BINOMIAL ARRAYS AND
GENERALIZED VANDERMONDE IDENTITIES

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ABSTRACT. In previous work on Clebsch-Gordan coefficients, certain remarkable hexagonal arrays of integers are constructed that display behaviors found in Pascal's Triangle. We explain these behaviors further using the binomial transform and discrete convolution. Here we begin by introducing the notion of a binomial array and develop several "hockey stick" rules. Then we give an algorithm that generalizes the classical Vandermonde Identity; this produces infinite families of summation formulas, which we use to expand and prove certain combinatorial identities for the Catalan numbers. Finally, we recast the theory in terms of the finite-dimensional representation theory of $SL(2, F)$.

1. INTRODUCTION

Recent work of the author on Clebsch-Gordan coefficients allows for wide application of the methods of combinatorial analysis, driven in particular by generating functions, recurrences, finite group actions, and Pascal's triangle. The basic combinatorial rules established in [4] receive a somewhat thorough application in [5], save for the orthogonality relations. The goal of this work is to draw further analogies from Pascal's triangle, in particular characterizing the first orthogonality relation as a special case of discrete convolution.

Coupling this observation with methods of Dwyer ([6], [7]) and Frankel ([9]), we also explain the invariance property of orthogonality relations under uniform inward or outward shifting of columns. This method yields vast extensions of the summation formulas associated to Pascal's recurrence.

Key to understanding these phenomena is the notion of a binomial array; this concept extends both Pascal's triangle and forward difference tables and has properties in accordance with Riordan arrays [2]. In turn, the binomial array is a natural implementation of the generalized binomial transform. Instead of producing a new sequence from a given sequence, a consideration of all possible binomial transforms and their inverses produce an array of values. The binomial transform here may be regarded as iteration of Pascal's recurrence and its inverse as iteration of alternating partial sums.

Examples of these arrays are both old and well-represented in the literature, including

(1) the extended Pascal triangle (Figure 1; compare with Table 1 in [9]), binomial coefficients, and cumulative (or figurate) numbers,

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TABLE 1. Shapiro's Catalan Triangle for $B_{n,k}$

$n\backslash k$	1	2	3	4	5	6
1	1					
2	2	1				
3	5	4	1			
4	14	14	6	1		
5	42	48	27	8	1	
6	132	165	110	44	10	1

(2) difference tables and the calculus of finite differences (for instance, [11], Ch. 5.3),

(3) moment and correlation calculations in statistics ([6], [7]),

(4) Catalan numbers (A000108) and Catalan triangles (A009766 - in order of increasing detail, see [15], [8], [12], and Figure 5 below; see also Figure 3.2 in [10]),

(5) Bell numbers (A000110) and Bell's triangle (A011971),

(6) Motzkin numbers (A001006) as finite differences of Catalan numbers ([18], p. 126.),

(7) rencontres numbers (A008290) and derangement triangles ([17]),

(8) Clark's triangles (A090850), and

(9) hexagons of Clebsch-Gordan coefficients ([4], [5]; see also [19], III.8.7, p. 188, for finite differences).

Here entries of the OEIS ([16]) are referenced as usual with digits following a leading A.

In addition to providing another setting for Clebsch-Gordan coefficients, these methods allow an extension of summation formulas due to Shapiro. In particular, if we define

$$B_{n,k} = \frac{k}{n}\binom{2n}{n-k},$$

then an extension of the Catalan numbers is given by Table 1. These numbers satisfy the equation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} = B_{n-1,k+1},$$

which may be interpreted as the square of Pascal's recurrence. This triangle occurs as a fundamental example of a Riordan array [2], and it is known ([15]; [14], Theorem 13.1) that the length squared of any row is a Catalan number, as is the dot product of any two rows. Extensions of these two results (Corollaries 8.4 and 8.5) and placement into a wider context, adjunct to the Riordan array model, are two concerns of the present work.

Other new features include

(1) definitions of binomial array and the general binomial transform (section 2),

(2) symmetries of binomial arrays (section 3),

(3) nine hockey stick rules - six long and three short (section 4),

(4) six Vandermonde identities for convolution (sections 5),

(5) examples and two families generalizing the sequence of Catalan numbers (sections 6 through 8), and

-5	-4	-3	-2	-1	0	1	2	3	4	5
1	1	1	1	1	1	1	1	1	1	1
-5	-4	-3	-2	-1		1	2	3	4	5
15	10	6	3	1			1	3	6	10
-35	-20	-10	-4	-1				1	4	10
70	35	15	5	1					1	5
-126	-56	-21	-6	-1						1

FIGURE 1. Extended Pascal's Triangle

(6) an identification of the model in terms of the representation theory for $SL(2, F)$ (section 9).

For notation, we remind the reader of the usual conventions for binomial coefficients, which may be invoked without comment. For any integers k and n,

(1.1)
$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & 0 \leq k \leq n, \\ \frac{(-1)^k(-n+k-1)!}{k!(-n-1)!} & n < 0, \ k \geq 0, \ and \\ 0 & otherwise. \end{cases}$$

See Figure 1, with columns indexed by n and rows indexed by $k \geq 0$. While most of our examples require only the integers, we will assume constants lie in a field F of characteristic zero.

All figures in this work were created using Excel.

2. Binomial Arrays

Let $V = F[[x]]$ be the algebra of power series in one variable with coefficients in F, and let $\mathcal{B} = \{1, x, x^2, \dots\}$ be a basis for the subalgebra of polynomials in V. Every element in V can be uniquely represented as a formal infinite sum of elements in $\mathcal{B}$. Using this representation, V is isomorphic to the ring of sequences $\{a_i\}_{i=0}^{\infty}$.

Definition 2.1. Suppose $\{a_i\}_{i=0}^{\infty}$ is a sequence. We define the **binomial array** $B(a_i)$ to be the lower half-plane array defined as follows:

(1) All entries in the top line equal a_0. We index the top line as row 0 and all rows below in increasing order,
(2) The entry in the zeroth column and i-th row is a_i, and
(3) All other entries are defined using Pascal's recurrence; that is, if the entry in row k and column n is denoted by $a_{k,n}$ then

$$a_{k,n+1} = a_{k-1,n} + a_{k,n}.$$

If $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ is a polynomial of degree m, we define $B(p(x))$ to be the binomial array $B(a_i)$ with $a_i = 0$ for all $i > m$. For general a_i with generating function $p(x)$, we may likewise use $B(p(x))$ to denote $B(a_i)$ when convenient.

Of course, $a_{i,0} = a_i$. Pascal's recurrence allows us to extend the middle column to both the left and right sides of $B(p(x))$. The recurrence occurs as a capital L pattern; the vertical entries sum to define the toe of the L. Knowing any two entries on the L allows calculation of the third.

Definition 2.2. For $n \geq 0$, we define the **(extended) binomial transform** $B^n a_k$ of a_i to be the sum

$$(2.1) \qquad B^n a_k = a_{k,n} = \sum_{i=0}^{n} \binom{n}{i} a_{k-i}.$$

Several definitions of binomial transform appear in the literature; see, for instance, [3] or [17]. One version is given as $B^n a_n$; our definition extends this binomial transform by collecting all such transforms for all consecutive subsequences of a_i.

Definition 2.3. For $n > 0$, the **inverse binomial transform** $B^{-n} a_k$ of a_i is defined by the sum

$$(2.2) \qquad B^{-n} a_k = a_{k,-n} = \sum_{i=0}^{n} (-1)^i \binom{n+i-1}{i} a_{k-i}.$$

If $p(x) = \sum_{i=0}^{\infty} a_i x^i$, then $B^n a_k$ is the coefficient of x^k in $(1+x)^n p(x)$, and the definition follows from the Binomial Theorem. Thus if each column of a binomial array represents coordinates for a power series, Pascal's recurrence is implemented by multiplying $p(x)$ by $(1+x)$ repeatedly, and this process is inverted with division by $(1+x)$. For instance, if

$$B^1 p(x) = (1+x)p(x) = \sum_{i=0}^{\infty} b_i x^i,$$

then

$$b_k = a_k + a_{k-1}.$$

On the other hand, inversion is given by multiplication of power series; if

$$B^{-1} p(x) = \frac{p(x)}{1+x} = (1 - x + x^2 - x^3 + \dots)p(x)$$

$$= \sum_{i=0}^{\infty} b_i x^i.$$

then

$$b_k = a_k - a_{k-1} + \dots + (-1)^k a_0.$$

The formula for the inverse binomial transform then follows from the binomial series expansion: if $n \geq 0$, then

$$(1+x)^{-n} = \sum_{i=0}^{\infty} (-1)^i \binom{n+i-1}{i} x^i$$

Proposition 2.4 (Hockey Stick Rule). *Let $B(a_i)$ be a binomial array. If column n is given, then column $n-1$ is given by alternating partial sums of column n. In particular,*

$$a_{k,n-1} = a_{k,n} - a_{k-1,n} + a_{k-2,n} + \dots + (-1)^k a_{0,n}.$$

In practice, it may be inconvenient to sum to the top line. The sum may be shortened if a higher entry in column $n-1$ is known. That is, one may compute an alternating partial sum by continuing a known alternating partial sum of the same entries.

Proposition 2.5 (Short Hockey Stick Rule)**.** *Suppose* $0 < k_1 < k_2$.

$$a_{k_2,n-1} = (a_{k_2,n} - a_{k_2-1,n} + \cdots + (-1)^{k_2-k_1}a_{k_2+1,n}) + (-1)^{k_2-k_1+1}a_{k_2,n-1}.$$

The following subarrays indicate the pattern of each rule:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & \boxed{1} & 1 \\
6 & 7 & 8 & 9 & \boxed{10} & 11 \\
15 & 21 & 28 & 36 & \boxed{45} & 55 \\
20 & 35 & 56 & 84 & \boxed{120} & 165 \\
29 & 49 & 84 & \boxed{140} & \boxed{224} & 344
\end{bmatrix},
\quad
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
6 & 7 & 8 & \boxed{9} & 10 & 11 \\
15 & 21 & 28 & 36 & \boxed{45} & 55 \\
20 & 35 & 56 & 84 & \boxed{120} & 165 \\
29 & 49 & 84 & \boxed{140} & \boxed{224} & 344
\end{bmatrix}.
$$

Of course,

$$140 = 224 - 120 + 45 - 10 + 1, \qquad 140 = (224 - 120 + 45) - 9.$$

It will often be convenient to change the origin of a binomial array. If column 0 is considered as an initial condition at time $t = 0$, then the rule

$$B^{t_1}B^{t_2}a_i = B^{t_1+t_2}a_i$$

holds; for the system to evolve beyond time t, only the values at time t are needed.

As a consequence, we may shift the entire array to the left or right when convenient, which amounts to a uniform shift in the second index of $a_{k,n}$. Uniform vertical shifting may be performed by multiplying the polynomials for each column by x^k; when k is negative, columns must then allow for values in the ring of Laurent power series. In this work, we require that all values above row 0 equal zero.

We now extend the vector space operations on powers series and sequences to binomial arrays.

Definition 2.6. Let $\{a_i\}_{i=0}^{\infty}$ and $\{b_i\}_{i=0}^{\infty}$ be sequences. If the sum and scalar multiplication operations are defined as usual for matrices, then we have the sum

$$B(a_i) + B(b_i) = B(a_i + b_i)$$

and the scalar multiple

$$rB(a_i) = B(ra_i).$$

In effect, the binomial transform is linear, so binomial arrays respect linear operations. For instance,

Proposition 2.7. *Any finite linear combination of binomial arrays is a binomial array.*

With this proposition in mind, each binomial array may be expressed in terms with the extended Pascal's triangle as the atomic element. Now let $p(x) = 1$; that is, define

$$(2.3) \qquad e_i = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases}, \quad \text{so that} \quad e_{k,n} = \begin{cases} \dbinom{n}{k} & n \geq 0 \\[2mm] (-1)^k \dbinom{-n+k-1}{k} & n < 0. \end{cases}$$

Then $P(0) = B(p(x))$ is just the binomial array with Pascal's triangle on each side; on the right, the rows of the usual Pascal's triangle are oriented as columns from the top line, and, on the left, the rows are oriented as diagonals from the top line

to column -1, now with signs alternating based on row number parity. See Figure 1 above.

Definition 2.8. For $j \geq 0$, define the sequence $\{e_i^j\}_{i=0}^{\infty}$ by

(2.4) $$e_i^j = \begin{cases} 1 & i = j, \\ 0 & otherwise \end{cases};$$

that is, e_i^j is the zero sequence with a single 1 in the j-th position.

Definition 2.9. Fix $j \geq 0$. Define $P(j) = B(e_i^j)$. That is, $P(j)$ is the binomial array corresponding to Pascal's triangle, but with the top row of ones moved to row k and zeros in all rows above.

It follows that every $B(a_i)$ is a possibly infinite linear combination of binomial arrays of type $P(k)$. Specifically,

$$B(a_i) = \sum_{i=0}^{\infty} a_i P(i),$$

and the sum for a given entry $a_{k,n}$ is always finite. This fact indicates that we should deduce general properties of binomial arrays from those of Pascal's triangle that are linear in nature - and perhaps some that are not.

If $p(x)$ is a polynomial of degree m, then the proper values of $B(p(x))$ are bordered by values with constant absolute value; the constant value is a_0 on the top line, the diagonal from column zero to the lower right has constant value a_m, and, past row m, column -1 alternates in sign and has constant absolute value $|\sum_{i=0}^{m}(-1)^i a_i|$.

3. SYMMETRIES

In this section, we consider symmetries of binomial arrays induced from triangular symmetries. Suppose $p(x)$ is a polynomial of degree m. Then the binomial array $B(p(x))$ is the join of three triangles: a binomial trapezoid progressing in the lower left direction, a overlapping difference table, and a binomial trapezoid progressing to the right. The right-hand side of the binomial array is clear.

To see that a binomial trapezoid occurs on the left-hand side, we may interpret the usual Pascal's recurrence, progressing to the right, as a Pascal's recurrence progressing to the lower left direction with sign corrections. With this rule, the corner of the L is the sum of the right value of the L plus the negation of the value at the top of the L. The initial condition of this trapezoid is the diagonal of entries starting at row 0 of column $-m-1$ and ending at the entry at row $m+1$ in column -1, with sign parity affected by row parity.

The left and right trapezoids are joined by a difference table with sides of length $m+2$ joining the initializing diagonal noted above to the origin. The columns of the triangle are the given by repeatedly dividing $p^*(x) = x^m p(1/x)$ by $x+1$; each column records the quotient with remainder. The remainders give the coefficients in the Taylor expansion of $p^*(x)$ at $x = -1$. For example, the difference table in Figure 2 below expresses the equality

$$p^*(x) = 2x^3 + 5x^2 + x - 6 = 2(x+1)^3 - (x+1)^2 - 3(x+1) - 4.$$

We have immediately

-5	-4	-3	-2	-1	0	1	2	3	4	5
2	2	2	2	2	2	2	2	2	2	2
-5	-3	-1	1	3	5	7	9	11	13	15
6	1	-2	-3	-2	1	6	13	22	33	46
-6	0	1	-1	-4	-6	-5	1	14	36	69
10	4	4	5	4	0	-6	-11	-10	4	40
-27	-17	-13	-9	-4			-6	-17	-27	-23
70	43	26	13	4				-6	-23	-50

FIGURE 2. $B(p(x))$ with $p(x) = -6x^3 + x^2 + 5x + 2$

Proposition 3.1. *The symmetry that interchanges the trapezoids on each side of $B(p(x))$ is induced by reflecting the difference table of $B(p(x))$ along rows, with sign changes along rows based on row parity.*

For example,

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ & -1 & 1 & 3 & 5 \\ & & -3 & -2 & 1 \\ & & & -4 & -6 \\ & & & & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ & -5 & -3 & -1 & 1 \\ & & 1 & -2 & -3 \\ & & & 6 & 4 \\ & & & & 0 \end{bmatrix}$$

Next

Proposition 3.2. *The involution $p(x) \mapsto p^*(x) = x^m p(1/x)$ inverts the columns of $B(p(x))$ with non-negative index. The difference table for $B(p^*(x))$ has values on its left-hand side given by the Taylor coefficients of $p(x)$ at $x = -1$.*

For example,

$$\begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ & -1 & 1 & 3 & 5 \\ & & -3 & -2 & 1 \\ & & & -4 & -6 \\ & & & & 0 \end{bmatrix} \mapsto \begin{bmatrix} -6 & -6 & -6 & -6 & -6 \\ & 19 & 13 & 7 & 1 \\ & & -15 & -2 & 5 \\ & & & 4 & 2 \\ & & & & 0 \end{bmatrix},$$

One readily checks that

$$p(x) = -6x^3 + x^2 + 5x + 2 = -6(x+1)^3 + 19(x+1)^2 - 15(x+1) + 4.$$

Finally one reflects across the diagonal of the left trapezoid by passing to to the right side, reflecting, and passing back to the left. To compute the right trapezoid, one inverts the left-hand side of the difference table with sign changes.

4. HOCKEY STICK RULES

We now describe the various hockey stick rules in any $B(p(x))$. If $p(x)$ is not a polynomial, then there are two hockey stick rules: Proposition 2.4 and its reflection under the symmetry that interchanges trapezoids. If $p(x)$ is a polynomial, we may apply the four-fold symmetries to obtain six hockey stick rules, with two additional rules for each lower edge of the proper region. In addition, each rule has a corresponding short version, yielding three short rules after multiplicity.

Proposition 4.1 (Top Line to Lower Right)**.** *In addition to the top line rule (Proposition 2.4), one also has*

$$a_{0,n} + a_{1,n+1} + \cdots + a_{k,n+k} = a_{k,n+k+1}.$$

With $0 < k_1 < k_2$, the short rule is given by

$$a_{k_1-1,n+k_1} + (a_{k_1,n+k_1} + a_{k_1+1,n+k_1+1} + \cdots + a_{k_2,n+k_2}) = a_{k_2,n+k_2+1}.$$

Proof. The formula follows immediately from Proposition 2.4 after applying the symmetry that interchanges trapezoids. We see readily that the alternation is removed in the case of Pascal's triangle; the case of general $B(a_i)$ follows from linearity. □

The following subarrays indicate the pattern of each rule:

$$
\begin{bmatrix}
\boxed{1} & 1 & 1 & 1 & 1 & 1 \\
6 & \boxed{7} & 8 & 9 & 10 & 11 \\
15 & 21 & \boxed{28} & 36 & 45 & 55 \\
20 & 35 & 56 & \boxed{84} & 120 & 165 \\
29 & 49 & 84 & 140 & \boxed{224} & \boxed{344}
\end{bmatrix},
\quad
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
6 & 7 & \boxed{8} & 9 & 10 & 11 \\
15 & 21 & \boxed{28} & 36 & 45 & 55 \\
20 & 35 & 56 & \boxed{84} & 120 & 165 \\
29 & 49 & 84 & 140 & \boxed{224} & \boxed{344}
\end{bmatrix}.
$$

Of course,

$$224 + 84 + 36 + 28 + 7 + 1 = 344, \qquad (224 + 84 + 28) + 8 = 344$$

Now suppose $p(x)$ is a polynomial of degree m. Using the symmetries, we obtain four additional rules, based off the diagonal on the right-hand side and the centerline of the left-hand side.

Proposition 4.2 (Right-hand Side Rules)**.** *Let $p(x)$ be a polynomial of degree m. Fix $n > 0$, and suppose $k > m$.*

(1) *Along rows from the central column of $B(p(x))$,*

$$a_{k,1} + \cdots + a_{k,n} = a_{k+1,n+1}.$$

(2) *Along columns to the right-hand side diagonal of $B(p(x))$,*

$$a_{k,n} - a_{k+1,n} + \cdots + (-1)^{m+n-k}a_{m+n,n} = a_{k-1,n-1}.$$

The following subarrays indicate the pattern of each rule with $p(x) = 6x^2 - 6x + 3$:

$$
\begin{bmatrix}
3 & 3 & 3 & 3 & 3 & 3 \\
-6 & -3 & 0 & 3 & 6 & 9 \\
6 & 0 & -3 & -3 & 0 & 6 \\
0 & \boxed{6} & \boxed{6} & \boxed{3} & 0 & 0 \\
0 & 0 & 6 & 12 & \boxed{15} & 15
\end{bmatrix},
\quad
\begin{bmatrix}
3 & 3 & 3 & 3 \\
-6 & \boxed{-3} & 0 & 3 \\
6 & 0 & \boxed{-3} & -3 \\
0 & 6 & \boxed{6} & 3 \\
0 & 0 & \boxed{6} & 12 \\
0 & 0 & 0 & 6
\end{bmatrix}.
$$

Of course,

$$6 + 6 + 3 = 15, \qquad -3 + 6 - 6 = -3.$$

Proposition 4.3 (Left-hand Side Rules)**.** *Let $p(x)$ be a polynomial of degree m. Fix $n > 0$ and suppose $k > m$.*

(1) *Along rows from the centerline of $B(p(x))$,*

$$a_{k,-1} + \cdots + a_{k,-n} = -a_{k+1,-n}.$$

(2) *Along diagonals from the central column to the upper left of $B(p(x))$,*

$$a_{k,-1} + a_{k-1,-2} + \cdots + a_{k-n+1,-n} = -a_{k-n,-n}.$$

The following subarrays indicate the pattern of each rule with $p(x) = 6x^2 - 6x + 3$:

$$
\begin{bmatrix}
3 & 3 & 3 & 3 & 3 & 3 \\
-21 & -18 & -15 & -12 & -9 & -6 \\
81 & 60 & 42 & 27 & 15 & 6 \\
-225 & \boxed{-144} & \boxed{-84} & \boxed{-42} & \boxed{-15} & 0 \\
510 & \boxed{285} & 141 & 57 & 15 & 0
\end{bmatrix},
\qquad
\begin{bmatrix}
3 & 3 & 3 & 3 \\
\boxed{-15} & -12 & -9 & -6 \\
\boxed{42} & 27 & 15 & 6 \\
-84 & \boxed{-42} & -15 & 0 \\
141 & 57 & \boxed{15} & 0 \\
-213 & -72 & -15 & 0
\end{bmatrix}.
$$

Of course,

$$-15 - 42 - 84 - 144 = -285, \qquad 42 - 42 + 15 = -(-15).$$

In general, the three short rules apply to any $B(a_i)$. For the remaining short rule,

Proposition 4.4 (Third Short Rule). *Let $B(a_i)$ be any binomial array. Suppose $n_1 < n_2$. Then*

$$a_{k+1,n_1} + (a_{k,n_1} + \cdots + a_{k,n_2}) = a_{k+1,n_2+1}.$$

The following subarray indicates the pattern of this rule:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
6 & 7 & 8 & 9 & 10 & 11 \\
15 & \boxed{21} & \boxed{28} & \boxed{36} & 45 & 55 \\
20 & \boxed{35} & 56 & 84 & \boxed{120} & 165 \\
29 & 49 & 84 & 140 & 224 & 344
\end{bmatrix},
$$

Of course, $35 + (21 + 28 + 36) = 120$.

Alternatively, the short rules may be derived by iterating Pascal's recurrence from a fixed entry in one of three directions: upwards, to the right, and to the upper left. The usual hockey stick rules result when these iterations intersect with a border.

5. Cauchy Product

We now recall the background of the Cauchy algebra for Theorem 6 of [9], recast to fit the present situation. Here we use the binomial transform instead of finite differences; these models are equivalent by interchanging $1 + x$ and $1 - x$.

Definition 5.1 (Convolution). Suppose a and b are sequences with generating functions $p(x)$ and $q(x)$, respectively. Then the **Cauchy product** (or **discrete convolution** if polynomials) of a and b is a new sequence $a * b$ defined by

$$(5.1) \qquad (a * b)_m = \sum_{i+j=m} a_i b_j = \sum_{i=0}^{m} a_i b_{m-i} = [x^m](p(x)q(x)),$$

where $[x^m](p(x)) = a_m$.

Properties of convolution follow naturally from algebraic properties of power series. For instance, we immediately note

Proposition 5.2. *Let a, b, and c be sequences, and let r be in F. Then*

(1) $a * b = b * a$,

(2) $(ra + b) * c = (ra) * c + b * c$, and

(3) $(a * b) * c = a * (b * c)$.

Definition 5.3. Define the sequence e by $e_i = \delta_{i,0}$, the Kronecker delta function.

It will be convenient to use non-standard notation for binomial coefficients in this section.

Definition 5.4. For $k \geq 0$ and any integer n, define $t_k^n = B^n e_k = \binom{n}{k}$.

It follows immediately that, for $n \geq 0$ and any integers n_1 and n_2,

(1) $\sum_{i=0}^{n} t_i^n x^i = (1 + x)^n$,

(2) $\sum_{i=0}^{\infty} t_i^{-n} x^i = (1 + x)^{-n}$, and

(3) $t^{n_1} * t^{n_2} = t^{n_1 + n_2}$.

In the context of binomial transforms,

(5.2) $B^n a_k = a_{k,n} = (t^n * a)_k$ and $B^{-n} a_k = a_{k,-n} = (t^{-n} * a)_k$.

We also have

(1) $e * e = e$,

(2) $e * a = a * e$ for any sequence a, and

(3) $t^n * t^{-n} = e$.

With the multiplication of convolution, the algebra of sequences has no zero divisors, has an identity e, and a sequence a_i is invertible if and only if $a_0 \neq 0$.

With respect to the binomial transform, we have

Theorem 5.5 (Dwyer, Frankel). *Suppose a_i and b_j are sequences. Then, for all integers n,*

(5.3) $B^n a * B^{-n} b = a * b.$

Proof. Formally,

$$B^n a * B^{-n} b = (t^n * a) * (t^{-n} * b)$$
$$= (t^n * t^{-n}) * (a * b)$$
$$= e * (a * b) = a * b.$$

Of course, if $p(x)$ and $q(x)$ represent a and b, respectively, then

$$[x^m]\big((1 + x)^n p(x)(1 + x)^{-n} q(x)\big) = [x]^m \big(p(x)q(x)\big).$$

$\square$

We recommend the interested reader to MR0033265, T. Fort's review of [9].

In the following sections, we apply this theorem to produce infinite families of summation formulas derived from binomial arrays. In section 9, we express this theorem in terms of finite-dimensional representations of $SL(2, F)$.

Using the binomial array construction, the following corollary describes an implementation of this theorem.

Corollary 5.6 (Generalized Vandermonde Identity). *Fix $m \geq 0$, and let a and b be sequences. For any n in $\mathbb{Z}$,*

$$(5.4) \qquad \sum_{i=0}^{m} a_i b_{m-i} = \sum_{i=0}^{m} B^n a_i \, B^{-n} b_{m-i}.$$

To compute using $B(a_i)$ and $B(b_i)$,

(1) *restrict $B(a_i)$ and $B(b_i)$ to the first $m+1$ rows,*
(2) *rotate the restricted $B(b_i)$ by 180 degrees about column zero, and*
(3) *dot products of columns in the same relative position have equal values.*

Example: Consider the finite subarrays centered about column zero with $m = 3$:

$$\begin{bmatrix} 3 & 3 & \boxed{3} & 3 & 3 \\ -2 & 1 & \boxed{4} & 7 & 10 \\ 0 & -2 & \boxed{-1} & 3 & 10 \\ 0 & 0 & \boxed{-2} & -3 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 2 & \boxed{2} & 2 & 2 \\ -2 & 0 & \boxed{2} & 4 & 6 \\ 1 & -1 & \boxed{-1} & 1 & 5 \\ -1 & 0 & \boxed{-1} & -2 & -1 \end{bmatrix}.$$

We rotate the second array to obtain

$$\begin{bmatrix} 3 & 3 & \boxed{3} & 3 & 3 \\ -2 & 1 & \boxed{4} & 7 & 10 \\ 0 & -2 & \boxed{-1} & 3 & 10 \\ 0 & 0 & \boxed{-2} & -3 & 0 \end{bmatrix}, \qquad \begin{bmatrix} -1 & -2 & \boxed{-1} & 0 & -1 \\ 5 & 1 & \boxed{-1} & -1 & 1 \\ 6 & 4 & \boxed{2} & 0 & -2 \\ 2 & 2 & \boxed{2} & 2 & 2 \end{bmatrix}.$$

We see immediately that each of the five dot products between matching columns equal -13.

Without the rotation, the matched columns move about the initial columns in opposite directions by the same increment. If both columns are from a single array, the matching is a telescoping effect.

We note further corollaries to Theorem 5.5:

Corollary 5.7. *Suppose a_i^j are sequences for $1 \leq j \leq k$. Then*

$$B^{n_1} a^1 * B^{n_2} a^2 * \cdots * B^{n_k} a^k = B^{n_1 + \cdots + n_k}(a^1 * a^2 * \cdots * a^k).$$

Corollary 5.8. *Suppose $(a * b)_m = 0$. Then, for n in $\mathbb{Z}$,*

$$(B^n a * B^{-n} b)_m = 0.$$

In particular, if $a_{k,n} = 0$, then $\sum_{i=0}^{k} \binom{l}{i} a_{k-i,n-l} = 0$ for all l in $\mathbb{Z}$.

Finally we may apply the symmetries from section 3 to obtain further Vandermonde identities. For general binomial arrays $B(a_i)$ and $B(b_i)$, a second convolution occurs along the top line, now along diagonals to the right.

Corollary 5.9. *Fix $m \geq 0$, and let $B(a_i)$ and $B(b_i)$ be binomial arrays. For any r, s, t in $\mathbb{Z}$,*

$$\sum_{i=0}^{m} a_{i,r+i} b_{m-i,s-i} = \sum_{i=0}^{m} a_{i,r+i+t} b_{m-i,s-i-t}.$$

In the case $B(a_i)$ and $B(b_i)$ correspond to polynomials, we obtain four additional convolution formulas along the lower edges of the proper regions, two per side. We leave the indexing in these cases to the reader.

6. Examples of Sequential Zeros

We use the following strategy for generating families of summation formulas:

(1) choose a binomial array with an interesting sequence near a regular progression of zeros in the array,
(2) pair with a binomial array with a simple initial condition, and
(3) expand a simple convolution using Corollary 5.6.

Definition 6.1. Suppose the entries of $B(a_i)$ are denoted $a_{k,n}$. If w, v, (resp. u, v') are non-negative (resp. non-zero) integers, we define a **progression of arithmetic type** (v, v') in $B(a_i)$ to be a sequence of the form $c_t = a_{u+tv, w+tv'}$ for all $t \geq 0$. Such a progression c_t^* is called a **sequence of near zeros** if $c_t = 0$ always, c_t and c_t^* have the same type, and $max(|u - u^*|, |w - w^*|) = 1$.

Thus we first identify elementary rules for generating binomial arrays with progressions of zeros. Of course, when $p(x)$ is a polynomial, there is always an infinite triangular region of zeros in $B(p(x))$ with a side on column 0. We note two further types of specific interest: those with initial condition

$$p(x) = sx + r$$

for some nonzero integers r and s, and those with a skew-palindromic initial condition.

Consider the first case with initial condition $p(x) = sx + r$. Using scalar multiplication, this case reduces further to two cases: r is positive and s is negative, or both r and s are positive and $r < s$. In the first case, the progression of zeros occurs on the right-hand side of $B(p(x))$, the second on the left-hand side. It also happens that these are the only proper zeros in $B(p(x))$.

Proposition 6.2. *Suppose r and s are non-zero integers. If $(|r|, |s|) = d > 1$, then $B(sx + r)$ and $B(\frac{sx+r}{d})$ have the same zeros.*

Otherwise suppose r and s are both positive with $(r, s) = 1$. Then

(1) *$a_{k,n}$ is a proper zero of $B(-sx+r)$ if and only if $n = l(r+s)-1$ and $k = lr$ for $l \geq 1$, and*
(2) *$a_{k,n}$ is a proper zero of $B(sx + r)$ if and only if $n = l(r - s) + 1$, $k = lr$, and $r < s$ for $l \leq -1$.*

Proof. The first statement follows from the scalar multiplication property.

Next suppose r and s are both positive and relatively prime. Then the general term of $B(-sx + r)$ is given by, with n any integer and $0 < k < n$,

$$(6.1) \qquad a_{k,n} = r \binom{n}{k} - s \binom{n}{k-1}.$$

When $n > 0$, this difference equals

$$\frac{r(n+1) - (r+s)k}{n-k+1} \binom{n}{k}.$$

and all zeros on the right-hand side of $B(-sx + r)$ are found. When $n < 0$, the difference becomes a sum of two terms with the same parity, so no zeros occur on the left-hand side.

The remaining case follows a similar argument. Now the zeros occur on the left-hand side only when $r < s$, and no zeros occur on the right. $\square$

Examples of each case are given in Figures 5 and 6, respectively. In the general case, $B(-sx + r)$ and $B((r + s)x + r)$ interchange trapezoids. Furthermore, we may consider the right-hand side of $B(-sx + r)$ to be a direct limit of hexagons of Clebsch-Gordan coefficients ([4], [5]) under inclusion:

$$(6.2) \qquad M(r, s, 1) \hookrightarrow \frac{1}{2} M(2r, 2s, 1) \hookrightarrow \frac{1}{3} M(3r, 3s, 1) \hookrightarrow \frac{1}{4} M(4r, 4s, 1) \hookrightarrow \dots .$$

We now identify some values adjacent to these zeros.

Definition 6.3 (Figure 3)**.** Fix positive integers r and s. Define the integer sequence

$$C_t^{(r,s)} = \frac{1}{t} \frac{(rt + st)!}{(rt + 1)!(st - 1)!} = \frac{1}{t} \binom{rt + st}{rt + 1}$$

for $t \geq 1$.

Alternatively, one has

$$C_t^{(r,s)} = \frac{r + s}{rt + 1} \binom{rt + st - 1}{rt}$$

This definition gives another generalization of the Catalan numbers C_t, which occur when $r = s = 1$. When $s = 1$, we obtain the Fuß-Catalan numbers.

Up to sign, the sequence $C_t^{(r,s)}$ follows the zeros of $B(-sx + r)$ from below and to the lower right, while the sequence $C_t^{(s,r)}$ follows from above and to the right. When $r = s$, $C_t^{(r,r)} = rC_{rt}$. This case is somewhat degenerate, as rC_t follows the zeros of $B(-rx + r) = rB(-x + 1)$.

Next we define the palindromic conditions:

Definition 6.4. Let $p(x)$ be a polynomial of degree m. $p(x)$ is called **palindromic** if $p(x) = x^m p(1/x)$. On the other hand, if $p(x) = -x^m p(1/x)$, we call $p(x)$ **skew-palindromic**. That is, $p(x)$ is palindromic (resp. skew-palindromic) if,

$$\text{for} \quad 0 \leq k \leq m, \quad a_k = a_{m-k} \text{ (resp. } a_k = -a_{m-k}).$$

If $p(x)$ is palindromic or skew-palindromic, then so is $(1 + x)^n p(x)$ for all $n \geq 0$; that is, a palindromic (resp. skew-palindromic) initial condition produces a binomial array with palindromic (resp. skew-palindromic) columns on the right. If m is even and $p(x)$ is skew-palindomic, then the middle entry equals zero, and the following proposition holds immediately.

Proposition 6.5. *Suppose $p(x)$ is skew-palindromic of degree m.*
  (1) *If $m = 2l$, then $a_l = 0$. There is a diagonal of zeros in $B(p(x))$ to the right in even-numbered columns; specifically, for all k in $\mathbb{N}$, $a_{l+k,2k} = 0$.*
  (2) *If $m = 2l+1$, then $a_{l+1,1} = 0$. There is a diagonal of zeros in $B(p(x))$ to the right in odd-numbered columns; specfically, for all k in $\mathbb{N}$, $a_{l+k+1,2k+1} = 0$.*

The simplest non-trivial example of the skew-palindromic case is given by the Catalan triangle array in Figure 5. Vertical symmetry guarantees that four values about a zero on the diagonal have the same absolute value (below, above, to the right, to the lower right). Using the symmetry that switches the initial condition to the trapezoid on the left, we obtain an unbroken diagonal of zeros on the left. See Figure 6.

We note two general sources of skew-palindromic initial conditions:

r	s	1	2	3	4	5	6	7	8	9	10
1	1	1	2	5	14	42	132	429	1430	4862	16796
1	2	3	10	42	198	1001	5304	29070	163438	937365	5462730
1	3	6	28	165	1092	7752	57684	444015	3506100	28242984	231180144
1	4	10	60	455	3876	35420	339300	3362260	34179860	354465254	3735373880
2	1	1	3	12	55	273	1428	7752	43263	246675	1430715
2	2	4	28	264	2860	33592	416024	5348880	70715340	955277400	13128240840
2	3	10	126	2145	41990	891480	19959975	463991880	11091547350	*	*
2	4	20	396	10608	326876	10925460	385131600	14096061088	*	*	*
3	1	1	4	22	140	969	7084	53820	420732	3362260	27343888
3	2	5	60	1001	19380	408595	9104550	210905400	5028168132	*	*
3	3	15	396	14586	624036	29084535	1432916100	73398801060	*	*	*
3	4	35	1716	117572	9360540	811985790	74462551800	*	*	*	*
4	1	1	5	35	285	2530	23751	231880	2330445	23950355	250543370
4	2	6	110	2856	86526	2861430	100134216	3645533040	*	*	*
4	3	21	1001	67830	5368545	463991880	42443654526	*	*	*	*
4	4	56	5720	832048	141430680	26256481680	*	*	*	*	*

FIGURE 3. Near-zero sequences $C_t^{(r,s)}$ for small parameters

-3	-2	-1	0	1	2	3	4	5	6	7
10	10	10	10	10	10	10	10	10	10	10
-48	-38	-28	-18	-8	2	12	22	32	42	52
132	84	46	18	0	-8	-6	6	28	60	102
-272	-140	-56	-10	8	8	0	-6	0	28	88
468	196	56		-10	-2	6	6	0	0	28
-720	-252	-56			-10	-12	-6	0	0	0
1028	308	56				-10	-22	-28	-28	-28

FIGURE 4. $M(5,5,3)$ as a subarray followed by triangle of zeros

(1) polynomials of the form $p_{r,s}(x) = (1 - x^r)^s$ for positive integers r and s with odd s, and
(2) hexagons of Clebsch-Gordan coefficients ([4], [5]) of the form $M(m,m,k)$ with k odd and $0 < k < m$.

The first case is just the aeration of the coefficients of $(1-x)^s$ with strings of $r-1$ zeros. Then $p_{r,s}$ is palindromic (resp. skew-palindromic) if and only if s is even (resp. odd).

For the second case, the initial condition is given by

$$a_i = (-1)^i \binom{m-i}{k-i} \binom{m-k+i}{i}$$

for $0 \le i \le k$, and $a_i = 0$ otherwise. Since the hexagon $M(r,r,1)$ is a subarray of $B(-rx + r)$, we obtain yet another generalization of the Catalan numbers.

Finally we note another source of zeros from hexagons of Clebsch-Gordan coefficients. If the hexagon $M(m,n,k)$ is subsection of the binomial array $B(a_i)$, then there is a right triangle of zeros of height k to the right of the hexagon. See Figure 4. For a survey of vanishing rules for Clebsch-Gordan coefficients, see [5].

7. EXAMPLE: PASCAL'S TRIANGLE

To set up a general framework, we review the classical case of Pascal's triangle. Recall that the extended Pascal's triangle is $B(e)$ with entries $e_{k,n}$. For fixed n and $k > 0$, we may section off the rectangle from the origin to $e_{k,n}$ to obtain three cases:

$$\begin{bmatrix} e_{0,-n} & \cdots & \boxed{1} \\ e_{1,-n} & \cdots & 0 \\ \cdots & \cdots & \cdots \\ \boxed{e_{k,-n}} & \cdots & 0 \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} \\ 0 \\ \cdots \\ \boxed{e_{k,0}} \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & \cdots & e_{0,n} \\ 0 & \cdots & e_{1,n} \\ \cdots & \cdots & \cdots \\ 0 & \cdots & \boxed{e_{k,n}} \end{bmatrix}.$$

Applying Corollary 5.6, we obtain

$$\sum_{i=0}^{k} e_{i,0-l} e_{k-i,n+l} = e_{k,n},$$

which summarizes as

Proposition 7.1 (Chu-Vandermonde)**.** *Fix integers m and n and positive integer k. Then*

$$\sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}.$$

Of course, we obtain four non-vanishing cases based on sign parity, and a vanishing sum occurs when $0 < m + n < k$. Of course, all cases may be obtained directly and simply using a product of binomial series expansions. In terms of Section 5, this is just item (3) before (5.2):

$$(B^m e * B^n e)_k = B^{m+n} e_k.$$

Now suppose we choose l in Corollary 5.6 such that the columns translate to either the same or adjacent columns and sum over the length of the right-hand column. Then we obtain the classical special formulas:

(7.1) $$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}, \qquad \sum_{i=0}^{n} \binom{n}{i}\binom{n+1}{i} = \binom{2n+1}{n}.$$

Two principles here serve as a model for the general case. We note the first as another corollary to Theorem 5.5.

Corollary 7.2. *For sequences a_i and b_i and integer n, we have*

$$B^n a * B^n b = a * B^{2n} b \quad and \quad B^n a * B^{n+1} b = a * B^{2n+1} b.$$

In general, if $p(x)$ has degree m, then the convolution of a column of $B(p(x))$ with itself is given by

$$\sum_{i=0}^{n+m} a_{i,n} a_{n+m-i,n} = \sum_{i=0}^{m} \binom{2n}{n+m-2i} a_i^2 + 2\sum_{i<j} \binom{2n}{n+m-i-j} a_i a_j,$$

and the convolution of adjacent columns of $B(p(x))$ is given by

$$\sum_{i=0}^{n+m+1} a_{i,n} a_{n+m+1-i,n+1} = \sum_{i=0}^{m} \binom{2n+1}{n+m+1-2i} a_i^2$$
$$+ 2\sum_{i<j} \binom{2n+1}{n+m+1-i-j} a_i a_j.$$

These equalities follow immediately when expressed as

$$[x^{n+m}]((1+x)^{2n}(p(x))^2) \quad and \quad [x^{n+m+1}]((1+x)^{2n+1}(p(x))^2).$$

We obtain simple closed formulas when $m = 1$:

Theorem 7.3. *Fix non-zero integers r and s. For $n > 0$, the columns of $B(sx+r)$ satisfy*

$$\sum_{i=0}^{n+1} a_{i,n} a_{n+1-i,n} = \left[(r+s)^2 n + 2rs\right] C_n$$

and

$$\sum_{i=0}^{n+2} a_{i,n} a_{n+2-i,n+1} = \frac{1}{2}\left[(r+s)^2 n + 4rs + 2s^2\right] C_{n+1}.$$

Of course, choosing odd r and even s in the second equality implies, for positive n, C_n odd implies n odd. In fact ([1]; [14], Theorem 13.1), C_n is odd if and only if $n = 2^k - 1$.

Next the palindromic or skew-palindromic conditions allow further simplification of our sums.

Proposition 7.4. *Suppose $p(x) = a_0 + \cdots + a_m x^m$ and $q(x) = b_0 + \cdots + b_m x^m$ have degree m, and denote the entries of $B(p(x))$ (resp. $B(q(x))$) by $a_{k,n}$ (resp. $b_{k,n}$). If $p(x)$ and $q(x)$ are palindromic, then, for all l in $\mathbb{Z}$,*

$$\sum_{k=0}^{m} a_{k,-l} b_{m-k,l} = \sum_{k=0}^{m} a_k b_{m-k} = 2 \sum_{k=0}^{\frac{m-1}{2}} a_k b_k \quad (m \text{ odd}),$$

$$= a_{\frac{m}{2}} b_{\frac{m}{2}} + 2 \sum_{k=0}^{\frac{m-2}{2}} a_k b_k \quad (m \text{ even}).$$

On the other hand, if $p(x)$ and $q(x)$ are both skew-palindromic, then

$$\sum_{k=0}^{m} a_{k,-l} b_{m-k,l} = \sum_{k=0}^{m} a_k b_{m-k} = -2 \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} a_k b_k.$$

Finally, if $p(x)$ is palindromic and $q(x)$ is skew-palindromic, then

$$\sum_{k=0}^{m} a_{k,-l} b_{m-k,l} = \sum_{k=0}^{m} a_k b_{m-k} = 0.$$

Proposition 7.5. *Suppose $p(x) = a_0 + \cdots + a_m x^m$ has degree m and $q(x) = b_0 + \cdots + b_{m+1} x^{m+1}$ has degree $m + 1$. If $p(x)$ is palindromic, then, for all l in $\mathbb{Z}$,*

$$\sum_{k=0}^{m} a_{k,-l} b_{m-k+1,l+1} = \sum_{k=0}^{m} a_k b_{m-k+1} = \sum_{k=0}^{m} a_k b_{k+1}.$$

If $p(x)$ is palindromic and $b_i = B a_i$, then

$$(B^{-l} a * B^{l+1} a)_{m+1} = (a * a)_m + (a * a)_{m-1}.$$

On the other hand, if $p(x)$ is skew-palindromic, then

$$\sum_{k=0}^{m} a_{k,-l} b_{m-k+1,l+1} = \sum_{k=0}^{m} a_k b_{m-k+1} = -\sum_{k=0}^{m} a_k b_{k+1}.$$

If $p(x)$ is skew-palindromic and $b_i = B a_i$, then

$$(B^{-l} a * B^{l+1} a)_{m+1} = -(a * a)_m + (a * a)_{m-1}.$$

8. Example: Catalan Numbers

In this section, we consider various binomial arrays concerning the Catalan numbers, both directly and indirectly. Following the extended Pascal's triangle, the next simplest initial conditions are $1 \pm x$. Now $1 + x$ is the second column of the extended Pascal's triangle, so we consider the array $B(1 - x)$ and its reflection $B(1 + 2x)$. For this section, we reserve $a_{k,n}$ (resp. $b_{k,n}$) for the values in the former (resp. latter).

One has immediately that

Proposition 8.1 (Figure 5). *For integers k and n,*

$$(8.1) \qquad a_{k,n} = \binom{n}{k} - \binom{n}{k-1}$$

Specifically, for $n > 0$ and $0 < k \leq n$,

$$(8.2) \qquad a_{k,n} = \frac{n - 2k + 1}{n - k + 1} \binom{n}{k}$$

-5	-4	-3	-2	-1	0	1	2	3	4	5
1	1	1	1	1	1	1	1	1	1	1
-6	-5	-4	-3	-2	-1	0	1	2	3	4
20	14	9	5	2		-1	-1	0	2	5
-50	-30	-16	-7	-2			-1	-2	-2	0
105	55	25	9	2				-1	-3	-5
-196	-91	-36	-11	-2					-1	-4
336	140	49	13	2						-1

FIGURE 5. Catalan number array $B(1-x)$: $C_{\frac{n+1}{2}}$ on the right, below/above zeros

and, for $n > 1$ and $k > 0$,

$$(8.3) \qquad a_{k,-1} = (-1)^{k+1}2, \quad a_{k,-n} = (-1)^k \, \frac{-n-2k+1}{n-1} \binom{n+k-2}{k}$$

By the skew-palindromic property or by direct substitution, we obtain a progression of zeros of type $(1,2)$ starting at $a_{1,1}$. If we consider the near zero sequence at points $(t, 2t)$, we have

$$a_{t,2t} = \frac{1}{t+1} \binom{2t}{t} = C_t,$$

the sequence of Catalan numbers. Alternatively, we note

Definition 8.2 (Segner's Recurrence). The Catalan numbers are defined recursively as follows:

$$(8.4) \qquad\qquad C_0 = 1, \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

See the first rows of Figures 3 or 8 for C_1 through C_{10}, or see [14], Appendix B, for the first 100 Catalan numbers. Interpretations of the Catalan numbers by way of enumeration are vast in number; see [18] for over 200 examples.

In Segner's recurrence, we see that the Catalan numbers are defined by iterated convolution. For comparison with Definition 6.3, we also note the alternative formula for $t \geq 1$,

$$(8.5) \qquad\qquad C_t = \frac{1}{t} \frac{(2t)!}{(t+1)!(t-1)!} = \frac{1}{t}\binom{2t}{t+1}.$$

In its right-hand side trapezoid, $B(1-x)$ is subject to all rules governing Clebsch-Gordan coefficients from [4] and [5]; that is, we may consider this trapezoid as a direct limit of hexagons of Clebsch-Gordan coefficients under inclusion:

$$(8.6) \quad \frac{1}{r}M(r,r,1) \hookrightarrow \frac{1}{r+1}M(r+1,r+1,1) \hookrightarrow \frac{1}{r+2}M(r+2,r+2,1) \hookrightarrow \dots.$$

If we now consider $B(1+2x)$, one has

Proposition 8.3 (Figure 6). *For positive integers n and k,*

$$b_{k,n} = \binom{n}{k} + 2\binom{n}{k-1} = \frac{n+k+1}{n-k+1}\binom{n}{k},$$

-5	-4	-3	-2	-1	0	1	2	3	4	5
1	1	1	1	1	1	1	1	1	1	1
-3	-2	-1	0	1	2	3	4	5	6	7
5	2	0	-1	-1		2	5	9	14	20
-5	0	2	2	1			2	7	16	30
0	-5	-5	-3	-1				2	9	25
14	14	9	4	1					2	11
-42	-28	-14	-5	-1						2

FIGURE 6. Catalan number array $B(1 + 2x)$: C_{-n-1} on the left, below zeros

and

$$b_{k,-n} = (-1)^k \left[\binom{n+k-1}{k} - 2\binom{n+k-2}{k-1} \right]$$

$$= (-1)^k \, \frac{n-k-1}{n+k-1} \binom{n+k-1}{k}$$

Since the initial condition is not skew-palindromic, we use Proposition 8.3 to see that the only proper zeros in this array are at points $(t, -t-1)$ for $t \geq 1$. To the left of these zeros, we obtain

$$B_{t,-t-2} = (-1)^t \frac{1}{2t+1} \binom{2t+1}{t} = (-1)^t C_t$$

Now we apply Theorem 7.3 to extend Shapiro's formulas for Catalan triangle numbers in Corollary 8.4 and 8.5. The skew-palindromic property with truncation yields the dot product of rows in Shapiro's Catalan triangle.

Corollary 8.4. *For $n \geq 0$, the sum of squares the values of the n-th column of $B(1-x)$ is $2C_n$. That is,*

$$(8.7) \qquad C_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} a_{i,n}^2,$$

and, for all l in $\mathbb{Z}$,

$$(8.8) \qquad C_n = -\frac{1}{2} \sum_{i=0}^{n+1} a_{i,n-l} a_{n+1-i,n+l}.$$

Corollary 8.5. *For $n \geq 0$, the convolution of the n-th and $(n+1)$-st columns of $B(1-x)$ is $-C_{n+1}$. That is,*

$$(8.9) \qquad C_{n+1} = -\sum_{i=0}^{n+1} a_{i,n} a_{n+1-i,n+1}$$

and, for all l in $\mathbb{Z}$,

$$(8.10) \qquad C_{n+1} = -\sum_{i=0}^{n+1} a_{i,n-l} a_{n+1-i,n+l+1}.$$

Likewise, for $B(1 + 2x)$,

Corollary 8.6. *For $n \geq 0$, the sum of squares the values of the n-th column of $B(1 + 2x)$ is $(9n + 4)C_n$. That is,*

$$(8.11) \qquad (9n + 4)C_n = \sum_{i=0}^{n+1} b_{i,n}^2,$$

and, for all l in $\mathbb{Z}$,

$$(8.12) \qquad (9n + 4)C_n = \sum_{i=0}^{n+1} b_{i,n-l} b_{n+1-i,n+l}.$$

Corollary 8.7. *For $n \geq 0$, the convolution of the n-th and $(n + 1)$-st columns of $B(1 + 2x)$ is $\frac{1}{2}(9n + 16)C_{n+1}$. That is,*

$$(8.13) \qquad (9n + 16)C_{n+1} = 2\sum_{i=0}^{n+1} b_{i,n} b_{n+1-i,n+1}$$

and, for all l in $\mathbb{Z}$,

$$(8.14) \qquad (9n + 16)C_{n+1} = 2\sum_{i=0}^{n+1} b_{i,n-l} b_{n+1-i,n+l+1}.$$

On the other hand, we may directly match the initial condition to Catalan numbers adjacent to zeros in three ways, as noted below:

$$C_{2n-1} = -\sum_{i=0}^{2n} b_{i,-n} b_{2n-i,-n} = -\frac{1}{2}\sum_{i=0}^{2n} b_{i,-n-1} b_{2n-i,-n}$$

$$= \sum_{i=0}^{2n-1} b_{i,-n-1} b_{2n-i-1,-n}.$$

$$C_{2n} = \sum_{i=0}^{2n+1} b_{i,-n} b_{2n-i+1,-n-1} = \frac{1}{2}\sum_{i=0}^{2n+1} b_{i,-n-1} b_{2n-i+1,-n-1}$$

$$= -\sum_{i=0}^{2n} b_{i,-n-1} b_{2n-i,-n-1}.$$

Now we consider the near zero sequence of the binomial array associated to $M(m, m, k)$ with k odd and $0 < k < m$.

Theorem 8.8 (Figure 7). *Suppose $1 < k < m$ with $k = 2k' + 1$. Define $C_t(m, k)$ to be the entry to the right of zero in column $2t - 1$ on the diagonal of the extended Clebsch-Gordan hexagon $M(m, m, k)$. For $t \geq 1$,*

$$C_t(m, k) = (m - k')\binom{m - k' - 1}{k'}\binom{t + 2k' - m}{k'}\binom{t + k' + 1}{k'}^{-1} C_t,$$

where C_t is the t-th Catalan number. When $k = 1$, $C_t(m, 1) = mC_t$.

m	k					t					
		1	2	3	4	5	6	7	8	9	10
5	3	-8	-6	0	28	144	594	2288	8580	31824	117572
6	3	-20	-20	-20	0	120	660	2860	11440	44200	167960
7	3	-40	-45	-60	-70	0	495	2860	12870	53040	209950
8	3	-70	-84	-126	-196	-252	0	2002	12012	55692	235144
9	3	-112	-140	-224	-392	-672	-924	0	8008	49504	235144
10	3	-168	-216	-360	-672	-1296	-2376	-3432	0	31824	201552
11	3	-240	-315	-540	-1050	-2160	-4455	-8580	-12870	0	125970
6	5	2	0	0	8	54	264	1144	4680	18564	72352
7	5	15	6	0	0	45	330	1716	7800	33150	135660
8	5	60	36	20	0	0	220	1716	9360	44200	193800
9	5	175	126	105	70	0	0	1001	8190	46410	226100
10	5	420	336	336	336	252	0	0	4368	37128	217056
11	5	882	756	840	1008	1134	924	0	0	18564	162792
8	7	-2	0	0	0	10	88	520	2600	11900	51680
9	7	-24	-6	0	0	0	66	624	3900	20400	96900
10	7	-140	-56	-20	0	0	0	364	3640	23800	129200
11	7	-560	-280	-160	-70	0	0	0	1820	19040	129200
10	9	2	0	0	0	0	12	130	900	5100	25840
11	9	35	6	0	0	0	0	91	1050	7650	45220

FIGURE 7. Near-zero sequences $C_t(m,k)$ with $1 < k < m$ and k odd

Proof. For an elementary proof, we follow the methods of Proposition 4.2 of [5], but two potential issues require attention: use of the reverse recurrence inside $M(m, m, k)$, and failure of recursion if we encounter the triangle of zeros on the right-hand side of the $M(m, m, k)$. To circumvent these issues, we proceed as follows.

First fix odd $k > 1$; if $k = 1$, then $M(m, m, 1)$ is a subarray of $B(m - mx)$. The initial condition is a column of length $k + 1$, so the shape of the array is fixed as m varies. Writing binomial coefficients as Pochhammer symbols, we may consider the entries of the initial condition as polynomials in m of degree k. Since Pascal's recurrence is linear, all proper values of the array may be considered as polynomials in m of degree less than or equal to k. Since the proper entries of $M(m, m, k)$ are nested by shape in $B(a_i)$ as m increases, the polynomial for a fixed entry in some $M(m, m, k)$ is fixed in the entire array. That is, we may restrict our calculations within some $M(m, m, k)$.

Now the proof follows by calculating the recurrence relation and the initial condition. By the position of zeros and the skew-palindromic property, we have the following subarray relating $C_t(m, k)$ to $C_{t+1}(m, k)$:

$$
\begin{bmatrix}
C_t & * & * & * \\
0 & \boxed{C_t} & \boxed{C_{t+1}} & * \\
-C_t & -C_t & \boxed{0} & C_{t+1} \\
* & * & -C_{t+1} & -C_{t+1}
\end{bmatrix}.
$$

Using the reverse recurrence on the boxed entries with $i = k' + t$ and $j = k' + t + 1$, we obtain

$$
(8.15) \qquad C_{t+1}(m, k) = \frac{2(2t+1)(m-t-k)}{(t+k'+2)(m-t-k'-1)} C_t(m, k).
$$

For the initial condition C_1, we sum the initial values $a_{k'-1}$ and $a_{k'}$

$$
(-1)^{k'-1} \binom{m-k'+1}{k'+2} \binom{m-k'-2}{k'-1} + (-1)^{k'} \binom{m-k'}{k'+1} \binom{m-k'-1}{k'}
$$

to get

$$
(8.16) \qquad C_1(m, k) = (-1)^{k'} \frac{2}{k'+2} \binom{m-k'-2}{k'} \binom{m-k'}{k'+1}.
$$

$\square$

Note that $C_t(m, k)$ is a polynomial in m of degree less than or equal to k, and that $C_t(m, k) = 0$ for $m - 2k' \leq t \leq m - k' - 1$. Otherwise, the t-th zero on the diagonal of $M(m, m, k)$ occurs at coordinates $(k' + t, 2t - 1)$ in $B(a_i)$. So $C_t(m, k)$ is the value at coordinates $(k' + t, 2t)$, which in the notation of [5] is given by

$$
(8.17) \quad c_{m,m,k}(k'+t, k'+t+1) = \sum_{l=0}^{k} (-1)^l \binom{2t}{k'+t-l} \binom{m-l}{k-l} \binom{m-k+l}{l}.
$$

Now we consider the near zero sequence of a general skew-palindromic binomial array $B(d_i)$ with degree $m = 2m' + 1$; if m is even, we may replace the initial condition d_i with $B^1 d_i$. We decompose $B(d_i)$ into a sum of skew-palindromic binomial arrays of the form $B(1 - x^{2i+1})$.

Lemma 8.9 (Figure 8). *Fix* $r \geq 0$. *The diagonal of entries of* $B(-x+1)$ *at coordinates* $(l, r+2l)$ *are given by the sequence*

$$a_{l,r+2l} = \frac{r+1}{r+l+1} \binom{r+2l}{l}.$$

Proof. This follows immediately from substitution into the formula for $a_{k,n}$ above.

$\square$

Remark. The r-th sequence is the $(r+1)$-th convolution power of C_t.

Proposition 8.10. *(a) For $t \geq 1$, let $c_{2i+1}(t)$ be the value to the right of the zero in column $2t-1$ on the diagonal of $B(1 - x^{2i+1})$. Then $c_1(t) = C_t$, and, for $i \geq 1$, $c_{2i+1}(t)$ is given by the values of Lemma 8.9 with $r = 2i$ and $l = t - i$. That is, for $i \geq 1$,*

$$(8.18) \qquad c_{2i+1}(t) = \frac{2i+1}{t+i+1} \binom{2t}{t-i}.$$

This sequence is the $2i$-th row of Figure 8, preceded by $i-1$ zeros.

(b) For $t \geq 0$, let $c_{2i}(t)$ be the value to the right of the zero in column $2t$ on the diagonal of $B(1 - x^{2i})$. Then $c_2(t) = C_{t+1}$, and, for $i \geq 1$, $c_{2i}(t)$ is given by the values of Lemma 8.9 with $r = 2i - 1$ and $l = t - i + 1$. That is, for $i \geq 1$,

$$(8.19) \qquad c_{2i}(t) = \frac{2i}{t+i+1} \binom{2t+1}{t-i+1}.$$

This sequence is the $(2i-1)$-th row of Figure 8, preceded by $i-1$ zeros.

Proof. (a) Suppose the entries of $B(1 - x^{2i+1})$ are denoted by $c_{k,n}$ and $i \geq 1$. Then the entry to the right of the t-th zero has coordinates $(i+t, 2t)$. Recall that $B(x^{2i+1}) = P(2i+1)$, the extended Pascal's triangle shifted down by $2i+1$ rows. Then

$$c_{k,n} = \binom{n}{k} - \binom{n}{k-2i-1}.$$

Thus

$$c_{i+t,2t} = \binom{2t}{i+t} - \binom{2t}{t-i-1} = \frac{(2t)!(2i+1)}{(t+i+1)!(t-i)!},$$

and the lemma holds.

A similar proof holds for part (b). $\square$

By linearity and the skew-palindromic property of the first column of $B(d_i)$, we may express the corresponding binomial array as

$$\sum_{i=0}^{m'} d_i B(x^i(1 - x^{m-2i})).$$

The diagonal zeros occur at coordinates $(m'+t, 2t-1)$, so the t-th value of interest has coordinates $(m'+t, 2t)$. Combining the initial condition with Proposition 8.10, we obtain

$B(1-x)$ (left table), column label l:

r	0	1	2	3	4	5	6	7	8	9
0	1	1	2	5	14	42	132	429	1430	4862
1	1	2	5	14	42	132	429	1430	4862	16796
2	1	3	9	28	90	297	1001	3432	11934	41990
3	1	4	14	48	165	572	2002	7072	25194	90440
4	1	5	20	75	275	1001	3640	13260	48450	177650
5	1	6	27	110	429	1638	6188	23256	87210	326876
6	1	7	35	154	637	2548	9996	38760	149226	572033
7	1	8	44	208	910	3808	15504	62016	245157	961400
8	1	9	54	273	1260	5508	23256	95931	389367	1562275
9	1	10	65	350	1700	7752	33915	144210	600875	2466750
10	1	11	77	440	2244	10659	48279	211508	904475	3798795

$c_k(t)$ (right table), column label t:

k	0	1	2	3	4	5	6	7	8	9
1		1	2	5	14	42	132	429	1430	4862
3		1	3	9	28	90	297	1001	3432	11934
5		0	1	5	20	75	275	1001	3640	13260
7		0	0	1	7	35	154	637	2548	9996
9		0	0	0	1	9	54	273	1260	5508
1	1	1	2	5	14	42	132	429	1430	4862
2	1	2	5	14	42	132	429	1430	4862	16796
4	0	1	4	14	48	165	572	2002	7072	25194
6	0	0	1	6	27	110	429	1638	6188	23256
8	0	0	0	1	8	44	208	910	3808	15504
10	0	0	0	0	1	10	65	350	1700	7752

FIGURE 8. $(1,2)$-progressions $a_{l,r+2l}$ in the array $B(1-x)$(left); sequences $c_k(t)$(right)

Theorem 8.11. *Suppose $p(x)$ is skew-palindromic of degree $m = 2m' + 1$ with corresponding sequence d_i, and $c_k(t)$ is defined as in Proposition 8.10. Then, for $t \geq 1$, the near zero sequence D_t of the binomial array $B(d_i)$ satisfies*

$$(8.20) \qquad D_t = d_{m'+t,2t} = \sum_{i=0}^{m'} d_i\, c_{m-2i}(t),$$

To implement this formula as a dot product, the left-hand side of Figure 8 is separated into rows by parity to give the table on the right-hand side. Note that the columns of the even rows are the rows of Shapiro's Catalan triangle.

Corollary 8.12. *(a) For $t \geq 1$ and $m = 2m' + 1$,*

$$(8.21) \qquad c_{m+1}(t+1) = \sum_{i=0}^{m'} (-1)^i\, c_{m-2i}(t).$$

(b) For $t \geq 1$ and $m = 2m'$,

$$(8.22) \qquad c_{m+1}(t) = (-1)^{m'} C_t + \sum_{i=0}^{m'-1} (-1)^i\, c_{m-2i}(t).$$

Proof. (a) Suppose $d_i = (-1)^i$ for $0 \leq i \leq m$ and zero otherwise. If $p(x)$ corresponds to d_i, then $(1+x)p(x) = 1 - x^{m+1}$, and the result follows.

(b) Suppose $p(x) = \sum_{i=0}^{m'-1} (-1)^i x^i (1 - x^{m-2i})$. Then

$$(1+x)p(x) = 1 - (-1)^{m'} (x^{m'} - x^{m'+1}) - x^{m+1},$$

and the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Of course, this corollary may be realized as a special case of the vertical short hockey stick rule (Proposition 2.5) for $B(1-x)$. Additionally, part (a) is the case of $C_t(m,m)$ when m is odd; compare with Theorem 8.8.

9. Realization through representation theory of $SL(2, F)$

In this section, we recast Corollary 5.6 in terms of the finite-dimensional representation theory of $SL(2, F)$. Recall that F is a field of characteristic zero; in general, we only consider irreducible representations of $SL(2, F)$ with a highest weight n. When $F = \mathbb{R}$ or $\mathbb{C}$, see, for instance, the earlier chapters of [13] for definitions and a convenient model (Chapter 2).

Define $M(2, F)$ to be the set of square matrices of size 2 with coefficients in F. Let $GL(2, F)$ be the group of invertible matrices in $M(2, F)$ under matrix multiplication, and let $G = SL(2, F)$ be the subgroup of matrices with determinant 1. Let $\mathfrak{g}_0 = \mathfrak{sl}(2, F)$ be the Lie algebra of traceless matrices in $M(2, F)$ under the bracket $[X, Y] = XY - YX$, and let $U(\mathfrak{g}_0)$ be the universal enveloping algebra of $\mathfrak{g}_0$. We express the model of previous sections in terms of the representation theory of $SL(2, F)$ and $\mathfrak{sl}(2, F)$.

We choose the usual basis for $\mathfrak{g}_0$ with

$$(9.1) \qquad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Fix $n \geq 0$, and let $(\pi_n, V(n))$ be the irreducible representation of G with highest weight n. When extending π_n to elements of $U(\mathfrak{g}_0)$, we typically omit π_n from the notation. If we choose a highest weight vector ϕ, then $V(n)$ has dimension $n + 1$ with basis given by

$$(9.2) \qquad \mathcal{B}_n = \{\phi, f\phi, \ldots, f^n\phi\}.$$

Up to scalar, $V(n)$ admits a unique non-degenerate invariant bilinear form $\langle \cdot, \cdot \rangle_n$, and we fix this form by the normalization

$$\langle f^n\phi, \phi \rangle_n = 1.$$

If X is in $\mathfrak{g}_0$, invariance implies that

$$\langle Xu, v \rangle_n = \langle u, -Xv \rangle_n.$$

Thus we have for all $0 \leq k, l \leq n$,

$$(9.3) \qquad \langle f^l\phi, f^k\phi \rangle_n = \begin{cases} (-1)^k & l = n - k, \\ 0 & \text{otherwise.} \end{cases}$$

Now this form is symmetric when n is even, alternating when n is odd, and the dual basis to $\mathcal{B}_n$ with respect to $\langle \cdot, \cdot \rangle_n$ is given by

$$(9.4) \qquad \mathcal{B}_n' = \{(-f)^n\phi, \ldots, (-f)\phi, \phi\}.$$

Next let

$$s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and consider the involution $S : SL(2, F) \to SL(2, F)$ defined by

$$Sg = sgs^{-1} \quad \text{or} \quad S\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}.$$

The induced involution on $\mathfrak{sl}(2, F)$ is given by extending

$$e \mapsto -e, \quad f \mapsto -f, \quad h \mapsto h$$

This involution induces an equivalent representation $(\pi_n', V(n))$ by

$$\pi_n'(g)v = \pi_n(Sg)v$$

with induced involution S given on $V(n)$ by extending $S\phi = \phi$ and $f^k\phi \mapsto (-f)^k\phi$. That is, we have an equivalence by $S\pi_n(g) = \pi_n'(g)S$.

Consider the form on $V(n)$ defined by

$$\langle u, v \rangle_n' = \langle u, Sv \rangle_n.$$

Now we have for all $0 \leq k, l \leq n$,

$$(9.5) \qquad \langle f^l\phi, f^k\phi \rangle_n' = \begin{cases} 1 & l = n - k, \\ 0 & \text{otherwise,} \end{cases}$$

and we shall see that this form implements the convolution pairing on polynomials. It is no longer invariant under the action by π_n but rather

$$\begin{aligned} \langle \pi_n(g)u, \pi_n'(g)v \rangle_n' &= \langle \pi_n(g)u, S\pi_n'(g)v \rangle_n \\ &= \langle \pi_n(g)u, \pi_n(g)Sv \rangle_n \\ &= \langle u, Sv \rangle_n \\ &= \langle u, v \rangle_n'. \end{aligned}$$

For v in $V(n)$, define

(9.6) $$B_f : V(n) \to V(n) \qquad \text{by} \qquad B_f v = (1 + f)v.$$

Since $\pi_n(f)$ is nilpotent, B_f is invertible on $V(n)$, and

(9.7) $$B_f^{-1} v = \frac{1}{1 + f} v = (1 + (-f) + \cdots + (-f)^n)v.$$

From above,

$$\langle \pi_n(f)u, v \rangle_n' = \langle u, \pi_n(f)v \rangle_n'.$$

Thus B_f is symmetric with respect to $\langle \cdot, \cdot \rangle_n'$, and it follows immediately that

$$\langle B_f u, B_f^{-1} v \rangle_n' = \langle u, v \rangle_n'.$$

If $p(X)$ and $q(X)$ are polynomials with coefficients in F of degree less than or equal to n, we define

$$\langle\langle p(X), q(X) \rangle\rangle_n = \langle p(f)\phi, q(f)\phi \rangle_n'$$

and reacquire

(9.8) $$\langle\langle B^1 p(X), B^{-1} q(X) \rangle\rangle_n = \langle\langle p(X), q(X) \rangle\rangle_n = [X^n](p(X)q(X)).$$

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