

THE SEQUENCES OF FIBONACCI AND LUCAS FOR EACH REAL QUADRATIC FIELDS $\mathbb{Q}(\sqrt{d})$

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ABSTRACT. We construct the sequences of Fibonacci and Lucas at any quadratic field $\mathbb{Q}(\sqrt{d})$ with $d > 0$ square free, noting in general that the properties remain valid as those given by the classical sequences of Fibonacci and Lucas for the case $d = 5$, under the respective variants. For this construction, we use the fundamental unit of $\mathbb{Q}(\sqrt{d})$ and then we observe the generalizations for any unit of $\mathbb{Q}(\sqrt{d})$ where, under certain conditions, some of this constructions correspond to k -Fibonacci sequence for some $k \in \mathbb{N}$. Of course, for both sequences, we obtain the generating function, Golden ratio, Binet's formula and some identities that they keep.

1. INTRODUCTION

The Fibonacci sequence was introduced by Leonardo of Pisa in 1202 in his book *Liber Abaci* (Book of Calculation) [23]. Many of the properties of the Fibonacci sequence were obtained by F. Édouard Lucas who appoints such sequence by “Fibonacci” [21, Section 3.1.2]. For more information about the history of the Fibonacci numbers, we can see [20]. But also, Lucas is who initiates the generalizations and their variants that have emerged from the Fibonacci sequence, as we can observe, for example, in [4], [7], [24], [25] and [26]. Vera W. de Spinadel introduced the Metallic Means family whose members of such a family have many wonderful and amazing properties, and applications to almost every areas of sciences and arts, such as in some areas of the physical, biology, astronomy and music (see [8], [9], [10] and [15]). On the other hand, Sergio Falcón and Ángel Plaza give properties of k -Fibonacci sequence in [4], [5], [6] and [7], and these are a particular case and general of metallic means families. Also in [3] M. El-Mikkawy and T. Sogabe given a new family of k -Fibonacci numbers. In [16], we can find hundreds of known identities, and Azarian presents in [1] some known identities as binomial sums for quick numerical calculations.

In this paper, we associate with each real quadratic field $\mathbb{Q}(\sqrt{d})$, with $d > 0$ square free, its owns sequences of Fibonacci and Lucas (Definition 5), which correspond to certain metallic means families (Theorem 8 and 13). These sequences of Fibonacci and Lucas are determined by their generating functions (Theorem 19) satisfying each Binet's formula (Theorem 22 and Corollary 23). This means that each real quadratic field $\mathbb{Q}(\sqrt{d})$ will have also associated its own Golden ratio (Definition 20), characteristic equation (5.2) and its Golden ratio will be the

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fundamental unit (Theorem 18). Finally, we will establish for each $k \in \mathbb{N}$, the k -Fibonacci sequence corresponds to Fibonacci sequence of the real quadratic field $\mathbb{Q}(\sqrt{d})$ for a unique $d > 0$ square free (Theorem 28).

At the time of submission, there is no description of the infinite family of sequences in the Online Encyclopedia of Integer Sequences, though some of the sequences do appear there, as indicated in Table 1 and Table 2.

This paper is organized as follows. In Section 2 we collect results of quadratic fields necessary for the development of the work. In Section 3 we construct the sequences of Fibonacci and Lucas at any real quadratic field. Also we prove that the properties remain valid as those given by the classical sequence of Fibonacci and Lucas for $d = 5$. In Section 4 the main goal is to prove that Fibonacci and Lucas sequence are determined by the generating functions. In Section 5 we give the Golden ratio associated with the real quadratic field and we obtain Binet's formula in $\mathbb{Q}(\sqrt{d})$. In Section 6 we extend our construction of the sequences of Fibonacci and Lucas over all integer numbers. Finally, in Section 7 we define the sequence of Fibonacci and Lucas of degree d with respect to an arbitrary unit η of $\mathbb{Q}(\sqrt{d})$ and we prove that the results of the previous sections are still met.

2. QUADRATIC FIELDS

In this section we collect fundamental results from quadratic fields. Throughout this paper, d denotes a square free integer, δ the discriminant of the quadratic field $\mathbb{Q}(\sqrt{d})$, \mathcal{O} the ring of integers of $\mathbb{Q}(\sqrt{d})$, and \mathcal{O}^* the multiplicative group of all invertible elements of the ring \mathcal{O} . When $d > 0$, we say that $\mathbb{Q}(\sqrt{d})$ is a **real quadratic field**, while if $d < 0$ then $\mathbb{Q}(\sqrt{d})$ is called an **imaginary quadratic field**. The following results are well known.

Theorem 1. *Keeping the previous notation.*

(i) *If $d \equiv 1 \pmod{4}$, then the set $\left\{ 1, \frac{1 + \sqrt{d}}{2} \right\}$ is an integral basis of $\mathbb{Q}(\sqrt{d})$,*

$$\delta = d, \mathcal{O} = \mathbb{Z} \left[\frac{1 + \sqrt{d}}{2} \right] = \mathbb{Z} + \mathbb{Z} \frac{1 + \sqrt{d}}{2} \text{ and}$$

$$\mathcal{O}^* = \left\{ \frac{a + b\sqrt{d}}{2} \mid a, b \in \mathbb{Z}, a^2 - db^2 = \pm 4 \right\}.$$

(ii) *If $d \equiv 2 \pmod{4}$ or $d \equiv 3 \pmod{4}$, then the set $\{ 1, \sqrt{d} \}$ is an integral basis of $\mathbb{Q}(\sqrt{d})$, $\delta = 4d$, $\mathcal{O} = \mathbb{Z}[\sqrt{d}] = \mathbb{Z} + \mathbb{Z}\sqrt{d}$ and*

$$\mathcal{O}^* = \left\{ a + b\sqrt{d} \mid a, b \in \mathbb{Z}, a^2 - db^2 = \pm 1 \right\}.$$

(iii) *If $d < 0$, then $\mathcal{O}^* = \{ -1, 1 \}$ when $d \neq -1, -3$, $\mathcal{O}^* = \langle i \rangle = \{ -1, 1, i, -i \}$ when $d = -1$ and $\mathcal{O}^* = \langle \zeta_6 \rangle$ if $d = -3$, where ζ_6 is a primitive 6-th root of unity.*

(iv) *If $d > 0$, then*

(a) *There exists a unit $\varepsilon > 1$ in \mathcal{O} such that $\mathcal{O}^* = \langle -1 \rangle \times \langle \varepsilon \rangle$.*

(b) *If $u > 1$ is a unit of \mathcal{O} , then $u = a + b\sqrt{d}$ for some $a > 0, b > 0$ in \mathbb{Q} .*

(c) *If $N(\varepsilon) = 1$, then $N(u) = 1$ for all $u \in \mathcal{O}^*$.*

Proof. See [13]. □

The unit ε of \mathcal{O} in the Theorem 1, (iv), is called the **fundamental unit** of \mathcal{O} . Hence the unit ε of \mathcal{O} completely determines the group \mathcal{O}^* . For example, we have for $d = 5$ ($d \equiv 1 \pmod{4}$), $\frac{1 + \sqrt{5}}{2}$ is the fundamental unit of $\mathbb{Q}(\sqrt{5})$. If $d = 17$, then $\frac{8 + 2\sqrt{17}}{2} = 4 + \sqrt{17}$ is a fundamental unit of $\mathbb{Q}(\sqrt{17})$. In general, if $d \equiv 1 \pmod{4}$, then $\varepsilon = \frac{a_0 + b_0\sqrt{d}}{2}$ where a_0 and b_0 are either both even or both odd. Of course, if a_0 and b_0 are both even, then $\varepsilon \in \mathbb{Z}[\sqrt{d}]$.

On the other hand, we denote by $\mathfrak{M}_{2 \times 2}(\mathbb{Z})$ the set of all matrices 2×2 with integer entries. Let $GL_2(\mathbb{Q})$ be the multiplicative group of invertible 2×2 matrices with rational entries, which is called the **general linear group of degree 2 over \mathbb{Q}** . The subset of all matrices of $GL_2(\mathbb{Q})$ with determinant 1 is a normal subgroup of $GL_2(\mathbb{Q})$ called the **special linear group of degree 2 over \mathbb{Q}** and denoted by $SL_2(\mathbb{Q})$.

For each $\lambda \in \mathbb{Q}$, let

$$G_\lambda = \left\{ A \in GL_2(\mathbb{Q}) \mid A = \begin{bmatrix} a & b\lambda \\ b & a \end{bmatrix} \right\}, \quad L_\lambda = \{A \in G_\lambda \mid \det(A) = \pm 1\} \quad \text{and}$$

$$T_d = \left\{ A \in \mathfrak{M}_{2 \times 2}(\mathbb{Z}) \mid A = \begin{bmatrix} a & bd \\ b & a \end{bmatrix} \right\}.$$

We have the follows results whose proofs can be seen in [17].

Theorem 2. *Keeping the previous notation we obtain*

- (i) T_d is a commutative subring with identity of $\mathfrak{M}_{2 \times 2}(\mathbb{Z})$.
- (ii) If T_d^* is the multiplicative group of units of T_d , then $T_d^* = L_d \cap \mathfrak{M}_{2 \times 2}(\mathbb{Z})$. In particular, T_d^* is a subgroup of L_d .
- (iii) The rings T_d and $\mathbb{Z}[\sqrt{d}]$ are isomorphic under the correspondence

$$\begin{bmatrix} a & bd \\ b & a \end{bmatrix} \mapsto a + b\sqrt{d}.$$

In particular, T_d is an integral domain.

- (iv) The isomorphism in (iii) induces an isomorphism between the multiplicative groups T_d^* and $(\mathbb{Z}[\sqrt{d}])^*$.
- (v) $T_d / (T_d \cap SL_2(\mathbb{Q})) \cong \{-1, 1\}$. □

Theorem 3. *Let Q_d be the set of all matrices of the form $A = \begin{bmatrix} a & bd \\ b & a \end{bmatrix}$ with $a, b \in \mathbb{Q}$.*

- (i) Q_d is a field isomorphic $\mathbb{Q}(\sqrt{d})$ under the correspondence $\begin{bmatrix} a & bd \\ b & a \end{bmatrix} \mapsto a + b\sqrt{d}$. This is, Q_d is the field of quotients of T_d .
- (ii) There exists a monomorphism of the multiplicative group $\mathbb{Q}(\sqrt{d})^*$ in the group $GL_2(\mathbb{Q})$.

(iii) The group $GL_2(\mathbb{Q})$ contains the chain of subgroups $Q_d^* \cap SL_2(\mathbb{Q}) < L_m < G_d = Q_d^* < GL_2(\mathbb{Q})$. \square

Theorem 4. Let $A = \begin{bmatrix} a & bd \\ b & a \end{bmatrix} \in Q_d$ where a, b are two rational numbers. Then the powers of A , $A^n = \begin{bmatrix} a_n & b_n d \\ b_n & a_n \end{bmatrix}$ with $n \in \mathbb{N}$, are given as follows:

$$(2.1) \quad a_n = \begin{cases} \sum_{0 \leq t \leq \frac{n}{2}} \binom{n}{2t} a^{2t} b^{n-2t} d^{\frac{n}{2}-t} & \text{if } n \text{ even} \\ \sum_{0 \leq t \leq \frac{n-1}{2}} \binom{n}{2t+1} a^{2t+1} b^{n-2t-1} d^{\frac{n-1}{2}-t} & \text{if } n \text{ odd} \end{cases}$$

and

$$(2.2) \quad b_n = \begin{cases} \sum_{0 \leq t \leq \frac{n-2}{2}} \binom{n}{2t+1} a^{2t+1} b^{n-2t-1} d^{\frac{n-2}{2}-t} & \text{if } n \text{ even} \\ \sum_{0 \leq t \leq \frac{n-1}{2}} \binom{n}{2t} a^{2t} b^{n-2t} d^{\frac{n-1}{2}-t} & \text{if } n \text{ odd.} \end{cases}$$

\square

3. THE SEQUENCES OF FIBONACCI AND LUCAS IN $\mathbb{Q}(\sqrt{d})$

In this section, we construct the sequences of Fibonacci and Lucas at any real quadratic field. We proof that the properties remain valid as those given by the classical sequence of Fibonacci and Lucas for $d = 5$. Being $d > 0$ a square free integer and ε the fundamental unit of $\mathbb{Q}(\sqrt{d})$, we will write $\varepsilon = a + b\sqrt{d}$ where $a, b \in \mathbb{Q}$ with its corresponding matrix $A_\varepsilon = \begin{bmatrix} a & bd \\ b & a \end{bmatrix}$ and the powers n -th of A_ε by $A_\varepsilon^n = \begin{bmatrix} a_n & b_n d \\ b_n & a_n \end{bmatrix}$ where a_n and b_n are given as in the equations (2.1) and (2.2) of Theorem 4. Also, Δ will be the determinant of A_ε , that is, $\Delta = a^2 - b^2 d = N(\varepsilon) = \pm 1$, where N is the **norm function** of the square field $\mathbb{Q}(\sqrt{d})$.

Keeping the previous notation, we have the follows:

Definition 5. The sequence of **Fibonacci** (resp. **Lucas**) of degree d with respect to the fundamental unit ε (or simply the sequence of **Fibonacci** (resp. **Lucas**), if there is no risk of confusion with respect to d and to its fundamental unit ε) is the sequence $\{F_{\varepsilon,n}\}_{n \in \mathbb{N}}$ (resp. $\{L_{\varepsilon,n}\}_{n \in \mathbb{N}}$) of positive numbers given as follows:

$$(3.1) \quad F_{\varepsilon,n} := \frac{b_n}{b} \quad \left(\text{resp. } L_{\varepsilon,n} := \frac{a_n}{a} \right) \quad (n \in \mathbb{N})$$

where the sequence $\{b_n\}_{n \in \mathbb{N}}$ (resp. $\{a_n\}_{n \in \mathbb{N}}$) is given as in the equation (2.2) (resp. (2.1)) of Theorem 4.

According to the equation (3.1) of the Definition 5, we have that $F_{\varepsilon,n}$ and $L_{\varepsilon,n}$ are given by the follows equations:

$$(3.2) \quad F_{\varepsilon,n} = \begin{cases} \sum_{0 \leq t \leq \frac{n-2}{2}} \binom{n}{2t+1} a^{2t+1} b^{n-2t-2} d^{\frac{n-2}{2}-t} & \text{if } n \text{ even} \\ \sum_{0 \leq t \leq \frac{n-1}{2}} \binom{n}{2t} a^{2t} b^{n-2t-1} d^{\frac{n-1}{2}-t} & \text{if } n \text{ odd} \end{cases}$$

and

$$(3.3) \quad L_{\varepsilon,n} = \begin{cases} \sum_{0 \leq t \leq \frac{n}{2}} \binom{n}{2t} a^{2t-1} b^{n-2t} d^{\frac{n}{2}-t} & \text{if } n \text{ even} \\ \sum_{0 \leq t \leq \frac{n-1}{2}} \binom{n}{2t+1} a^{2t} b^{n-2t-1} d^{\frac{n-1}{2}-t} & \text{if } n \text{ odd} \end{cases}$$

for each $n \in \mathbb{N}$.

In the Table 1 expresses some terms of the sequences $\{F_{\varepsilon,n}\}_{n \in \mathbb{N}}$ and $\{L_{\varepsilon,n}\}_{n \in \mathbb{N}}$ for some d 's square free. Unless otherwise noted, the sequences are not in the Online Encyclopedia of Integer Sequences at the time of publication, though some of the sequences do appear there, as indicated in Table 2.

d	ε	Δ	Terms of the sequence $\{F_{\varepsilon,n}\}_{n \in \mathbb{N}}$	Terms of the sequence $\{L_{\varepsilon,n}\}_{n \in \mathbb{N}}$
2	$1 + \sqrt{2}$	-1	1, 2, 5, 12, 29, 70, 169, 408, ... OEIS sequence A000129	1, 3, 7, 17, 41, 99, ... OEIS sequence A001333
3	$2 + \sqrt{3}$	1	1, 4, 15, 56, 209, 780, ... OEIS sequence A001353	$1, \frac{7}{2}, 13, \frac{97}{2}, 181, \frac{1351}{2}, \dots$ OEIS sequence A001075
5	$\frac{1 + \sqrt{5}}{2}$	-1	1, 1, 2, 3, 5, 8, ... OEIS sequence A000045	1, 3, 4, 7, 11, 18, ... OEIS sequence A000032
6	$5 + 2\sqrt{6}$	1	1, 10, 99, 980, 9701, 96030, ... OEIS sequence A004189	$1, \frac{49}{5}, 97, \frac{4801}{5}, 9505, \frac{470449}{5}, \dots$ OEIS sequence A001079
7	$8 + 3\sqrt{7}$	1	1, 16, 255, 4064, 64769, ... OEIS sequence A077412	$1, \frac{127}{8}, 253, \frac{32257}{8}, 64261, \dots$ OEIS sequence A001081
10	$3 + 1\sqrt{10}$	-1	1, 6, 37, 228, 1405, 8658, ... OEIS sequence A005668	$1, \frac{19}{3}, 39, \frac{721}{3}, 1481, \dots$ OEIS sequence A005667
11	$10 + 3\sqrt{11}$	1	1, 20, 399, 7960, 158801, ... OEIS sequence A075843	$1, \frac{199}{10}, 397, \frac{79201}{10}, 158005, \dots$ OEIS sequence A001085
13	$\frac{3 + \sqrt{13}}{2}$	-1	1, 3, 10, 33, 109, 360, ... OEIS sequence A006190	$1, \frac{11}{3}, 12, \frac{119}{3}, 131, \frac{1298}{3}, \dots$ OEIS sequence A006497

TABLE 1. Sequence of Fibonacci and Lucas of degree d .

Observation 6. Note that when $d = 5$, we have $\{F_{\varepsilon,n}\}_{n \in \mathbb{N}}$ and $\{L_{\varepsilon,n}\}_{n \in \mathbb{N}}$ are exactly the classical sequences of Fibonacci and Lucas, respectively.

d	ε	Δ	Terms of the sequence $\{F_{\varepsilon,n}\}_{n \in \mathbb{N}}$	OEIS integer sequence
37	$6 + \sqrt{37}$	-1	1, 12, 145, 1752, 21169, 255780, ...	A041061
38	$37 + 6\sqrt{38}$	1	1, 74, 5475, 405076, 29970149, 2217385950, ...	
39	$25 + 4\sqrt{39}$	1	1, 50, 2499, 124900, 6242501, 312000150, ...	
41	$32 + 5\sqrt{41}$	-1	1, 64, 4097, 262272, 16789505, 1074790592, ...	
42	$13 + 2\sqrt{42}$	1	1, 26, 675, 17524, 454949, 11811150, ...	A097309

 TABLE 2. Sequence of Fibonacci of degree d .

In the rest of the work, by abuse of notation, we write F_n and L_n instead of $F_{\varepsilon,n}$ and $L_{\varepsilon,n}$ if there is no risk of confusion with respect to the classical sequences of Fibonacci and Lucas.

Theorem 7. For each $m, n \in \mathbb{N}$,

- (i) $F_{n+1} = a(L_n + F_n)$.
- (ii) $L_{n+1} = aL_n + \frac{b^2d}{a}F_n$.
- (iii) $F_n = \frac{a}{\Delta} (F_{n+1} - L_{n+1}) = \begin{cases} a(L_{n+1} - F_{n+1}) & \text{if } \Delta = -1 \\ a(F_{n+1} - L_{n+1}) & \text{if } \Delta = 1. \end{cases}$
- (iv) $L_n = \frac{1}{\Delta} \left(aL_{n+1} - \frac{b^2d}{a}F_{n+1} \right) = \begin{cases} \frac{b^2d}{a}F_{n+1} - aL_{n+1} & \text{if } \Delta = -1 \\ aL_{n+1} - \frac{b^2d}{a}F_{n+1} & \text{if } \Delta = 1. \end{cases}$
- (v) $F_{n+1} - a^n F_1 = \sum_{t=0}^{n-1} a^{t+1} L_{n-t}$.
- (vi) $L_{n+1} - a^n L_1 = b^2d \sum_{t=0}^{n-1} a^{t-1} F_{n-t}$.
- (vii) $F_{m+n} = a(F_m L_n + F_n L_m)$.
- (viii) $L_{m+n} = \frac{b^2d}{a} \cdot F_m F_n + a L_m L_n$.
- (ix) $b^2d F_n^2 - a^2 L_n^2 = -\Delta^n$.
- (x) $F_n = \sum_{t=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2t+1} a^{n-2t-1} b^{2t} d^t = \sum_{t=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{n-2t-1} a^{n-2t-1} b^{2t} d^t$.
- (xi) $L_n = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2t} a^{n-2t-1} b^{2t} d^t = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{n-2t} a^{n-2t-1} b^{2t} d^t$.

Here $[x]$ is the integral part of $x \in \mathbb{R}$, i.e., is the greatest integer n such that $n \leq x < n+1$.

Proof. (i) and (ii) are obtained directly from the equations (3.2) and (3.3). (iii) and (iv) are deduced from (i) and (ii). For induction, we obtain (v) and (vi). (vii) and (viii) are obtained from the relationship $A_\varepsilon^{m+n} = A_\varepsilon^m \cdot A_\varepsilon^n$. The relation $\det(A_\varepsilon^n) = \Delta^n$ implies the relation (ix). Finally, (x) y (xi) are obtained of the relationships

$$a_n + b_n \sqrt{d} = (a + b\sqrt{d})^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i} (\sqrt{d})^{n-i} = \sum_{i=0}^n \binom{n}{i} a^{n-i} b^i (\sqrt{d})^i.$$

□

Theorem 8. *There exist unique $r, s \in \mathbb{Q}^*$ such that $F_{n+2} = rF_n + sF_{n+1}$ for all $n \in \mathbb{N}$. More precisely, $F_{n+2} = (-\Delta)F_n + 2aF_{n+1}$ for all $n \in \mathbb{N}$.*

Proof. We have for each $n \in \mathbb{N}$,

$$\begin{aligned} (-\Delta)F_n + 2aF_{n+1} &= -(a^2 - b^2d)F_n + 2aF_{n+1} = a \left(\frac{b^2d}{a} \right) F_n - a^2F_n + 2aF_{n+1} \\ &= a(L_{n+1} - aL_n) - a^2F_n + 2aF_{n+1} = a(L_{n+1} + F_{n+1}) \\ &= F_{n+2}. \end{aligned}$$

On the other hand, let $r, s \in \mathbb{Q}^*$ be such that

$$(3.4) \quad F_{n+2} = rF_n + sF_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

As $b^2d = a^2 - \Delta$, implies that $F_1 = 1$, $F_2 = 2a$, $F_3 = 4a^2 - \Delta$ and $F_4 = 8a^3 - 4a\Delta$. In particular, by the equation (3.4) for $n = 1$ and $n = 2$, we obtain the system of equations

$$(3.5) \quad \begin{aligned} r + 2as &= 4a^2 - \Delta \\ 2ar + (4a^2 - \Delta)s &= 8a^3 - 4a\Delta \end{aligned}$$

which it has an unique solution, namely $r = -\Delta$ and $s = 2a$. This complete the proof of theorem.

□

Corollary 9. *The Fibonacci sequence $\{F_n\}_{n \in \mathbb{N}}$ is a k -Fibonacci sequence for some $k \in \mathbb{N}$ (namely, $k = 2a$) if and only if $\Delta = -1$.*

Proof. It immediate by Theorem 8.

□

Corollary 10. *The following conditions are equivalent:*

- (i) $F_{n+2} = F_n + F_{n+1}$ for all $n \in \mathbb{N}$;
- (ii) $F_3 = F_1 + F_2$;
- (iii) $d = 5$ and $\varepsilon = \frac{1 + \sqrt{5}}{2}$.

Proof. (i) \implies (ii): It is immediate.

(ii) \implies (iii): We have that $-\Delta + 4a^2 = (-\Delta)F_1 + 2aF_2 = F_3 = F_1 + F_2 = 1 + 2a$, then $4a^2 - 2a - (\Delta + 1) = 0$. If $\Delta = 1$, then $2a^2 - a - 1 = 0$; since $a \neq 1$, necessarily $a = -1/2$. But this implies that $4b^2d = -3$; contradiction. Therefore $\Delta = -1$, $a = 1/2 = b$ and $d = 5$.

(iii) \implies (i): It is clear.

□

Corollary 11. *If $\{F_n\}_{n \in \mathbb{N}}$ is the Fibonacci sequence classical, that is $d = 5$, then*

$$F_{n+2} = F_n + F_{n+1}$$

for each $n \in \mathbb{N}$.

Proof. It is immediate. □

We recall if $d \equiv 2$ or $3 \pmod{4}$, then $\varepsilon = a + b\sqrt{d}$ where $a, b \in \mathbb{Z}$. In this case, it is obvious that $F_n \in \mathbb{N}$ for all $n \in \mathbb{N}$. If $d \equiv 1 \pmod{4}$, then $\varepsilon = a + b\sqrt{d} = \frac{a_0 + b_0\sqrt{d}}{2}$ with $a_0, b_0 \in \mathbb{N}$, where either are both even or both odd. When they are both even, we have that $a, b \in \mathbb{N}$ and, hence, $F_n \in \mathbb{N}$. But, in any case, $2a \in \mathbb{N}$. Therefore, we obtain the following result.

Corollary 12. $F_n \in \mathbb{N}$ for all $n \in \mathbb{N}$.

Proof. By Theorem 8, we have $F_{n+2} = (-\Delta)F_n + 2aF_{n+1}$ for all $n \in \mathbb{N}$, where $F_1 = 1$ and $F_2 = 2a \in \mathbb{N}$. Then, the show follows by induction on n . □

Theorem 13. *There exist unique $r, s \in \mathbb{Q}^*$ such that $L_{n+2} = rL_n + sL_{n+1}$ for all $n \in \mathbb{N}$. More precisely, $L_{n+2} = (-\Delta)L_n + 2aL_{n+1}$ for all $n \in \mathbb{N}$.*

Proof. We have that for each $n \in \mathbb{N}$

$$\begin{aligned} (-\Delta)L_n + 2aL_{n+1} &= -\left(aL_{n+1} - \frac{b^2d}{a}F_{n+1}\right) + 2aL_{n+1} = aL_{n+1} + \frac{b^2d}{a}F_{n+1} \\ &= L_{n+2}. \end{aligned}$$

Now we prove the uniqueness. As $b^2d = a^2 - \Delta$, it follows that

$$\begin{aligned} L_1 &= 1 \\ L_2 &= 2a - \frac{\Delta}{a} \\ L_3 &= 4a^2 - 3\Delta \\ L_4 &= 8a^3 - 8a\Delta + \frac{1}{a} \\ &\vdots \qquad \qquad \qquad \vdots \end{aligned}$$

Let $r, s \in \mathbb{Q}^*$ be such that

$$(3.6) \qquad L_{n+2} = rL_n + sL_{n+1} \quad \text{for all } n \in \mathbb{N}.$$

In particular, for $n = 1$ and $n = 2$, we have the system of equations

$$(3.7) \qquad \begin{aligned} r + \left(2a - \frac{\Delta}{a}\right)s &= 4a^2 - 3\Delta \\ \left(2a - \frac{\Delta}{a}\right)r + (4a^2 - 3\Delta)s &= 8a^3 - 8a\Delta + \frac{1}{a} \end{aligned}$$

which it has a unique solution, namely $r = -\Delta$ and $s = 2a$; so that, this system of equations has the same solution that the system of equations (3.5) given in the proof of Theorem 8. Therefore, the theorem is true. □

Similarly to the corollaries to Theorem 8 for Fibonacci sequence, we obtain corollaries to Theorem 13 for Lucas sequence.

Corollary 14. *The Lucas sequence $\{L_n\}_{n \in \mathbb{N}}$ is a k -Lucas sequence for some $k \in \mathbb{N}$ (namely, $k = 2a$) if and only if $\Delta = -1$.*

Proof. It immediate by Theorem 13. □

Corollary 15. *The following conditions are equivalent:*

- (i) $L_{n+2} = L_n + L_{n+1}$ for all $n \in \mathbb{N}$;
- (ii) $L_3 = L_1 + L_2$;

$$(iii) \quad d = 5 \text{ and } \varepsilon = \frac{1 + \sqrt{5}}{2}.$$

Proof. (i) \implies (ii): It is immediate.

(ii) \implies (iii): Since $L_3 = L_1 + L_2$, that is, $4a^2 - 3\Delta = 1 + 2a - \frac{\Delta}{a}$, we have that $4a^3 - 2a^2 - a + (1 - 3a)\Delta = 0$. If $\Delta = 1$, then $4a^3 - 2a^2 - 4a + 1 = 0$ and a can not be a rational number, contradiction. Hence, $\Delta = -1$ and $(2a^2 + 1)(2a - 1) = 0$. This implies that $a = 1/2$ and $4b^2d = 5$. Therefore, $d = 5$ and $a = 1/2 = b$.

(iii) \implies (i): It is clear. \square

Corollary 16. *If $\{L_n\}_{n \in \mathbb{N}}$ is the Lucas sequence classical, that is $d = 5$, then*

$$L_{n+2} = L_n + L_{n+1}$$

for each $n \in \mathbb{N}$.

Proof. It is immediate. \square

Corollary 17. *For all $k \in \mathbb{N}$,*

$$(i) \quad L_{2k-1} \in \mathbb{N};$$

$$(ii) \quad \text{if } a \in \mathbb{N}, \text{ then } aL_{2k} \in \mathbb{N} \text{ and } (a, aL_{2k}) = 1;$$

$$(iii) \quad \text{if } a = \frac{a_0}{2}, \text{ with } a_0 \text{ odd, then } a_0L_{2k} \in \mathbb{N} \text{ and } (a_0, a_0L_{2k}) = 1.$$

Proof. Applying the Theorem 13, the proof follows by induction over all the pairs (L_{2k-1}, L_{2k}) , $k \in \mathbb{N}$. \square

4. GENERATING FUNCTION

The main goal of this section is to show that the Fibonacci and Lucas sequences given in (4) and (5) are determined by the generating functions.

Theorem 18. *We obtain*

$$(i) \quad \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \varepsilon = \lim_{n \rightarrow \infty} \frac{L_{n+1}}{L_n}.$$

(ii) *The series $\sum_{n=1}^{\infty} F_n x^{n-1}$ and $\sum_{n=1}^{\infty} L_n x^{n-1}$ both have the same radius of convergence, namely $R = 1/\varepsilon$.*

Proof. (i): By Theorem 7, we have $\frac{F_{n+1}}{F_n} = \frac{a(L_n + F_n)}{F_n} = a + a \cdot \frac{L_n}{F_n} = a + b \cdot \frac{a_n}{b_n}$, and

$\frac{L_{n+1}}{L_n} = \frac{aL_n + \frac{b^2d}{a}F_n}{L_n} = a + \frac{b^2d}{a} \cdot \frac{F_n}{L_n} = a + b \cdot \frac{b_nd}{a_n}$ where $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \sqrt{d} = \lim_{n \rightarrow \infty} \frac{b_nd}{a_n}$, see [17, Theorem 3.1]. Thus, $\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \varepsilon = \lim_{n \rightarrow \infty} \frac{L_{n+1}}{L_n}$.

(ii): For each $x \in \mathbb{R}$, $x \neq 0$, we have that $\lim_{n \rightarrow \infty} \frac{F_{n+1}x^n}{F_n x^{n-1}} = \varepsilon x = \lim_{n \rightarrow \infty} \frac{L_{n+1}x^n}{L_n x^{n-1}}$. Then $\lim_{n \rightarrow \infty} \frac{F_{n+1}|x|^n}{F_n|x|^{n-1}} < 1$ if and only if $|x| < \frac{1}{\varepsilon}$. Similarly, $\lim_{n \rightarrow \infty} \frac{L_{n+1}|x|^n}{L_n|x|^{n-1}} < 1$ if and only if $|x| < \frac{1}{\varepsilon}$. Therefore, both series have the same radius of convergence $R = 1/\varepsilon$. This complete the proof of the theorem. \square

Theorem 19. (Generating function) *Let $x \in \mathbb{R}$ be such that $|x| < 1/\varepsilon$.*

$$(i) \quad \text{If } f(x) = \sum_{n=1}^{\infty} F_n x^{n-1}, \text{ then } f(x) = \frac{1}{\Delta x^2 - 2ax + 1}.$$

$$(ii) \text{ If } g(x) = \sum_{n=1}^{\infty} L_n x^{n-1}, \text{ then } g(x) = \left(\frac{a - \Delta x}{a} \right) f(x) = \frac{a - \Delta x}{a(\Delta x^2 - 2ax + 1)}.$$

Proof. (i): For each $x \in \mathbb{R}$ with $|x| < 1/\varepsilon$, we have that

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} F_n x^{n-1} = 1 + 2ax + \sum_{n=1}^{\infty} F_{n+2} x^{n+1} = 1 + 2ax + \sum_{n=1}^{\infty} (-\Delta F_n + 2aF_{n+1}) x^{n+1} \\ &= 1 + 2ax - \Delta x^2 f(x) + 2ax(f(x) - 1) = 1 + f(x)(2ax - \Delta x^2) \end{aligned}$$

$$\text{this implies that } f(x) = \frac{1}{\Delta x^2 - 2ax + 1}.$$

(ii): We observe that, for each $x \in \mathbb{R}$ with $|x| < 1/\varepsilon$

$$f(x) = 1 + \sum_{n=1}^{\infty} F_{n+1} x^n = 1 + \sum_{n=1}^{\infty} a(L_n + F_n) x^n = 1 + axg(x) + axf(x).$$

If $x \neq 0$, then

$$g(x) = \frac{(1 - ax)f(x) - 1}{ax} = \frac{a - \Delta x}{a(\Delta x^2 - 2ax + 1)} = \left(\frac{a - \Delta x}{a} \right) f(x).$$

□

5. GOLDEN RATIO AND BINET'S FORMULA IN $\mathbb{Q}(\sqrt{d})$

In this section we give Golden ratio associated as the quadratic field $\mathbb{Q}(\sqrt{d})$. Also we obtain Binet's formula in $\mathbb{Q}(\sqrt{d})$. We start with

Definition 20. Let $x, y \in \mathbb{R}$ be such that $0 < y < x$. We say that x and y are in **Golden ratio with respect to the quadratic field** $\mathbb{Q}(\sqrt{d})$ (or simply that they are in **Golden ratio**, if there is no risk of confusion with respect to the quadratic field $\mathbb{Q}(\sqrt{d})$), if

$$(5.1) \quad \frac{2ax - \Delta y}{x} = \frac{x}{y}.$$

Thus, if x and y are in Golden ratio and we write $\varphi := \frac{x}{y}$, then we have that

$$\begin{aligned} 2a - \frac{\Delta}{\varphi} &= 2a - \Delta \cdot \frac{y}{x} = \frac{2ax - \Delta y}{x} = \frac{x}{y} \\ &= \varphi. \end{aligned}$$

This is, φ satisfies the equation

$$(5.2) \quad \varphi^2 - 2a\varphi + \Delta = 0.$$

But $x^2 - 2ax + \Delta$ is the irreducible polynomial of ε over \mathbb{Q} with $\bar{\varepsilon}$ its other root, where $\bar{\varepsilon}$ is the conjugate of ε . Therefore, $\varphi = \varepsilon$ or $\varphi = \bar{\varepsilon}$. As $x > y > 0$ and $\bar{\varepsilon} = \Delta/\varepsilon$, necessarily $\varphi = \varepsilon$. In consequence, we have the equation

$$(5.3) \quad \varepsilon^2 = 2a\varepsilon - \Delta.$$

Theorem 21. For each $n \in \mathbb{N}$, with $n \geq 2$,

$$(5.4) \quad \varepsilon^n = F_n \varepsilon - F_{n-1} \Delta.$$

Proof. The show is by induction on n . It is clear for $n = 2$, that is, $\varepsilon^2 = 2a\varepsilon - \Delta = F_2 \varepsilon - F_1 \Delta$. Hence,

$$\begin{aligned} \varepsilon^{n+1} &= \varepsilon(F_n \varepsilon - F_{n-1} \Delta) = F_n(2a\varepsilon - \Delta) - F_{n-1}\varepsilon\Delta = (-\Delta F_{n-1} + 2aF_n)\varepsilon - F_n\Delta \\ &= F_{n+1} \varepsilon - F_n\Delta. \end{aligned}$$

□

Since $\bar{\varepsilon}$ also satisfies the equation (5.3), we have the equation

$$(5.5) \quad (\bar{\varepsilon})^n = F_n \bar{\varepsilon} - F_{n-1} \Delta,$$

for each $n \geq 2$.

Theorem 22. For each $n \in \mathbb{N}$,

$$(5.6) \quad F_n = \frac{\varepsilon^n - (\bar{\varepsilon})^n}{\varepsilon - \bar{\varepsilon}}.$$

Proof. It follows to make the difference of the equations (5.4) and (5.5). □

The equation (5.6) is known as the **Binet's formula**

Corollary 23. For each $n \in \mathbb{N}$,

$$(5.7) \quad L_n = \frac{\varepsilon^n + (\bar{\varepsilon})^n}{\varepsilon + \bar{\varepsilon}}.$$

Proof. It is immediate from the following

$$\varepsilon^n + (\bar{\varepsilon})^n = 2aF_n - 2\Delta F_{n-1} = 2a \left(F_n - \frac{\Delta}{a} \cdot F_{n-1} \right) = 2aL_n = (\varepsilon + \bar{\varepsilon})L_n.$$

□

The following two theorems give us other version of the generating functions of the sequences of Fibonacci and Lucas in $\mathbb{Q}(\sqrt{d})$.

Theorem 24. Let $f_1(x) = \sum_{n=0}^{\infty} \Delta^n F_{n+1} x^n$ and $g_1(x) = \sum_{n=0}^{\infty} \Delta^n L_{n+1} x^n$. Then, the series $f_1(x)$ and $g_1(x)$ are convergent for $|x| < \min\{|\varepsilon|, |\bar{\varepsilon}|\}$. Furthermore,

$$(5.8) \quad f_1(x) = \frac{\Delta}{x^2 - 2ax + \Delta}$$

and

$$(5.9) \quad g_1(x) = \frac{\Delta(a-x)}{a(x^2 - 2ax + \Delta)} = \left(\frac{a-x}{a} \right) f_1(x).$$

Proof. We have for $|x| < \min\{|\varepsilon|, |\bar{\varepsilon}|\}$

$$\begin{aligned} \frac{2b\sqrt{d}}{(x-\varepsilon)(x-\bar{\varepsilon})} &= \frac{1}{x-\varepsilon} - \frac{1}{x-\bar{\varepsilon}} = \frac{1}{\bar{\varepsilon} \left(1 - \frac{x}{\bar{\varepsilon}}\right)} - \frac{1}{\varepsilon \left(1 - \frac{x}{\varepsilon}\right)} \\ &= \frac{1}{\bar{\varepsilon}} \sum_{n=0}^{\infty} \left(\frac{x}{\bar{\varepsilon}}\right)^n - \frac{1}{\varepsilon} \sum_{n=0}^{\infty} \left(\frac{x}{\varepsilon}\right)^n = \sum_{n=0}^{\infty} \left(\frac{\varepsilon^{n+1} - \bar{\varepsilon}^{n+1}}{(\varepsilon\bar{\varepsilon})^{n+1}}\right) x^n \\ &= 2b\sqrt{d} \left(\sum_{n=0}^{\infty} (a^2 - b^2d)^{n+1} \left(\frac{\varepsilon^{n+1} - \bar{\varepsilon}^{n+1}}{\varepsilon - \bar{\varepsilon}}\right) x^n \right) \\ &= 2b\sqrt{d} \left(\sum_{n=0}^{\infty} \Delta^{n+1} F_{n+1} x^n \right). \end{aligned}$$

This implies that

$$\begin{aligned} \frac{1}{x^2 - 2ax + \Delta} &= \frac{1}{x^2 - 2ax + (a^2 - b^2d)} = \frac{1}{(x - \varepsilon)(x - \bar{\varepsilon})} \\ &= \sum_{n=0}^{\infty} \Delta^{n+1} F_{n+1} x^n \end{aligned}$$

or equivalently

$$(5.10) \quad \frac{\Delta}{x^2 - 2ax + \Delta} = \sum_{n=0}^{\infty} \Delta^{n+2} F_{n+1} x^n = \sum_{n=0}^{\infty} \Delta^n F_{n+1} x^n.$$

Therefore

$$f_1(x) = \frac{\Delta}{x^2 - 2ax + \Delta}.$$

On the other hand, we have

$$\begin{aligned} (a-x)f_1(x) &= (a-x) \left(\sum_{n=0}^{\infty} \Delta^{n+2} F_{n+1} x^n \right) \\ &= a F_1 + \sum_{n=1}^{\infty} \Delta^{n+1} \left(\frac{a}{\Delta} F_{n+1} - F_n \right) x^n \\ &= a L_1 + \sum_{n=1}^{\infty} \Delta^n a L_{n+1} x^n \\ &= a \sum_{n=0}^{\infty} \Delta^n L_{n+1} x^n. \end{aligned}$$

Therefore

$$g_1(x) = \left(\frac{a-x}{a} \right) f_1(x) = \frac{\Delta(a-x)}{a(x^2 - 2ax + \Delta)}.$$

□

6. SOME OTHER PROPERTIES

Using the equations (5.6) and (5.7), we can extend the definition of the sequences of Fibonacci and Lucas over all integer number. This is, we use the Binet's formula for all $n \in \mathbb{Z}$, Theorem 22 and Corollary 23 we obtain

$$(6.1) \quad F_{-n} = \begin{cases} 0 & \text{if } n = 0 \\ -\Delta^n F_n & \text{if } n \geq 1 \end{cases}$$

and

$$(6.2) \quad L_{-n} = \begin{cases} \frac{1}{a} & \text{if } n = 0 \\ \Delta^n L_n & \text{if } n \geq 1. \end{cases}$$

Thus, it holds that for all $n \in \mathbb{Z}$

$$(6.3) \quad F_{n+2} = (-\Delta)F_n + 2aF_{n+1}$$

and

$$(6.4) \quad L_{n+2} = (-\Delta)L_n + 2aL_{n+1}.$$

But also we can obtain, in our case, the identities established by Catalan, Cassini, D'Ocagne, and Hosnberger which are hold for all $n \in \mathbb{Z}$, that is

Theorem 25. For all $m, n \in \mathbb{Z}$, the follows identities are holds:

$$\begin{aligned}
(i) \quad & F_n^2 - F_{n+m}F_{n-m} = \Delta^{n-m}F_m^2. \\
(ii) \quad & F_n^2 - F_{n-1}F_{n+1} = \Delta^{n-1}. \\
(iii) \quad & L_n^2 - L_{n+r}L_{n-r} = \frac{\Delta^n}{2a^2} - \left(\frac{\Delta^{n-r}}{2a}\right)L_{2r}. \\
(iv) \quad & F_mF_{n+1} - F_nF_{m+1} = \Delta^n F_{m-n}. \\
(v) \quad & F_{m-1}F_n + F_mF_{n+1} = \begin{cases} F_{m+n} & \text{if } \Delta = -1 \\ \frac{a}{2b^2d} \cdot (2aL_{m+n} - L_{m-n-1}) & \text{if } \Delta = 1. \end{cases} \\
(vi) \quad & L_nL_{n+r} = \left(\frac{1}{2a}\right)L_{2n+r} + \left(\frac{\Delta^n}{2a}\right)L_r.
\end{aligned}$$

Proof. The show for each of the identities can be performed using the Binet's Formula. So we will prove only (iv). Hence we have

$$\begin{aligned}
F_mF_{n+1} - F_nF_{m+1} &= \left(\frac{\varepsilon^m - (\bar{\varepsilon})^m}{\varepsilon - \bar{\varepsilon}}\right) \left(\frac{\varepsilon^{n+1} - (\bar{\varepsilon})^{n+1}}{\varepsilon - \bar{\varepsilon}}\right) \\
&\quad - \left(\frac{\varepsilon^n - (\bar{\varepsilon})^n}{\varepsilon - \bar{\varepsilon}}\right) \left(\frac{\varepsilon^{m+1} - (\bar{\varepsilon})^{m+1}}{\varepsilon - \bar{\varepsilon}}\right) \\
&= \frac{\varepsilon^m(\bar{\varepsilon})^n - \varepsilon^n(\bar{\varepsilon})^m}{\varepsilon - \bar{\varepsilon}} = \frac{\varepsilon^{m-n+n}(\bar{\varepsilon})^n - \varepsilon^n(\bar{\varepsilon})^{m-n+n}}{\varepsilon - \bar{\varepsilon}} \\
&= \Delta^n \left(\frac{\varepsilon^{m-n} - (\bar{\varepsilon})^{m-n}}{\varepsilon - \bar{\varepsilon}}\right) = \Delta^n F_{m-n}.
\end{aligned}$$

□

7. THE SEQUENCE OF FIBONACCI AND OF LUCAS OF DEGREE d WITH RESPECT TO AN ARBITRARY UNIT

The unit group of $\mathbb{Q}(\sqrt{d})$, with $d > 0$, is isomorph to group $\langle -1 \rangle \times \langle \varepsilon \rangle$ where ε is the fundamental unit of $\mathbb{Q}(\sqrt{d})$, generator of the infinity cyclic subgroup. This cyclic subgroup also is generated by $1/\varepsilon$, $-\varepsilon$ and $-1/\varepsilon$. This is, each unit of $\mathbb{Q}(\sqrt{d})$ has the form $\pm \varepsilon^l$ for some $l \in \mathbb{Z}$. Observing the previous development, we can define the sequence of Fibonacci and Lucas of degree d with respect to an arbitrary unit η of $\mathbb{Q}(\sqrt{d})$, and the results of the previous sections are still met. Essentially this is because $N(\eta) = \pm 1$. This allows us to build even more an infinity of sequences in $\mathbb{Q}(\sqrt{d})$ meeting similar properties of the sequences of Fibonacci and Lucas. For example, we consider the unit $\eta = \frac{-1 + \sqrt{5}}{2}$ of $\mathbb{Q}(\sqrt{5})$, we have that the first terms of the sequence of Fibonacci of degree 5 with respect to the unit η are:

$$F_{\eta,1} = 1, F_{\eta,2} = -1, F_{\eta,3} = 2, F_{\eta,4} = -3, \dots$$

where $N(\eta) = -1$. Comparing the terms of the sequence of Fibonacci with negative index, F_{-n} with $n \geq 1$, we have that $F_{\eta,n} = F_{-n}$ for all $n \in \mathbb{N}$. This is, the sequence of Fibonacci with negative index of degree 5 with respect to the fundamental unit $\varepsilon = \frac{1 + \sqrt{5}}{2}$ is the sequence of Fibonacci of degree 5 respect to the unit $\eta = \frac{-1 + \sqrt{5}}{2}$. This it is not a coincidence, that is, this fact is generalized in the following.

Theorem 26. The Fibonacci sequence of degree d with respect to the unit $1/\varepsilon$ and $\Delta = -1$ is the Fibonacci sequence with negative index of degree d with respect to the fundamental unit ε .

Proof. We write $\eta = 1/\varepsilon = \Delta\bar{\varepsilon}$. Hence, $\bar{\eta} = \Delta\varepsilon$. Using the Binet's formula, we have that for all $n \in \mathbb{N}$

$$F_{\eta,n} = \frac{\eta^n - (\bar{\eta})^n}{\eta - \bar{\eta}} = \frac{(\Delta\bar{\varepsilon})^n - (\Delta\varepsilon)^n}{\Delta\bar{\varepsilon} - \Delta\varepsilon} = \Delta^{n-1} \frac{\varepsilon^n - (\bar{\varepsilon})^n}{\varepsilon - \bar{\varepsilon}} = \Delta^{n-1} F_n = -\Delta^n F_n = F_{-n} .$$

□

Observation 27. We note that if $\Delta = 1$ then the Fibonacci sequence of degree d with respect to the unit $1/\varepsilon$ coincides with the Fibonacci sequence of degree d with respect to the unit ε .

We finish our work with the following result.

Theorem 28. For each $k \in \mathbb{N}$ there exist unique $d, r \in \mathbb{N}$ such that d is square free and $\frac{k}{2} + \frac{r}{2}\sqrt{d}$ is a unit of the quadratic field $\mathbb{Q}(\sqrt{d})$ with norm -1 . Therefore, in this case, the k -Fibonacci sequence is the Fibonacci sequence of degree d with respect to a unit of $\mathbb{Q}(\sqrt{d})$.

Proof. Let $k \in \mathbb{N}$ be arbitrary. We have that $k^2 + 4$ is not a perfect square. Hence, there exist $d, r \in \mathbb{N}$ such that $k^2 + 4 = r^2 d$ where d is positive square free. This implies that $\left(\frac{k}{2}\right)^2 - \left(\frac{r}{2}\right)^2 d = -1$. Hence, $\frac{k}{2} + \frac{r}{2}\sqrt{d}$ is a unit of $\mathbb{Q}(\sqrt{d})$ with norm -1 . On the other hand, if $d, d_1, r, r_1 \in \mathbb{N}$ such that $\frac{k}{2} + \frac{r}{2}\sqrt{d}$ and $\frac{k}{2} + \frac{r_1}{2}\sqrt{d_1}$ are units of the quadratic field $\mathbb{Q}(\sqrt{d})$ both with norm -1 , then $\left(\frac{k}{2}\right)^2 - \left(\frac{r}{2}\right)^2 d = -1 = \left(\frac{k}{2}\right)^2 - \left(\frac{r_1}{2}\right)^2 d_1$, thus $\left(\frac{r}{2}\right)^2 d = \left(\frac{r_1}{2}\right)^2 d_1$. That is, $r^2 d = r_1^2 d_1$, where d and d_1 are square free. Therefore, $d_1 = d$ and $r_1 = r$. In consequence, the k -Fibonacci sequence is the Fibonacci sequence of degree d with respect to a unit of $\mathbb{Q}(\sqrt{d})$. □

Corollary 29. For each $k \in \mathbb{N}$, the k -Fibonacci sequence is the Fibonacci sequence of degree d with respect to a unit of $\mathbb{Q}(\sqrt{d})$ for some d square free.

Proof. It is immediately of Theorem 28 and Corollary 9. □

8. CONCLUSIONS

In this work we have established that every real quadratic field $\mathbb{Q}(\sqrt{d})$ has its own Fibonacci sequence and Lucas sequence, and variants of these, through the fundamental unit, being this the golden ratio. Therefore, the real quadratic field $\mathbb{Q}(\sqrt{d})$ has its own gold ratio. Under these conditions, it is possible that may arise further research aimed at obtaining properties, both algebraic and geometric, related with the intrinsic properties of the real quadratic field $\mathbb{Q}(\sqrt{d})$.

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