ASYMPTOTIC PERIOD RELATIONS FOR JACOBIAN ELLIPTIC SURFACES

N. I. SHEPHERD-BARRON

Abstract

We find an asymptotic description of the period locus of simply connected Jacobian elliptic surfaces and of the period locus of hyperelliptic curves. The two descriptions are essentially the same, and are given by the alkanes of organic chemistry.

1 Introduction

The classical Schottky problem is that of describing the period locus $\mathfrak{J}_g$ of period matrices of complex algebraic curves of genus g as a subvariety of Siegel space $\mathfrak{H}_g$.

This problem naturally extends to higher dimensions; for example, a simply connected algebraic surface of positive geometric genus g has a period matrix, and then the Schottky problem becomes that of describing the image of the moduli space under the multi-valued period map. This image we refer to as the period locus. As described below, we are only concerned here with local aspects of the geometry of the situation, for which the fact that the period map is multi-valued is irrelevant.

We consider this problem for simply connected Jacobian elliptic surfaces of geometric genus g. In the classification of surfaces these are the simplest beyond those which are covered by K3 or abelian surfaces (and for such surfaces the Schottky problem is well understood). Of course, beyond these lie the surfaces of general type, the well known complexity of whose moduli spaces suggests that it is reasonable to focus on some particular classes of surfaces such as the ones considered here.

The main result (summarized in Theorem 1.1 below) is that the picture for these elliptic surfaces is analogous to that for hyperelliptic curves and that the period loci PL_g of elliptic surfaces and $\mathfrak{H}\mathfrak{y}\mathfrak{p}_g$ of hyperelliptic curves are described by the *alkanes* of elementary organic chemistry. These are the acyclic saturated hydrocarbons and their molecular formula is C_gH_{2g+2}. They were first enumerated by Cayley, whose calculation is given (with some corrections) in [OEIS]. The connexion with algebraic geometry is that on a hyperelliptic curve of genus g the hyperelliptic involution has $2g + 2$ fixed points and that when such curves degenerate to trees of elliptic curves then each elliptic curve E plays the rôle of a carbon atom C whose bonds correspond to the 2-torsion points of E.

In fact, there is a subdomain $\mathcal{W}_{1^g}$ of the period domain for elliptic surfaces which corresponds to trees of g special Kummer surfaces and which is isomorphic to $\mathfrak{H}_1^{g+1}$. We regard this as the analogue of the locus $\mathfrak{Diag}_g$ of diagonal matrices in $\mathfrak{H}_g$, which corresponds to trees of elliptic curves.

Theorem 1.1 *(1) There are one-to-one correspondences between the alkanes $C_g H_{2g+2}$, the branches through $\mathfrak{Diag}_g$ of $\mathfrak{Hyp}_g$ and the branches through $\mathcal{W}_{1^g}$ of PL_g.*

(2) Each of these branches has, to first order, a straightforward and explicit linear description in terms of matrices.

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2   Some further details

We begin by recalling some constructions that involve algebraic curves. Later we shall extend these constructions to include algebraic surfaces.

Fay [F1] constructed certain degenerating families of complex algebraic curves (that is, compact Riemann surfaces) via means of explicit plumbing constructions that are recalled below and then derived formulae for the derivative of the period matrix of each of these families.

Akira Yamada [Y] then pointed out that Fay's formulae are wrong, and gave correct formulae for Fay's constructions.

In [F2] (bottom of page 123), Fay corrects his error by pointing out that his plumbing constructions should have been done differently, and that for these different plumbing constructions his formulae are correct.

In other words, Fay in [F1] in fact made two different plumbing constructions of 1-parameter degenerating families of curves without monodromy (so the curve degenerates but its Jacobian does not) for which there are explicit formulae for the derivative of the period matrix. For one construction the correct formula is that given in [F1] and for the other the correct formula is given in [Y]. (There are also two different plumbing constructions of families of curves with monodromy but we shall not use such constructions in this paper.) We shall refer to the plumbing constructions for which Yamada's formula [Y] is correct as Yamada plumbings, and those for which Fay's formula [F1] is correct as Fay plumbings.

We shall recall the detailed construction of Fay plumbings below. It is convenient to point out here that in our formulae there is a minus sign that does not appear in [F1]; this is because we have chosen a different normalization which slightly increases the flexibility of the construction.

In fact, and this is the crux of this paper, we do this in higher dimensions; one advantage of Fay plumbings is that they can be generalized to plumb not

only curves but also, at least in certain circumstances, morphisms from curves to stacks.

We next recover Fay's version [F1] of Poincaré's "asymptotic period relations". These were discovered by Poincaré [P] when $g = 4$ and generalized by Fay to all values of g. According to Igusa's account (see p. 167 of [I]) Poincaré exploited the geometry of the theta divisor (specifically, that it is a hypersurface of translation type), so his argument cannot extend to the case of surfaces, while Fay uses his plumbing construction. These relations describe, both intrinsically and in terms of co-ordinates, the tangent cone to the closure $\mathfrak{J}_g^c$ of the Jacobian locus along the locus $\mathfrak{Diag}_g$ of diagonal matrices in Siegel space $\mathfrak{H}_g$ in terms of the Grassmannian $Grass(2, g)$.

These Poincaré–Fay asymptotic period relations have also been recovered by Farkas, Grushevsky and Salvati Manni [FGSM] in the course of proving their global weak solution to the Schottky problem. More precisely, they show that differentiating the identities obtained by substituting the Schottky-Jung proportionalities into Riemann's quartic theta identities leads to the Poincaré–Fay relations; they deduce their global weak solution as an immediate consequence.

Then we describe, to first order along $\mathfrak{Diag}_g$, the closure $\mathfrak{Hyp}_g^c$ of the hyperelliptic locus in $\mathfrak{H}_g$. This description is given in terms of the *alkanes* of elementary organic chemistry, which were first enumerated by Cayley; see sequence A000602 in [OEIS] for corrections. These are the hydrocarbons whose molecular formula is of the type $C_g H_{2g+2}$ (and so are exactly the saturated acyclic hydrocarbons) and have uses ranging from fuel to furniture polish, depending on their molecular weight. The g-alkane is the one whose carbon skeleton is a chain of length g. It is distinguished from the others of the same molecular formula by having a higher boiling point. We shall refer to g as the genus of the alkane. The number $2g+2$ is the number of fixed points of the hyperelliptic involution on a hyperelliptic curve of genus g. It turns out that $\mathfrak{Hyp}_g^c$ has one branch along $\mathfrak{Diag}_g$ for each alkane, each branch is smooth and we give explicit first-order equations in terms of the entries τ_{ij} of the period matrix. Here are more precise statements. Recall that a square matrix (a_{ij}) is *tridiagonal* if $a_{ij} = 0$ whenever $|i - j| \geq 2$.

Theorem 2.1 *(a special case of Theorem 9.6) To first order (that is, modulo the square of the defining ideal) the branch of $\mathfrak{Hyp}_g^c$ in $\mathfrak{H}_g$ that corresponds to the g-alkane is the locus of tridiagonal matrices.*

There is a similar, although slightly more complicated, description of the branches corresponding to the other alkanes.

Now turn to surfaces. We use Fay's plumbing to make similar constructions and calculations for degenerating families of simply connected Jacobian elliptic surfaces, which can be regarded as the simplest surfaces of strictly positive Kodaira dimension and also as the simplest surfaces for which, thanks mainly to our understanding of the period map for K3 surfaces, the Schottky problem is not vacuous.

Fix an integer $h \geq 2$ and consider Jacobian elliptic surfaces X over $\mathbb{P}^1$ of geometric genus h. These have $10h + 8$ moduli, and the coarse moduli space is rational. The primitive (co)homology $H = H_{prim}$ of X is the orthogonal complement of the section and a fibre. We can describe a chart of the period domain

$$\mathcal{V}_h = \{\xi \in Grass(h, H_{\mathbb{C}}) : (u, v) = 0 \text{ and } (u, \overline{u}) > 0 \; \forall u, v \in \xi\}$$

as follows.

Pick a totally isotropic sublattice L of H whose rank is h.

Definition 2.2 *The surface X is in L-general position if the pairing $L \otimes \mathbb{C} \times H^0(X, \Omega_X^2) \to \mathbb{C}$ given by integration is non-degenerate.*

Assume that X is in L-general position. Then a basis $(A_1, ..., A_h)$ of L defines a basis $(\omega_1, ..., \omega_h)$ of $H^0(X, \Omega_X^2)$ by the requirement $\int_{A_i} \omega_j = \delta_{ij}$. Extend $(A_1, ..., A_h)$ first to a basis of $L^\perp$ and then to a basis of H such that the induced basis of $H/L^\perp$ is dual to $(A_1, ..., A_h)$ and H is decomposed as

$$H = L \oplus (H/L^\perp)) \oplus (L^\perp/L).$$

Then the normalized period matrix of X is, when we ignore the $h \times h$ identity matrix that arises from integrating around the cycles A_i, an $h \times (11h + 8)$ matrix whose final $h \times h$ block is skew-symmetric. So, if $h = 1$, this last block is zero and can be ignored to yield a vector of length 18. Different isotropic lattices L will give different charts.

Let PL_h, the *period locus*, denote the image of the moduli stack in the period domain $\mathcal{V}_h$ under the period map. We shall recall the definition of the domain of the period map later; the presence of RDPs creates a slight subtlety.

Definition 2.3 *A Jacobian elliptic surface is special if it is birational to a geometric quotient $[C \times E/\iota]$ where C is a hyperelliptic curve, E is an elliptic curve and ι acts on C as its hyperelliptic involution and on E as (-1_E).*

Special surfaces X are characterized as the simply connected Jacobian elliptic surfaces whose global monodromy on the cohomology of the elliptic fibration is of order 2. We have $p_g(X) = g(C)$.

Here are some trivial remarks and some notation.

(1) If $h = \sum h_i$ then the period domain $\mathcal{V}_h$ contains a copy of $\mathcal{V}_{h_1} \times \cdots \times \mathcal{V}_{h_r}$.

(2) Each $\mathcal{V}_{h_i}$ contains a copy $PL_{h_i,special}$ of the period locus of special elliptic surfaces of genus h_i. $PL_{h_i,special}$ is isomorphic to $\mathfrak{Hyp}_{h_i} \times \mathfrak{H}_1$.

(3) If $j \in [0, 4]$ then $\mathcal{K}_j$ is the period domain for Jacobian elliptic K3 surfaces with j fibres of type $\widetilde{D}_4$. So $\mathcal{K}_4 = PL_{1,special}$.

(4) Fix an alkane Γ and let γ_j, for $j \in [1, 4]$, denote the number of vertices (carbon atoms) in Γ that are joined to j other carbon atoms in Γ. Then there is a closed subvariety $\mathcal{V}_\Gamma$ of the product $\mathcal{K}_1^{\gamma_1} \times \cdots \times \mathcal{K}_4^{\gamma_4}$ that is defined by the requirement that the $\widetilde{D}_4$-fibres on K3s that are adjacent in Γ should be isomorphic. Induction shows that $\dim \mathcal{V}_\Gamma = 9h + 9$.

(5) In $PL_{h_1,\ special} \times \cdots \times PL_{h_r,\ special}$, which is isomorphic to $\prod \mathfrak{Hyp}_{h_i} \times \mathfrak{H}_1^r$, there is a subvariety $\mathcal{W}_{h_1,...,h_r}$ defined by the property that the factors in each copy of $\mathfrak{H}_1$ are equal. This is the period locus for unions of special elliptic surfaces of genera $h_1, ..., h_r$ where the relevant elliptic curves are isomorphic.

(6) For each Γ, $\mathcal{W}_{1^h}$ is the subvariety of $\mathcal{V}_\Gamma$ defined by the condition that each K3 surface should be special.

We regard $\mathcal{W}_{1^h}$, which is isomorphic to $(\mathfrak{H}_1)^{h+1}$, as the analogue in the period domain $\mathcal{V}_h$ of the diagonal locus $\mathfrak{Diag}_g$ in $\mathfrak{H}_g$.

The next result is stated in terms of a certain vector bundle $E_\Gamma \to \mathcal{V}_\Gamma$ of rank $h - 1$. The fibre of E_Γ over a point $(Y_1, ..., Y_h)$ of $\mathcal{V}_\Gamma$ is the vector space spanned by certain matrices Π_e of rank 1, where $e = (i, j)$ runs over the edges (the carbon-carbon bonds) of Γ. Each Π_e is a tensor product $\Pi_e = \underline{\omega}_e \otimes \underline{I}_e$ of two vectors, where each vector is computed from the surfaces Y_i and Y_j. The vector $\underline{\omega}_e$ is comprised of projective data while $\underline{I}_e$ is a vector of integrals, so consists of transcendental data. These vectors are described explicitly in Proposition 6.20.

Theorem 2.4 *(= Theorem 11.12) Fix an alkane Γ.*
(1) There is a branch B_Γ of PL_h that contains $\mathcal{V}_\Gamma$.
(2) To first order, B_Γ is, in a neighbourhood of $\mathcal{W}_{1^h}$, the vector bundle E_Γ.
(3) The zero section of E_Γ is $\mathcal{V}_\Gamma$ embedded in B_Γ.
This leads to the main result. It is an analogue of Theorem 9.6.

Theorem 2.5 *(= Theorem 11.13) (1) The branch B_Γ above is the unique branch that contains $\mathcal{V}_\Gamma$.*
(2) In a neighbourhood of $\mathcal{W}_{1^h}$, the period locus PL_h is, to first order, the union of the vector bundles E_Γ.
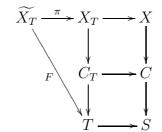
3 The domain of the period map for Jacobian elliptic surfaces

We shall refer to a given Jacobian elliptic surface $f : X \to C$ with section C_0 as smooth if it is smooth and also relatively minimal and as an RDP surface if it has only RDPs, C_0 lies in the smooth locus of f and C_0 is f-ample. That is, if every geometric fibre of f is a reduced and irreducible curve of arithmetic genus 1. They are objects of two of the stacks that we shall consider:

(1) $\mathcal{JE}^{smooth}$ (resp., $\mathcal{JE}_h^{smooth}$) is the stack of smooth Jacobian elliptic surfaces (resp., simply connected such surfaces of geometric genus h);

(2) $\mathcal{JE}^{RDP}$ (resp., $\mathcal{JE}_h^{RDP}$) is the stack of RDP Jacobian elliptic surfaces (resp., simply connected such surfaces of geometric genus h).

So $\mathcal{JE}^{RDP}$ is separated and there is a natural morphism $\mathcal{JE}^{smooth} \to \mathcal{JE}^{RDP}$ given by passing to the relative canonical model. According to Artin's results [Ar74], which we now recall, this morphism is representable, 1-to-1 on field-valued points but not separated.

Suppose that $(X \to C \to S, C_0)$ is an object of $\mathcal{JE}^{RDP}$ over S. Then we have Artin's functor $Res_{X/S}$, whose T-points, for an S-scheme T, are isomorphism classes of diagrams

$$
\begin{array}{ccccc}
\widetilde{X_T} & \xrightarrow{\pi} & X_T & \longrightarrow & X \\
& & \downarrow & & \downarrow \\
F \diagdown & & C_T & \longrightarrow & C \\
& & \downarrow & & \downarrow \\
& & T & \longrightarrow & S
\end{array}
$$

where F is smooth, projective and relatively minimal (this is equivalent to saying that $\widetilde{X_T} \to C_T$ is an object of $\mathcal{JE}^{smooth}$), and π is projective and birational, in the sense that $\pi_*\mathcal{O} = \mathcal{O}$. Then $Res_{X/S}$ is represented by a locally quasi-separated algebraic space R over S, such that $R \to S$ is a bijection on all field-valued points. The morphism $\mathcal{JE}^{smooth} \to \mathcal{JE}^{RDP}$ described above can be localized to yield an isomorphism $\mathcal{JE}^{smooth} \times_{\mathcal{JE}^{RDP}} S \xrightarrow{\cong} R$.

Now restrict attention to simply connected Jacobian elliptic surfaces of geometric genus $p_g = h$ and denote the corresponding stacks by $\mathcal{JE}_h^{smooth}$ and $\mathcal{JE}_h^{RDP}$. Fix a unimodular lattice $\Lambda = \Lambda_h$ of rank $12h + 10$ and signature $(2h + 1, 10h + 9)$ and elements $\sigma, \phi \in \Lambda$ such that $\sigma^2 = -(h + 1), \sigma.\phi = 1$ and $\phi^2 = 0$. So $\Lambda = \mathbb{Z}\{\sigma, \phi\} \perp H$ where $H = H_h$ is unimodular, its rank is $12h + 8$ and its signature is $(2h, 10h + 8)$. Assume also that H is even; these requirements specify Λ_h and H_h uniquely. Consider the subgroup $\mathfrak{G}$ of the orthogonal group $O_\Lambda(\mathbb{Z})$ given by

$$\mathfrak{G} = \{\gamma \in O_\Lambda(\mathbb{Z}) | \gamma(\sigma) = \sigma \text{ and } \gamma(\phi) = \phi\};$$

then $\mathfrak{G}$ is naturally identified with $O_H(\mathbb{Z})$. There is a $\mathfrak{G}$-torsor $\mathcal{JE}_\Lambda \to \mathcal{JE}_h^{smooth}$ defined by the fact that a T-point of $\mathcal{JE}_\Lambda$ consists of a T-point of $\mathcal{JE}_h^{smooth}$ and an isometry $\Psi : \Lambda_T \to R^2 F_* \mathbb{Z}$ such that Ψ maps σ to the class of the section $(C_0)_T$ of $\widetilde{X_T} \to C_T$ and ϕ to the class of a fibre.

The discussion in the third paragraph on p. 228 of [C1] can be translated into the language of stacks to say that $\mathcal{JE}_\Lambda$ is the domain of the period map. That

is, the period map is a $\mathfrak{G}$-equivariant holomorphic morphism

$$\widetilde{per} : \mathcal{JE}_\Lambda \to \mathcal{V}_h,$$

where $\mathcal{V}_h$ is the period domain. Equivalently, the period map is the quotient morphism

$$per : \mathcal{JE}_h^{smooth} = \mathcal{JE}_\Lambda/\mathfrak{G} \to \mathcal{V}_h/\mathfrak{G}$$

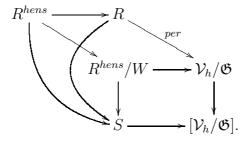of quotient stacks. This fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{JE}_h^{smooth} & \xrightarrow{per} & \mathcal{V}_h/\mathfrak{G} \\ \downarrow & & \downarrow \\ \mathcal{JE}_h^{RDP} & \longrightarrow & [\mathcal{V}_h/\mathfrak{G}]. \end{array}$$
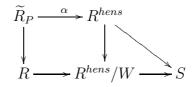
If S is the henselization of $\mathcal{JE}_h^{RDP}$ at a closed point, so that S is the base of a miniversal deformation of an RDP Jacobian elliptic surface X_s, then [Ar74] the henselization R^{hens} of $R = \mathcal{JE}_h^{smooth} \times_{\mathcal{JE}_h^{RDP}} S$ at its unique closed point is the base of a miniversal deformation of the minimal resolution $\widetilde{X}_s$ of X_s. (In fact Artin shows that R^{hens} is the base of a versal deformation, but in characteristic zero his argument proves the miniversality of R^{hens}.) So, for local purposes such as those of this paper, we can take the domain of the period map to be a miniversal deformation space of a smooth Jacobian elliptic surface.

In fact, slightly more is true. The finite Weyl group W associated to the configuration of singularities on X_s acts on R^{hens} in such a way that $[R^{hens}/W] = S$ and there is a commutative diagram

$$\begin{array}{ccc} R^{hens} & \longrightarrow & R \\ & & \\ & R^{hens}/W & \xrightarrow{per} & \mathcal{V}_h/\mathfrak{G} \\ & \downarrow & & \downarrow \\ & S & \longrightarrow & [\mathcal{V}_h/\mathfrak{G}]. \end{array}$$

This is because there is a non-separated union $\widetilde{R}_P = \cup_\mathcal{C} R_\mathcal{C}^{hens}$ of copies of R^{hens} on which W acts freely, where there is one copy $R_\mathcal{C}^{hens}$ of R^{hens} for each chamber $\mathcal{C}$ that defined in the usual way and two charts $R_\mathcal{C}^{hens}, R_\mathcal{E}^{hens}$ in $\widetilde{R}_P$ are glued in a way that depends upon the relative position of the chambers $\mathcal{C}, \mathcal{E}$ with respect to W. (This glueing always induces an isomorphism over the complement of the discriminant locus in S.) Moreover W permutes the charts $R_\mathcal{C}^{hens}$ simply transitively, and $\widetilde{R}_P/W = \widetilde{\mathcal{JE}}_h \times_{\mathcal{JE}_h} S = R$. The isomorphisms $R_\mathcal{C}^{hens} \to R^{hens}$ glue to a morphism $\alpha : \widetilde{R}_P \to R^{hens}$ that exhibits R^{hens} as the maximal separated

quotient of $\widetilde{R}_P$ in the category of schemes. So W also acts on R^{hens}, α is W-equivariant and $S = [R^{hens}/W]$. All this is documented in [SB17], to which we add one comment: since, in the commutative diagram

$$
\begin{array}{ccc}
\widetilde{R}_P & \xrightarrow{\ \alpha\ } & R^{hens} \\
\downarrow & & \downarrow \\
R & \longrightarrow & R^{hens}/W \longrightarrow S
\end{array}
$$

the vertical maps are quotients by W, so that the square is 2-Cartesian, it follows that the scheme S is the maximal separated quotient of the algebraic space R in the category of algebraic spaces and that the stack R^{hens}/W is the maximal separated quotient of R in the 2-category of algebraic stacks.

Finally, consider the forgetful morphism $\alpha : S \to \prod_x Def_{X_s,x}$ from S to the product of the miniversal deformation spaces of the germs (X_s, x) of X_s at its singularities x and the morphisms $Def_{\widetilde{X}_s,x} \to Def_{X_s,x}$. These latter are geometric quotients by the relevant finite Weyl group and R^{hens} is isomorphic to the fibre product

$$
S \times_{\prod_x Def_{X_s,x}} \prod_x Def_{\widetilde{X}_s,x},
$$

so that if α is smooth then so is R^{hens}.

4 Fay's plumbing for curves and stacky curves

In this section we give a detailed description of Fay plumbings. All of this is taken from [F1] and [F2] but we repeat it here to avoid confusion.

Fix a real number $\delta > 0$. Let $\Delta = \Delta_t$ be the open disc in the complex plane $\mathbb{C}$ with co-ordinate t defined by $|t| < \delta^2$ and let F be the open submanifold of $\mathbb{C}^2$ with co-ordinates q, v defined by $|q| < \delta^{1/2}$, $|v| < \delta$ and $|v \pm q| < \delta$ (so that, in particular, $|v^2 - q^2| < \delta^2$). Set $t = v^2 - q^2$. Note that the morphism $t : F \to \Delta$ is smooth outside the origin in Δ and the fibre $t^{-1}(0)$ consists of two discs U'_a and U'_b, where U'_a is given by $q - v = 0$ and U'_b by $q + v = 0$. These discs cross normally at the point 0 given by $q = v = 0$.

We shall use F, with these co-ordinates, as the basic plumbing fixture.

Now suppose that C_a, C_b are curves of genus g_a, g_b, that $a \in C_a, b \in C_b$ and that if $i = a, b$ then we have chosen a local co-ordinate z_i on some simply connected neighbourhood U_i of i in C_i such that U_i is defined by the inequality $|z_i| < \delta^{1/2}$. In particular, U_i is an open disc that is embedded in C_i.

The suffix i will stand for either a or b in what follows.

Let U_i^c denote the closure of U_i in C_i. We shall assume that each U_i^c is a (closed) disc; this is slightly stronger than assuming U_i is an open disc. We shall

assume also *either* that C_a and C_b are distinct *or* that $C_a = C_b$ and the closed discs U_a^c and U_b^c are disjoint.

Of course, all of these assumptions can be fulfilled after decreasing δ if necessary.

In F we have open subsets V_a and V_b defined, respectively, by the inequalities

$$
\begin{aligned}
|v - q| &< |q|, \ |t| < \delta|q|^2 \text{ and} & (4.1) \\
|v + q| &< |q|, \ |t| < \delta|q|^2. & (4.2)
\end{aligned}
$$

Note that V_a and V_b are disjoint and that V_i contains $U_i' - \{0\}$.

Define $\pi_i : V_i \to U_i \times \Delta$ by $z_i = q$, $t = q^2 - v^2$. It is easy to check that π_i is unramified (since the ramification locus is defined by $v = 0$) and is injective. Therefore π_i is an isomorphism from V_i to its image Y_i, which is open in $U_i \times \Delta$ and so open in $C_i \times \Delta$.

Let Y_i denote the image of V_i. Then Y_i is contained in the region $|t| < \delta|z_i|^2$ and contains $(U_i - \{i\}) \times \{0\}$. Note that π_i maps $U_i' - \{0\}$ isomorphically onto $U_i - \{i\}$.

In $C_i \times \Delta$ consider the closed subset $\{(P, t) \ : \ P \in U_i^c \text{ and } (P, t) \notin Y_i\}$. Define W_i to be the open subset of $C_i \times \Delta$ obtained by deleting this closed set.

Note that $W_i \cap (C_i \times \{0\}) = C_i - \{i\}$.

Proposition 4.3 *We can glue the three charts W_a, W_b and F via the identifications $V_i \xrightarrow{\cong} Y_i$ to get a separated 2-dimensional complex manifold $\mathcal{C}$ with a proper holomorphic map $\pi : \mathcal{C} \to \Delta_t$ such that $\pi^{-1}(0) = C_a \cup_{a \sim b} C_b$, where C_a and C_b cross normally at a single point.*

PROOF: This is routine. $\square$

We call this a *Fay plumbing* because it is, implicitly, constructed and considered by Fay in the final paragraph of p. 37 of [F1].

If C_a and C_b are distinct then for all $t \neq 0$ the fibre $\mathcal{C}_t$ is a curve of genus $g = g_a + g_b$ and there is no monodromy on its cohomology. This is what Fay calls "pinching a cycle homologous to zero" [F1], p. 37 *et seq.* We shall call it a *homologically trivial Fay plumbing of C_a to C_b that identifies a with b* or just a *Fay plumbing of C_a to C_b* without explicit mention of the points or co-ordinates that are chosen.

If $C_a = C_b$, of genus g, and also $U_a^c \cap U_b^c = \emptyset$ (so that the plumbing is possible), then for all $t \neq 0$ the fibre $\mathcal{C}_t$ is of genus $g + 1$, $\mathcal{C}_0$ is the nodal curve $C/a \sim b$ and there is non-trivial monodromy on the cohomology of $\mathcal{C}_t$. Fay calls this "pinching a non-zero homology cycle" [F1], p. 50.

Note that on $F \cap W_i$ we have $z_i = q$.

In a homologically trivial Fay plumbing the derivative at $t = 0$ of the period matrix of $\mathcal{C}_t$ is given as Corollary 3.2 on p. 41 of [F1]. We shall prove this, or, rather, a version of it in a higher-dimensional context, later on.

When a non-zero homology cycle is pinched in a Fay plumbing the period matrix is described by by Corollary 3.8 on p. 53 of [F1].

However, the families that are written down on pp. 37 and 50 of *loc. cit.*, which we call the *Yamada plumbing* because they are considered explicitly by Yamada, are given by the glueing $t = z_a z_b$ and (provided that $g_a + g_b \geq 1$) are different families because, for example, their period matrices are different. Their expansions at $t = 0$ are given as Corollary 2 on p. 129 and Corollary 6 on pp. 137-138 of [Y].

In this paper we shan't have any need to pinch cycles that are not homologous to zero. For one application of such pinchings, however, see [CSB13]. Note that there Yamada plumbings are used, although an argument could also be based on the use of Fay plumbings.

It will also be useful to be able to plumb certain stacky curves.

Definition 4.4 *A $\mathbb{Z}/2$-curve is a reduced connected proper 1-dimensional Deligne–Mumford stack $\widetilde{C}$ such that at all geometric points $\widetilde{C}$ has at worst nodes, at every generic point the stabilizer (the isotropy group) is trivial, at every point the stabilizer is a subgroup of $\mathbb{Z}/2$ and that if P is a node with non-trivial stabilizer then the stabilizer preserves each branch of the node.*

For example, if $i \in C_i$ then there is a unique $\mathbb{Z}/2$-curve $\widetilde{C}_i$ such that the geometric quotient $[\widetilde{C}_i] = C_i$, the quotient map $\widetilde{C}_i \to C_i$ is an isomorphism over $C_i - \{i\}$ and there is a unique geometric point on $\widetilde{C}_i$ lying over i. There are local co-ordinates $\tilde{z}_i$ and z_i on $\widetilde{C}_i$ and on C_i, respectively, such that the non-trivial element ι of $\mathbb{Z}/2$ acts via $\iota^* \tilde{z}_i = -\tilde{z}_i$ and $z_i = \tilde{z}_i^2$.

Take the plumbing fixture F, with co-ordinates q, v as before, and let ι act on F by $\iota^* q = -q$, $\iota^* v = -v$. So the quotient stack $\widetilde{F} = F/\langle \iota \rangle$ is a smooth separated 2-dimensional Deligne–Mumford stack.

Suppose $\widetilde{C}_i$, for $i = a, b$, are $\mathbb{Z}/2$-curves, that i is a smooth point of $\widetilde{C}_i$ for each i and that each of a, b the stabilizer is $\langle \iota \rangle \cong \mathbb{Z}/2$. Fix a local co-ordinate $\tilde{z}_i$ on $\widetilde{C}_i$ at i such that $\iota^* \tilde{z}_i = -\tilde{z}_i$.

Proposition 4.5 *There are suitable open substacks $\widetilde{\mathcal{W}}_i$ of $\widetilde{C}_i \times \Delta$ such that the charts $\widetilde{\mathcal{W}}_a$, $\widetilde{\mathcal{W}}_b$ and $\widetilde{F}$ can be glued via the formulae $\tilde{z}_i = q$ and $t = q^2 - v^2$ to get a smooth 2-dimensional Deligne–Mumford stack $\widetilde{\mathcal{C}}$ with a proper separated morphism $\widetilde{\mathcal{C}} \to \Delta_t$.*

PROOF: As before.   □

For future reference, put $\Delta' = \mathbf{Spec}\,\mathbb{C}[t]/(t^2)$, $\widetilde{\mathcal{C}}' = \widetilde{\mathcal{C}} \times_\Delta \Delta'$ and $\widetilde{\mathcal{W}}_i' = \widetilde{\mathcal{W}}_i \times_\Delta \Delta'$.

Taking geometric quotients gives the following result. Define $G = [F/\langle \iota \rangle]$.

Proposition 4.6 *The curves C_i can be plumbed via the plumbing fixture G to $\mathcal{B} \to \Delta_t$. On $\mathcal{B}$ there is an A_1-singularity at 0 and $\mathcal{B}_0 = C_a \cup C_b/a \sim b$.*

PROOF: This is clear.   □

The minimal resolution of $\mathcal{B}$ gives a semi-stable family of curves over Δ_t.

5 Plumbing families of curves

The plumbing construction of the previous section can be extended so as to plumb families of curves.

So suppose that $C_a \to S_a$, $C_b \to S_b$ are analytic families of semi-stable curves of genera g_a and g_b, respectively, with sections $\sigma_i : S_i \to C_i$ each of which lies in the relevant smooth locus. Assume also that U_i is a tubular neighbourhood of the section $i = \sigma_i(S_i)$ and that z_i is a fibre co-ordinate on U_i such that U_i is isomorphic to $S_i \times \Delta_{z_i}$, where Δ_{z_i} is the disc with co-ordinate z_i such that $|z_i| < \delta$. Such a neighbourhood and such a co-ordinate will exist if each S_i is sufficiently small, for example a small polydisc.

Take the same 2-dimensional plumbing fixture F and disjoint open subsets V_i as before and consider the morphisms $\pi_i : V_i \times S_i \to U_i \times \Delta_t$ defined by

$$\pi_i(q, v, s_i) = (\sigma_i(s_i), z_i, t)$$

where $z_i = q$ and $t = q^2 - v^2$.

As before, π_i is an isomorphism to an open analytic subvariety Y_i of $U_i \times \Delta_t$.

In $C_i \times \Delta_t$ consider the closed subset that is the intersection of $U_i^c \times \Delta_t$ and the complement of Y_i, and define W_i to be the open subvariety of $C_i \times \Delta_t$ obtained by deleting this closed subset.

Now glue the chart $W_a \times S_b$ to $F \times S_a \times S_b$ via the isomorphism $V_a \times S_a \times S_b \to Y_a \times S_b$, and glue $W_b \times S_a$ to $F \times S_a \times S_b$ via the isomorphism $V_b \times S_a \times S_b \to Y_b \times S_a$.

The result of this glueing is a analytic space $\mathcal{C}$ with a proper morphism $\mathcal{C} \to S_a \times S_b \times \Delta_t$ that is a family of semi-stable curves of genus $g_a + g_b$. If $D_i \subset S_i$ is the discriminant locus of $C_i \to S_i$ then the discriminant locus of $\mathcal{C} \to S_a \times S_b \times \Delta_t$ is $D_a \times S_b \times \Delta_t \cup D_b \times S_a \times \Delta_t \cup S_a \times S_b \times \{0\}$.

We can concatenate plumbings as follows: suppose that $C_i \to S_i$ are families of stable curves for $i = a, b, c$ and that $\sigma_a, \sigma_{b_1}, \sigma_{b_2}, \sigma_c$ are sections of C_a, C_b, C_b, C_c respectively and that σ_{b_1} and σ_{b_2} are disjoint.

Then there are two choices: we can first plumb C_a to C_b, obtaining $\mathcal{C}' \to S_a \times S_b \times \Delta$, in a way that identifies the sections σ_a and σ_{b_1}, and then plumb $\mathcal{C}'$ to C_c to get a family over $S_a \times S_b \times \Delta \times S_c \times \Delta$, or we can first plumb C_b to C_c, obtaining $\mathcal{C}''$, by identifying σ_{b_2} with σ_c, and then plumb $\mathcal{C}''$ to C_a to get another family over $S_a \times S_b \times \Delta \times S_c \times \Delta$. It will be important for us to notice that these two families are the same; that is, the final result is independent of the order of the plumbings.

Similarly we can construct stacks $\widetilde{C}_i \to S_i$ by introducing isotropy groups $\mathbb{Z}/2$ along each section $\sigma(S_i)$ and then plumb together the stacks $\widetilde{C}_i$.

6 Homologically trivial plumbings of surfaces

Here we construct families of Jacobian elliptic surfaces that degenerate in a homologically trivial fashion, so that the period matrix specializes to a matrix lying in the interior of the period domain and there is an explicit formula for the derivative of the period matrix that involves no derivatives but is instead a matrix of rank one, just as for curves.

This leads to a result analogous to the asymptotic relations derived earlier for the locus of hyperelliptic curves.

We begin with a modification of Kodaira's degenerate fibres and his notation for them.

(1) $I_{n-4}^* = \widetilde{D}_n$ for $n \geq 4$: contract the four (-2) curves at the extremities and call the result $\overline{D}_n$. This has four A_1 singularities.

(2) $II = Cu$: blow up the cusp three times, then contract the resulting $(-2), (-3)$ and (-6) curves, and call the result $\overline{II}$. This has three singularities, one of each type $\frac{1}{2}(1,1) = A_1, \frac{1}{3}(1,1)$ and $\frac{1}{6}(1,1)$.

(3) $III = Ta$: blow up the tacnode twice, then contract the resulting (-2) curve and two (-4) curves and call the result $\overline{III}$. This has one A_1 singularity and two singularities of type $\frac{1}{4}(1,1)$.

(4) $IV = Tr$: blow up the triple point, then contract the three resulting (-3) curves and call the result $\overline{IV}$. This has three singularities of type $\frac{1}{3}(1,1)$.

(5) $II^* = \widetilde{E}_8, III^* = \widetilde{E}_7$ and $IV^* = \widetilde{E}_6$: contract all curves except the central one and call the result $\overline{R}^*$ for $R = II, III, IV$. This has three singularities, all of type A.

Now suppose that $Y_i \to C_i$ are smooth simply connected Jacobian elliptic surfaces, each of which has a fibre ϕ_i (with its structure as a scheme) of type $\widetilde{D}_4$ over i. Let $\pi_i : Y_i \to \overline{Y}_i$ denote the contraction of the four (-2)-curves at the extremities, so that $\overline{Y}_i \to C_i$ has a fibre $\overline{\phi}_i$, with its structure as a scheme, of type $\overline{D}_4$. So $\overline{\phi}_i$ is a copy of $\mathbb{P}^1$ but with multiplicity 2. Say $p_g(Y_i) = h_i$.

Let $\mathcal{E}\ell\ell^{Kod}$ denote the stack of Kodaira fibres, so that $Y_i \to C_i$ defines a classifying morphism $C_i \to \mathcal{E}\ell\ell^{Kod}$.

Lemma 6.1 *For each i there is a smooth $\mathbb{Z}/2$-curve $\widetilde{C}_i$ and a smooth proper Deligne–Mumford surface $\widetilde{Y}_i$ with a projective (in particular, representable) morphism $\widetilde{Y}_i \to \widetilde{C}_i$ such that*

(1) the geometric quotients $[\widetilde{C}_i]$ and $[\widetilde{Y}_i]$ are $[\widetilde{C}_i] = C_i$ and $[\widetilde{Y}_i] = \overline{Y}_i$,

(2) $\widetilde{C}_i \to C_i$ is an isomorphism outside i and the stabilizer group at the unique point $\tilde{i}$ of $\widetilde{C}_i$ that lies over i is $\mathbb{Z}/2$,

(3) $\widetilde{Y}_i \to \widetilde{C}_i$ is smooth over $\tilde{i}$ and the fibre $\widetilde{\phi}_{\tilde{i}}$ is $E/(-1)$ for some elliptic curve E,

(4) the quotient morphism $\rho_i : \widetilde{Y}_i \to \overline{Y}_i$ is an isomorphism outside the four points that map to the four nodes on $\overline{Y}_i$, where the stabilizer group is $\mathbb{Z}/2$,

(5) the morphism $\widetilde{Y}_i \to \overline{Y}_i \times_{C_i} \widetilde{C}_i$ is finite and birational and

(6) the induced morphisms $\widetilde{C}_i \to \mathcal{E}\ell\ell^{Kod}$ are isomorphic to first order at a, b.

PROOF: Everything follows from the basic property of a $\overline{D}_4$ fibre, which is that, if $B_i \to C_i$ is a double cover that is ramified at i, then the normalization of $\overline{Y}_i \times_{C_i} B_i$ is smooth over B_i at i. $\qquad\square$

Lemma 6.2 *(1) There are natural isomorphisms*

$$
\begin{aligned}
\pi_i^* \omega_{\overline{Y}_i} &\xrightarrow{\cong} \Omega^2_{Y_i}, \\
\pi_i^* \omega_{\overline{Y}_i}(\overline{\phi}_i) &\xrightarrow{\cong} \Omega^2_{Y_i}(\phi_i), \\
\rho_i^* \omega_{\overline{Y}_i} &\xrightarrow{\cong} \Omega^2_{\widetilde{Y}_i} \text{ and} \\
\rho_i^* \omega_{\overline{Y}_i}(\phi_i) &\xrightarrow{\cong} \Omega^2_{\widetilde{Y}_i}(2\widetilde{\phi}_{\tilde{i}}).
\end{aligned}
$$

(2) These isomorphisms induce isomorphisms

$$
\begin{aligned}
\pi_i^* : H^0(\overline{Y}_i, \omega_{\overline{Y}_i}) &\xrightarrow{\cong} H^0(Y_i, \Omega^2_{Y_i}), \\
\pi_i^* : H^0(\overline{Y}_i, \omega_{\overline{Y}_i}(\overline{\phi}_i)) &\xrightarrow{\cong} H^0(Y_i, \Omega^2_{Y_i}(\phi_i)), \\
\rho_i^* : H^0(\overline{Y}_i, \omega_{\overline{Y}_i}) &\xrightarrow{\cong} H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}) \text{ and} \\
\rho_i^* : H^0(\overline{Y}_i, \omega_{\overline{Y}_i}(\overline{\phi}_i)) &\xrightarrow{\cong} H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}(2\widetilde{\phi}_{\tilde{i}})).
\end{aligned}
$$

PROOF: This is a well known local calculation depending on the fact that the non-trivial element ι of a stabilizer group acts trivially on a local generator $d\tilde{z}_i \wedge dw_i$ of $\Omega^2_{\widetilde{Y}_i}$ where w_i is a fibre co-ordinate. $\qquad\square$

Now assume that the two $\overline{D}_4$ fibres are isomorphic. That is, two cross-ratios are equal.

Corollary 6.3 *Choose local co-ordinates $\tilde{z}_i$ on $\widetilde{C}_i$ at $\tilde{i}$ such that $\iota^* \tilde{z}_i = -\tilde{z}_i$ where ι is a generator of the stabilizer of the point $\tilde{i}$.*

(1) In terms of these local co-ordinates there is a Fay plumbing of the stacks $\widetilde{Y}_i$ modulo t^2 via the morphisms $\widetilde{C}_i \to \mathcal{E}\ell\ell^{Kod}$ to give a Deligne–Mumford stack $\widetilde{\mathcal{Y}}'$ with a flat morphism $\widetilde{\mathcal{Y}}' \to \Delta'$.

(2) If $\widetilde{\mathcal{Y}} \to \Delta$ is any lifting of $\widetilde{\mathcal{Y}}' \to \Delta'$ then $\widetilde{\mathcal{Y}}$ is a smooth 3-dimensional Deligne–Mumford stack, whose only non-trivial stabilizers are copies of $\mathbb{Z}/2$ at each of the four 2-torsion points of $\widetilde{Y}_a \cap \widetilde{Y}_b$ (which is isomorphic to the quotient E/ι of an elliptic curve E by (-1)).

(3) $p_g(\widetilde{\mathcal{Y}}_t) = p_g(Y_a) + p_g(Y_b)$ *and there is no monodromy on* $H^2(\widetilde{Y}_t, \mathbb{Z})$.

PROOF: (1): by Lemma 6.1 (6) the classifying morphisms $\widetilde{C}_i \to \mathcal{E}\ell\ell^{Kod}$ are isomorphic to first order at the points a, b. That is, there is a 2-commutative diagram

$$U'_a = \mathbf{Spec}\,\mathbb{C}[[\tilde{z}_a]]/(\tilde{z}_a^2)/\iota \xrightarrow[\cong]{\alpha} \mathbf{Spec}\,\mathbb{C}[[\tilde{z}_b]]/(\tilde{z}_b^2)/\iota = U'_b$$

$$\phi'_a \searrow \qquad \swarrow \phi'_b$$

$$\mathcal{E}\ell\ell^{Kod}$$

where α is an ι-equivariant isomorphism such that $\alpha^* \tilde{z}_b = \tilde{z}_a$. Suppose that $(w_1, ..., w_n)$ are local co-ordinates on some smooth chart $X \to \mathcal{E}\ell\ell^{Kod}$ and that a local lift $\widetilde{\phi}'_i : U'_i \to X$ of ϕ'_i is given, in terms of local co-ordinates, by a formula

$$\widetilde{\phi}'_i(\tilde{z}_i) = (\phi'_{i,1}(\tilde{z}_i), ..., \phi'_{i,n}(\tilde{z}_i)).$$

By hypothesis $\widetilde{\phi}'_a(\tilde{z}_a)$ is equivalent to $\widetilde{\phi}'_b(\tilde{z}_b)$, so that we can define $\Psi : \widetilde{F}' \to \mathcal{E}\ell\ell^{Kod}$ by

$$\Psi'(q, v) = (\phi'_{a,1}(q), ..., \phi'_{a,n}(q)) \sim (\phi'_{b,1}(q), ..., \phi'_{b,n}(q)).$$

Then define $\Phi' : \widetilde{\mathcal{W}}'_i \to \mathcal{E}\ell\ell^{Kod}$ as $\Phi'_i = \phi'_i \circ pr_i$ where $pr_i : \widetilde{\mathcal{W}}'_i \to \widetilde{C}_i$ is the projection. Since the morphisms Ψ' and Φ'_i agree on the overlaps (1) is proved.

(2): in local co-ordinates the morphism $\widetilde{\mathcal{Y}}' \to \Delta'$ is given by $t = xy$, which proves smoothness. The rest is immediate.

(3): a holomorphic 2-form ω_t on $\widetilde{Y}_t$ will specialize to a pair (ω_a, ω_b) where ω_i lies in $H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}(E/\iota))$ Since ι acts as (-1) on E while ω_i is ι-invariant the residue of ω_i along E/ι is zero, so ω_i is holomorphic. The absence of monodromy is a well known consequence. □

Now suppose that there is a lifting

$$\widetilde{\mathcal{Y}} \to \Delta, \tag{6.4}$$

and fix one. Then there is a C^0 collapsing map $\gamma : \widetilde{\mathcal{Y}}_t \to \widetilde{\mathcal{Y}}_0$, in the context of orbifolds, unique up to homotopy. There is a totally isotropic sublattice L of $H_2(\widetilde{\mathcal{Y}}_t, \mathbb{Z})$ with a basis $(A_1, ..., A_{h_a}, A_{h_a+1}, ..., A_{h_a+h_b})$ such that the image of $(A_1, ..., A_{h_a})$ under γ_* is a basis of an isotropic lattice L_a in $H_2(\widetilde{Y}_a, \mathbb{Z})$ and the image of $(A_{h_a+1}, ..., A_{h_a+h_b})$ is a basis of an isotropic lattice L_b in $H_2(\widetilde{Y}_b, \mathbb{Z})$. We shall assume that each $\widetilde{Y}_i$ is in L_i-general position.

Lemma 6.5 $\widetilde{\mathcal{Y}}_t$ *is in L-general position for* $t \neq 0$.

PROOF: If not, then there is a holomorphic 2-form ω_t on $\widetilde{Y}_t$ such that $\int_A \omega_t = 0$ for every $A \in L$. By the argument used in the proof of Lemma 6.3 3 ω_t specializes

to a pair (ω_a, ω_b), where ω_i is holomorphic on $\widetilde{Y}_i$ and $\int_{A_i} \omega_i = 0$ for every $A_i \in L_i$, and the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Poincaré duality on $\widetilde{Y}_i$ identifies $H_2(\widetilde{Y}_i, \mathbb{Z})$ with a subgroup of $H^2(\widetilde{Y}_i, \mathbb{Z})$ whose quotient is 2-elementary. Correspondingly, by Lemma 6.5, there is a unique normalized basis $(\omega^{(1)}(t), ..., \omega^{(h)}(t))$ of $H^0(\widetilde{\mathcal{Y}}_t, \Omega^2_{\widetilde{\mathcal{Y}}_t})$ such that

$$\int_{A_l} \omega^{(j)}(t) = \delta_l^j. \qquad (6.6)$$

Also, there are holomorphic 3-forms $\Omega^{(j)}$ on the smooth 3-fold stack $\widetilde{\mathcal{Y}}$ such that

$$\omega^{(j)}(\lambda) = \mathrm{Res}_{\widetilde{\mathcal{Y}}_\lambda} \Omega^{(j)}/(t - \lambda)$$

for every j and λ. We can expand $\Omega^{(j)}$ on the inverse image in $\widetilde{Y}$ of the plumbing fixture $\widetilde{F}$ as

$$\Omega^{(j)} = \sum_{m,n \geq 0} c_{m,n}^{(j)} q^m v^n dq \wedge dv \wedge dw,$$

so that, since $t \equiv q^2 - v^2 \pmod{t^2}$,

$$\omega^{(j)}(t) \equiv -\tfrac{1}{2} \sum_{m,n \geq 0} c_{m,n}^{(j)} q^m v^{n-1} dq \wedge dw.$$

Here and from now on all congruences are taken modulo t^2.

In a neighbourhood of $\widetilde{Y}_a - (\widetilde{Y}_a \cap \widetilde{Y}_b)$ we have, from the construction of the plumbing described in (4.1), (4.2) and the discussion immediately afterwards,

$$v \equiv q(1 - tq^{-2})^{1/2},$$

so that

$$v^{n-1} = q^{n-1}(1 - tq^{-2})^{(n-1)/2} \equiv q^{n-1}(1 - (n-1)tq^{-2}/2).$$

Therefore $\omega^{(j)}(t)|_{\widetilde{Y}_a} \equiv \omega^{(j)}_{\widetilde{Y}_a} + t\eta^{(j)}_{\widetilde{Y}_a}$ where

$$\begin{aligned}
\omega^{(j)}_{\widetilde{Y}_a} &= -\tfrac{1}{2} \sum c_{m,n}^{(j)} q^{m+n-1} dq \wedge dw \text{ and} \\
\eta^{(j)}_{\widetilde{Y}_a} &= \tfrac{1}{4} \sum c_{m,n}^{(j)} (n-1) q^{m+n-3} dq \wedge dw.
\end{aligned}$$

On $\widetilde{Y}_a$ we have $q = \tilde{z}_a$ and $w = w_a$, so that

$$\begin{aligned}
\omega^{(j)}_{\widetilde{Y}_a} &= -\tfrac{1}{2} \sum_{p \geq 0} \left(\sum_{m+n=p} c_{m,n}^{(j)} \right) \tilde{z}_a^{p-1} d\tilde{z}_a \wedge dw_a \text{ and} \\
\eta^{(j)}_{\widetilde{Y}_a} &= \tfrac{1}{4} \sum_{p \geq 0} \left(\sum_{m+n=p} (n-1) c_{m,n}^{(j)} \right) \tilde{z}_a^{p-3} d\tilde{z}_a \wedge dw_a.
\end{aligned}$$

In a neighbourhood of $\widetilde{Y}_b - (\widetilde{Y}_a \cap \widetilde{Y}_b)$, on the other hand, we have $v \equiv -q(1-tq^{-2})^{1/2}$ and on $\widetilde{Y}_b$ we have $q = \tilde{z}_b$ and $w = w_b$, so that $\omega^{(j)}(t)|_{\widetilde{Y}_b} \equiv \omega^{(j)}_{\widetilde{Y}_b} + t\eta^{(j)}_{\widetilde{Y}_b}$ where

$$\omega^{(j)}_{\widetilde{Y}_b} = -\tfrac{1}{2}\sum_{p\geq 0}\left(\sum_{m+n=p}(-1)^{n-1}c^{(j)}_{m,n}\right)\tilde{z}_b^{p-1}d\tilde{z}_b \wedge dw_b \text{ and}$$

$$\eta^{(j)}_{\widetilde{Y}_b} = \tfrac{1}{4}\sum_{p\geq 0}\left(\sum_{m+n=p}(-1)^{n-1}(n-1)c^{(j)}_{m,n}\right)\tilde{z}_b^{p-3}d\tilde{z}_b \wedge dw_b.$$

The involution ι acts via $\iota^*\tilde{z}_i = -\tilde{z}_i$ and $\iota^*w_i = -w_i$, so that, since each $\omega^{(j)}_{\widetilde{Y}_i}$ and $\eta^{(j)}_{\widetilde{Y}_i}$ is ι-invariant, we need only consider odd values of the index p in the expansions above. That is,

$$\omega^{(j)}_{\widetilde{Y}_a} = -\tfrac{1}{2}\sum_{r\geq 0}\left(\sum_{m+n=2r+1}c^{(j)}_{m,n}\right)\tilde{z}_a^{2r}d\tilde{z}_a \wedge dw_a, \tag{6.7}$$

$$\eta^{(j)}_{\widetilde{Y}_a} = \tfrac{1}{4}\sum_{r\geq 0}\left(\sum_{m+n=2r+1}(n-1)c^{(j)}_{m,n}\right)\tilde{z}_a^{2r-2}d\tilde{z}_a \wedge dw_a, \tag{6.8}$$

$$\omega^{(j)}_{\widetilde{Y}_b} = -\tfrac{1}{2}\sum_{r\geq 0}\left(\sum_{m+n=2r+1}(-1)^{n-1}c^{(j)}_{m,n}\right)\tilde{z}_b^{2r}d\tilde{z}_b \wedge dw_b \text{ and} \tag{6.9}$$

$$\eta^{(j)}_{\widetilde{Y}_b} = \tfrac{1}{4}\sum_{r\geq 0}\left(\sum_{m+n=2r+1}(-1)^{n-1}(n-1)c^{(j)}_{m,n}\right)\tilde{z}_b^{2r-2}d\tilde{z}_b \wedge dw_b \tag{6.10}$$

and $c^{(j)}_{m,n} = 0$ if $m + n \equiv 0 \pmod 2$.

The next lemma is well known. It holds for any one-parameter degeneration of surfaces, not only for the families that we have constructed via Fay plumbings. However, the results such as Lemma 6.12 below that follow for the derivatives $\eta^{(j)}_{\widetilde{X}_i}$ do not hold in such generality, because for more general glueing data a formula such as $t \equiv q^2 - v^2 \pmod{t^2}$ will not hold. It follows that for a general one-parameter degeneration it is not possible to control the orders of the poles of the differentiated forms $\eta^{(j)}_{X_i}$ along E.

Lemma 6.11 (1) $\omega^{(j)}_{\widetilde{Y}_i}$ is holomorphic on $\widetilde{Y}_i$ for every $j \in [1, h_a + h_b]$.

(2) $\omega^{(j)}_{\widetilde{Y}_b} = 0$ for every $j \in [1, h_a]$ and $\omega^{(j)}_{\widetilde{Y}_a} = 0$ for every $j \in [h_a + 1, h_a + h_b]$.

(3) $(\omega^{(1)}_{\widetilde{Y}_a}, ..., \omega^{(h_a)}_{\widetilde{Y}_a})$ and $(\omega^{(h_a+1)}_{\widetilde{Y}_b}, ..., \omega^{(h_a+h_b)}_{\widetilde{Y}_b})$ are bases of the vector spaces $H^0(\widetilde{Y}_a, \omega_{\widetilde{Y}_a})$ and $H^0(\widetilde{Y}_b, \omega_{\widetilde{Y}_b})$, respectively.

(4) These bases are normalized with respect to the given A-cycles on $\widetilde{Y}_a$ and $\widetilde{Y}_b$.

PROOF: This is a restatement of Lemmas 6.3 3 and 6.5.   □

Lemma 6.12 *(1) Up to scalars there is a unique meromorphic 2-form $\widetilde{\eta}_{\widetilde{Y}_i} \in H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}(2\widetilde{\phi}_{\widetilde{i}}))$ such that every $\int_{A_l} \widetilde{\eta}_{\widetilde{Y}_i} = 0$*

(2) Every $\eta^{(j)}_{\widetilde{Y}_i}$ is a multiple of $\widetilde{\eta}_{\widetilde{Y}_i}$.

PROOF: (1) follows from the facts that $\dim H^0(\widetilde{Y}_i, \Omega^2_{\widetilde{Y}_i}(2\widetilde{\phi}_{\widetilde{i}})) = p_g(\widetilde{Y}_i) + 1$ and that $\widetilde{Y}_i$ is in L_i-general position. For (2), differentiate (6.6) and use (1).   □

Construct row vectors

$$\begin{aligned}
\underline{\eta}_{\widetilde{Y}_i} &= [\eta^{(1)}_{\widetilde{Y}_i}, ..., \eta^{(h_a+h_b)}_{\widetilde{Y}_i}], \\
\underline{\omega}_{\widetilde{Y}_a} &= [\omega^{(1)}_{\widetilde{Y}_a}, ..., \omega^{(h_a)}_{\widetilde{Y}_a}] \text{ and} \\
\underline{\omega}_{\widetilde{Y}_b} &= [\omega^{(h_a+1)}_{\widetilde{Y}_b}, ..., \omega^{(h_a+h_b)}_{\widetilde{Y}_b}]
\end{aligned}$$

whose entries are differentials. Via the identifications of Lemma 6.2 we also regard these as vectors of differentials on Y_i and on $\overline{Y}_i$ and then write them as $\underline{\eta}_{Y_i}$, etc.

Abbreviate the notation by defining

$$\omega^{(j)}_{\widetilde{Y}_i}(i) = \frac{\omega^{(j)}_{\widetilde{Y}_i}}{d\widetilde{z}_i \wedge dw_i}(i).$$

Proposition 6.13 *(1) There is an equality of row vectors*

$$\underline{\eta}_{\widetilde{Y}_i} = \widetilde{\eta}_{\widetilde{Y}_i}[\underline{\omega}_{\widetilde{Y}_a}(a), -\underline{\omega}_{\widetilde{Y}_b}(b)].$$

(2) In terms of the local co-ordinates $\widetilde{z}_i$ the forms $\widetilde{\eta}_{\widetilde{Y}_i}$ are given by

$$\begin{aligned}
\widetilde{\eta}_{\widetilde{Y}_a} &= \tfrac{1}{4}(\widetilde{z}_a^{-2} + \text{ h.o.t.})d\widetilde{z}_a \wedge dw_a \text{ and} \\
\widetilde{\eta}_{\widetilde{Y}_b} &= -\tfrac{1}{4}(\widetilde{z}_b^{-2} + \text{ h.o.t.})d\widetilde{z}_b \wedge dw_b.
\end{aligned}$$

PROOF: Examining the first few terms in the expansion provided by (6.7)-(6.10) gives

$$\omega^{(j)}_{\widetilde{Y}_a}(a) = \tfrac{1}{2}(c^{(j)}_{1,0} + c^{(j)}_{0,1}), \tag{6.14}$$

$$\omega^{(j)}_{\widetilde{Y}_b}(b) = \tfrac{1}{2}(-c^{(j)}_{1,0} + c^{(j)}_{0,1}), \tag{6.15}$$

$$\eta^{(j)}_{\widetilde{Y}_a} = \tfrac{1}{4}(-c^{(j)}_{1,0}\widetilde{z}_a^{-2} + \text{h.o.t.})d\widetilde{z}_a \wedge dw_a \text{ and} \tag{6.16}$$

$$\eta^{(j)}_{\widetilde{Y}_b} = \tfrac{1}{4}(c^{(j)}_{1,0}\widetilde{z}_b^{-2} + \text{h.o.t.})d\widetilde{z}_b \wedge dw_b. \tag{6.17}$$

If $j \le h_a$ then $\omega^{(j)}_{\widetilde{Y}_b} = 0$, so that $c^{(j)}_{1,0} = c^{(j)}_{0,1} = \omega^{(j)}_{\widetilde{Y}_a}(a)$.

If $j \ge h_a + 1$ then $\omega^{(j)}_{\widetilde{Y}_a} = 0$, so that $c^{(j)}_{1,0} + c^{(j)}_{0,1} = 0$ and $\omega^{(j)}_{\widetilde{Y}_b}(b) = -c^{(j)}_{1,0}$. The result now follows from (6.14)-(6.17).   □

Let Ψ_{Y_i} denote the period matrix of Y_i with respect to the 2-cycles $A_1, ..., A_{h_i}$ above.

Proposition 6.18 *(1) The period matrix Ψ_t of $\widetilde{\mathcal{Y}}_t$ is given by*

$$\Psi_t = [\Psi_{Y_a}, \Psi_{Y_b}] + t[\underline{\omega}_{Y_a}(a), -\underline{\omega}_{Y_b}(b)] \otimes [\underline{I}_a, \underline{I}_b] + h.o.t.$$

where $\underline{I}_i$ is a vector of integrals of the meromorphic differential $\widetilde{\eta}_{Y_i}$ over cycles on Y_i.

(2) The derivative $(d\Psi_t/dt)|_{t=0}$ of the period matrix of $\widetilde{\mathcal{Y}}_t$ at $t = 0$ is the rank 1 matrix $[\underline{\omega}_{Y_a}(a), -\underline{\omega}_{Y_b}(b)] \otimes [\underline{I}_a, \underline{I}_b]$.

PROOF: This is an immediate consequence of the calculations that we have made. Note that the vector $[\underline{I}_a, \underline{I}_b]$ can be seen to have the correct length (so that the matrix that describes $(d\Psi_t/dt)|_{t=0}$ has the correct shape) by omitting the zeroes, four in each $[\underline{I}_i]$, that arise from integrating $\widetilde{\eta}_{Y_i}$ around the four exceptional (-2)-curves on Y_i that are contracted in $\overline{Y}_i$. $\qquad\square$

Remark: (1) Recall also that the final $h \times h$ block (with respect to an appropriate choice of basis) of the period matrix, and so of the derivative $(d\Psi/dt)|_{t=0}$, is skew-symmetric. But a skew-symmetric matrix of rank 1 vanishes identically; for $(d\Psi/dt)|_{t=0}$ this is predicted by Griffiths transversality. In terms of the bases that we have used, this means that the final piece of length h_i in each vector $\underline{I}_i$ vanishes.

(2) (An analogue of Example 3.5 on p. 45 of [F1].)

Suppose that Y_b is rational. Then $(d\Psi_t/dt)|_{t=0}$ is given by

$$(d\Psi_t/dt)|_{t=0} = [\underline{\omega}_{Y_a}(a)] \otimes [\underline{I}_a, \underline{I}_b].$$

Note that Y_b has 4 moduli if the constraint that its $\widetilde{D}_4$ fibre ϕ_b should be isomorphic to ϕ_a is ignored. This constraint reduces the number of moduli to 3.

There is a resolution $\mathcal{Z} \to [\widetilde{\mathcal{Y}}]$ of the geometric quotient such that the closed fibre $\mathcal{Z}_0$ of $\mathcal{Z} \to \Delta$ is semi-stable and has six components: two are the surfaces Y_i for $i = a, b$ and four are Veronese surfaces $V_1, ..., V_4$. So $\overline{\mathcal{Y}} \to \Delta$ has a birational model $\mathcal{Z}' \to \Delta$ with good reduction, which is obtained from $\mathcal{Z}$ after flopping four times (each time in a curve in Y_b that is a section of the elliptic fibration and meets one of the V_i), then contracting the strict transforms of the V_i to curves and finally contracting the strict transform of Y_b to a curve. The closed fibre $\mathcal{Z}'_0$ is isomorphic to Y_a. That is, a Fay plumbing gives a one-parameter deformation of a surface Y_a where the derivative of the period matrix is given explicitly, provided that Y_a contains a $\widetilde{D}_4$-fibre. Since Y_b has three moduli the vector $\underline{I}_b$ has three moduli, so $(d\Psi_t/dt)|_{t=0}$ has three moduli.

Now suppose that $f_1 : Y_1 \to C_1, ..., f_r : Y_r \to C_r$ are smooth simply connected Jacobian elliptic surfaces each of which is in L_i-general position for an appropriate L_i. Assume also that $Y_i \to C_i$ has $\widetilde{D}_4$-fibres over a finite non-empty set $\{P_{ij}\}$ of points in C_i. Let $Y_i \to \overline{Y}_i$ denote the contraction of each of these designated $\widetilde{D}_4$-fibres to a $\overline{D}_4$-fibre.

Suppose also that Γ is a connected tree with r vertices and that we can associate the surfaces $\overline{Y}_i$ to the vertices of Γ such that two vertices i, j are joined in Γ if and only if the $\overline{D}_4$-fibre on $\overline{Y}_i$ that lies over P_{ij} is isomorphic to the $\overline{D}_4$-fibre on $\overline{Y}_j$ that lies over P_{ji}.

Then the plumbings modulo t^2 of the stacks $\widetilde{Y}_i$ that has just been described can be iterated to give

$$\widetilde{\mathcal{Y}}' = \widetilde{\mathcal{Y}}'_\Gamma \to \Delta'_{r-1} = \mathbf{Spec}\,\mathbb{C}[\{t_e\}]/(t_e^2) \tag{6.19}$$

where e runs over the edges of Γ and the closed fibre is $\sum \widetilde{Y}_i$ arranged according to the tree Γ. Take the normalized basis $\underline{\omega}_{Y_i}$ of $H^0(Y_i, \Omega^2_{Y_i})$ and $\widetilde{\eta}_{ij}$ a normalized element of $H^0(Y_i, \Omega^2_{Y_i}(\phi_{ij}))$. That is, if (A_k) is the given basis of L_i, then

$$\int_{A_k} \omega_{Y_i}^{(l)} = \delta_k^l, \quad \int_{A_k} \widetilde{\eta}_{ij} = 0$$

and the leading terms of the forms $\widetilde{\eta}_{ij}$ and $\widetilde{\eta}_{ji}$ have opposite signs; recall that these leading terms are $\pm\frac{1}{4}\tilde{z}_i^{-2}$.

Proposition 6.20 *Assume that $\widetilde{\mathcal{Y}}' \to \Delta'_{r-1}$ can be lifted to a family $\widetilde{\mathcal{Y}} \to S_{t_e}$ over an $(r-1)$-dimensional polydisc.*

Then $\widetilde{\mathcal{Y}}$ is smooth and the general fibre $\widetilde{\mathcal{Y}}_t$ is in L-general position for an appropriate L, there is no monodromy on $H^2(\widetilde{\mathcal{Y}}_t, \mathbb{Z})$, the closed fibre $\widetilde{\mathcal{Y}}_0$ is $\widetilde{\mathcal{Y}}_0 = \sum \widetilde{Y}_i$ and the period matrix $\Psi(\widetilde{\mathcal{Y}}_t)$ is given by

$$\Psi(\widetilde{\mathcal{Y}}_t) = [\Psi(Y_1), ..., \Psi(Y_h)] + \sum_e t_e[\underline{\omega}_{Y_i}(P_{ij}), -\underline{\omega}_{Y_j}(P_{ji})] \otimes [\underline{L}_{ij}, \underline{L}_{ji}] + h.o.t.$$

where $\underline{L}_{ij}$ is a vector of integrals of $\widetilde{\eta}_{ij}$ around 2-cycles on Y_i and the sum is over the edges of Γ.

PROOF: This is a straightforward consequence of Proposition 6.18.   □

We shall see, in Lemma 11.8, that the liftings $\widetilde{\mathcal{Y}} \to S_{t_e}$ exist when each surface Y_i is a special elliptic surface. Then, in Theorem 11.12, which will lead to the main result, Theorem 11.13, we shall take all the surfaces Y_i to be K3 surfaces and Γ will be an alkane. Note that K3 surfaces are in L-general position for all appropriate L, so that the general fibre $\widetilde{\mathcal{Y}}_t$ will also be in L-general position.

7   Fay's formulae for homologically trivial plumbings of curves

For a homologically trivial Fay plumbing of curves, the arguments of the previous section go through to recover Fay's Corollary 3.2, as follows. No change is required in the argument except that the fibre co-ordinates w and w_i should be suppressed.

Suppose that $\mathcal{C} \to \Delta_t$ is a homologically trivial Fay plumbing of C_a to C_b that identifies a with b and $(\omega^{(1)}(t), ..., \omega^{(g)}(t))$ is a normalized basis of $H^0(\mathcal{C}_t, \omega_{\mathcal{C}_t})$.

Proposition 7.1
$$\omega^{(j)}(t) \equiv \omega_{C_i}^{(j)} + t\eta_{C_i}^{(j)} \pmod{t^2}$$

where

(1) $(\omega_{C_a}^{(j)})_{1 \leq j \leq g_a}$ *is a normalized basis of* $H^0(C_a, \omega_{C_a})$, $(\omega_{C_b}^{(j)})_{g_a+1 \leq j \leq g_a+g_b}$ *is a normalized basis of* $H^0(C_b, \omega_{C_b})$ *and* $\omega_{C_i}^{(j)} = 0$ *otherwise, and*

(2) *there is a unique element* $\tilde{\eta}_i \in H^0(C_i, \omega_{C_i}(2i))$, *normalized by the requirements that, firstly,* $\int_{A_k} \tilde{\eta}_i = 0$ *for every A-cycle* A_k *on* C_i *and that, secondly,* $\tilde{\eta}_a = \frac{1}{4}(z_a^{-2} + \text{ h.o.t.})dz_a$ *while* $\tilde{\eta}_b = -\frac{1}{4}(z_b^{-2} + \text{ h.o.t.})dz_b$, *such that there is an equality*
$$[\underline{\eta}_{C_i}] = \tilde{\eta}_i \left[\underline{\omega}_{C_a}(a), -\underline{\omega}_{C_b}(b)\right]$$
of row vectors of length g.

It follows that
$$\tau(\mathcal{C}_t) \equiv \begin{bmatrix} \tau(C_a) & 0 \\ 0 & \tau(C_b) \end{bmatrix} + t\left[\underline{\omega}_{C_a}(a), -\underline{\omega}_{C_b}(b)\right] \otimes \underline{v} \pmod{t^2}$$

where $\underline{v} = [\underline{v}_a, \underline{v}_b]$ and $\underline{v}_i$ is the vector of integrals of the form $\tilde{\eta}_i$ around the B-cycles on C_i. Since, by the bilinear relations for integrals of the first kind, $\tau(\mathcal{C}_t)$ is symmetric, it follows that $v = \lambda \left[\underline{\omega}_{C_a}(a), -\underline{\omega}_{C_b}(b)\right]$ for some scalar λ and
$$\tau(\mathcal{C}_t) \equiv \begin{bmatrix} \tau(C_a) & 0 \\ 0 & \tau(C_b) \end{bmatrix} + \lambda t \left[\underline{\omega}_{C_a}(a), -\underline{\omega}_{C_b}(b)\right]^{\otimes 2} \pmod{t^2}$$

We can calculate λ from the bilinear relations for integrals of the second kind on C_a with only poles at a:
$$\sum_{l=1}^{g_a} \left(\int_{A_l} \phi \int_{B_l} \psi - \int_{A_l} \psi \int_{B_l} \phi\right) = \int_\gamma f\psi$$

where γ is a loop in C_a around a and $\phi = df$. Taking $\psi = \omega_{C_a}^{(j)}$ and $\phi = \tilde{\eta}_a$ gives, via the expansions above in terms of power series of $\omega_{C_a}^j$ and $\eta_{C_a}^j$,
$$\int_{B_j} \tilde{\eta}_a = \frac{\pi\sqrt{-1}}{2} \omega_{C_a}^{(j)}(a),$$

so that $\lambda = \frac{2\pi\sqrt{-1}}{4}$. This differs from the formula given in Fay's Corollary 3.2, in which $\lambda = \frac{1}{4}$, because our 1-forms are normalized by the requirement that $\int_{A_l} \omega^{(j)} = \delta_{jl}$ and not $2\pi\sqrt{-1}\delta_{jl}$.

Remark: As already mentioned, on p. 41 of [F1] the minus sign in front of $\underline{\omega}_{C_b}(b)$ is missing. The reason is a different choice of normalization in the plumbing construction, which replaces z_b by $-z_b$. Logically, however, there is no difference.

8 Poincaré's asymptotic period relations

Suppose that $E_1, ..., E_g$ are disjoint elliptic curves and that D is a copy of $\mathbb{P}^1$. Fix points $a_i \in E_i$ and a local co-ordinate z_i on E_i at a_i. On D fix a point ∞ and a global co-ordinate u on $D - \{\infty\}$. Fix distinct points $b_1, ..., b_g \in D - \{\infty\}$ given by $u - b_j = 0$.

Then successively making Fay plumbings of the E_i to D using these data in a way that identifies $a_j \in E_j$ to $b_j \in D - \{\infty\}$ leads to a family $\mathcal{C} \to S$ of genus g curves over a g-dimensional polydisc $S = S_g = \Delta_{t_1,...,t_g}$ with co-ordinates $t_1, ..., t_g$. We fix a symplectic basis (A_j, B_j) of each $H_1(E_j, \mathbb{Z})$ and let v_j be the corresponding normalized 1-form on E_j. We write $v_i(a_i) = \frac{v_i}{dz_i}(a_i)$; this notation will be used many times.

The next result is due to Poincaré when $g = 4$ and to Fay in general. According to Igusa ([I], p. 167) Poincaré's proof exploited the fact that the theta divisor on a Jacobian is of translation type, and so can not be obviously extended to an analysis of period matrices in higher dimensions. However, Fay's approach, of which we include some details that he omitted, goes via his plumbing construction and so can be extended. It turns out that hyperelliptic curves are particularly interesting from this point of view.

As already mentioned, this result is derived in [FGSM] from their global results.

Theorem 8.1 *("Poincaré's asymptotic period relations", [F1], p.45) For $i \neq j$ the entries τ_{ij} of the period matrix τ of $\mathcal{C}_t$ can be written as $\tau_{ij} = \overline{\tau}_{ij} u_{ij}$ where $u_{ij} \equiv 1 \pmod{(t)}$ and*

$$\overline{\tau}_{ij} = \frac{2\pi\sqrt{-1}}{16} t_i t_j v_i(a_i) v_j(a_j)/(b_i - b_j)^2.$$

PROOF: Induction on g.

Construct a family $\mathcal{C}' \to S_1$ of curves of genus 1 by plumbing $a_1 \in E_1$ to $b_1 \in D$. This has a normalized 1-form $\omega_1(\mathcal{C}'_{t_1})$. Then near a point of $D - \{b_1\}$ we have

$$\omega_1(\mathcal{C}'_{t_1}) = \tfrac{1}{4} t_1 v_1(a_1) \frac{dz}{(z - b_1)^2} + O(t_1^2).$$

Now construct a genus 2 family $\mathcal{C} \to \Delta_{t_1, t_2}$ by plumbing $\mathcal{C}'$ to E_2 in a way that identifies $b_2 \in D$ with $a_2 \in E_2$. There are normalized 1-forms $\Omega_j(\mathcal{C}_{t_1, t_2})$ on $\mathcal{C}_{t_1, t_2}$, where $j = 1, 2$. Then, near a point in E_2, we have

$$\Omega_1(\mathcal{C}_{t_1, t_2}) = t_2 \omega_1(\mathcal{C}')(b_2) \eta_2 + O(t_2^2),$$

where η_2 is a meromorphic 1-form on E_2 with a double pole at a_2 and $\eta_2 = \frac{1}{4} v_2(a_2)(z_2^{-2} + \text{h.o.t.}) dz_2$. Moreover, $\int_{A_2} \eta_2 = 0$. So

$$\tau_{12} = \int_{B_2} \Omega_1(\mathcal{C}) \equiv t_2 \tfrac{1}{4} t_1 v_1(a_1) \frac{1}{(b_2 - b_1)^2} \int_{B_2} \eta_2 \pmod{(t_1^2, t_2^2)}$$

$$\equiv \frac{2\pi\sqrt{-1}}{16} t_1 t_2 v_1(a_1) v_2(a_2)/(b_2 - b_1)^2.$$

Now assume that $g \geq 3$ and that the result is true for plumbings of genus $\leq g-1$. Write

$$\frac{2\pi\sqrt{-1}}{16} v_i(a_i) v_j(a_j)/(b_i - b_j)^2 = c_{ij}.$$

Note that $c_{ij} \in \mathbb{C}^*$ since $v_i(a_i) \neq 0$.

Suppose that $k \in [3, g]$ and that $j \in [1, 2]$. Then, by induction,

$$\tau_{12} = c_{12} t_1 t_2 u_{12k} + t_k X_{12k}$$

where $u_{12k} \equiv 1 \pmod{(t)}$ and X_{12k} is some function.

Moreover, $\tau_{12} \equiv 0 \pmod{t_j}$ since setting $t_j = 0$ gives a plumbing where the curve $\mathcal{C}_t$ remains singular and one of its irreducible components is the non-varying elliptic curve E_j.

Therefore X_{12k} is divisible by t_j and then we can write

$$\tau_{12} = t_1 t_2 (c_{12} u_{12k} + t_k Y_{12k}).$$

Since $c_{12} \neq 0$ the induction is complete. $\qquad\qquad\qquad\square$

It follows that the off-diagonal quantities $y_{ij} = \overline{\tau}_{ij}^{-1/2}$ satisfy the Plücker relations

$$y_{ij} y_{kl} - y_{ik} y_{jl} + y_{il} y_{jk} = 0.$$

These translate into octic relations amongst the $\overline{\tau}_{ij}$, which are "asymptotic relations" amongst the τ_{ij}. Let $\mathcal{J}_g^c$ denote the closure of the Jacobian locus $\mathcal{J}_g$ inside the stack $\mathcal{A}_g$ along the closed substack $\mathcal{D}iag_g$ of $\mathcal{A}_g$ that parametrizes products of elliptic curves. Note that $\mathcal{D}iag_g$ is isomorphic to the quotient $(\mathcal{A}_1)^g/\mathfrak{S}_g$ of $\mathcal{A}_1^g$ by the symmetric group. In transcendental terms, $\mathcal{A}_g = \mathfrak{H}_g/Sp_{2g}(\mathbb{Z})$ and $\mathcal{D}iag_g = \mathfrak{Diag}_g/(SL_2(\mathbb{Z}) \wr \mathfrak{S}_g)$. Let $\overline{\mathcal{M}}_g^c$ denote the open substack of the stack $\overline{\mathcal{M}}_g$ of stable curves of genus g that parametrizes curves of compact type. Then the Jacobian morphism $\mathcal{M}_g \to \mathcal{A}_g$ extends to a proper morphism $\overline{\mathcal{M}}_g^c \to \mathcal{A}_g$ whose image is $\mathcal{J}_g^c$. We let $\mathfrak{J}_g$ and $\mathfrak{J}_g^c$ denote the inverse images in $\mathfrak{H}_g$ of $\mathcal{J}_g$ and $\mathcal{J}_g^c$.

We show next, in Proposition 8.2, that these "asymptotic relations" are exactly the defining equations, in terms of the entries τ_{ij} of the period matrix, of the associated graded ring belonging to the closed subvariety $\mathfrak{Diag}_g$ of $\mathfrak{J}_g^c$.

We shall use the term *multi-elliptic* to refer to a stable curve of genus g that contains g elliptic components (and maybe some smooth rational components). Such a curve is necessarily of compact type. Then $\mathcal{D}iag_g$ is the image of the stack of multi-elliptic stable curves.

Let $\mathcal{G}r_{\mathcal{J}_g^c}$ denote the sheaf of graded $\mathcal{O}_{\mathcal{D}iag_g}$-algebras that is associated to the closed embedding $\mathcal{D}iag_g \hookrightarrow \mathcal{J}_g^c$ and $\mathcal{G}r_{\mathfrak{J}_g^c}$ its pullback to $\mathcal{O}_{\mathfrak{Diag}_g}$.

The Grassmannian $Grass(2, g)$ is embedded in $\operatorname{Proj} \mathbb{C}[\{y_{ij}\}] = \mathbb{P}_y^{\binom{g}{2}-1}$ via the Plücker co-ordinates y_{ij}. Let X denote the closure of the image of $Grass(2, g)$ under the generically finite rational map

$$\mathbb{P}_y^{\binom{g}{2}-1} - \to \mathbb{P}_{\overline{\tau}}^{\binom{g}{2}-1}$$

given by $y_{ij} \mapsto y_{ij}^{-2} = \overline{\tau}_{ij}$ and let $\widehat{X} \subseteq \mathbb{A}^{\binom{g}{2}}$ be the affine cone over X with affine co-ordinate ring $\mathbb{C}[\widehat{X}]$. The quadratic Plücker identities relating the y_{ij} give rise to octic relations between the $\overline{\tau}_{ij}$ that are the defining equations of $\widehat{X}$ (or of X).

Proposition 8.2 $\mathcal{G}r_{\mathfrak{J}_g^c}$ is isomorphic, as a sheaf of graded $\mathcal{O}_{\mathfrak{Diag}_g}$-algebras, to the constant sheaf $\mathcal{O}_{\mathfrak{Diag}_g} \otimes_{\mathbb{C}} \mathbb{C}[\widehat{X}]$.

PROOF: Let $\mathcal{E} \to \mathfrak{Diag}_g$ be the g-fold universal elliptic curve over $\mathfrak{Diag}_g$, so that $\dim \mathcal{E} = 2g$. Set $U = ((\mathbb{P}^1)^g - \Delta)/PGL_2$ where Δ is the union of all the diagonals. Let S be a g-dimensional polydisc with co-ordinates $t_1, ..., t_g$ and put $L = \mathcal{E} \times U \times S$. Then a suitable Fay plumbing is a family of curves over L which then gives a period map

$$h : L \to \mathfrak{H}_g$$

such that $\tau_{ij} = \overline{\tau}_{ij} u_{ij}$ as above. Moreover, along the sublocus of L defined by $t_1 = \cdots = t_g = 0$ the space L is everywhere locally the base of a miniversal deformation of a singular stable curve C_0 which is of the form $C_0 = \mathbb{P}^1 + \sum_1^g E_i'$ for varying elliptic curves E_i'. Therefore, by the local Torelli theorem for smooth curves, there is a dense open subspace of L along which the derivative of h is injective. So the image of h is open inside some branch of $\mathfrak{J}_g^c$ along $\mathfrak{Diag}_g$. In particular, its dimension is $3g - 3$.

Lemma 8.3 $\mathcal{J}_g^c$ is unibranched along $\mathcal{Diag}_g$.

PROOF: Suppose that $E_1, ..., E_g$ are elliptic curves and that $\overline{\mathcal{M}}_{\sum E_i} \subset \overline{\mathcal{M}}_g$ is the locus of multi-elliptic curves C that contain each E_i. That is, C is a tree whose components are the curves E_i together with some copies of $\mathbb{P}^1$.

To prove the lemma it is enough to show that $\overline{\mathcal{M}}_{\sum E_i}$ is connected. We do this by induction on g.

This is clearly true when $g = 1$. Say $\overline{\mathcal{M}}' = \overline{\mathcal{M}}_{E_g}$ and $\overline{\mathcal{M}}'' = \overline{\mathcal{M}}_{\sum_1^{g-1} E_i}$. These are connected, by the induction hypothesis. Let $\overline{\mathcal{M}}_1'$ and $\overline{\mathcal{M}}_1''$ be their inverse images in the stack $\overline{\mathcal{M}}_{g,1}$ of 1-pointed stable curves of genus g. There are forgetful morphisms $\overline{\mathcal{M}}_1' \to \overline{\mathcal{M}}'$ and $\overline{\mathcal{M}}_1'' \to \overline{\mathcal{M}}''$. Since their fibres are connected, both $\overline{\mathcal{M}}_1'$ and $\overline{\mathcal{M}}_1''$ are connected. There is a clutching morphism $\overline{\mathcal{M}}_1' \times_{\mathbb{C}} \overline{\mathcal{M}}_1'' \to \overline{\mathcal{M}}_{\sum_1^g E_i}$; since this is surjective the lemma is proved. $\square$

Set $R = \mathcal{O}_{\mathfrak{H}_g}$. Let T denote the ideal in R generated by $\{\tau_{ij} | i \neq j\}$. This is the defining ideal of $\mathfrak{Diag}_g$. Set $S = \mathcal{G}r_T R$, the associated sheaf of graded $\mathcal{O}_{\mathfrak{Diag}_g}$-algebras. Since the spaces concerned are Stein we shall not emphasize the

distinction between a ring and a sheaf of rings. Given a graded $\mathcal{O}_{\mathfrak{Diag}_g}$-algebra such as S, Proj S will denote the projectivization relative to $\mathfrak{Diag}_g$ in the analytic category.

Let I be the ideal of R that defines $\mathfrak{J}_g^c$ and K the ideal of R that defines $(\mathfrak{Diag}_g \times \widehat{X}) \cap \mathfrak{H}_g$. Let $\overline{I}$, $\overline{K}$ be the associated graded ideals of S.

Note that $\overline{K}$ is the defining ideal of $\mathfrak{Diag}_g \times X$ inside Proj S and so is prime.

The existence of Fay's octic identities of Plücker type given above mean that there exist $f_1, ...f_r \in I$ such that, if $g_i = f_i \pmod{I}$, then $g_1, ..., g_r$ generate $\overline{K}$. Therefore $\overline{K} \subseteq \overline{I}$.

Suppose that $\overline{K} \neq \overline{I}$. Now $\mathrm{Proj}(S/\overline{K}) = \mathfrak{Diag}_g \times X$, which is reduced and irreducible of dimension $3g - 4$. So $\dim \mathrm{Proj}(S/\overline{I}) \leq 3g - 5$. On the other hand, $\mathrm{Proj}(S/\overline{I})$ is the exceptional divisor in the blow-up $\mathrm{Bl}_{\mathfrak{Diag}_g} \mathfrak{J}_g$ and so has dimension $3g - 4$. Therefore $\overline{K} = \overline{I}$ and the proposition follows. $\square$

Corollary 8.4 *Up to graded equivalence the singularity of $\mathcal{J}_g^c$ at a given point x of $\mathcal{D}iag_g$ is isomorphic to the cone $\widehat{X}$ and is independent of x.*

When $g = 4$ (the case considered in detail by Poincaré) $\widehat{X}$ is then an octic hypersurface in $\mathbb{A}^6$ so that $\mathcal{J}_4^c$ does not have rational singularities along $\mathcal{D}iag_4$.

9 Hyperelliptic curves and alkanes

We continue with the notation of the previous section.

Suppose instead that we choose points a_1 on E_1, b_{i-1} and a_i on E_i for $i = 2, ..., g-1$ and b_{g-1} on E_g and a local co-ordinate z_x at each point x, and then plumb E_i to E_{i+1} by identifying a_i to b_i in a chain. This leads to a family $\mathcal{C} \to S$ of genus g curves where $S = \Delta_{t_1,...,t_{g-1}}$ is a $(g-1)$-dimensional polydisc with co-ordinates $t_1, ..., t_{g-1}$. Let τ_i be the period of E_i, defined, as before, after the choice of a symplectic basis (A_j, B_j) of $H_1(E_j, \mathbb{Z})$ and normalized 1-form ω_j on E_j. Construct a symplectic basis of $H_1(\mathcal{C}_t, \mathbb{Z})$ as a union of the symplectic bases (A_j, B_j).

Lemma 9.1 *If each difference $b_{i-1} - a_i$ is 2-torsion on E_i and at each point b_{i-1}, a_i on E_i a local co-ordinate is chosen that is anti-invariant under the involution $[-1_{E_i}]$ then the family $\mathcal{C} \to S$ is hyperelliptic.*

PROOF: The constraint on the points $a_1, ..., a_{g-1}$ implies that the chain $\mathcal{C}_0 = E_1 \cup \cdots \cup E_g$ is a double cover of a chain $\mathcal{B}_0 = \Gamma_1 \cup \cdots \cup \Gamma_g$ of copies of $\mathbb{P}^1$ and that the map $\mathcal{C}_0 \to \mathcal{B}_0$ is not étale at the nodes. So, via Proposition 4.6, Fay plumbings can be constructed simultaneously to give a double cover $\mathcal{C} \to \mathcal{B}$ over S where $\mathcal{B} \to S$ is a family of genus zero curves. $\square$

We next give a description of the closure $\mathcal{HYP}_g^c$ of the hyperelliptic locus in $\overline{\mathcal{M}}_g^c$ near $\mathcal{D}iag_g$ in combinatorial terms.

Lemma 9.2 *Suppose that C is multi-elliptic. Choose a node on each component E_i and take it to be the origin of E_i.*

(1) C is hyperelliptic if and only if C contains no rational curves, each component E_i of C contains at most 4 nodes and each node of C that lies on E_i is a 2-torsion point of E_i.

(2) If C is hyperelliptic then the hyperelliptic involution ι_C preserves each component of C and the fixed point locus of ι_C consists of the nodes of C and the 2-torsion points of each component E_i.

PROOF: This is an elementary exercise.   □

Recall from elementary chemistry that carbon has valence 4, hydrogen has valence 1 and that an *alkane* is a hydrocarbon whose molecular formula is of the form $C_g H_{2g+2}$. Note that the hyperelliptic involution of a hyperelliptic curve of genus g has $2g + 2$ fixed points. Alkanes were first enumerated by Cayley and again, more recently, by Rains and Sloane (who corrected some errors by Cayley); see [OEIS], sequence A000602. We shall refer to g as the genus of the alkane. We can describe stable multi-elliptic hyperelliptic curves C in terms of alkanes of the same genus.

Corollary 9.3 *Each fibre in $\mathcal{HYP}_g^c$ over a point in $\mathcal{Diag}_g$ is a finite set that corresponds naturally to the set of alkanes of genus g.*

PROOF: This is a translation of the preceding lemma. The carbon atoms correspond to the elliptic curves and the hydrogen atoms to the 2-torsion points on each elliptic curve that are not nodes. That is, the the hydrogen atoms correspond to the fixed points of the hyperelliptic involution that are not nodes.
□

Corollary 9.4 *The branches of $\mathcal{HYP}_g^c$ through $\mathcal{Diag}_g$ in $\mathcal{A}_g$ correspond to the alkanes of genus g.*

We shall also use the term alkane to refer to a hyperelliptic multi-elliptic stable curve.

Suppose that $C = \sum_1^g E_i$ is an alkane. Suppose that K is the set of edges, so that $\{i, j\} \in K$ if and only if $i \neq j$ and E_i meets E_j. Choose a local co-ordinate on each E_i at each node of C (not necessarily anti-invariant under the involution $[-1_{E_i}]$) and let $\mathcal{C} \to S_{g-1} = \Delta_{\{t_k | k \in K\}}$ be the result of plumbing together the curves E_i according to these data.

Theorem 9.5 *Modulo $(\{t_l | l \in K\})^2$, the off-diagonal entry $\tau(\mathcal{C}_t)_k$ of the period matrix of $\mathcal{C}_t$ is a multiple of t_k if $k \in K$. The other off-diagonal entries vanish.*

PROOF: We can suppose the curves E_i ordered so that E_g meets only E_{g-1} and then argue by induction on g.

Suppose that $a \in E_{g-1}$ is identified with $b \in E_g$. Suppose that $\mathcal{C}' \to S' = \Delta_{t_1, ..., t_{g-2}}$ is the result of plumbing $E_1, ..., E_{g-1}$ according to the data, so that $\mathcal{C} \to S$ is the plumbing of $\mathcal{C}$ to $E_g \times S'$ that identifies $\{a\} \times S'$ with $\{b\} \times S'$.

Write $(t') = (t_1, ..., t_{g-2})$ and $(t) = (t', t_{g-1})$ and suppose that $(\Omega'_1, ..., \Omega'_{g-1})$ is a basis of normalized 1-forms on $\mathcal{C}'$. By Fay's formula, $\tau(\mathcal{C}_t)$ is congruent to

$$\begin{bmatrix} \tau(\mathcal{C}'_{t'}) & 0 \\ 0 & \tau_g \end{bmatrix} + \frac{\pi\sqrt{-1}}{2}t_{g-1}[\Omega'_1(\{a\} \times S'), ..., \Omega'_{g-1}(\{a\} \times S'), -\omega_g(b)]^{\otimes 2}$$

modulo $(t)^2$. The restriction $\Omega'_i|_{E_{g-1}}$ is identically zero for all $i \leq g - 2$, and so $\Omega'_i(\{a\} \times S') \equiv 0 \pmod{t'}$ for $i \leq g - 2$. Moreover, according to the induction hypothesis, $\tau(\mathcal{C}'_{t'})$ has the required form, and the result follows. $\qquad\square$

Let $\mathfrak{Hyp}^c_g$ denote the closure of the hyperelliptic locus in $\mathfrak{H}_g$.

Theorem 9.6 *(Asymptotic period relations for the hyperelliptic locus) To first order in a neighbourhood of $\mathfrak{D}_g$ each branch of $\mathfrak{Hyp}^c_g$ is described as a subvariety of $\mathfrak{H}_g$ by the vanishing of the entries τ_k in the period matrix τ where k runs over the set of pairs that are not edges of the corresponding alkane.*

In particular, the branch corresponding to the linear alkane equals, to first order, the locus $\mathfrak{T}_g$ of tridiagonal matrices.

PROOF: We prove this only for the linear alkane. The proof in general is the same but with more complicated notation.

Let $E_\tau = \mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z})$ be the elliptic curve corresponding to $\tau \in \mathfrak{H}_1$. Pick a co-ordinate z on $\mathbb{C}$ such that the involution $[-1_{E_\tau}]$ acts via $z \mapsto -z$. Let $\mathcal{F} \to \mathfrak{D}_g$ be the family whose fibre over $(\tau_1, ..., \tau_g)$ is $E_{\tau_2} \times \cdots \times E_{\tau_{g-1}}$. Then the Fay plumbing just described, using the co-ordinates provided, gives a family $\mathcal{C} \to \mathcal{F} \times S_{g-1}$ of stable curves which is minimally versal at each point where $t_1 = \cdots = t_{g-1} = 0$ and so is minimally versal everywhere.

Fixing a non-zero 2-torsion point on each of $E_2, ..., E_{g-1}$ defines a section of $\mathcal{F} \to \mathfrak{D}_g$. Then the restriction of $\mathcal{C}$ to the corresponding subvariety $\mathfrak{D}_g \times S_{g-1}$ of $\mathcal{F} \times S_{g-1}$ is then, over the complement of the discriminant, a family of hyperelliptic curves.

Since $\dim(\mathfrak{D}_g \times S_{g-1}) = 2g - 1$ this is everywhere a minimally versal family of hyperelliptic curves. Then, by the local Torelli theorem for hyperelliptic curves, the image T of $\mathfrak{D}_g \times S_{g-1}$ in $\mathfrak{H}_g$ is of dimension $2g - 1$. To first order T lies in $\mathfrak{T}_g$, and the theorem is proved. $\qquad\square$

10   Stable reduction

Chakiris proved [C1] [C2] a generic Torelli theorem for Jacobian elliptic surfaces over $\mathbb{P}^1$ by reducing the problem to special elliptic surfaces.

To achieve this reduction he extended the domain of the period map to make the map proper. In turn, he did this by first showing that a one-parameter degeneration of Jacobian elliptic surfaces over $\mathbb{P}^1$ without monodromy can be put into a certain standard form. See, e.g., the statement $(**)$ in [C2], top of p.

174 or the "stable reduction" theorem on p. 231 of [C1]. Note, however, that this version of a stable reduction theorem gives a closed fibre that contains curves of cusps, and so is not semi log canonical (slc) in the sense of the Minimal Model Program (MMP).

In this section we shall refine his result, so that the period map becomes proper over each locus $\mathcal{W}_{h_1,\ldots,h_r}$. For this, we use his ideas strengthened by the MMP, which was not available to him.

Theorem 10.1 *Suppose that $\mathcal{X} \to \Delta$ is a 1-parameter degeneration of simply connected Jacobian elliptic surfaces of geometric genus $h \geq 1$ which is semi-stable (in the usual sense that the closed fibre $\mathcal{X}_0$ is reduced with normal crossings) and that there is no monodromy on the cohomology of the geometric generic fibre $\mathcal{X}_{\bar{\eta}}$. Assume also that, under the period map, the image of $0 \in \Delta$ is the direct sum of the period matrices of special elliptic surfaces $\widetilde{V}_1, \ldots, \widetilde{V}_r$.*

Then there is a birationally equivalent model $\mathcal{Y} \to \Delta$ with the following properties:

(1) $\mathcal{Y}$ has $\mathbb{Q}$-factorial canonical singularities;

(2) the closed fibre $\mathcal{Y}_0$ has slc singularities;

(3) the irreducible components of $\mathcal{Y}_0$ are the singular models V_i of the $\widetilde{V}_i$ with $\overline{D}_4$-fibres;

(4) if $V_i \cap V_j$ is not empty then it is a copy of $\mathbb{P}^1$ and contains 4 points at all of which V_i and V_j each has a node;

(5) each triple intersection $V_i \cap V_j \cap V_k$ is empty;

(6) $\mathcal{Y}_0$ is formed by arranging the surfaces V_i in a tree.

PROOF: First assume only that the generic fibre $\mathcal{X}_{\bar{\eta}}$ is a smooth minimal elliptic surface and is not ruled.

To begin, run the MMP in two steps, as follows.

Step 1: Run a $K_{\mathcal{X}/\Delta}$ MMP on $\mathcal{X} \to \Delta$ and let $\mathcal{X}_1 \to \Delta$ be the result. Then $\mathcal{X}_1$ has $\mathbb{Q}$-factorial terminal singularities and $K_{\mathcal{X}_1/\Delta}$ is semi-ample So some relative pluricanonical system $|mK_{\mathcal{X}_1/\Delta}|$ defines an algebraic fibre space $f : \mathcal{X}_1 \to S$ where $S \to \Delta$ is a semi-stable family of curves (so that S has singularities of type A) and $K_{\mathcal{X}_1/\Delta}$ pulls back from an ample $\mathbb{Q}$-line bundle on S. Moreover, the closed fibre $\mathcal{X}_{1,0}$ has slc singularities. Replace $\mathcal{X}$ by $\mathcal{X}_1$.

Step 2: If there are surfaces E_i in $\mathcal{X}$ such that $f(E_i)$ is a point then there are only finitely many such. For suitable $\alpha_i \in \mathbb{Q}$ with $0 < \alpha_i \ll 1$ run a $(K_{\mathcal{X}/S}, \sum \alpha_i E_i)$ MMP on $\mathcal{X} \to S$. The result is a birational map $\mathcal{X}- \to \mathcal{X}_1$ under which the strict transform of each E_i is of dimension at most 1. Since $K_{\mathcal{X}}$ is trivial in a neighbourhood of $\sum E_i$, the rational map $\mathcal{X}_1- \to S$ is a morphism, all its fibres are 1-dimensional, $\mathcal{X}_1$ has canonical singularities and $\mathcal{X}_{1,0}$ has slc singularities. Moreover, $\mathcal{X}_1$ is $\mathbb{Q}$-factorial, by general properties of the MMP. Replace $\mathcal{X}$ by $\mathcal{X}_1$.

At this point $\mathcal{X}$ has $\mathbb{Q}$-factorial canonical singularities and $\mathcal{X}_0$ has slc singularities. Moreover, there is a surface S, a semi-stable morphism $S \to \Delta$ (so

that S has singularities of type A) and a morphism $f : \mathcal{X} \to S$ with only one-dimensional fibres such that $K_{\mathcal{X}/\Delta}$ is the pullback of an ample $\mathbb{Q}$-line bundle on S.

Now we assume also that $\mathcal{X}_{\bar{\eta}}$ is Jacobian and simply connected. Then $\mathcal{X}_\eta$ is also Jacobian (maybe after a finite base change $\Delta \to \Delta$, which we can and do ignore), so that $\mathcal{X} \to S$ has a generic section (that is, a closed subscheme $\widetilde{S}$ such that the induced morphism $\widetilde{S} \to S$ is proper and birational), the closed fibre S_0 is a tree of copies C_i of $\mathbb{P}^1$ and the inverse image $f^{-1}(C_i) \to C_i$ has a section. Moreover, the generic fibre of $f^{-1}(C_i) \to C_i$ is either elliptic or a cycle of rational curves.

Lemma 10.2 *The irreducible components of $\mathcal{X}_0$ are either models of the special elliptic surfaces $\widetilde{V}_i$ or are ruled.*

PROOF: Recall that a Jacobian elliptic surface over $\mathbb{P}^1$ whose geometric genus is zero is either rational or (elliptic) ruled. So we only have to consider components Z of $\mathcal{X}_0$ that are Jacobian elliptic surfaces of positive genus. Now the Hodge structure of Z embeds into the product of the Hodge structures of the special elliptic surfaces $\widetilde{V}_i$; this is enough [C1] to ensure that the global monodromy of the elliptic fibration $Z \to \mathbb{P}^1$ that is provided by the fact that Z is a component of $\mathcal{X}_0$ is of order at most 2, so that Z is special, and then the Torelli theorem for hyperelliptic curves shows that Z is birational to one of the $\widetilde{V}_i$.   □

From now on let C_i denote a typical irreducible component of S_0, put $X_i = f^{-1}(C_i)$ and let $f_i : X_i \to C_i$ denote the restriction of $f : \mathcal{X} \to S$. Let $\nu_i : X_i^\nu \to X_i$ be the normalization and $\widetilde{X}_i \to X_i^\nu$ the minimal resolution; *a priori*, these objects might be disconnected. We assume that there is no monodromy on $H^2(\mathcal{X}_{\bar{\eta}}, \mathbb{Z})$; this is equivalent to the statement that $p_g(\mathcal{X}_{\bar{\eta}}) = \sum p_g(\widetilde{X}_i)$ (recall that every surface that has been contracted is ruled).

Lemma 10.3 *The maps $H^2(\mathcal{X}_0, \mathcal{O}) \to \oplus H^2(\widetilde{X}_i, \mathcal{O})$, $H^2(X_i, \mathcal{O}) \to H^2(X_i^\nu, \mathcal{O})$ and $H^2(X_i^\nu, \mathcal{O}) \to H^2(\widetilde{X}_i, \mathcal{O})$ are all isomorphisms.*

PROOF: $H^2(\mathcal{X}_0, \mathcal{O}) \to \oplus H^2(\widetilde{X}_i, \mathcal{O})$ is surjective. Since the formation of the groups $H^i(\mathcal{X}_t, \mathcal{O})$ commutes with specialization the lemma follows from the assumption that $p_g(\mathcal{X}_{\bar{\eta}}) = \sum p_g(\widetilde{X}_i)$.   □

Fix $P \in S_0$ and denote localizations at P and along fibres over P by a superscript h.

Lemma 10.4 *If P is a node of S_0 then $H^1(f^{-1}(P), \mathcal{O}) = 0$ and $f^{-1}(P)_{red}$ is a simply connected configuration (a tree) of $\mathbb{P}^1$'s.*

PROOF: Say $\Gamma_{ij} = X_i \cap X_j$ if this is not empty. Then the cohomology of the short exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}_0} \to \oplus \mathcal{O}_{X_i} \to \oplus \mathcal{O}_{\Gamma_{jk}} \to 0$$

shows that $\oplus H^1(\mathcal{O}_{X_i}) \to \oplus H^1(\mathcal{O}_{\Gamma_{jk}})$ is an isomorphism. Now, if the map $H^1(\mathcal{O}_{X_i}) \to H^1(\mathcal{O}_{\Gamma_{jk}})$ is non-zero for some Γ_{jk}, then there is a birational morphism $\pi : X_i \to \Gamma_{jk} \times \mathbb{P}^1$ with $\pi_*\mathcal{O} = \mathcal{O}$, and so $H^1(\mathcal{O}_{X_i}) \to H^1(\mathcal{O}_{\Gamma_{jk}})$ is an isomorphism for every $\Gamma_{jk} \subset X_i$. Also, if $H^1(\mathcal{O}_{\Gamma_{jk}}) \neq 0$, then there exists X_i containing Γ_{jk} such that $H^1(\mathcal{O}_{X_i}) \to H^1(\mathcal{O}_{\Gamma_{jk}})$ is an isomorphism. So, if $\#\{\Gamma_{jk} | H^1(\mathcal{O}_{\Gamma_{jk}}) \neq 0\} = r$, then (since a disjoint union of trees has more vertices than edges) $\#\{X_i | H^1(\mathcal{O}_{X_i}) \neq 0\} \geq r + 1$, which is impossible. $\square$

Now suppose that C, D are irreducible components of S_0 that meet at a point P. Say $Y = f^{-1}(C)$, $Z = f^{-1}(D)$, so that $Y \cap Z = f^{-1}(P)$, a tree of $\mathbb{P}^1$s.

Recall that S has an RDP of type A_{N-1} at P for some $N \geq 1$. Then the henselization S^{hens} can be written as a geometric quotient $S^{hens} = [S'/(\mathbb{Z}/N)]$ where S' is smooth and local and $S' \to \Delta$ is semi-stable. Let $\mathcal{X}'$ denote the normalization of $\mathcal{X}^{hens} \times_{S^{hens}} S'$ with induced morphism $f' : \mathcal{X}' \to S'$ and let P' denote the inverse image of P in S'.

Since $\mathbb{Z}/N$ acts freely in codimension 1 on S', it also does so on $\mathcal{X}'$. So the quotient map $\pi : \mathcal{X}' \to [\mathcal{X}'/(\mathbb{Z}/N)] = \mathcal{X}^{hens}$ is étale outside the 1-dimensional sublocus $f^{-1}(P)$ of $\mathcal{X}^{hens}$ and so, in particular, is étale in codimension 1. Therefore $\mathcal{X}'$ has canonical singularities, so is Cohen–Macaulay. Since $f' : \mathcal{X}' \to S'$ is dominant with equi-dimensional fibres and S' is smooth, f' is flat.

Suppose that $\Delta \to \Delta$ is a finite base change, say of order L. Then the same argument shows that $\mathcal{X}' \times_\Delta \Delta$ also has canonical singularities, and so $\mathcal{X}'_0$ has slc singularities.

Note that $\mathcal{X}^{hens}_0 = [\mathcal{X}'_0/(\mathbb{Z}/N)]$.

We have $S'_0 = C' \cup D'$, where (C', P') and (D', P') are irreducible smooth germs on which $\mathbb{Z}/N$ acts effectively in such a way that $C^{hens} = [C'/(\mathbb{Z}/N)]$ and $D^{hens} = [D'/(\mathbb{Z}/N)]$. Then $\mathcal{X}'_0 = Y' \cup Z'$ where $Y' = f'^{-1}(C')$ and $Z' = f'^{-1}(D')$. Of course, the induced morphisms $p : Y' \to C'$ and $q : Z' \to D'$ are proper; they are partial normalizations of $Y \times_C C'$ and $Z \times_D D'$. Since $\mathbb{Z}/N$ acts freely outside a finite set, and since Y, Z are Cohen–Macaulay, we can identify $Y = [Y'/(\mathbb{Z}/N)]$ and $Z = [Z'/(\mathbb{Z}/N)]$.

Since C', D' are principal divisors on S', the divisors Y' and Z' are principal on $\mathcal{X}'$. We can write $Y' = (x = 0)$ and $Z' = (y = 0)$, where x, y are eigenfunctions of $\mathbb{Z}/N$ and xy is $\mathbb{Z}/N$-invariant.

Extend the ADE notation in the usual way, to include $A_0 = \mathbb{A}^2$, $A_\infty = (xy = 0)$ and $D_\infty = (x^2 = y^2z)$.

Suppose that ξ is a closed point of $f'^{-1}(P')$. Localize $\mathcal{X}', Y', Z'$ at ξ to get a 3-fold germ $\mathcal{X}''$ and principal divisors Y'', Z'' on it. The stabilizer of ξ is a subgroup $H \cong \mathbb{Z}/M$ of $\mathbb{Z}/N$ that acts freely on $\mathcal{X}'' - \{\xi\}$. Put $\mathcal{X}^{loc} = [\mathcal{X}''/H]$ etc.; these are localizations of $\mathcal{X}, Y$ and Z.

Lemma 10.5 *There are just two possibilities.*

(1) $\mathcal{X}''$ is smooth, $\mathcal{X}''_0 = A_\infty$, $Y'' \cap Z''$ is smooth and $\mathcal{X}^{loc}_0 = [A_\infty/\frac{1}{M}(a, -a, 1)]$ where $M \in \{1, 2, 3, 4, 6\}$ and a is prime to M.

(2) $\mathcal{X}_0''$ is a degenerate cusp of multiplicity at most 4, Y'', Z'' are of type A, $Y'' \cap Z''$ is a nodal curve, $M \in \{1,2\}$ and $\mathcal{X}_0^{loc}$ is either a degenerate cusp or the geometric quotient of a degenerate cusp by $\mathbb{Z}/2$.

PROOF: We know that $\mathcal{X}''$ is canonical, $\mathcal{X}_0'' = Y'' \cup Z''$ is slc and that Y'', Z'' are principal divisors on $\mathcal{X}''$.

Next, the classification of slc singularities with at least two branches shows that the curve $Y'' \cap Z''$ is either smooth or a plane node. Then in the first case $\mathcal{X}''$ is smooth, $\mathcal{X}_0'' = A_\infty$ and $\mathcal{X}_0^{loc} = A_\infty/\frac{1}{M}(a, -a, 1)$ while in the second case $\mathcal{X}_0''$ is a degenerate cusp and Y'', Z'' are of type A.

Because $p : Y' \to C'$ is a Jacobian semi-stable family of elliptic curves and H acts effectively on Y', the classification of automorphism groups of elliptic curves shows that $M \in \{1,2,3,4,6\}$. If the fibre $Y' \cap Z'$ over P' of $Y' \to C'$ is singular, then $M \in \{1,2\}$. $\qquad\square$

Recall that $p^{-1}(P') = q^{-1}(P') = Y' \cap Z'$ and that $f^{-1}(P)$ is set-theoretically, equal to $[Y' \cap Z'/(\mathbb{Z}/N)]$.

Corollary 10.6 (1) $\mathcal{X}'$ is Gorenstein, Y' and Z' have only singularities of type A and their canonical classes are linearly equivalent to zero.
(2) The fibre $p^{-1}(P')$ is nodal and is either an elliptic curve or a cycle of rational curves.
(3) If $f^{-1}(P)$ is irreducible then $\mathcal{X}$ has only quotient singularities along $f^{-1}(P)$. The possible configurations are $\{\frac{1}{6}(1,5,1) + \frac{1}{3}(1,2,1) + \frac{1}{2}(1,1,1)\}, \{2 \times \frac{1}{4}(1,3,1) + \frac{1}{2}(1,1,1)\}, \{3 \times \{\frac{1}{3}(1,2,1)\}$ and $\{4 \times \frac{1}{2}(1,1,1)\}$.

PROOF: (1): By Lemma 10.5 $\mathcal{X}_0''$ has Gorenstein singularities so $\mathcal{X}_0'$ is everywhere locally Gorenstein. Since Y' and Z' have singularities of type A so do Y' and Z'.

Now $f' : \mathcal{X}' \to S'$ is generically Jacobian, and so Jacobian over $S' - \{P'\}$. Therefore $K_{\mathcal{X}'-f'^{-1}(P')}$ is linearly equivalent to the pull back of a line bundle on $S' - \{P'\}$, and so $K_{\mathcal{X}'-f'^{-1}(P')} \sim 0$. Since $\mathcal{X}'$ is Gorenstein, $K_{\mathcal{X}'} \sim 0$; the triviality of $K_{Y'}$ and $K_{Z'}$ is then immediate.

(2): The fact that $p^{-1}(P')$ is nodal follows from the description given in Lemma 10.5. The rest follows from the triviality of $K_{Y'}$.

(3): This follows from Lemma 10.5. $\qquad\square$

Lemma 10.7 There are two possibilities.
(1) The fibres over P of Y and Z are both of type $\overline{D}_n$ for some $n \geq 4$.
(2) One fibre is of type $\overline{R}$ and the other of type $\overline{R}^*$ for $R = II, III$ or IV.

PROOF: This follows from the discussion just given. $\qquad\square$

Lemma 10.8 Y and Z are Cohen–Macaulay everywhere and are smooth at each generic point of $f^{-1}(P)$.

PROOF: This is a consequence of the classification of slc singularities. $\qquad\square$

Lemma 10.9 *Suppose that $p : Y \to C$ is generically smooth and that $p^{-1}(P)$ is of type either $\overline{D}_4$ or $\overline{R}$ or $\overline{R}^*$. Then $Z \to D$ is also generically smooth.*

PROOF: If $q : Z \to D$ were generically a cycle of rational curves then $q^{-1}(P)$ would be of type $\overline{D}_n$ for some $n \geq 5$. $\square$

For any irreducible component C_i of S_0 let C_i^0 denote the complement in C_i of the singular points of S_0.

Lemma 10.10 *Suppose that $X_i = f^{-1}(C_i) \to C_i$ is generically smooth. Then X_i is normal, and over C_i^0 it is Gorenstein.*

PROOF: Let $P \in C_i^0$. Note that $S_0^{hens} = C_i^{hens}$. Then $\mathcal{X}^{hens} - X_i^{hens} \to S^{hens} - S_0^{hens}$ is a Jacobian elliptic fibration, so that $K_{\mathcal{X}^{hens} - X_i^{hens}}$ pulls back from a line bundle on $S^{hens} - S_0^{hens}$ and so is trivial. Since X_i^{hens} is an irreducible and principal Weil divisor in $\mathcal{X}^{hens}$, the homomorphism $\mathrm{Cl}(\mathcal{X}^{hens}) \to \mathrm{Cl}(\mathcal{X}^{hens} - X_i^{hens})$ is an isomorphism. Therefore $K_{\mathcal{X}^{hens}}$ is trivial. So $\mathcal{X}$ is Gorenstein along $f^{-1}(P)$ and $K_{\mathcal{X}^{hens}}$ pulls back from a line bundle on S^{hens}. Therefore X_i^{hens} is also Gorenstein and $\omega_{X_i^{hens}}$ also pulls back from a line bundle on C_i^{hens}.

Suppose that $\widetilde{X}_i \to \widetilde{X}_i^{min}$ is the minimal model. Then $K_{\widetilde{X}_i^{min}}$ pulls back from a line bundle on C_i; comparison of this with the description of $\omega_{X_i^{hens}}$ just given shows that $h^2(\widetilde{X}_i, \mathcal{O}) < h^2(X_i, \mathcal{O})$ if X_i is not normal along $f^{-1}(P)$. So X_i is normal and Gorenstein over C_i^0.

Normality follows from Lemma 10.8. $\square$

Lemma 10.11 *X_i has rational singularities.*

PROOF: From our knowledge of the structure of $\mathcal{X}_0$ along the curves $X_i \cap X_j$ we only need to prove that X_i has RDPs over C_i^0. Since $\mathcal{X}_0$ has slc singularities, we need only exclude cusps and simply elliptic singularities.

So suppose that there is at least one such irrational singularity. Let $\widetilde{X}_i \to X_i$ be the minimal resolution. Since $\widetilde{X}_i$ is rational or elliptic ruled, it follows that $H^1(X_i, \mathcal{O}_{X_i}) = 0$. Consider the natural morphism $\coprod \widetilde{X}_i \to \mathcal{X}_0$. From the assumption on monodromy, the induced homomorphism $\widetilde{\phi} : H^2(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}) \to \oplus H^2(\widetilde{X}_i, \mathcal{O}_{\widetilde{X}_i})$ is an isomorphism. It factors through the natural homomorphism $\phi : H^2(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}) \to \oplus H^2(X_i, \mathcal{O}_{X_i})$; this latter map is surjective, and so both ϕ and the natural map $H^2(X_i, \mathcal{O}_{X_i}) \to H^2(\widetilde{X}_i, \mathcal{O}_{\widetilde{X}_i})$ is an isomorphism. Since $H^1(\widetilde{X}_i, \mathcal{O}_{\widetilde{X}_i}) = 0$ the lemma is proved. $\square$

Lemma 10.12 *Suppose S_0 is regarded as a tree and that C_i is an end of S_0. Suppose that C_i meets C_j with $P = C_i \cap C_j$, that $X_i = f^{-1}(C_i) \to C_i$ is generically smooth and that $f^{-1}(P)$ is irreducible. Then X_i is a special elliptic surface.*

PROOF: Assume not; then X_i is rational. Pick a (-1)-curve $\widetilde{E}$ on the minimal resolution $\widetilde{X}_i$ of X_i, and let E denote its image on X_i.

Now $f^{-1}(P)$ is in fact of type $\overline{D}$, $\overline{R}$ or $\overline{R}^*$. Examination of each case leads to an evaluation of $K_{X_i}.E$ and (from Corollary 10.1 3) $X_j.E$; since $X_i.E = -X_j.E$ the adjunction formula then shows that $K_\chi.E < 0$, which is absurd.

For example, if $f^{-1}(P)$ is of type $\overline{D}$, then $K_{X_i}.E = -1$ and $X_j.E = 1/2$, so that $K_\chi.E = -1/2$. The other six cases are handled similarly. $\qquad\square$

In particular, if X_i is an end (that is, if C_i is an end of the tree S_0), is generically smooth over C_i and is rational, then X_i must meet its neighbour X_j in a fibre of type $\overline{D}_{\geq 5}$.

Suppose that X_0 is special elliptic and that $X_0, X_1, ..., X_n$ is a maximal chain such that each of $X_1, ..., X_n$ is generically smooth over its base curve and is rational. Say $\phi_i = X_i \cap X_{i+1} = f^{-1}(P_i)$ when $C_i \cap C_{i+1} = P_i$.

For $T = D_n, R, R^*$, define $\overline{T}^* = \overline{D}_n, \overline{R}^*, \overline{R}$, respectively. So each ϕ_i is of type $\overline{T}$ on X_i and of type $\overline{T}^*$ on X_{i+1}, for some $\overline{T}$. Say that ϕ is of type $(\overline{T}, \overline{T}^*)$.

Let $\widetilde{X}_i \to C_i$ be the minimal smooth model of $X_i \to C_i$. Say that the total inverse image of ϕ_i on $\widetilde{X}_i$ has Euler characteristic a_i and that the total inverse image of ϕ_i on $\widetilde{X}_{i+1}$ has Euler characteristic b_i. Inspection shows that $a_i + b_i \geq 12$ and that $a_i + b_i = 12$ if and only if ϕ is of type $(\overline{T}, \overline{T}^*)$ for $T = D_4, II, III, IV$. Since ϕ_0 is of type $\overline{D}_4, \overline{D}_4$, $a_0 = b_0 = 6$. Moreover, $c_2(\widetilde{X}_i) \geq b_{i-1} + a_i$, and equality holds of and only if $\widetilde{X}_i$ has no other singular fibres.

Then the preceding discussion shows that there are, *a priori*, three possibilities.

(1) X_n is an end and ϕ_{n-1} is of type $(\overline{D}_{\geq 5}, \overline{D}_{\geq 5})$. Then $a_{n-1} \geq 7$, while

$$12(n-1) = \sum_1^{n-1} c_2(\widetilde{X}_i) \geq \sum_1^{n-1}(b_{i-1} + a_i) = b_0 + 12(n-2) + a_{n-1},$$

which is impossible.

(2) Besides X_{n-1}, X_n only meets special elliptic surfaces and meets at least one such surface. Then, besides ϕ_{n-1}, X_n contains a $\overline{D}_4$ fibre ψ. So

$$12n = \sum_1^n c_2(\widetilde{X}_i) \geq b_0 + 12(n-1) + 6,$$

which shows that none of $X_1, ..., X_n$ meets any other X_j and that each of $\widetilde{X}_1, ..., \widetilde{X}_n$ has only two singular fibres.

(3) X_n meets a further surface X_{n+1} which is not generically smooth over its base and $X_n \cap X_{n+1} = \phi_n$, say, is of type $(\overline{D}_{\geq 5}, \overline{D}_{\geq 5})$. Then

$$12n = \sum_1^n c_2(\widetilde{X}_i) \geq b_0 + 12(n-1) + a_n,$$

which is impossible.

So only (2) can occur.

Lemma 10.13 *If $n \geq 2$ then X_n meets exactly one special elliptic surface. If $n = 1$ then X_n meets exactly two special elliptic surfaces.*

PROOF: Assume that $n \geq 2$. If X_n meets more one special elliptic surface then X_n contains at least three singular fibres ϕ, at least two of which are of type $(\overline{D}_{\geq 4}, \overline{D}_{\geq 4})$ This contradicts $c_2(\widetilde{X}_n) = 12$. □

It follows that $\mathcal{X}_0$ is a tree of singular models of special elliptic surfaces with $\overline{D}_4$ fibres and some rational surfaces $Y_1, ..., Y_m$. The surfaces intersect in $\overline{D}_4$ fibres and the $Y_1, ..., Y_m$ contain exactly two such fibres. The Y_i are arranged in disjoint chains that join pairs of special surfaces, and no Y_i is an end of the tree. Each Y_i is of the form $Y_i = [(E \times \mathbb{P}^1)/\iota]$ and so possesses a projection $r_i : Y_i \to [E/\iota] \cong \mathbb{P}^1$ that is, generically, a ruling whose fibres ϕ_i are sections of $f_j : Y_j \to C_j$. Then $\phi_i.K_{\mathcal{X}/\Delta} = 0$. However, $K_{\mathcal{X}/\Delta}$ pulls back from an ample $\mathbb{Q}$-line bundle on S, so this is absurd and there are no rational surfaces in $\mathcal{X}_0$.

Theorem 10.1 is now proved. □

Note that $\mathcal{Y} \to \Delta$ factors as $\mathcal{Y} \to T$ where $S \to T$ is obtained by contracting those curves $C_j \subset S_0$ that are images of the surfaces Y_j.

Let $\pi_i : \Gamma_i \times E \to \overline{V}_i = [(\Gamma_i \times E)/\iota]$ denote the quotient map. Note that we now know that the elliptic factor E is independent of the index i. Let $\widetilde{V}_i \to \overline{V}_i$ be the minimal resolution; this factorizes as $\widetilde{V}_i \to V_i \to \overline{V}_i$ is the resolution of those nodes that do not lie on any double curve in $\mathcal{Y}_0$.

Say that V_i meets s_i other surfaces V_j in $\mathcal{Y}_0$. That is, V_i contains s_i double curves δ_{ij}, each of which is of type $\overline{D}_4$.

Since each V_i has only nodes, we can assume that, after some finite base change $\Delta \to \Delta$ if necessary, each V_i is smooth outside the double curves δ_{ij}.

Lemma 10.14 *If $B_i \subset V_i$ is a section of $f_i : V_i \to C_i$ then B_i meets each δ_{ij} in a node and $B_i^2 = -(h_i + 1) + s_i/2$.*

PROOF: The strict transform $\widetilde{B}_i$ of B_i on $\widetilde{V}_i$ is a section, so meets each $\widetilde{D}_4$ fibre in an end curve of the $\widetilde{D}_4$ configuration. So B_i meets δ_{ij} is a node. Since $\widetilde{B}_i^2 = -(h_i + 1)$ the lemma follows. □

Proposition 10.15 $\mathcal{Y} \to T$ *has a section.*

PROOF: There is an irreducible Weil divisor $D \subset \mathcal{Y}$ that restricts to a section of $\mathcal{Y} \to T$ over the generic point of Δ. We can write $D \cap V_i = D_i + \psi_i$ where D_i is a section of $f_i : V_i \to C_i$ and ψ_i is an f_i-vertical curve that meets D_i. To show that D is a section it is enough to show that each $\psi_i = 0$.

Recall that $p_g(V_i) = h_i$ and $p_g(\mathcal{Y}_t) = h$ for $t \neq 0$. Then $D^2.\mathcal{Y}_t = -(h+1)$ and $(D_i)_{V_i}^2 = -(h_i + 1) + s_i/2$, since D_i passes through s_i nodes on V_i. So

$$\begin{aligned} -(h+1) &= D^2.\mathcal{Y}_t = \sum D^2.V_i = \sum (D_i + \psi_i)_{V_i}^2 \\ &= -\sum (h_i + 1) + \sum s_i/2 + 2\sum (D_i.\psi_i)_{V_i}. \end{aligned}$$

Now $\sum s_i = 2(r-1)$ and $\sum h_i = h$, so that $\sum (D_i.\psi_i)_{V_i} = 0$. It follows that each $\psi_i = 0$. $\quad\square$

Lemma 10.16 *The singularities of $\mathcal{Y}$ are of index at most 2.*

PROOF: Inspection. $\quad\square$

11 The main results

Our goal is to give a first order description of the period locus PL_h in a neighbourhood of $\mathcal{W}_{1^h}$.

Definition 11.1 *Denote by $\mathcal{JE}^{RDP,n.m.}$ (resp., $\mathcal{JE}^{smooth,n.m.}$) the stack of generalized RDP (resp., smooth) semi-stable Jacobian elliptic surfaces with no monodromy. Its objects over a scheme B are pairs $(\mathcal{X} \xrightarrow{f} S \to B, S')$ with the following properties:*

(1) $f : \mathcal{X} \to S$ and $S \to B$ are projective;

(2) $\mathcal{X} \to B$ and $S \to B$ are flat, Cohen–Macaulay and of relative dimensions 2 and 1, respectively;

(3) S' is a section of $\mathcal{X} \to S$;

(4) $S \to B$ is a pre-semi-stable family of curves of genus zero;

(5) for every geometric point b of B the fibre $\mathcal{X}_b$ of $\mathcal{X} \to B$ is a reduced sum $\mathcal{X}_b = \sum V_i$ of irreducible components V_i each of which is of the form $V_i = f^{-1}(C_i)$ for some irreducible component C_i of S_b;

(6) over the complement of the nodes of S_b each V_i has only RDPs (resp., is smooth) and $V_i \to C_i$ is relatively minimal;

(7) $2S'$ is a Cartier divisor on $\mathcal{X}$;

(8) $2S'$ is f-ample (resp., is f-nef);

(9) the restriction $f_i = f|_{V_i} : V_i \to C_i$ gives the minimal resolution V_i' of V_i the structure of an elliptic surface of genus h_i over C_i on which the strict transform of $S' \cap V_i$ is the identity section;

(10) $\mathcal{X}_b$ has slc singularities;

(11) if $P = C_i \cap C_j$ is a node of S_b then $f^{-1}(P)$ is a fibre of type $\overline{D}_4$ on each of V_i and V_j and the section S' meets $f^{-1}(P)$ in one of the four points on it that is an A_1 singularity on both V_i and V_j;

(12) the rank one sheaf $\omega^{[2]}_{\mathcal{X}/B}$ is the pullback of an invertible sheaf on S.

Lemma 11.2 *If the geometric generic fibre $\mathcal{X}_t$ of $\mathcal{X} \to B$ is smooth then $p_g(\mathcal{X}_t) = \sum h_i$. If also B is a disc then there is no monodromy on $H^2(\mathcal{X}_t, \mathbb{Z})$.*

PROOF: We can assume that B is a disc. The closed fibre is then a tree of surfaces with slc singularities of index 2, and the curves of intersection are rational. Now the lemma follows from well known criteria.   □

$\mathcal{JE}^{RDP,n.m.}$ (resp., $\mathcal{JE}^{smooth,n.m.}$) is a partial compactification of the stack $\mathcal{JE}^{RDP}$ (resp., $\mathcal{JE}^{smooth}$).

Proposition 11.3 *(1) The object $\mathcal{Y} \to T \to \Delta$ of Theorem 10.1 (an output of the MMP) is an object over Δ of $\mathcal{JE}^{RDP,n.m.}$.*

(2) The geometric quotient $\overline{\mathcal{Y}} \to \Delta$ of the stack $\widetilde{\mathcal{Y}} \to \Delta$ of (6.4) (a lifting to Δ of an output over Δ' of a plumbing construction) is an object over Δ of $\mathcal{JE}^{RDP,n.m.}$ or of $\mathcal{JE}^{smooth,n.m.}$ according to whether, away from the curve $\overline{Y}_a \cap \overline{Y}_b = \overline{\phi}$, each surface $\overline{Y}_i$ is an RDP or a smooth elliptic surface.

PROOF: For $\overline{\mathcal{Y}}$ of Proposition 6.18 this is clear. For $\mathcal{Y}$ this follows from Theorem 10.1, Proposition 10.15 and Lemma 10.16.   □

Let the superscript $*$ denote either *smooth* or *RDP*.

Lemma 11.4 *Given an object $(\mathcal{X} \xrightarrow{f} S \to B, S')$ of $\mathcal{JE}^{*,n.m.}$, there are Deligne–Mumford stacks $\widetilde{\mathcal{X}}$ and $\widetilde{S}$ such that*

(1) *there is a flat and projective (so representable) morphism $\tilde{f} : \widetilde{\mathcal{X}} \to \widetilde{S}$ such that $f : \mathcal{X} \to S$ is the geometric quotient of $\tilde{f}$,*

(2) *there is a section $\widetilde{S}'$ of $\tilde{f}$ whose quotient is S',*

(3) *$\widetilde{S} \to B$ is a family of $\mathbb{Z}/2$-curves,*

(4) *each geometric fibre $\widetilde{\mathcal{X}}_b$ is a reduced tree $\widetilde{\mathcal{X}}_b = \sum \widetilde{V}_i$ where each $\widetilde{V}_i$ is a 2-dimensional stack that is an elliptic surface over a component of $\widetilde{S}_b$,*

(5) *$\widetilde{\mathcal{X}}_b$ has normal crossings except (when $* = RDP$) for RDPs,*

(6) *each non-empty intersection $\widetilde{V}_i \cap \widetilde{V}_j$ is of the form $E_{ij}/(-1)$ for some elliptic curve E_{ij}.*

PROOF: This is merely an observation.   □

Proposition 11.5 *$\mathcal{JE}^{RDP,n.m.}$ is smooth.*

PROOF: According to Lemma 11.4 it is enough to show that the deformation theory of one of the 2-dimensional stacks $\widetilde{\mathcal{X}}_b$ is unobstructed. Write $\widetilde{\mathcal{X}}_b = \widetilde{\mathcal{X}}$; then there is a morphism $\tilde{f} : \widetilde{\mathcal{X}} \to \widetilde{S}$ with a section $\widetilde{S}_0$ that describes $\widetilde{\mathcal{X}}$ as an RDP Jacobian elliptic surface over a $\mathbb{Z}/2$-curve S. The section $\widetilde{S}_0$ is then a relatively ample Cartier divisor on $\widetilde{\mathcal{X}}$. Define $\mathcal{L} = \tilde{f}_* \mathcal{O}_{\widetilde{\mathcal{X}}}(\widetilde{S}_0)$ and then define

$\mathbb{P}_{\mathcal{L}} = [(\mathcal{L} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{L}^{\otimes 3}) - \{0\}/\mathbb{G}_m]$. This is a $\mathbb{P}(1, 2, 3)$-bundle over $\widetilde{S}$ and $\widetilde{\mathcal{X}}$ is a sextic divisor in $\mathbb{P}_{\mathcal{L}}$.

Therefore to give the stack $\widetilde{\mathcal{X}}$ lying over $\widetilde{S}$ is to give a line bundle $\mathcal{L}$ on $\widetilde{S}$ and a sextic divisor in $\mathbb{P}_{\mathcal{L}}$. It follows that the deformation theory of $\widetilde{\mathcal{X}}$ is unobstructed.
□

Denote by subscript h the value of the geometric genus.

Lemma 11.6 *The morphism $\mathcal{JE}_h^{smooth} \to \mathcal{JE}_h^{RDP}$ given by passing to the relative canonical model extends to a morphism $\mathcal{JE}_h^{smooth,n.m.} \to \mathcal{JE}_h^{RDP,n.m.}$.*

Proposition 11.7 *The period map $per : \mathcal{JE}_h^{smooth} \to \mathcal{V}_h/\mathfrak{G}$ extends to a morphism $per^+ : \mathcal{JE}_h^{smooth,n.m.} \to \mathcal{V}_h/\mathfrak{G}$ that is proper over a neighbourhood of each $\mathcal{W}_{h_1,...,h_r}$ and that fits into a 2-commutative diagram*

$$
\begin{array}{ccc}
\mathcal{JE}_h^{smooth,n.m.} & \xrightarrow{\ per^+\ } & \mathcal{V}_h/\mathfrak{G} \\
\downarrow & & \downarrow \\
\mathcal{JE}_h^{RDP,n.m.} & \longrightarrow & [\mathcal{V}_h/\mathfrak{G}].
\end{array}
$$

PROOF: The properness follows from Proposition 11.3.   □

Lemma 11.8 *For each $(h_1, ..., h_r)$ and each point x in $\mathcal{W}_{h_1,...,h_r}$, the stack $\mathcal{JE}_h^{smooth,n.m.}$ is smooth at each point lying over x.*

PROOF: Suppose that $(X \to S, S')$ is a geometric point of $\mathcal{JE}_h^{smooth,n.m.}$ that maps to x. Let $r : X \to Y$ be the contraction of all vertical (-2)-curves that lie in the smooth locus of X and are disjoint from S'. Recall the combinatorial description of Y.

(1) $X = \sum_1^r X_i$ and $Y = \sum_1^r Y_i$ where, by assumption, each Y_i is birational to a geometric quotient $[(E_i \times C_i)/\iota]$.

(2) The elliptic curves E_i are all isomorphic since Y is connected.

(3) The configurations $X = \sum X_i$, $Y = \sum Y_i$ and $S = \sum C_i$ are isomorphic trees.

(4) $S'' = r(S')$ is a section of $Y \to S$. $2S''$ is Cartier and is ample relative to S.

(5) Say $\sigma_{ij} = X_i \cap X_j$ if this is non-empty and $\phi_{ij} = r(\sigma_{ij})$. Then σ_{ij} and ϕ_{ij} are fibres of type $\overline{D}_4$ on each of X_i, X_j, Y_i, Y_j, as appropriate.

(6) Y_i has 4 singularities of type A_1 on each ϕ_{ij} and has D_4-singularities disjoint from S'' and from the ϕ_{ij}.

The divisor $2S''$ defines a finite morphism $\rho : Y \to Z = \sum_1^r Z_i$ of degree 2. Say $\psi_{ij} = \rho(\phi_{ij})$. Then, because of the nature of special Jacobian elliptic surfaces,

(1) $Z \to S$ is a $\mathbb{P}^1$-bundle,

(2) the branch locus $B \subset Z$ is $B = B_0 + B_1 + \sum_{ij} \psi_{ij}$ where

(3) $B_0 = \rho(S'')$ is a section of $Z \to S$ and a Cartier divisor on Z,

(4) B_1 is disjoint from B_0,

(5) B_1 is a sum $B_1 = \sum_1^3 D_i$ of three sections D_i of $Z \to S$,

(6) each D_i is a Cartier divisor on Z,

(7) the D_i are linearly equivalent and

(8) B_1 has ordinary triple points over the complement of the nodes in S and, as does B_0, meets each ψ_{ij} transversely.

Conversely, given (Z, B), we recover Y as the double cover of Z branched along $B + \sum_{ij} \psi_{ij}$.

We next prove that we can deform the triple points of B_1 independently, while fixing Z. For this, let Σ denote the set of triple points of B_1 and $\mathcal{I}_\Sigma$ its sheaf of ideals, and consider the short exact sequence

$$0 \to \mathcal{I}_\Sigma^3(B_1) \to \mathcal{O}_Z(B_1) \to \mathcal{O}_Z/\mathcal{I}_\Sigma^3 \to 0$$

of coherent sheaves on Z. It is straightforward to verify that $H^0(Z, \mathcal{I}_\Sigma^3(B_1)) = \mathrm{Sym}^3 H^0(Z, \mathcal{I}_\Sigma(D_1))$ and that then a count of dimensions shows that the map $H^0(Z, \mathcal{O}_Z(B_1)) \to H^0(Z, \mathcal{O}_Z/\mathcal{I}_\Sigma^3)$ is surjective.

It follows that the morphism $\mathrm{Def}_Y \to \prod \mathrm{Def}_{Y,P}$ of deformation spaces, where P runs over the D_4 singularities of Y, is formally smooth. $\qquad\square$

Corollary 11.9 *(1) The second order plumbings $\widetilde{\mathcal{Y}}' \to \Delta'_{r-1}$ of (6.19) can be lifted to a family $\widetilde{\mathcal{Y}} \to S_{t_e}$ over an $(r-1)$-dimensional polydisc.*

(2) The conclusions of Proposition 6.20 hold for $\widetilde{\mathcal{Y}} \to S_{t_e}$.

Now fix an alkane Γ of genus h. Recall the $9h + 9$-dimensional closed subvariety $\mathcal{V}_\Gamma$ of the period space $\mathcal{V}_h$. Via the Torelli theorem and the surjectivity of the period map for K3 surfaces, we regard the points of $\mathcal{V}_\Gamma$ as h-tuples $(Y_1, ..., Y_h)$ of Jacobian elliptic K3 surfaces, one surface for each vertex of Γ, where adjacent surfaces (that is, surfaces that correspond to adjacent vertices of Γ) have isomorphic $\widetilde{D}_4$-fibres.

Let $\mathcal{JE}_\Gamma$ denote the reduced closed substack of $\mathcal{JE}_h^{smooth,n.m.}$ whose geometric points are configurations $x = (V_i | i \in \Gamma)$. So each V_i is K3 and is smooth outside the double locus $D_i = V_i \cap (\cap_{j \neq i} V_j)$. Each component of D_i is a fibre of type $\overline{D}_4$. So $\mathcal{JE}_\Gamma$ equals $(per^+)^{-1}(\mathcal{V}_\Gamma)$. Let $\mathcal{JE}_{1^h}$ denote the substack defined by the condition that each V_i is a special Kummer surface. So $\mathcal{JE}_\Gamma$ equals $(per^+)^{-1}(\mathcal{V}_\Gamma)$ and $\mathcal{JE}_{1^h}$ equals $(per^+)^{-1}(\mathcal{W}_{1^h})$.

Lemma 11.10 $\mathcal{JE}_\Gamma$ *is smooth along* $\mathcal{JE}_{1^h}$.

PROOF: Recall first the easy fact that dim $\mathcal{JE}_\Gamma = 9h + 9$.

Let $x = (V_i | i \in \Gamma)$ be a point in the subvariety $\mathcal{JE}_{1^h}$ of $\mathcal{JE}_\Gamma$. Let $\widetilde{V}_i \to V_i$ be the minimal resolution and $\widetilde{D}_i \subset \widetilde{V}_i$ denote the total transform of D_i. So $\widetilde{D}_i$ is a sum of $\widetilde{D}_4$ fibres. Let $\widetilde{\sigma}_i \subset \widetilde{V}_i$ be the given section. Then the Zariski tangent space $T_x \mathcal{JE}_\Gamma$ is given by

$$T_x \mathcal{JE}_\Gamma = \left\{ (\xi_i) \in H^1(\widetilde{V}_i, T_{\widetilde{V}_i}(-\log(\widetilde{D}_i + \widetilde{\sigma}_i))) : \xi_i|_{V_i \cap V_j} = \xi_j|_{V_i \cap V_j} \right\};$$

this is because each H^1 classifies first order deformations of $\widetilde{V}_i$ that preserve the combinatorial structure of the configuration $\widetilde{D}_i + \widetilde{\sigma}_i$, and then we must impose the condition that when the cross-ratio of the four marked points on each $\overline{D}_4$ fibre varies, it does so in a way that is compatible with the fact that it lies on V_i and V_j.

Then, if V_i is a vertex of Γ whose valency is r,

$$\dim H^1(\widetilde{V}_i, T_{\widetilde{V}_i}(-\log(\widetilde{D}_i + \widetilde{\sigma}_i))) = 20 - (1 + 5r) + r - 1 = 18 - 4r;$$

this is because the first Chern classes of the curves in the configuration $\widetilde{D}_i + \widetilde{\sigma}_i$ on $\widetilde{V}_i$ are not linearly independent in $H^1(\widetilde{V}_i, \Omega^1_{\widetilde{V}_i})$, but rather satisfy $r - 1$ linear conditions. So, if Γ has γ_j vertices of valency j, then

$$\dim T_x \mathcal{JE}_\Gamma = \sum_1^4 \gamma_j(18 - 4j) - (h - 1) = 9h + 9,$$

since $\sum j\gamma_j = 2e$, where $e = h - 1$ is the number of edges in Γ. Therefore $\dim T_x \mathcal{JE}_\Gamma = \dim \mathcal{JE}_\Gamma$ and the smoothness is established. $\square$

Lemma 11.11 *(1) The period map* per^+ *restricts to a morphism* $per^+_\Gamma : \mathcal{JE}_\Gamma \to \mathcal{V}_\Gamma$ *that is an isomorphism over a neighbourhood of* $\mathcal{W}_{1^h}$.

(2) $\mathcal{V}_\Gamma$ *is smooth along* $\mathcal{W}_{1^h}$.

PROOF: We show first that the derivative per^+_* of per^+ is injective at all points x of $\mathcal{JE}_\Gamma$.

As above, $x = (V_i | i \in \Gamma)$ and

$$T_x \mathcal{JE}_\Gamma = \left\{ (\xi_i) \in H^1(\widetilde{V}_i, T_{\widetilde{V}_i}(-\log(\widetilde{D}_i + \widetilde{\sigma}_i))) : \xi_i|_{V_i \cap V_j} = \xi_j|_{V_i \cap V_j} \right\}.$$

Choose a generator ω_i of $H^0(\widetilde{V}_i, \Omega^2_{\widetilde{V}_i}) = H^0(V_i, \omega_{V_i})$; then per^+_* is the linear map defined by contraction of (ξ_i) against the various vectors $(0, ..., 0, \omega_i, 0, ..., 0)$, so is clearly injective.

It follows that per^+_Γ is étale over a neighbourhood of $\mathcal{W}_{1^h}$ and then that $\mathcal{V}_\Gamma$ is smooth along $\mathcal{W}_{1^h}$. Finally, the surjectivity of the period map for K3 surfaces completes the proof of the lemma. $\square$

Define the vector bundle $E_\Gamma \to \mathcal{V}_\Gamma$ by the property that its fibre over the point $(Y_1, ..., Y_h)$ of $\mathcal{V}_\Gamma$ is the vector space spanned by the $h-1$ matrices Π_e, each of rank one, where $e = (i,j)$ runs over the edges of Γ and, in the notation of Proposition 6.20,

$$\Pi_e = [\omega_{Y_i}(P_{ij}), -\omega_{Y_j}(P_{ji})] \otimes [\underline{L}_i, \underline{L}_j].$$

This is a vector bundle of rank $h-1$.

Theorem 11.12 *(1) There is a branch B_Γ of PL_h that contains $\mathcal{V}_\Gamma$.*

(2) To first order B_Γ is, in a neighbourhood of $\mathcal{W}_{1^h}$, the vector bundle E_Γ.

PROOF: Choose a smooth neighbourhood $\mathcal{V}_\Gamma^0$ of $\mathcal{W}_{1^h}$ in $\mathcal{V}_\Gamma$ and let E_Γ^0 denote the restriction of E_Γ to $\mathcal{V}_\Gamma^0$. Then Proposition 6.20 gives a family of surfaces of genus h parametrized by $\mathcal{V}_\Gamma^0 \times S_{h-1}$, where S_{h-1} is an $h-1$-dimensional polydisc, and the image of $\mathcal{V}_\Gamma^0 \times S_{h-1}$ under the period map equals, to first order, precisely E_Γ^0. Since $\dim E_\Gamma = 10h + 8$, which is the number of moduli, and, by the results of [C1] and [C2] (see also the remark below) the period map is generically injective, E_Γ^0 is, to first order, the image of some open piece of the moduli space. $\square$

Theorem 11.13 *(1) The branch B_Γ of PL_h is the unique branch of PL_h that contains $\mathcal{V}_\Gamma$.*

(2) To first order, the period locus PL_h equals the union $\cup_\Gamma E_\Gamma$ of the vector bundles $E_\Gamma \to \mathcal{V}_\Gamma$ in a neighbourhood of $\mathcal{W}_{1^h}$.

PROOF: Proposition 11.7 shows that, in particular, per^+ is proper over some neighbourhood $\mathcal{N}_h$ of $\mathcal{W}_{1^h}$ in PL_h. Set $\mathcal{JE}_h^0 = (per^+)^{-1}(\mathcal{N}_h)$. Note that $\mathcal{V}_\Gamma \subset PL_h$ and put $\mathcal{V}_\Gamma^0 = \mathcal{N}_h \cap \mathcal{V}_\Gamma$. (We could have used this choice of $\mathcal{V}_\Gamma^0$ in the proof of Theorem 11.12.)

Let $E_\Gamma^0 \to \mathcal{V}_\Gamma^0$ be the restriction of E_Γ to $\mathcal{V}_\Gamma^0$. The plumbing construction of Section 6 and the formula of Proposition 6.18 for the derivative of the period matrix show that E_Γ^0 is, to first order, a closed subvariety of $\mathcal{N}_h$. That is, there is, for each Γ, a closed substack $\mathcal{F}_\Gamma$ of $\mathcal{JE}_h^0$ such that per^+ induces an isomorphism $\mathcal{F}_\Gamma \to E_\Gamma^0$ to first order.

Now on one hand $\mathcal{JE}_h^{smooth, n.m.}$, and so $\mathcal{JE}_h^0$, is smooth and, on the other hand, for each $x = (V_1, ..., V_h) \in \mathcal{W}_{1^h}$, Theorem 10.1 shows that $(per^+)^{-1}(x)$ is a finite set which consists of exactly one point for each alkane Γ of genus h. Therefore $(per^+)^{-1}(\mathcal{W}_{1^h}) \cong \coprod_\Gamma \mathcal{W}_{1^h}$ and, in a neighbourhood of $\mathcal{W}_{1^h}$, $\mathcal{JE}_h^0 = \coprod_\Gamma \mathcal{F}_\Gamma$.

So $PL_h = \cup_\Gamma E_\Gamma^0$ in a neighbourhood of $\mathcal{W}_{1^h}$. $\square$

Remark: Note also that the properness of per^+ over a neighbourhood of $\mathcal{W}_{1^h}$ is enough to provide a slight variant of Chakiris' proof [C1], [C2] of generic Torelli for simply connected Jacobian elliptic surfaces.

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King's College, Strand, London WC2R 2LS, U.K.
E-mail address: Nicholas.Shepherd-Barron@kcl.ac.uk