral condition of order d (5), cf. [19, Corollary 2.12]. Furthermore, $\mathscr{E}_{\boldsymbol{B}}$ is spanned by $\boldsymbol{e}_d$. Therefore $\boldsymbol{V} = \boldsymbol{I}_{d+1}$ is an $\mathscr{E}_{\boldsymbol{B}}$-generator and by Lemma 1 there exists a subdivision operator $S_{\boldsymbol{C}}$ such that

$$D\,S_{\boldsymbol{B}} = S_{\boldsymbol{C}}\,D.$$

The latter implies

$$\widetilde{T}_d S_{\boldsymbol{A}} = D\,T_d S_{\boldsymbol{A}} = 2^{-d} D S_{\boldsymbol{B}} T_d = 2^{-d} S_{\boldsymbol{C}} D T_d = 2^{-d} S_{\boldsymbol{C}} \widetilde{T}_d,$$

which is the complete Taylor factorization of [18, Theorem 4]:

$$\widetilde{T}_d S_{\boldsymbol{A}} = 2^{-d}\, S_{\boldsymbol{C}} \widetilde{T}_d. \tag{9}$$

In this paper we prove a generalization of (9) to operators $S_{\boldsymbol{A}}$ which satisfy the spectral condition (5) for $n > d$ (Theorem 16). In particular we prove that every such operator factorizes with respect to the *augmented Taylor operator* of order n:

Definition 2 (Augmented Taylor operators)**.** *For* $d \geq 1$ *and* $n \geq d$ *we define the* augmented Taylor operator *of order* n *by*

$$\widetilde{T}_d^n := \begin{pmatrix} \widetilde{T}_{d-1} & -\sum_{k=0}^{n-d} G_k^{d:1} \Delta^k \\ & \Delta^{n+1-d} \end{pmatrix} = \begin{pmatrix} \Delta & -1 & -\frac{1}{2} & \dots & -\frac{1}{(d-1)!} & -\sum_{k=0}^{n-d} G_k^d \Delta^k \\ & \ddots & \ddots & & \vdots & \vdots \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \Delta & -1 & -\sum_{k=0}^{n-d} G_k^2 \Delta^k \\ & & & & \Delta & -\sum_{k=0}^{n-d} G_k^1 \Delta^k \\ & & & & & \Delta^{n+1-d} \end{pmatrix},$$

where $G_k^{d:1} = \left(G_k^d, G_k^{d-1}, \dots, G_k^1\right)^T$*, and* $G_k^\ell, k \geq 0, \ell \geq 1$ *are the* coefficients for repeated integration with forward differences *[27].*

Remark 3. *Normalizing the coefficients* G_k^n *as in* (17) *leads to the* p–Cauchy numbers of the first kind *, see [26]. Since* G_k^1 *are known, among others, as* Gregory coefficients*, cf. [1], one could call these numbers* generalized Gregory coefficients*. We discuss them in more detail in Section 4.*

The existence of such a factorization follows from combining the Taylor factorization (8) of [18] with iterated factorizations for rank 1 schemes (Lemma 1) of [21, 22]. The contribution of this paper is the explicit computation of the augmented Taylor operators via computing $\mathscr{E}_{\boldsymbol{B}_j}$ for every iteration $j = d, \dots, n$ of rank 1 factorizations. In particular, we show that the spectral condition (5), but *not* the choice of spectral polynomials, already determines all $\mathscr{E}_{\boldsymbol{B}_j}, j = d, \dots, n$. We thus also extend the results of [24].

4 Stirling and p–Cauchy numbers

Following [12], we recall the definition of Stirling numbers.

The *Stirling numbers of the first kind*, denoted by $\left[{n \atop m}\right]$, count the numbers of ways to arrange n elements into m cycles. From the initial conditions

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0, \quad n \geq 1,$$

they can be computed via the following recurrence relation:

$$\begin{bmatrix} n+1 \\ m \end{bmatrix} = n \begin{bmatrix} n \\ m \end{bmatrix} + \begin{bmatrix} n \\ m-1 \end{bmatrix}, \quad m \geq 1.$$

The *signed Stirling numbers of the first kind* are defined by

$$s(n,m) = (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}. \tag{10}$$

They satisfy the recurrence relation

$$s(n+1,m) = s(n,m-1) - n\, s(n,m), \tag{11}$$

with initial conditions

$$s(n,n) = 1, \quad s(n,m) = 0 \quad \text{if } k = m < n \text{ or } n < m.$$

The *Stirling numbers of the second kind*, denoted by $\left\{ {n \atop m} \right\}$, count the number of ways to split a set of n elements into m non-empty subsets. They satisfy the following recurrence relation

$$\left\{ {n+1 \atop m} \right\} = m \left\{ {n \atop m} \right\} + \left\{ {n \atop m-1} \right\}, \quad m \geq 1. \tag{12}$$

with initial conditions

$$\left\{ {0 \atop 0} \right\} = 1, \quad \left\{ {n \atop 0} \right\} = \left\{ {0 \atop n} \right\} = 0, \quad n \geq 1.$$

The Stirling numbers of the second kind can be computed using Binomial coefficients

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} j^n.$$

We also need the following relation between the Stirling numbers of the second kind and the Binomial coefficients (see [12, Eq. 6.15]):

$$\left\{ {n+1 \atop m+1} \right\} = \sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\}. \tag{13}$$

Following [27], we define the *coefficients for repeated integration with forward differences*, G_n^k for $k, n \geq 1$, by

$$G_n^1 = \frac{1}{n!} \int_0^1 x(x-1)\cdots(x-n+1)\,dx, \quad n \geq 1, \tag{14}$$

and

$$G_n^k = \frac{1}{n!}\int_0^1 \int_0^{x_2} \cdots \int_0^{x_k} x(x-1)\cdots(x-n+1)\, dx dx_k \cdots dx_2, \quad n \geq 1, k \geq 2. \tag{15}$$

We also define

$$G_0^k = \frac{1}{k!}, \quad k \geq 1. \tag{16}$$

The coefficients G_n^k are connected to the *p–Cauchy numbers of the first kind*, $\mathscr{C}_{n,p}$, defined in [26], via

$$\mathscr{C}_{n,p-1} = n!\, p!\, G_n^p. \tag{17}$$

The sequence G_n^1 are the *Gregory coefficients*, since (14) is their well-known integral representation, see e.g. [16]. The Gregory coefficients are a well-studied sequence in number theory and are also known as the *Cauchy numbers of the first kind*, the *Bernoulli numbers of the second kind* and the *reciprocal logarithmic numbers*, see e.g. [2, 15, 16]. In this sense, the coefficients in (15) are a generalization of the Gregory coefficients. Another generalization of the Gregory coefficients can be found in [3, Eq. (63)].

In [27], the following recursion is shown to hold:

$$G_n^k = \frac{1}{1-k}\left((n-1)G_n^{k-1} + (n+1)G_{n+1}^{k-1}\right), \quad k \geq 2, n \geq 1, \tag{18}$$

compare also to the equivalent recursion for p–Cauchy numbers in [26, Theorem 2.5]. Via (17), Corollary 2.3 & Theorem 2.2 of [26] imply

$$\sum_{r=1}^{j} \left\{ {j \atop r} \right\} r!\, G_r^k = \frac{1}{(j+1)\cdots(j+k)}, \quad j,k \geq 1 \tag{19}$$

and

$$G_n^k = \frac{1}{n!}\sum_{j=1}^{n} \frac{s(n,j)}{(j+1)\cdots(j+k)}, \quad j,k \geq 1. \tag{20}$$

For $k=1$, (19) and (20) are proved in [16].

Remark 4. *The case $k=2$ of* (20) *can also be found on oeis.org (sequence A002687 resp. A002688) under "formula".*

5 Auxiliary results

We start by proving that the Stirling numbers of the second kind relate forward differences to derivatives:

Lemma 5. *For* $p \in \Pi_n, \ell \le n, 1 \le k \le n-\ell$ *we have*

$$\frac{1}{k!}\Delta^k p^{(\ell)} = \sum_{m=k}^{n-\ell} \frac{1}{m!} \begin{Bmatrix} m \\ k \end{Bmatrix} p^{(m+\ell)}.$$

Proof: We prove this by induction on k. For $k=1$ the Taylor formula gives

$$\Delta p(x) = p(x+1) - p(x) = \sum_{m=1}^{n} \frac{1}{m!} p^{(m)}(x)$$

and for $\ell \le n$

$$\Delta p^{(\ell)}(x) = \sum_{m=1}^{n-\ell} \frac{1}{m!} p^{(\ell+m)}(x). \tag{21}$$

We assume the statement is true for k and prove it for $k+1$ using (12), (13) and (21):

$$\begin{aligned}
\Delta^{k+1} p^{(\ell)} &= \Delta \Delta^k p^{(\ell)} = \Delta \sum_{m=k}^{n-\ell} \frac{k!}{m!} \begin{Bmatrix} m \\ k \end{Bmatrix} p^{(m+\ell)} = \sum_{m=k}^{n-\ell} \frac{k!}{m!} \begin{Bmatrix} m \\ k \end{Bmatrix} \sum_{s=1}^{n-m-\ell} \frac{1}{s!} p^{(s+m+\ell)} \\
&= \sum_{m=k}^{n-\ell} \sum_{s=m+1}^{n-\ell} \frac{k!}{s!} \binom{s}{m} \begin{Bmatrix} m \\ k \end{Bmatrix} p^{(s+\ell)} = \sum_{s=k+1}^{n-\ell} \sum_{m=k}^{s-1} \frac{k!}{s!} \binom{s}{m} \begin{Bmatrix} m \\ k \end{Bmatrix} p^{(s+\ell)} \\
&= \sum_{s=k+1}^{n-\ell} \frac{(k+1)!}{s!} \begin{Bmatrix} s \\ k+1 \end{Bmatrix} p^{(s+\ell)}.
\end{aligned}$$

This concludes the induction. □

Definition 6. *Define the following vector-valued sequences for* $j \ge 0$*:*

$$\begin{aligned}
\boldsymbol{a}_j &:= \left(\frac{1}{(j+d)!}, \frac{1}{(j+d-1)!}, \dots, \frac{1}{(j+1)!}, \frac{1}{j!} \right)^T, \\
\boldsymbol{y}_j &:= \left(G_j^d, \dots, G_j^1, 0 \right)^T.
\end{aligned}$$

The following lemma is essential for the main result of this paper, Theorem 16, since it identifies the sequence $\boldsymbol{y}_j$ as the correct coefficients for factorization.

Lemma 7. *The sequences* $(\boldsymbol{y}_j, j \ge 0)$*, and* $(\boldsymbol{a}_j, j \ge 0)$*, from Definition 6 satisfy the following property*

$$\boldsymbol{y}_0 = \hat{\boldsymbol{a}}_0 \tag{22}$$

$$\sum_{m=1}^{j} \gamma_m^j \boldsymbol{y}_m = \hat{\boldsymbol{a}}_j, \quad j \ge 1. \tag{23}$$

where

$$\gamma_m^j := \frac{m!}{j!} \begin{Bmatrix} j \\ m \end{Bmatrix}.$$

Proof. Equation (22) follows from the definition of $G_0^k, k = 1, \dots, d$, in (16).

For $j \geq 1$ and $k = 1, \dots, d$ equation (23) is equivalent to

$$\sum_{m=1}^{j} \gamma_m^j \boldsymbol{y}_{m,k} = \frac{1}{(j+k)!} \Longleftrightarrow \sum_{m=1}^{j} \left\{ \begin{matrix} j \\ m \end{matrix} \right\} m! \boldsymbol{y}_{m,k} = \frac{j!}{(j+k)!} = \frac{1}{(j+1)\cdots(j+k)}$$

Since $\boldsymbol{y}_{m,k} = G_m^k$ for $k = 1, \dots, d$, (23) is true by (19). For $k = 0$, (23) is correct because both sides equal 0. □

Remark 8. *Lemma 7 implies* $\widetilde{T}_d^d = \widetilde{T}_d$.

Lemma 9. *For* $d \geq 1$ *and* $j \geq d$*, the augmented Taylor operator satisfies*

$$\widetilde{T}_d^j = D\left(\boldsymbol{I}_d - \boldsymbol{y}_{j-d} \boldsymbol{e}_d^T\right) \cdots D\left(\boldsymbol{I}_d - \boldsymbol{y}_0 \boldsymbol{e}_d^T\right) T_d',$$

with $(\boldsymbol{y}_j, j \geq 0)$ *from Definition 6.*

Proof: Recall from Definition 6 that

$$\boldsymbol{y}_j = \left(G_j^d, \dots, G_j^1, 0\right)^T = \left(G_j^{d:1}, 0\right)^T$$

and from Lemma 7 that $\boldsymbol{y}_0 = \hat{\boldsymbol{a}}_0$. Furthermore, note that for any vector $\boldsymbol{c} \in \mathbb{R}^{d+1}$ with $\boldsymbol{c}_d = 0$ we have

$$D\left(\boldsymbol{I}_d - \boldsymbol{c}\boldsymbol{e}_d^T\right) = \begin{pmatrix} \boldsymbol{I}_{d-1} & \\ & \Delta \end{pmatrix} \begin{pmatrix} \boldsymbol{I}_{d-1} & -\boldsymbol{c}_{0:d-1} \\ & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_{d-1} & -\boldsymbol{c}_{0:d-1} \\ & \Delta \end{pmatrix}.$$

We prove the Lemma by induction on j. For $j = d$, by Remark 8 we have

$$\widetilde{T}_d^d = \widetilde{T}_d = \begin{pmatrix} \widetilde{T}_{d-1} & -\boldsymbol{a}_{0,0:d-1} \\ & \Delta \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_{d-1} & -\boldsymbol{y}_{0,0:d-1} \\ & \Delta \end{pmatrix} \begin{pmatrix} \widetilde{T}_{d-1} & \\ & 1 \end{pmatrix} = D\left(\boldsymbol{I}_d - \boldsymbol{y}_0 \boldsymbol{e}_d^T\right) T_d'.$$

Assume that the Lemma is true for j, we prove it for $j+1$.

$$\begin{aligned} \widetilde{T}_d^{j+1} &= \begin{pmatrix} \widetilde{T}_{d-1} & -\sum_{k=0}^{j+1-d} G_k^{d:1} \Delta^k \\ & \Delta^{j+2-d} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_{d-1} & -\boldsymbol{y}_{j+1-d,0:d-1} \\ & \Delta \end{pmatrix} \begin{pmatrix} \widetilde{T}_{d-1} & -\sum_{k=0}^{j-d} G_k^{d:1} \Delta^k \\ & \Delta^{j+1-d} \end{pmatrix} \\ &= D\left(\boldsymbol{I}_d - \boldsymbol{y}_{j+1-d} \boldsymbol{e}_d^T\right) D\left(\boldsymbol{I}_d - \boldsymbol{y}_{j-d} \boldsymbol{e}_d^T\right) \cdots D\left(\boldsymbol{I}_d - \boldsymbol{y}_0 \boldsymbol{e}_d^T\right) T_d', \end{aligned}$$

which concludes the induction step. □

The next lemma follows from [18] and Lemma 5:

Lemma 10. *For* $p \in \Pi$ *with* $\deg(p) = n > d$ *we have*

$$\widetilde{T}_d \boldsymbol{v}(p) = \sum_{k=1}^{n-d} \boldsymbol{a}_k p^{(k+d)}.$$

If $n \leq d$ *then* $\widetilde{T}_d \boldsymbol{v}(p) = 0$.

We write the polynomial of Lemma 10 in the following form

$$\sum_{k=1}^{n-d} \boldsymbol{a}_k p^{(k+d)} = \boldsymbol{e}_d q + \sum_{k=1}^{n-d} \hat{\boldsymbol{a}}_k p^{(k+d)},$$

where

$$q = \sum_{k=1}^{n-d} \boldsymbol{a}_{k,d} p^{(k+d)}. \tag{24}$$

If $\deg(p) = n > d$ then $\deg(q) = n - d - 1$.

Lemma 11. *For* $n > d$, $0 \leq k < n - d$ *and the polynomial* q *from* (24) *we have:*

$$\Delta^k q = \sum_{s=k+1}^{n-d} \gamma_{k+1}^s p^{(s+d)},$$

with γ *defined in Lemma 7.*

Proof. Note that the result is true for $k = 0$. For $k \geq 1$ we use Definition 6, Lemma 5, (12), and (13):

$$\begin{aligned}
\frac{1}{k!}\Delta^k q &= \frac{1}{k!} \sum_{\ell=1}^{n-d-k} \boldsymbol{a}_{\ell,d} \Delta^k p^{(\ell+d)} = \sum_{\ell=1}^{n-d-k} \boldsymbol{a}_{\ell,d} \sum_{m=k}^{n-d-\ell} \frac{1}{m!} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} p^{(m+\ell+d)} \\
&= \sum_{\ell=1}^{n-d-k} \sum_{m=k}^{n-d-\ell} \frac{1}{\ell!\, m!} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} p^{(m+\ell+d)} = \sum_{\ell=1}^{n-d-k} \sum_{s=k+\ell}^{n-d} \frac{1}{\ell!\,(s-\ell)!} \left\{ \begin{matrix} s-\ell \\ k \end{matrix} \right\} p^{(s+d)} \\
&= \sum_{s=k+1}^{n-d} \sum_{\ell=1}^{s-k} \frac{1}{\ell!\,(s-\ell)!} \left\{ \begin{matrix} s-\ell \\ k \end{matrix} \right\} p^{(s+d)} = \sum_{r=1}^{n-d-k} \sum_{\ell=1}^{r} \frac{1}{\ell!\,(r+k-\ell)!} \left\{ \begin{matrix} r+k-\ell \\ k \end{matrix} \right\} p^{(r+k+d)} \\
&= \sum_{r=1}^{n-d-k} \sum_{s=k}^{r+k-1} \frac{1}{(r+k-s)!\,s!} \left\{ \begin{matrix} s \\ k \end{matrix} \right\} p^{(r+k+d)} = \sum_{r=1}^{n-d-k} \frac{1}{(r+k)!} \sum_{s=k}^{r+k-1} \binom{r+k}{s} \left\{ \begin{matrix} s \\ k \end{matrix} \right\} p^{(r+k+d)} \\
&= \sum_{r=1}^{n-d-k} \frac{1}{(r+k)!} \left(\left\{ \begin{matrix} r+k+1 \\ k+1 \end{matrix} \right\} - \left\{ \begin{matrix} r+k \\ k \end{matrix} \right\} \right) p^{(r+k+d)} = \sum_{r=1}^{n-d-k} \frac{(k+1)}{(r+k)!} \left\{ \begin{matrix} r+k \\ k+1 \end{matrix} \right\} p^{(r+k+d)}.
\end{aligned}$$

This implies

$$\Delta^k q = \sum_{r=1}^{n-d-k} \gamma_{k+1}^{r+k} p^{(r+k+d)} = \sum_{s=k+1}^{n-d} \gamma_{k+1}^s p^{(s+d)}. \qquad \square$$

Lemma 12. *For $p \in \Pi, \deg(p) = n$, $n > d$ and $(\boldsymbol{c}_k, k \geq 1)$ such that $\boldsymbol{c}_{k,d} = 0$ for all k, we have*

$$D\left(\boldsymbol{I}_d - \boldsymbol{c}_j \boldsymbol{e}_d^T\right) \cdots D\left(\boldsymbol{I}_d - \boldsymbol{c}_1 \boldsymbol{e}_d^T\right) \sum_{k=1}^{n-d} \boldsymbol{a}_k \, p^{(k+d)} = \boldsymbol{e}_d \Delta^j q - \sum_{k=0}^{j-1} \hat{\boldsymbol{c}}_{k+1} \Delta^k q + \sum_{k=1}^{n-d} \hat{\boldsymbol{a}}_k \, p^{(k+d)},$$

for some $1 \leq j \leq n-d$.

Proof. We prove this lemma by induction on j. First note that the operator $D\left(\boldsymbol{I} - \boldsymbol{c}\boldsymbol{e}_d^T\right)$ for any $\boldsymbol{c}$ with $\boldsymbol{c}_d = 0$, acts as the identity operator on vectors with last component equal to 0. Therefore

$$\begin{aligned} D\left(\boldsymbol{I} - \boldsymbol{c}\boldsymbol{e}_d^T\right) \sum_{k=1}^{n-d} \boldsymbol{a}_k p^{(k+d)} &= D\left(\boldsymbol{I} - \boldsymbol{c}\boldsymbol{e}_d^T\right) \boldsymbol{e}_d q + \sum_{k=1}^{n-d} \hat{\boldsymbol{a}}_k = \begin{pmatrix} \boldsymbol{I}_{d-1} & -\boldsymbol{c}_{0:d-1} \\ & \Delta \end{pmatrix} \begin{pmatrix} 0 \\ q \end{pmatrix} + \sum_{k=1}^{n-d} \hat{\boldsymbol{a}}_k \\ &= \begin{pmatrix} -\boldsymbol{c}_{0:d-1} q \\ \Delta q \end{pmatrix} + \sum_{k=1}^{n-d} \hat{\boldsymbol{a}}_k \\ &= \boldsymbol{e}_d \Delta q - \hat{\boldsymbol{c}} q + \sum_{k=1}^{n-d} \hat{\boldsymbol{a}}_k. \end{aligned}$$

This proves the case $j = 1$. Assume that the lemma is true for j, we prove it for $j+1$.

$$\begin{aligned} &D\left(\boldsymbol{I}_d - \boldsymbol{c}_{j+1} \boldsymbol{e}_d^T\right) \cdots D\left(\boldsymbol{I}_d - \boldsymbol{c}_1 \boldsymbol{e}_d^T\right) \sum_{k=1}^{n-d} \boldsymbol{a}_k \, p^{(k+d)} \\ &= D\left(\boldsymbol{I}_d - \boldsymbol{c}_{j+1} \boldsymbol{e}_d^T\right) \left(\boldsymbol{e}_d \Delta^j q - \sum_{k=0}^{j-1} \hat{\boldsymbol{c}}_{k+1} \Delta^k q + \sum_{k=1}^{n-d} \hat{\boldsymbol{a}}_k \, p^{(k+d)} \right) \\ &= \begin{pmatrix} \boldsymbol{I}_{d-1} & -\boldsymbol{c}_{j+1,0:d-1} \\ & \Delta \end{pmatrix} \begin{pmatrix} 0 \\ \Delta^j q \end{pmatrix} - \sum_{k=0}^{j-1} \hat{\boldsymbol{c}}_{k+1} \Delta^k q + \sum_{k=1}^{n-d} \hat{\boldsymbol{a}}_k \, p^{(k+d)} \\ &= \begin{pmatrix} -\boldsymbol{c}_{j+1,0:d-1} \Delta^j q \\ \Delta^{j+1} q \end{pmatrix} - \sum_{k=0}^{j-1} \hat{\boldsymbol{c}}_{k+1} \Delta^k q + \sum_{k=1}^{n-d} \hat{\boldsymbol{a}}_k \, p^{(k+d)} \\ &= \boldsymbol{e}_d \Delta^{j+1} q - \sum_{k=0}^{j} \hat{\boldsymbol{c}}_{k+1} \Delta^k q + \sum_{k=1}^{n-d} \hat{\boldsymbol{a}}_k \, p^{(k+d)}, \end{aligned}$$

which concludes the induction step. □

Lemma 11 also has the following consequence.

Corollary 13. *With notation as in Lemma 12 we have*

$$D\left(\boldsymbol{I}_d - \boldsymbol{c}_j \boldsymbol{e}_d^T\right) \cdots D\left(\boldsymbol{I}_d - \boldsymbol{c}_1 \boldsymbol{e}_d^T\right) \sum_{k=1}^{n-d} \boldsymbol{a}_k \, p^{(k+d)} = \boldsymbol{e}_d \Delta^j q + \sum_{s=1}^{n-d} \left(\hat{\boldsymbol{a}}_s - \sum_{k=1}^{\min\{s,j\}} \gamma_k^s \hat{\boldsymbol{c}}_k \right) p^{(s+d)}.$$

Lemma 14. *For* $p \in \Pi, \deg(p) = n$*,* $n > d$*, normalized such that* $p(x) = \frac{1}{n!}x^n + \dots$*, and* $(\boldsymbol{y}_k, k \geq 1)$ *from Definition 6, we have*

$$D\left(\boldsymbol{I}_d - \boldsymbol{y}_{n-d-1}\boldsymbol{e}_d^T\right)\cdots D\left(\boldsymbol{I}_d - \boldsymbol{y}_1\boldsymbol{e}_d^T\right)\sum_{k=1}^{n-d}\boldsymbol{a}_k\, p^{(k+d)} = \boldsymbol{e}_d + \boldsymbol{y}_{n-d}.$$

Proof: Lemma 11 implies $\Delta^{n-d-1}q = p^{(n)} = 1$, since p is normalized. Corollary 13 and Lemma 7 now imply

$$\begin{aligned}
&D\left(\boldsymbol{I} - \boldsymbol{y}_{n-d-1}\boldsymbol{e}_d^T\right)\cdots D\left(\boldsymbol{I} - \boldsymbol{y}_1\boldsymbol{e}_d^T\right)\sum_{k=1}^{n-d}\boldsymbol{a}_k\, p^{(k+d)} \\
&= \boldsymbol{e}_d\Delta^{n-d-1}q + \sum_{s=1}^{n-d}\left(\hat{\boldsymbol{a}}_s - \sum_{k=1}^{\min\{s,n-d-1\}}\gamma_k^s\,\boldsymbol{y}_k\right)p^{(s+d)} \\
&= \boldsymbol{e}_d + \sum_{s=1}^{n-d-1}\left(\hat{\boldsymbol{a}}_s - \sum_{k=1}^{s}\gamma_k^s\,\boldsymbol{y}_k\right)p^{(s+d)} + \left(\hat{\boldsymbol{a}}_{n-d} - \sum_{k=1}^{n-d-1}\gamma_k^{n-d}\,\boldsymbol{y}_k\right)p^{(n)} \\
&= \boldsymbol{e}_d + \boldsymbol{y}_{n-d}.
\end{aligned}$$

This concludes the proof. □

Finally, Lemma 9 and Lemma 14 imply the following result.

Corollary 15. *With notation as in Lemma 14 we have*

$$\widetilde{T}_d^{n-1}\boldsymbol{v}(p) = \boldsymbol{e}_d + \boldsymbol{y}_{n-d}.$$

6 Factorization with respect to the augmented Taylor operator

Theorem 16 (Main result)**.** *If* $S_{\boldsymbol{A}}$ *satisfies the spectral condition* (5) *with* $n \geq d$*, then there exist subdivision operators* $S_{\boldsymbol{B}_j}, j = d, \dots, n$*, such that we can factorize*

$$\widetilde{T}_d^j S_{\boldsymbol{A}} = 2^{-j}\, S_{\boldsymbol{B}_j}\widetilde{T}_d^j, \tag{25}$$

with the augmented Taylor operator $\widetilde{T}_d^j$ *from Definition 2. Furthermore* $\dim \mathscr{E}_{\boldsymbol{B}_j} = 1, j = d \dots, n$*, and the factorization* (25) *is independent of the concrete spectral polynomials in* (5)*.*

Proof: Denote by $p_k, k = 0, \dots, n$, the spectral polynomials from (5). Due to their normalization we have $p_k^{(k)} = 1$.

We prove this result by induction on j. From Remark 8 we have $\widetilde{T}_d^d = \widetilde{T}_d$ and the existence of $S_{\boldsymbol{B}_d}$ follows from [18], see (8). Also $\dim \mathscr{E}_{\boldsymbol{B}_d} = 1$ follows from [18]. This shows the case $j = d$.

We assume that the theorem is true for j and prove it for $j+1$. Lemma 10 and Corollary 15 imply

$$\widetilde{T}_d^j \boldsymbol{v}(p_{j+1}) = \boldsymbol{e}_d + \boldsymbol{y}_{j+1-d}.$$

The spectral condition implies

$$\begin{aligned}2^{-j-1}(\boldsymbol{e}_d + \boldsymbol{y}_{j+1-d}) &= 2^{-j-1}\widetilde{T}_d^j \boldsymbol{v}(p_{j+1}) = \widetilde{T}_d^j S_{\boldsymbol{A}} \boldsymbol{v}(p_{j+1}) = 2^{-j} S_{\boldsymbol{B}_j} \widetilde{T}_d \boldsymbol{v}(p_{j+1}) \\ &= 2^{-j} S_{\boldsymbol{B}_j}(\boldsymbol{e}_d + \boldsymbol{y}_{j+1-d}),\end{aligned}$$

and thus

$$2 S_{\boldsymbol{B}_j}\left(\boldsymbol{e}_d + \boldsymbol{y}_{j+1-d}\right) = \boldsymbol{e}_d + \boldsymbol{y}_{j+1-d}.$$

Therefore $\boldsymbol{e}_d + \boldsymbol{y}_{j+1-d}$ lies in $\mathscr{E}_{2\boldsymbol{B}_j}$ and since by assumption the dimension of this space is 1, it is spanned by $\boldsymbol{e}_d + \boldsymbol{y}_{j+1-d}$. Now we use Lemma 1 to factorize further. The Gauß matrix

$$\boldsymbol{I}_d + \boldsymbol{y}_{j+1-d}\boldsymbol{e}_d^T = \begin{pmatrix} 1 & & & \boldsymbol{y}_{j+1-d,0} \\ & \ddots & & \vdots \\ & & 1 & \boldsymbol{y}_{j+1-d,d-1} \\ & & & 1 \end{pmatrix}, \tag{26}$$

is an $\mathscr{E}_{2\boldsymbol{B}_j}$-generator. It is easy to check that $\left(\boldsymbol{I}_d + \boldsymbol{y}_{j+1-d}\boldsymbol{e}_d^T\right)^{-1} = \boldsymbol{I}_d - \boldsymbol{y}_{j+1-d}\boldsymbol{e}_d^T$. Lemma 1 thus implies that there exists a subdivision operator $S_{\boldsymbol{B}_{j+1}}$ such that

$$2D\left(\boldsymbol{I}_d - \boldsymbol{y}_{j+1-d}\boldsymbol{e}_d^T\right) S_{\boldsymbol{B}_j} = S_{\boldsymbol{B}_{j+1}} D\left(\boldsymbol{I}_d - \boldsymbol{y}_{j+1-d}\boldsymbol{e}_d^T\right) \tag{27}$$

and such that $\dim \mathscr{E}_{\boldsymbol{B}_{j+1}} = 1$. The factorization (27) further implies

$$\begin{aligned}D\left(\boldsymbol{I}_d - \boldsymbol{y}_{j+1-d}\boldsymbol{e}_d^T\right)\widetilde{T}_d^j S_{\boldsymbol{A}} &= 2^{-j} D\left(\boldsymbol{I}_d - \boldsymbol{y}_{j+1-d}\boldsymbol{e}_d^T\right) S_{\boldsymbol{B}_j}\widetilde{T}_d^j \\ &= 2^{-j-1} S_{\boldsymbol{B}_{j+1}} D\left(\boldsymbol{I}_d - \boldsymbol{y}_{j+1-d}\boldsymbol{e}_d^T\right)\widetilde{T}_d^j.\end{aligned}$$

From Lemma 9 we know that $D\left(\boldsymbol{I}_d - \boldsymbol{y}_{j+1-d}\boldsymbol{e}_d^T\right)\widetilde{T}_d^j = \widetilde{T}_d^{j+1}$. This concludes the induction. □

Remark 17. *Theorem 16 for $d=1$ and Definition 2 give*

$$\widetilde{T}_1^j = \begin{pmatrix} \Delta & -\sum_{k=0}^{n-1} G_k^1 \Delta^k \\ 0 & \Delta^j \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathscr{G}^{[j]},$$

where G_k^1 are the Gregory coefficients, see Section 4, and $\mathscr{G}^{[j]}$ is the Gregory operator *derived in [24]. Therefore, Theorem 16 generalizes [24]. Note that the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ appears since we use* (26) *to transform to* $\boldsymbol{e}_1$ *while [24] uses an equivalent factorization as in Lemma 1 where a transform to* $\boldsymbol{e}_0$ *is needed. The factorization is correct in both cases.*

Remark 18. *The paper [14] proves factorization and convergence results for level-dependent Hermite subdivision schemes of dimension* $d=1$*. In particular it considers schemes* (2)*, where the operators* $S_{\boldsymbol{A}^{[j]}}, j \in \mathbb{N}$*, are* not *restricted to the form* (3)*. From results 5.6 – 5.8 in [14] we can deduce an interesting connection to the augmented Taylor operator.*

Consider a subdivision operator $S_{\boldsymbol{A}^{[j]}}$ *of dimension* $d=1$ *which reproduces* $\{1, x, e^{\lambda x}\}$ *(this implies that it satisfies the spectral condition* (5) *with the functions* $1, x$ *and* $e^{\lambda x}$*). Then there exists a subdivision operator* $S_{\boldsymbol{B}^{[j]}}$ *such that*

$$R^{[j+1]} S_{\boldsymbol{A}^{[j]}} = 2^{-2} \zeta(j) S_{\boldsymbol{B}^{[j]}} R^{[j]},$$

where $R^{[j]}$ *is given by*

$$R^{[j]} = \begin{pmatrix} 0 & \delta_j \Delta \\ \Delta & -1 - \eta(j) \Delta \end{pmatrix},$$

with ζ, η *from [14, Proposition 5.8 (ii)]:*

$$\zeta(j) = \frac{2}{e^{\lambda 2^{-j-1}} + 1}, \quad \eta(j) = \frac{e^{\lambda 2^{-j}} - 1 - \lambda 2^{-j}}{\lambda 2^{-j} (e^{\lambda 2^{-j}} - 1)}$$

and

$$(\delta_j \boldsymbol{c})(\alpha) = e^{-\lambda 2^{-j}} \boldsymbol{c}(\alpha+1) - \boldsymbol{c}(\alpha), \quad \boldsymbol{c} \in \ell^2(\mathbb{Z}).$$

Furthermore, with Definition 2, (14) *and* (16)*, we obtain*

$$\lim_{j \to \infty} R^{[j]} = \begin{pmatrix} 0 & \Delta^2 \\ \Delta & -1 - 2^{-1} \Delta \end{pmatrix} = \mathscr{G}^{[2]} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widetilde{T}_1^2. \tag{28}$$

The transformation $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *and the Gregory operator* $\mathscr{G}^{[2]}$ *(cf. [24]) appear for the same reason as in Remark 17.*

Eq. (28) *implies that factorizing level-dependent schemes of dimension* $d=1$ *reproducing* $\{1, x, e^{\lambda x}\}$ *is connected to factorizing stationary schemes of the same dimension reproducing* $\{1, x, x^2\}$ *via limits. The level-dependent factorizations of [14] thus depend on* $S_{\boldsymbol{A}}$ *satisfying a type of overreproduction, in contrary to the factorizations of [8].*

Through this overreproduction, the connection to the augmented Taylor operator is not surprising, considering that the cancellation *operator for level-dependent Hermite schemes reproducing exponentials of [7] converges to the Taylor operator, cf. [7, Corollary 2]. This also indicates that a generalization of [14] to $d > 1$ and multiple exponentials, has to be an operator which converges to $\widetilde{T}_d^j$.*

A generalization of the spectral condition (5) to so-called *spectral chains* is proposed in [20]. We mention two special spectral chain for which the augmented Taylor operator can be computed easily. Consider a subdivision operator $S_{\boldsymbol{A}}$ with spectral chain

$$\boldsymbol{v}(p_k) = \left(\Delta^j p_k : j = 0, \dots, d\right)^T, \quad k = 0, \dots n. \tag{29}$$

This implies that $S_{\boldsymbol{A}}$ satisfies (5) with (29). In this case $S_{\boldsymbol{A}}$ factorizes with respect to a *complete Taylor operator* of the form

$$\begin{pmatrix} \Delta & -1 & & \\ & \ddots & \ddots & \\ & & \Delta & -1 \\ & & & \Delta \end{pmatrix},$$

cf. [20]. Applying the augmented Taylor construction, analogous to Theorem 16, we obtain that $S_{\boldsymbol{A}}$ factorizes with respect to the operators

$$\begin{pmatrix} \Delta & -1 & & \\ & \ddots & \ddots & \\ & & \Delta & -1 \\ & & & \Delta^{j+1-d} \end{pmatrix}, \quad j = d, \dots, n.$$

Note that in this case all vectors $\boldsymbol{y}$ are zero.

We also consider the following spectral chain which is connected to B-Splines:

$$\boldsymbol{v}(p_k) = \left(\Delta^j p_k(\cdot - j) : j = 0, \dots, d\right)^T, \quad k = 0, \dots n, \tag{30}$$

see [20]. In [20] it is proved that a subdivision operator $S_{\boldsymbol{A}}$ with spectral chain (30) factorizes with respect to the generalized Taylor operator

$$\begin{pmatrix} \Delta & -1 & \cdots & -1 \\ & \ddots & \ddots & \vdots \\ & & \Delta & -1 \\ & & & \Delta \end{pmatrix}.$$

With the augmented Taylor construction we obtain that S_A factorizes with respect to

$$\begin{pmatrix} \Delta & -1 & \cdots & -1 & -1-\Delta \\ & \ddots & \ddots & \vdots & \vdots \\ & & \Delta & -1 & -1-\Delta \\ & & & \Delta & -1-\Delta \\ & & & & \Delta^{j+1-d} \end{pmatrix}, \quad j = d+1, \ldots, n.$$

Note that in this case $\boldsymbol{y}_0 = (1, \ldots, 1, 0)^T$ and $\boldsymbol{y}_j = \boldsymbol{0}, j > 0$.

7 Interpretation of the augmented Taylor operator

The coefficients G_n^k appear in the following approximations for integrating functions f (see [25, 27]):

$$\int_{x_0}^{x_1} \int_{x_0}^{x_2} \cdots \int_{x_0}^{x_k} f(x) dx dx_k \cdots dx_2 = (x_1 - x_0)^k \sum_{n=0}^{m} G_n^k \Delta^n f(x_0) + R_m^k f(x_1; x_0), \quad (31)$$

where $R_m^k f(x_0; x_1)$ denotes the remainder term. Via (31) we derive an interpretation of the augmented Taylor operator $\widetilde{T}_d^n$ (Theorem 19).

Let $f \in C^d(\mathbb{R})$ and denote by $\mathcal{T}_n f(x_1; x_0)$ its n-th Taylor polynomial, i.e.

$$\mathcal{T}_n f(x_1; x_0) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x_1 - x_0)^k, \quad n = 0, \ldots, d.$$

In analogy we define

$$\mathcal{I}_n^k f(x_1; x_0) := \sum_{m=0}^{n} G_m^k \Delta^m f(x_0)(x_1 - x_0)^k. \quad (32)$$

Thus (31) becomes

$$\int_{x_0}^{x_1} \int_{x_0}^{x_2} \cdots \int_{x_0}^{x_k} f(x) dx dx_k \cdots dx_2 = \mathcal{I}_n^k f(x_1; x_0) + R_n^k f(x_1, x_0).$$

It is easy to see that

$$\int_{x_0}^{x_1} \int_{x_0}^{x_2} \cdots \int_{x_0}^{x_{d-j}} f^{(d)}(x) dx dx_{d-j} \cdots dx_2 = f^{(j)}(x_1) - \mathcal{T}_{d-j-1} f^{(j)}(x_1; x_0),$$

for $j = 0, \ldots, d-1$. Thus we get

$$\mathcal{I}_n^k f^{(d)}(x_1; x_0) = f^{(d-k)}(x_1) - \mathcal{T}_{k-1} f^{(d-k)}(x_1; x_0) - R_n^k f^{(d)}(x_1, x_0). \quad (33)$$

From [18] we know

$$\left(\widetilde{T}_d \boldsymbol{v}(f)(x)\right)_j = f^{(j)}(x+1) - \mathcal{T}_{d-j} f^{(j)}(x+1;x), \quad j = 0,\dots,d-1,$$

i.e. the remainder term, when Taylor expanding $f^{(j)}(x+1)$ at x with order $d-j$. Now consider the augmented Taylor operator in view of (32) and (33):

$$\begin{aligned}\left(\widetilde{T}_d^n \boldsymbol{v}(f)(x)\right)_j &= f^{(j)}(x+1) - \mathcal{T}_{d-j-1} f^{(j)}(x+1;x) - \sum_{k=0}^{n-d} G_k^{d-j} \Delta^k f^{(d)} \\ &= f^{(j)}(x+1) - \mathcal{T}_{d-j-1} f^{(j)}(x+1;x) - \mathcal{I}_{n-d}^{d-j} f^{(d)}(x+1;x) \\ &= R_{n-d}^{d-j} f^{(d)}(x+1;x),\end{aligned}$$

that is, the remainder term, when integrating $f^{(d)}$, $(d-j)$-times with precision $n-d$. We summarize this result in the following theorem.

Theorem 19. *Let* $f \in C^d(\mathbb{R})$*. Then*

$$\widetilde{T}_d^n \boldsymbol{v}(f)(x) = \widetilde{T}_d^n \begin{pmatrix} f(x) \\ f'(x) \\ \vdots \\ f^{(d)}(x) \end{pmatrix} = \begin{pmatrix} R_{n-d}^{d} f^{(d)}(x+1;x) \\ R_{n-d}^{d-1} f^{(d)}(x+1;x) \\ \vdots \\ R_{n-d}^{0} f^{(d)}(x+1;x) \end{pmatrix}, \quad x \in \mathbb{R},$$

with the remainder terms $R_d^{d-j} f$, $j = 0,\dots,d$, *given in* (31)*.*

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