THE ALTERNATING RUN POLYNOMIALS OF PERMUTATIONS

SHI-MEI MA, JUN MA, AND YEONG-NAN YEH

Abstract. In this paper, we first consider a generalization of the David-Barton identity which relate the alternating run polynomials to Eulerian polynomials. By using context-free grammars, we then present a combinatorial interpretation of a family of q-alternating run polynomials. Furthermore, we introduce the definition of semi-γ-positive polynomial and we show the semi-γ-positivity of the alternating run polynomials of dual Stirling permutations. A connection between the up-down run polynomials of permutations and the alternating run polynomials of dual Stirling permutations is established.

Keywords: Alternating runs; Eulerian polynomials; Semi-γ-positivity; Stirling permutations

1. Introduction

The enumeration of permutations by number of alternating runs was first studied by André [1]. Knuth [19, Section 5.1.3] has discussed this topic in connection to sorting and searching. Over the past few decades, the study of alternating runs of permutations was initiated by David and Barton [12, 157-162].

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]=\{1,2,\ldots,n\}$. Let $\pi=\pi(1)\pi(2)\cdots\pi(n)\in\mathfrak{S}_n$. An *alternating run* of π is a maximal consecutive subsequence that is increasing or decreasing (see [1, 22]). An *up-down run* of π is an alternating run of π endowed with a 0 in the front (see [13, 22]). Let $\operatorname{altrun}(\pi)$ (resp. $\operatorname{udrun}(\pi)$) be the number of alternating runs (resp. up-down runs) of π. For example, if $\pi=324156$, then $\operatorname{altrun}(\pi)=4$, $\operatorname{udrun}(\pi)=5$. We define

$$R_{n,k}=\#\{\pi\in\mathfrak{S}_n:\operatorname{altrun}(\pi)=k\},$$
$$T_{n,k}=\#\{\pi\in\mathfrak{S}_n:\operatorname{udrun}(\pi)=k\}.$$

It is well known that these numbers satisfy the following recurrence relations

$$R_{n+1,k}=kR_{n,k}+2R_{n,k-1}+(n-k+1)R_{n,k-2},$$

$$T_{n+1,k}=kT_{n,k}+T_{n,k-1}+(n-k+2)T_{n,k-2}, \tag{1}$$

with the initial conditions $R_{1,0}=1$ and $R_{1,k}=0$ for $k\geq 1$, $T_{0,0}=1$ and $T_{0,k}=0$ for $k\geq 1$ (see [1, 13]). The *alternating run polynomial* and *up-down run polynomial* are respectively defined by $R_n(x)=\sum_{k=0}^{n-1}R_{n,k}x^k$ and $T_n(x)=\sum_{k=0}^{n}T_{n,k}x^k$.

A *descent* of $\pi\in\mathfrak{S}_n$ is an index $i\in[n-1]$ such that $\pi(i)>\pi(i+1)$. Denote by $\operatorname{des}(\pi)$ the number of descents of π. The classical *Eulerian polynomial* is defined by $A_n(x)=\sum_{\pi\in\mathfrak{S}_n}x^{\operatorname{des}(\pi)+1}$.

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By solving a differential equation, David and Barton [12, 157-162] established the identity:

$$R_n(x) = \left(\frac{1+x}{2}\right)^{n-1}(1+w)^{n+1}A_n\left(\frac{1-w}{1+w}\right) \tag{2}$$

for $n \geq 2$, where $w = \sqrt{\frac{1-x}{1+x}}$. Using (2), Bóna proved that the polynomial $R_n(x)$ has only real zeros (see [4]). Moreover, one can prove that $R_n(x)$ has the zero $x = -1$ with the multiplicity $\lfloor \frac{n}{2} \rfloor - 1$ by using (2), which can also be obtained based on the recurrence relation of $R_n(x)$ (see [25]). Motivated by (2), Zhuang [31] proved several identities expressing polynomials counting permutations by various descent statistics in terms of Eulerian polynomials.

Let us now recall another combinatorial interpretation of $T_n(x)$. An *alternating subsequence* of π is a subsequence $\pi(i_1)\cdots\pi(i_k)$ satisfying

$$\pi(i_1) > \pi(i_2) < \pi(i_3) > \cdots \pi(i_k),$$

where $i_1 < i_2 < \cdots < i_k$ (see [28]). Denote by $\mathrm{as}(\pi)$ the number of terms of the longest alternating subsequence of π. By definition, we see that $\mathrm{as}(\pi) = \mathrm{udrun}(\pi)$. Thus

$$T_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{as}(\pi)}.$$

There has been much recent work related to the numbers $R_{n,k}$ and $T_{n,k}$. In [3], Bóna and Ehrenborg proved that $R_{n,k}^2 \geq R_{n,k-1}R_{n,k+1}$. Subsequently, Bóna [4, Section 1.3.2] noted that

$$T_n(x) = \frac{1}{2}(1+x)R_n(x) \tag{3}$$

for $n \geq 2$. Set $\rho = \sqrt{1-x^2}$. Stanley [28, Theorem 2.3] showed that

$$T(x,z) =: \sum_{n=0}^{\infty} T_n(x)\frac{z^n}{n!} = (1-x)\frac{1+\rho+2xe^{\rho z}+(1-\rho)e^{2\rho z}}{1+\rho-x^2+(1-\rho-x^2)e^{2\rho z}}. \tag{4}$$

By using (3) and (4), Stanley [28] obtained explicit formulas of $T_{n,k}$ and $R_{n,k}$. Canfield and Wilf [6] presented an asymptotic formula for $R_{n,k}$. In [21], another explicit formula of $R_{n,k}$ was obtained by combining the derivative polynomials of tangent function and the following generating function obtained by Carlitz [7]:

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{k=0}^{n}R_{n+1,k}x^{n-k} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin(z\sqrt{1-x^2})}{x-\cos(z\sqrt{1-x^2})}\right)^2.$$

In [22], several convolution formulas of the polynomials $R_n(x)$ and $T_n(x)$ are obtained by using Chen's grammars. By generalizing a reciprocity formula of Gessel, Zhuang [30] obtained generating function for permutation statistics that are expressible in terms of alternating runs. Very recently, Josuat-Vergès and Pang [18] showed that alternating runs can be used to define subalgebras of Solomon's descent algebra.

In this paper, we continue the work initiated by David and Barton [12]. In Section 2, we consider a generalization of (2). In Section 3, we present a combinatorial interpretation of a family of q-alternating run polynomials by using Chen's grammars. In Section 4, we show the semi-γ-positivity of the alternating run polynomials of dual Stirling permutations.

2. The David-Barton type identity

Let $f(x)=\sum_{i=0}^{n} f_i x^i$ be a symmetric polynomial, i.e., $f_i=f_{n-i}$ for any $0\le i\le n$. Then $f(x)$ can be expanded uniquely as

$$f(x)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\gamma_k x^k(1+x)^{n-2k},$$

and it is said to be *γ-positive* if $\gamma_k\ge 0$ for $0\le k\le\lfloor\frac{n}{2}\rfloor$ (see [15]). The γ-positivity provides an approach to study symmetric and unimodal polynomials and has been extensively studied (see [2, 5, 10, 20] for instance).

The first main result of our paper is the following, which shows that the David-Barton type identities often occur in combinatorics and geometry.

Theorem 1. *Let*

$$M_n(x)=\sum_{k=0}^{\lfloor (n+\delta)/2\rfloor} M(n,k)x^k(1+x)^{n+\delta-2k}$$

be a symmetric polynomial, where δ is a fixed integer. Set $w=\sqrt{\frac{1-x}{1+x}}$. Then

$$N_n(x)=\left(\frac{1+x}{2}\right)^{n-\delta}(1+w)^{n+\delta}M_n\left(\frac{1-w}{1+w}\right) \tag{5}$$

if and only if

$$N_n(x)=\sum_{k=0}^{\lfloor (n+\delta)/2\rfloor}\frac{1}{2^{k-2\delta}}M(n,k)x^k(1+x)^{n-\delta-k}. \tag{6}$$

Proof. Set $\alpha=\frac{1+x}{2}$. Note that

$$\begin{aligned}
1-w^2&=\frac{x}{\alpha},\\
\frac{1-w}{1+w}&=\frac{1-w^2}{(1+w)^2}=\frac{1}{(1+w)^2}\frac{x}{\alpha},\\
1+\frac{1-w}{1+w}&=\frac{2}{1+w}.
\end{aligned}$$

It follows from (5) that

$$\begin{aligned}
N_n(x)&=\left(\frac{1+x}{2}\right)^{n-\delta}(1+w)^{n+\delta}M_n\left(\frac{1-w}{1+w}\right)\\
&=\alpha^{n-\delta}(1+w)^{n+\delta}\sum_k M(n,k)\frac{1}{(1+w)^{2k}}\frac{x^k}{\alpha^k}\left(\frac{2}{1+w}\right)^{n+\delta-2k}\\
&=\sum_k M(n,k)x^k\alpha^{n-\delta-k}2^{n+\delta-2k}\\
&=\sum_k M(n,k)x^k\left(\frac{1+x}{2}\right)^{n-\delta-k}2^{n+\delta-2k}\\
&=\sum_k\frac{1}{2^{k-2\delta}}M(n,k)x^k(1+x)^{n-\delta-k},
\end{aligned}$$

and vice versa. This completes the proof. □

The reader is referred to [2] for a survey of some recent results on γ-positivity. For any γ-positive polynomial $M_n(x)$, we can define an associated polynomial $N_n(x)$ by using (6). And then we get a David-Barton type identity (5). As illustrations, in the rest of this section, we shall present two examples.

For example, Foata and Schützenberger [14] discovered that

$$A_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} a(n,k)x^k(1+x)^{n+1-2k}$$

for $n \geq 1$, where the numbers $a(n,k)$ satisfy the recurrence relation

$$a(n,k) = ka(n-1,k) + (2n-4k+4)a(n-1,k-1),$$

with the initial conditions $a(1,1) = 1$ and $a(1,k) = 0$ for $k \neq 1$ (see [10, 26] for instance). By using the David-Barton identity (2) and Theorem 1, we immediately get the following result.

Proposition 2. *For $n \geq 2$, we have*

$$R_n(x) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{1}{2^{k-2}} a(n,k)x^k(1+x)^{n-1-k}.$$

Let $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. Let B_n be the hyperoctahedral group of rank n. Elements of B_n are signed permutations of $\pm[n]$ with the property that $\pi(-i) = -\pi(i)$ for all $i \in [n]$. In the sequel, we always assume that signed permutations in B_n are prepended by 0. That is, we identify a signed permutation $\pi = \pi(1)\cdots\pi(n)$ with the word $\pi(0)\pi(1)\cdots\pi(n)$, where $\pi(0) = 0$. A type B descent is an index $i \in \{0,1,\ldots,n-1\}$ such that $\pi(i) > \pi(i+1)$. Let $\operatorname{des}^B(\pi)$ be the number of type B descents of π. The *type B Eulerian polynomials* are defined by

$$B_n(x) = \sum_{\pi \in B_n} x^{\operatorname{des}_B(\pi)}.$$

It is well known that

$$B_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n,k)x^k(1+x)^{n-2k},$$

where the numbers $b(n,k)$ satisfy the recurrence relation

$$b(n,k) = (1+2k)b(n-1,k) + 4(n-2k+1)b(n-1,k-1), \tag{7}$$

with the initial conditions $b(1,0) = 1$ and $b(1,k) = 0$ for $k \neq 0$ (see [2, 10, 26]).

Define

$$b_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2^k} b(n,k)x^k(1+x)^{n-k}. \tag{8}$$

Then by Theorem 1, we get the following result.

Proposition 3. *For $n \geq 1$, we have*

$$b_n(x) = \left(\frac{1+x}{2}\right)^n (1+w)^n B_n\left(\frac{1-w}{1+w}\right).$$

Combining (7) and (8), we see that the polynomials $b_n(x)$ satisfy the recurrence relation

$$b_{n+1}(x) = (1 + x + 2nx^2)b_n(x) + 2x(1 - x^2)b_n'(x), \tag{9}$$

with the initial conditions $b_0(x) = 1,\ b_1(x) = 1 + x$. For $n \geq 1$, we define $b_n(x) = \frac{1+x}{x}c_n(x)$. It follows from (9) that the polynomials $c_n(x)$ satisfy the recurrence relation

$$c_{n+1}(x) = (2nx^2 + 3x - 1)c_n(x) + 2x(1 - x^2)c_n'(x).$$

Let $\widehat{B}_n = \{\pi \in B_n \mid \pi(1) > 0\}$. There is a combinatorial interpretation of $c_n(x)$ (see [11, 29]):

$$c_n(x) = \sum_{\pi \in \widehat{B}_n} x^{\mathrm{altrun}\,(\pi)}.$$

3. The q-alternating runs polynomials

For an alphabet A, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal power series in monomials formed from letters in A. A *Chen's grammar* (which is known as context-free grammar) over A is a function $G : A \to \mathbb{Q}[[A]]$ that replaces a letter in A by an element of $\mathbb{Q}[[A]]$, see [8, 9, 24] for details. The formal derivative $D := D_G$ is a linear operator defined with respect to a context-free grammar G. Following [9], a *grammatical labeling* is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar.

Let us now recall two results on context-free grammars.

Proposition 4 ([22, Theorem 6]). *If* $G = \{a \to ab,\ b \to bc,\ c \to b^2\}$, *then*

$$D^n(a) = a\sum_{k=0}^{n} T_{n,k}b^kc^{n-k},\ D^n(a^2) = a^2\sum_{k=0}^{n} R_{n+1,k}b^kc^{n-k}.$$

Proposition 5 ([22, Theorem 9]). *If* $G = \{a \to 2ab,\ b \to bc,\ c \to b^2\}$, *then*

$$D^n(a) = a\sum_{k=0}^{n} R_{n+1,k}b^kc^{n-k}.$$

Combining Leibniz's formula and Proposition 4, we see that

$$R_{n+1}(x) = \sum_{k=0}^{n} \binom{n}{k} T_k(x)T_{n-k}(x).$$

Motivated by Propositions 4 and 5, it is natural to consider the grammar

$$G_1 = \{a \to qab,\ b \to bc,\ c \to b^2\}. \tag{10}$$

Note that $D_{G_1}(a) = qab,\ D_{G_1}^2(a) = a(q^2b^2 + qbc)$. By induction, it is easy to verify that

$$D_{G_1}^n(a) = a\sum_{k=0}^{n} R_{n,k}(q)b^kc^{n-k}. \tag{11}$$

It follows from (10) that

$$\begin{aligned} D_{G_1}^{n+1}(a) &= D_{G_1}\left(a\sum_{k=0}^{n} R_{n,k}(q)b^kc^{n-k}\right) \\ &= a\sum_k R_{n,k}(q)\left(kb^kc^{n-k+1}+qb^{k+1}c^{n-k}+(n-k)b^{k+2}c^{n-k-1}\right), \end{aligned}$$

which leads to the recurrence relation

$$R_{n+1,k}(q) = kR_{n,k}(q) + qR_{n,k-1}(q) + (n-k+2)R_{n,k-2}(q). \tag{12}$$

The *q-alternating run polynomials* are defined by

$$R_n(x;q) = \sum_{k=0}^{n} R_{n,k}(q)x^k.$$

In particular, $R_n(x;1) = T_n(x)$, $R_n(x;2) = R_{n+1}(x)$. The first few $R_n(x;q)$ are given as follows:

$$R_0(x;q)=1,\ R_1(x;q)=qx,\ R_2(x;q)=qx(1+qx),\ R_3(x;q)=qx(1+3qx+x^2+q^2x^2).$$

We define

$$R(x,z;q) := \sum_{n=0}^{\infty} R_n(x;q)\frac{z^n}{n!}.$$

Proposition 6. *We have $R(x,z;q) = T^q(x,z)$, where $T(x,z)$ is given by (4). Therefore,*

$$\sum_{n=0}^{\infty} D_{G_1}^{n}(a)\frac{z^n}{n!} = aR\left(\frac{b}{c}, cz; q\right) = aT^q\left(\frac{b}{c}, cz\right). \tag{13}$$

Moreover, we have $R_n(x;-q) = R_n(-x;q)$ and $R_n(-x;-q) = R_n(x;q)$.

Proof. By rewriting (12) in terms of generating function $R(x,z;q)$, we obtain

$$(1-x^2z)\frac{\partial}{\partial z}R(x,z;q) = x(1-x^2)\frac{\partial}{\partial x}R(x,z;q) + qxR(x,z;q). \tag{14}$$

It is routine to check that the generating function $T^q(x,z)$ satisfies (14). Also, this generating function gives $T^q(0,z) = T^q(x,0) = 1$. Hence $R(x,z;q) = T^q(x,z)$. It is routine to check that

$$R(x,z;-q) = R(-x,z;q),\ R(-x,z;-q) = R(x,z;q)$$

which leads to the desired result. □

We say that $\pi \in \mathfrak{S}_n$ is a circular permutation if it has only one cycle. Let $A = \{x_1, x_2, \ldots, x_k\}$ be a finite set of positive integers, and let $\mathcal{C}_A$ be the set of all circular permutations of A. We will write a permutation $w \in \mathcal{C}_A$ by using its canonical presentation $w = y_1y_2\cdots y_k$, where $y_1 = \min A, y_i = w^{i-1}(y_1)$ for $2 \le i \le k$ and $y_1 = w^k(y_1)$. A *cycle peak* (resp. *cycle double ascent*, *cycle double descent*) of w is an entry y_i, $2 \le i \le k$, such that $y_{i-1} < y_i > y_{i+1}$ (resp. $y_{i-1} < y_i < y_{i+1}$, $y_{i-1} > y_i > y_{i+1}$), where we set $y_{k+1} = \infty$. Let $\operatorname{cpk}(w)$ (resp. $\operatorname{cdasc}(w)$, $\operatorname{cddes}(w)$, $\operatorname{cyc}(w)$) be the number of cycle peaks (resp. cycle double ascents, cycle double descents, cycles) of w.

Definition 7. *A cycle run of a circular permutation w is an alternating run of w endowed with a ∞ in the end. Let $\operatorname{crun}(w)$ be the number of cycle runs of w.*

It is clear that $\operatorname{crun}(w) = 2\operatorname{cpk}(w) + 1$. In the following discussion we always write $\pi \in \mathfrak{S}_n$ in standard cycle decomposition: $\pi = w_1 \cdots w_k$, where the cycles are written in increasing order of their smallest entry and each of these cycles is expressed in canonical presentation. We define

$$\operatorname{crun}(\pi) := \sum_{i=1}^{k} \operatorname{crun}(w_i).$$

In particular, $\operatorname{crun}((1)(2)\cdots(n)) = \sum_{i=1}^{n} \operatorname{crun}(i) = \sum_{i=1}^{n} \operatorname{altrun}(i\infty) = n$. We can now present the second main result.

Theorem 8. *For $n \geq 1$, we have*

$$R_n(x;q) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{crun}(\pi)} q^{\operatorname{cyc}(\pi)}. \tag{15}$$

Proof. For $\pi \in \mathfrak{S}_n$, we first put a ∞ in the end of each cycle. We then introduce a grammatical labeling of π as follows:

- (L_1) Put a subscript label q at the end of each cycle of π;
- (L_2) Put a superscript label a at the end of π;
- (L_3) Put a superscript label b before each ∞;
- (L_4) If $\pi(i)$ is a cycle peak, then put a superscript label b before $\pi(i)$ and a superscript label b right after π;
- (L_5) If $\pi(i)$ is a cycle double ascents, then put the superscript label c before $\pi(i)$;
- (L_6) If $\pi(i)$ is a cycle double descents, then put the superscript label c right after $\pi(i)$.

The weight of π is the product of its labels. When $n = 1, 2$, we have

$$\mathfrak{S}_1 = \{(1^b\infty)_q^a\},\ \mathfrak{S}_2 = \{(1^b\infty)_q(2^b\infty)_q^a,\ (1^c2^b\infty)_q^a\}.$$

Then the weight of $(1^b)_q^a$ is given by $D_{G_1}(a)$, and the sum of weights of the elements in $\mathfrak{S}_2$ is given by $D_{G_1}^2(a)$. Hence the result holds for $n = 1, 2$. Let

$$r_n(i,j) = \{\pi \in \mathfrak{S}_n : \operatorname{crun}(\pi) = i,\ \operatorname{cyc}(\pi) = j\}.$$

Suppose we get all labeled permutations in $r_{n-1}(i,j)$, where $n \geq 3$. Let π' be obtained from $\pi \in r_{n-1}(i,j)$ by inserting the entry n. We distinguish the following four cases:

- (c_1) If we insert n as a new cycle, then $\pi' \in r_{n-1}(i+1, j+1)$. This case corresponds to the substitution rule $a \to qab$.
- (c_2) If we insert n before a ∞, then $\pi' \in r_{n-1}(i,j)$. This case corresponds to the substitution rule $b \to bc$;
- (c_3) If we insert n before or right after a cycle peak, then $\pi' \in r_{n-1}(i,j)$. This case corresponds to the substitution rule $b \to bc$;
- (c_4) If we insert n before a cycle double ascents or right after a cycle double descents, then $\pi' \in r_{n-1}(i+2, j)$. This case corresponds to the substitution rule $c \to b^2$.

In each case, the insertion of n corresponds to one substitution rule in the grammar (10). It is easy to check that the action of D_{G_1} on elements of $\mathfrak{S}_{n-1}$ generates all elements of $\mathfrak{S}_n$. Using (11) and by induction, we present a constructive proof of (15). This completes the proof. □

We define

$$R_n(x,y;q)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathrm{crun}\,(\pi)}y^{\mathrm{fix}\,(\pi)}q^{\mathrm{cyc}\,(\pi)},$$

$$R(x,y,z;q)=\sum_{n=0}^{\infty}R_n(x,y;q)\frac{z^n}{n!}.$$

By using the principle of inclusion-exclusion, it is routine to verify that

$$R_n(x,y;q)=\sum_{i=0}^{n}\binom{n}{i}(qxy-qx)^iR_{n-i}(x;q).$$

Hence

$$R(x,y,z;q)=e^{qx(y-1)z}R(x,z;q)=e^{qx(y-1)z}T^q(x,z). \tag{16}$$

A permutation $\pi\in\mathfrak{S}_n$ is a *derangement* if $\pi(i)\neq i$ for any $i\in[n]$. Let $\mathcal{D}_n$ denote the set of derangements in $\mathfrak{S}_n$. Then

$$R_n(x,0;1)=\sum_{\pi\in\mathcal{D}_n}x^{\mathrm{crun}\,(\pi)}.$$

Proposition 9. *Set $d_n(x)=R_n(x,0;1)$. Then the polynomials $d_n(x)$ satisfy the recurrence*

$$d_{n+1}(x)=nx^2d_n(x)+x(1-x^2)d_n'(x)+nxd_{n-1}(x), \tag{17}$$

with the initial conditions $d_0(x)=1,\ d_1(x)=0$. In particular, $d_n(-1)=-(n-1)$ for $n\geq 1$.

Proof. Let $d(x,z)=\sum_{n=0}^{\infty}d_n(x)\frac{z^n}{n!}$. It follow from (16) that

$$d(x,z)=e^{-xz}T(x,z). \tag{18}$$

By rewriting (1) in terms of generating function $T(x,z)$, we obtain

$$(1-x^2z)\frac{\partial}{\partial z}T(x,z)=xT(x,z)+x(1-x^2)\frac{\partial}{\partial x}T(x,z).$$

Hence

$$(1-x^2z)\frac{\partial}{\partial z}d(x,z)=xzd(x,z)+x(1-x^2)\frac{\partial}{\partial x}d(x,z),$$

which yields the desired recurrence relation. □

Let $d_n(x)=\sum_{k=0}^{n}d_{n,k}x^k$. By using (18), it is not hard to verify that

$$\sum_{n=0}^{\infty}d_{n,n}\frac{z^n}{n!}=\frac{e^{-x}}{\tan x+\sec x}.$$

4. Semi-γ-positive polynomials

Let $g(x)=\sum_{i=0}^{2n}g_ix^i$ be a symmetric polynomial. Note that

$$\begin{aligned}g(x)&=\sum_{i=0}^{n}\gamma_ix^i(1+x)^{2(n-i)}\\&=\sum_{i=0}^{n}\gamma_ix^i(1+2x+x^2)^{n-i}\\&=\sum_{i=0}^{n}\sum_{\ell=0}^{n-i}\binom{n-i}{\ell}2^{\ell}\gamma_ix^{i+\ell}(1+x^2)^{n-i-\ell}.\end{aligned}$$

Hence $g(x)$ can be expanded as

$$g(x)=\sum_{k=0}^{n}\lambda_k x^k(1+x^2)^{n-k}.$$

It is clear that if $\gamma_i\geq 0$ for all $0\leq i\leq n$, then $\lambda_k\geq 0$ for all $0\leq k\leq n$. Furthermore, we have

$$\begin{aligned}g(x)&=\sum_{k=0}^{\lfloor n/2\rfloor}\lambda_{2k}x^{2k}(1+x^2)^{n-2k}+\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\lambda_{2k+1}x^{2k+1}(1+x^2)^{n-2k-1}\\&=g_1(x^2)+xg_2(x^2).\end{aligned}$$

Similarly, if $h(x)=\sum_{i=0}^{2n+1}h_ix^i$ a symmetric polynomial, then we have

$$\begin{aligned}h(x)&=\sum_{i=0}^{n}\beta_i x^i(1+x)^{2n+1-2i}\\&=(1+x)\sum_{i=0}^{n}\sum_{\ell=0}^{n-i}\binom{n-i}{\ell}2^{\ell}\beta_i x^{i+\ell}(1+x^2)^{n-i-\ell}.\end{aligned}$$

Hence $h(x)$ can be expanded as

$$h(x)=(1+x)\sum_{k=0}^{n}\mu_k x^k(1+x^2)^{n-k}.$$

Definition 10. *If $f(x)=(1+x)^{\nu}\sum_{k=0}^{n}\lambda_k x^k(1+x^2)^{n-k}$ and $\lambda_k\geq 0$ for all $0\leq k\leq n$, then we say that $f(x)$ is semi-γ-positive, where $\nu=0$ or $\nu=1$.*

It should be noted that a semi-γ-positive polynomial is not always γ-positive. From the above discussion it follows that we have the following result.

Proposition 11. *If $f(x)=(1+x)^{\nu}\left(f_1(x^2)+xf_2(x^2)\right)$ is a semi-γ-positive polynomial, then both $f_1(x)$ and $f_2(x)$ are γ-positive.*

In the following, we shall show the semi-γ-positivity of the alternating run polynomials of dual Stirling permutations. Following [16], a *Stirling permutation* of order n is a permutation of the multiset $\{1,1,\ldots,n,n\}$ such that for each i, $1\leq i\leq n$, all entries between the two occurrences of i are larger than i. There has been much recent work on Stirling permutations, see [17, 24] and references therein.

Denote by $\mathcal{Q}_n$ the set of *Stirling permutations* of order n. Let $\sigma=\sigma_1\sigma_2\cdots\sigma_{2n}\in\mathcal{Q}_n$. Let Φ be the injection which maps each first occurrence of entry j in σ to $2j$ and the second j to $2j-1$, where $j\in[n]$. For example, $\Phi(221331)=432651$. Let $\Phi(\mathcal{Q}_n)=\{\pi\mid\sigma\in\mathcal{Q}_n,\Phi(\sigma)=\pi\}$ be the set of *dual Stirling permutations* of order n. Clearly, $\Phi(\mathcal{Q}_n)$ is a subset of $\mathfrak{S}_{2n}$. For $\pi\in\Phi(\mathcal{Q}_n)$, the entry $2j$ is to the left of $2j-1$, and all entries in π between $2j$ and $2j-1$ are larger than $2j$, where $1\leq j\leq n$. Noted that $\pi\in\Phi(\mathcal{Q}_n)$ always ends with a descending run. The alternating runs polynomials of dual Stirling permutations are defined by

$$F_n(x)=\sum_{\sigma\in\Phi(\mathcal{Q}_n)}x^{\mathrm{altrun}\,(\sigma)}=\sum_{k=1}^{2n-1}F_{n,k}x^k.$$

According to [23], the numbers $F_{n,k}$ satisfy the recurrence relation

$$F_{n+1,k} = kF_{n,k} + F_{n,k-1} + (2n-k+2)F_{n,k-2}. \tag{19}$$

with the initial conditions $F_{0,0}=1,\ F_{1,1}=1$ and $F_{n,0}=0$ for $n\geq 1$. It follows from (19) that

$$F_{n+1}(x) = (x+2nx^2)F_n(x) + x(1-x^2)F_n'(x).$$

The first few $F_n(x)$ are given as follows:

$$\begin{aligned}
F_1(x) &= x,\\
F_2(x) &= x + x^2 + x^3,\\
F_3(x) &= x + 3x^2 + 7x^3 + 3x^4 + x^5,\\
F_4(x) &= x + 7x^2 + 29x^3 + 31x^4 + 29x^5 + 7x^6 + x^7.
\end{aligned}$$

Let

$$r(x) = \sqrt{\frac{1+x}{1-x}}.$$

By induction, it is to verify that

$$\begin{aligned}
\left(x\frac{d}{dx}\right)^{2n} r(x) &= \frac{r(x)F_{2n}(x)}{(1-x^2)^{2n}},\\
\left(x\frac{d}{dx}\right)^{2n+1} r(x) &= \frac{F_{2n+1}(x)}{r(x)(1-x^2)^{2n}(1-x)^2}.
\end{aligned}$$

Lemma 12 ([23])**.** *If*

$$G_2 = \{x \to xyz, y \to yz^2, z \to y^2z\}, \tag{20}$$

then we have

$$D_{G_2}^n(x) = x \sum_{\sigma\in\Phi(\mathcal{Q}_n)} y^{\operatorname{altrun}(\sigma)} z^{2n-\operatorname{altrun}(\sigma)} = x\sum_{k=0}^{2n-1} F_{n,k}y^kz^{2n-k}. \tag{21}$$

We now recall another combinatorial interpretation of $F_n(x)$. An occurrence of an *ascent-plateau* of $\sigma\in\mathcal{Q}_n$ is an index i such that $\sigma_{i-1}<\sigma_i=\sigma_{i+1}$, where $i\in\{2,3,\ldots,2n-1\}$. An occurrence of a *left ascent-plateau* is an index i such that $\sigma_{i-1}<\sigma_i=\sigma_{i+1}$, where $i\in\{1,2,\ldots,2n-1\}$ and $\sigma_0=0$. Let $\operatorname{ap}(\sigma)$ and $\operatorname{la}(\sigma)$ be the numbers of ascent-plateaus and left ascent-plateaus of σ, respectively. The number of flag ascent-plateaus of σ is defined by

$$\operatorname{fap}(\sigma) = \begin{cases} 2\operatorname{ap}(\sigma)+1, & \text{if } \sigma_1=\sigma_2;\\ 2\operatorname{ap}(\sigma), & \text{otherwise.}\end{cases}$$

Clearly, $\operatorname{fap}(\sigma)=\operatorname{ap}(\sigma)+\operatorname{la}(\sigma)$. Following [24, Section 3], we have

$$D_{G_2}^n(x) = x\sum_{\sigma\in\mathcal{Q}_n} y^{\operatorname{fap}(\sigma)}z^{2n-\operatorname{fap}(\sigma)}.$$

Thus,

$$F_n(x) = \sum_{\sigma\in\mathcal{Q}_n} x^{\operatorname{fap}(\sigma)}.$$

In fact, it is easy to verify that $\operatorname{fap}(\sigma)=\operatorname{altrun}(\Phi(\sigma))$ for any $\sigma\in\mathcal{Q}_n$.

Proposition 13. *For $n \geq 1$, we have*

$$F_n(x) = \sum_{k=1}^{n} \gamma_{n,k} x^k (1+x)^{2n-2k},$$

where the numbers $\gamma_{n,k}$ satisfy the recurrence relation

$$\gamma_{n+1,k} = k\gamma_{n,k} + (2n-4k+5)\gamma_{n,k-1}, \tag{22}$$

with the initial conditions $\gamma_{1,1} = 1$ and $\gamma_{1,k} = 0$ for $k \neq 1$. In particular,

$$\gamma_{n+1,n+1} = (-1)^n (2n-1)!! \text{ for } n \geq 1.$$

Proof. We first consider a change of the grammar (20). Set $a = yz$ and $b = y+z$. Then we have $D(x) = xa,\ D(a) = a(b^2 - 2a),\ D(b) = ab$. If

$$G_3 = \{x \to xa,\ a \to a(b^2 - 2a),\ b \to ab\},$$

then by induction, we see that there exist integers $\gamma_{n,k}$ such that

$$D_{G_3}^n(x) = x \sum_{k=0}^{n} \gamma_{n,k} a^k b^{2n-2k}. \tag{23}$$

Note that

$$\begin{aligned} D_{G_3}^{n+1}(x) &= D_{G_3}\left(x \sum_{k=1}^{n} \gamma_{n,k} a^k b^{2n-2k} \right) \\ &= x \sum_{k} \gamma_{n,k} a^k b^{2n-2k} \left(a + kb^2 - 2ka + (2n-2k)a \right) \end{aligned}$$

By comparing the coefficients of $a^k b^{2n-2k+2}$, we immediately get (22). Moreover, it is clear that $\gamma_{n,0} = 0$ for $n \geq 1$. By using (23), upon taking $a = yz$ and $b = y+z$, we get

$$D_{G_2}^n(x) = x \sum_{k=0}^{n} \gamma_{n,k} (yz)^k (y+z)^{2n-2k}. \tag{24}$$

Then comparing (24) with (21), we see that $F_n(x) = \sum_{k=1}^{n} \gamma_{n,k} x^k (1+x)^{2n-2k}$ for $n \geq 1$. By using (22), we obtain

$$\gamma_{n+1,n+1} = -(2n-1)\gamma_{n,n},$$

which yields the desired explicit formula. □

For $n \geq 1$, let $\gamma_n(x) = \sum_{k=1}^{n} \gamma_{n,k} x^k$. It follows from (22) that

$$\gamma_{n+1}(x) = (2n+1)x\gamma_n(x) + x(1-4x)\gamma_n'(x).$$

The first few $\gamma_n(x)$ are $\gamma_0(x) = 1,\ \gamma_1(x) = x,\ \gamma_2(x) = x - x^2,\ \gamma_3(x) = x - x^2 + 3x^3$. From Proposition 13, we see that for any positive even integer n, the polynomial $F_n(x)$ is not γ-positive.

We can now present the third main result of this paper.

Theorem 14. *The polynomial $F_n(x)$ is semi-γ-positive. More precisely, we have*

$$F_n(x)=\sum_{k=0}^{n} f_{n,k}x^k(1+x^2)^{n-k},$$

where the numbers $f_{n,k}$ satisfy the recurrence relation

$$f_{n+1,k}=kf_{n,k}+f_{n,k-1}+4(n-k+2)f_{n,k-2}, \tag{25}$$

with the initial conditions $f_{0,0}=1$ and $f_{n,0}=0$ for $n\geq 1$. Let $f_n(x)=\sum_{k=0}^{n} f_{n,k}x^k$. Then

$$f(x,z)=\sum_{n=0}^{\infty} f_n(x)\frac{z^n}{n!}=\sqrt{T(2x,z)}, \tag{26}$$

where $T(x,z)$ is given by (4).

Proof. We first consider the grammar (20). Note that

$$D(x)=xyz,\ D(yz)=yz(y^2+z^2),\ D(y^2+z^2)=4y^2z^2.$$

Set $u=yz$ and $v=y^2+z^2$. Then we have $D(x)=xu,\ D(u)=uv$ and $D(v)=4u^2$. If

$$G_4=\{x\rightarrow xu,\ u\rightarrow uv,\ v\rightarrow 4u^2\}, \tag{27}$$

then by induction we see that there exist nonnegative integers $f_{n,k}$ such that

$$D_{G_4}^n(x)=x\sum_{k=0}^{n} f_{n,k}u^kv^{n-k}. \tag{28}$$

Note that

$$\begin{aligned} D_{G_4}^{n+1}(x) &= D_{G_4}\left(x\sum_{k=1}^{n} f_{n,k}u^kv^{n-k}\right)\\ &= x\sum_k f_{n,k}\left(u^{k+1}v^{n-k}+ku^kv^{n-k+1}+4(n-k)u^{k+2}v^{n-k-1}\right). \end{aligned}$$

By comparing the coefficients of u^kv^{n+1-k}, we get (25). Moreover, it follows from (27) that $f_{0,0}=1$ and $f_{n,0}=0$ for $n\geq 1$. By using (28), upon taking $u=yz$ and $v=y^2+z^2$, we get

$$D_{G_2}^n(x)=x\sum_{k=0}^{n} f_{n,k}(yz)^k(y^2+z^2)^{n-k}. \tag{29}$$

By comparing (29) with (21), we get

$$F_n(x)=\sum_{k=0}^{n} f_{n,k}x^k(1+x^2)^{n-k}. \tag{30}$$

We now consider a change of the grammar (10). Set $q=\frac{1}{2},\ a=x,\ b=2u,\ c=v$. Then

$$D(x)=xu,\ D(u)=uv,\ D(v)=4u^2,$$

which are the substitution rules in the grammar (27). Hence it follows from (13) that

$$\sum_{n=0}^{\infty} D_{G_4}^n(x)\frac{z^n}{n!}=x\sum_{n=0}^{\infty}\sum_{k=0}^{n} f_{n,k}u^kv^{n-k}\frac{z^n}{n!}=xR\left(\frac{2u}{v},vz;\frac{1}{2}\right),$$

which leads to $f(x,z)=R(2x,z;1/2)=\sqrt{T(2x,z)}$. This completes the proof. □

Combining (26) and (30), we immediately get the following result.

Corollary 15. *We have*

$$F(x,z)=\sum_{n=0}^{\infty}F_n(x)\frac{z^n}{n!}=\sqrt{T\left(\frac{2x}{1+x^2},(1+x^2)z\right)}.$$

It would be interesting to present a combinatorial interpretation of Corollary 15. By using (26), it is not hard to verify that

$$\sum_{n=0}^{\infty}f_{n,n}\frac{x^n}{n!}=\sqrt{\frac{1+\tan x}{1-\tan x}}.$$

It should be noted that the numbers $f_{n,n}$ appear as A012259 in [27].

5. Concluding remarks

This paper gives a survey of some results related to alternating runs of permutations. We present a method to construct David-Barton type identities, and based on the survey [2], one can derive several David-Barton type identities. Moreover, we introduce the definition of semi-γ-positive polynomial. The γ-positivity of a polynomial $f(x)$ is a sufficient (not necessary) condition for the semi-γ-positivity of $f(x)$. In particular, we show that the alternating run polynomials of dual Stirling permutations are semi-γ-positive.

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School of Mathematics and Statistics, Northeastern University at Qinhuangdao, Hebei 066000, P.R. China

E-mail address: `shimeimapapers@163.com (S.-M. Ma)`

Department of mathematics, Shanghai Jiao Tong University, Shanghai, P.R. China

E-mail address: `majun904@sjtu.edu.cn(J. Ma)`

Institute of Mathematics, Academia Sinica, Taipei, Taiwan

E-mail address: `mayeh@math.sinica.edu.tw (Y.-N. Yeh)`