On the poset of king-non-attacking permutations

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1 Introduction

The Hertzsprung's problem is to find the number of ways to arrange n non-attacking kings on an $n \times n$ chess board such that each row and each column contains exactly one king. Let S_n be the symmetric group on n elements. In the squeal, we switch between some notations for permutations. Occasionally, we omit the commas in the writing of $\sigma = [\sigma_1, \ldots, \sigma_n] \in S_n$ as in $[3142] \in S_4$ or even the brackets, as in $31425 \in S_5$.

By identifying a permutation $\sigma = [\sigma_1, \ldots, \sigma_n] \in S_n$ with its plot, i.e. the set of all lattice points of the form (i, σ_i) where $1 \leq i \leq n$, the problem of placing n non-attacking kings reduces to finding the number of permutations $\sigma \in S_n$ such that for each $1 < i \leq n$, $|\sigma_i - \sigma_{i-1}| > 1$. This set is counted in OEIS A002464.

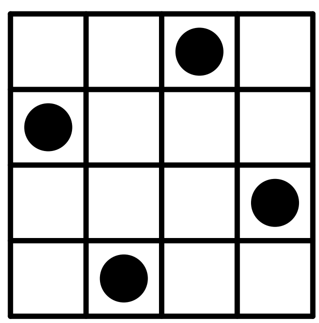


Figure 1: the plot of [3142]

Let K_n be the set of all such permutations in S_n and let us denote $\mathcal{K} = \cup_{n \in \mathbb{N}} K_n$. In this paper we call them simply *king permutations* or just *kings*. For example: $K_1 = S_1, K_2 = K_3 = \emptyset$, $K_4 = \{[3142], [2413]\}$. Observe that K_n is closed to the reverse and inverse actions.

An explicit formula for the number of king permutations was given by Robbins [4]. He also showed that when n tends to infinity, the probability of picking such a permutation from S_n approaches e^{-2}.

In the table below, we present the number of king permutations of order n for low values of n.

n	1	2	3	4	5	6	7	8	9
$\lvert K_n \rvert$	1	0	0	2	14	90	646	5242	47622

Let $\sigma, \pi \in \bigcup_{n \in \mathbb{N}} S_n$. We say that σ *contains* π if there is a sub-sequence of elements of σ that is order-isomorphic to π.

As an example, [3624715] contains [3142] as both the sub-sequences 6275 and 6475 testify. If π is contained in σ, then we write $\pi \preceq \sigma$, while if in addition $\pi \neq \sigma$, it will be written as $\pi \prec \sigma$.

The set of all permutations $\cup_{n\in\mathbb{N}} S_n$ is a poset under the partial order given by containment.

We are interested in the sub-poset $\mathcal{K} = \cup_{n\in\mathbb{N}} K_n$ containing only the king permutations. Its minimal element is the identity permutation [1], and in the next level appear [2413] and [3142].

In order to analyse properties of the poset we are dealing with, one can use the Manhattan or taxicab distance which is defined as follows:

Definition 1.1. *Let $\sigma \in S_n$ and let $i, j \in [n]$. The (Manhattan) distance between the entries σ_i and σ_j is defined to be the L_1 distance between the corresponding points in the plot of σ:*

$$d_\sigma(i,j) = ||(i,\sigma_i) - (j,\sigma_j)||_1 = |i-j| + |\sigma_i - \sigma_j|.$$

The breadth *of an element $\sigma \in S_n$ is defined in [1] to be:*

$$br(\sigma) = min_{i,j\in[n], i\neq j} d_\sigma(i,j).$$

It is easy to observe that for $n > 1$ we have $\pi \in K_n$ if and only if $br(\pi) \geq 3$.

In a paper by Bevan, Homborger and Tenner, [1], the authors define the notion of a *k-prolific permutation.* A permutation $\pi \in S_n$ is called k-prolific if each subset of the letters of $\pi = [\pi_1, \ldots, \pi_n]$ of order $n-k$ forms a unique pattern.

It is shown there that σ is k-prolific if and only if $br(\sigma) \geq k+2$. Hence, $\sigma \in K_n$ if and only if it is 1- prolific.

1.1 Main results

The first main result in this paper claims that the permutations [2413] and [3142] serve as building blocks of the poset of king permutations.

Theorem 1.2. *(See Theorem 3.9) For every $\pi \in K_n$ $(n \geq 4)$, either* $[2413] \preceq \pi$ *or* $[3142] \preceq \pi$.

The following result is a basic ingredient in a series of theorems which explore the structure of the poset $\mathcal{K}$.

Theorem 1.3. *(See Theorem 3.15) For each two king permutations $\pi \prec \sigma$ there exists a chain of king permutations $\pi = \pi^0 \prec \pi^1 \cdots \prec \pi^k = \sigma$ such that $|\pi^i| - |\pi^{i-1}| \in \{1,3\}$.*

We observe that the chains in the poset $\mathcal{K}$ might contain holes. In order to characterize the permutations of which this phenomenon occurs we define $\pi \in K_{n-1}$ to be a *regent* of $\sigma \in K_n$ if $\pi \prec \sigma$.

We give a complete description of all the permutations which have no regents in the following:

Theorem 1.4. *(See Theorem 3.18) The following conditions are equivalent for each $\sigma \in K_n$ with $n \geq 4$.*

1. *There are $\sigma^1, \ldots, \sigma^k \in \{[3142], [2413]\}$ and $\sigma' \in S_k$ such that $\sigma = \sigma'[\sigma^1, \ldots, \sigma^k]$.*
2. *For each $i \in \{1, \ldots, n\}$, by removing i from σ, we get a block of length* 3.
3. *σ has no regents.*

Theorem 1.5. *(See Theorem 3.21) Let $n > 4$. For each $\sigma \in K_n$ there exists $\pi \in K_5$ s.t. $\pi \preceq \sigma$.*

Theorem 1.6. *(See Theorem 4.1) Let $\pi \in K_n$, with $n > 4$. If $[2413] \not\preceq \pi$ or $[3142] \not\preceq \pi$ then $\mu(\pi) = 0$ in the poset of king permutations.*

The rest of the paper is organized as follows. Section 2 contains background material including blocks, simple permutations, inflation, and the Möbius function. In Section 3 we present our main results regarding the structure of the poset of the king permutations. In Section 4 we consider the Möbious function of the poset of the king permutations and we introduce some results regarding this poset.

2 Background

In order to better understand the structure of the poset of king permutations, we start by presenting some preliminaries concerning simple permutations and the Möbius function. Original papers will be mentioned occasionally, but terminology and notation will follow (with a few convenient exceptions) the recent survey [5].

Definition 2.1. *Let $\pi = [\pi_1, \ldots, \pi_n] \in S_n$. A* block *(or* interval*) of π is a nonempty contiguous sequence of entries $\pi_i \pi_{i+1} \ldots \pi_{i+k}$ whose values also form a contiguous sequence of integers. A block of length k will be also called a k*-block*, and it will be called a* strict k-block *if it is not contained in a $k+1$ block.*

Example 2.2. *If $\pi = [2647513]$ then 6475 is a block but 64751 is not.*

Each permutation can be decomposed into singleton blocks, and also forms a single block by itself; these are the *trivial blocks* of the permutation. All other blocks are called *proper*.

Definition 2.3. *A permutation is* simple *if it has no proper blocks.*

Example 2.4. *The permutation [3517246] is simple.*

Definition 2.5. *A* block decomposition *of a permutation is a partition of it into disjoint blocks.*

For example, the permutation $\sigma = [67183524]$ can be decomposed as 67 1 8 3524. In this example, the relative order between the blocks forms the permutation [3142], i.e., if we take for each block one of its digits as a representative then the sequence of representatives is order-isomorphic to [3142]. Moreover, the block 67 is order-isomorphic to [12], and the block [3524] is order-isomorphic to [2413]. These are instances of the concept of *inflation*, defined as follows.

Definition 2.6. *Let $n_1, \ldots, n_k$ be positive integers with $n_1 + \cdots + n_k = n$. The* inflation *of a permutation $\pi \in S_k$ by permutations $\alpha^i \in S_{n_i}$ $(1 \le i \le k)$ is the permutation $\pi[\alpha^1, \ldots, \alpha^k] \in S_n$ obtained by replacing the i-th entry of π by a block which is order-isomorphic to the permutation α^i on the numbers $\{s_i + 1, \ldots, s_i + n_i\}$ instead of $\{1, \ldots, n_i\}$, where $s_i = n_1 + \cdots + n_{i-1}$ $(1 \le i \le k)$.*

Example 2.7. *The inflation of [2413] by [213], [21], [132] and [1] is*

$$2413[213, 21, 132, 1] = 546\ 98\ 132\ 7.$$

Definition 2.8. *Let $\sigma \in K_n$. An element $\tau \in K_{n-1}$ will be called a regent of σ if $\tau \prec \sigma$.*

Example 2.9. *The king permutation $\tau = [41352]$ is a regent of the king permutation $\sigma = [524613]$. $\pi = [3142]$ is regent of τ, but not a regent of σ.*

In order to define the Möbius function of the poset $\mathcal{K}$, we recall the definition of an interval.

Definition 2.10. *The closed interval $[\tau, \sigma]$ is defined as:*

$$[\tau, \sigma] = \{\pi \in \mathcal{K} \mid \tau \preceq \pi \preceq \sigma\}.$$

The half open interval is defined as:

$$[\tau, \sigma) = \{\pi \in \mathcal{K} \mid \tau \preceq \pi \prec \sigma\}.$$

Now, we can define the Möbius function for the poset $\mathcal{K}$:

Definition 2.11.

$$\mu(\tau, \sigma) = \begin{cases} 0, & \text{if } \tau \npreceq \sigma; \\ 1, & \text{if } \tau = \sigma; \\ -\sum\limits_{\pi \in [\tau, \sigma)} \mu(\tau, \pi), & \text{Otherwise.} \end{cases}$$

If $\tau = [1]$, the identity permutation of length 1, then we write $\mu(\pi) := \mu([1], \pi)$.

3 The structure of the poset of kings permutations

In this section we study the structure of the poset of king permutations with respect to the containment relation. We start with the building blocks.

The elements [2413] and [3142] in the poset of simple permutations have a special role as every simple permutation must contain at least one of them (See [3]). As we show below, the same is true for king permutations. In order to do this, we start with the following definitions.

Definition 3.1. *For each permutation $\pi \in S_n$, $\pi = [\pi_1, \pi_2, \dots, \pi_n]$ and $i \in \{1, 2, \dots, n\}$, define the permutation $\nabla_i(\pi) \in S_{n-1}$ by deleting the $i-$th entry of π and standardizing the remaining entries. More precisely: for each $1 \le k \le n-1$, the $k-$th entry of $\nabla_i(\pi)$ is*

for $k < i$:

$$\big(\nabla_i(\pi)\big)_k = \begin{cases} \pi_k, & \pi_k < \pi_i \\ \pi_k - 1, & \pi_k > \pi_i \end{cases}$$

and for $k \ge i$:

$$\big(\nabla_i(\pi)\big)_k = \begin{cases} \pi_{k+1}, & \pi_{k+1} < \pi_i \\ \pi_{k+1} - 1, & \pi_{k+1} > \pi_i \end{cases}$$

Note that in order to omit from π the digit i rather then the entry i we use $\nabla_{\pi^{-1}(i)}(\pi)$.
In order to facilitate the writing, we use $\nabla_i^*(\pi)$ instead of $\nabla_{\pi^{-1}(i)}(\pi)$.
For example, if $\pi = [641325]$ then to omit the digit 4, we use $\nabla_4^*(\pi) = \nabla_2(\pi) = [51324]$.

If one omits two digits from a permutation π in a sequence, it is more convenient notation-wise to omit first the most rightest of the two (in the case of ∇) or the biggest between them (in the case of ∇^*). This is the content of the next observation:

Observation 3.2. *Let $\pi \in S_n$. Then for each $1 \le j < i \le n$:*

1. $\nabla_j(\nabla_i(\pi)) = \nabla_{i-1}(\nabla_j(\pi))$.
2. $\nabla^*_j(\nabla^*_i(\pi)) = \nabla^*_{i-1}(\nabla^*_j(\pi))$

Example 3.3.

$$\begin{tikzcd}
 & {[7\mathbf{4}26\mathbf{1}53]} \arrow[dl, "\nabla^*_4 = \nabla_2"'] \arrow[dr, "\nabla^*_1 = \nabla_5"] & \\
{[625\mathbf{1}43]} \arrow[dr, "\nabla^*_1 = \nabla_4"'] & & {[6\mathbf{3}1542]} \arrow[dl, "\nabla^*_3 = \nabla_2"] \\
 & {[51432]} &
\end{tikzcd}$$

It is easy to see that omitting any digit of a 2-block results in the same permutation. More precisely, we have:

Observation 3.4. *If σ has a 2-block $\sigma_j = a$, $\sigma_{j+1} = a \pm 1$, then $\nabla_j(\sigma) = \nabla_{j+1}(\sigma)$.*

In order to investigate the structure of the poset of king permutations we introduce a new concept called *separator*, as defined here:

Definition 3.5. *For $\sigma = [\sigma_1, \ldots, \sigma_n] \in S_n$ we say that $\boldsymbol{\sigma_i}$* separates *σ_{j_1} from σ_{j_2} in σ if by omitting σ_i from σ we get a* **new** *$2-$block. This happens if and only if one of the following cases holds:*

1. *Separator of type I :j_1, i, j_2 are subsequent numbers and $|\sigma_{j_1} - \sigma_{j_2}| = 1$, i.e*

$$\sigma = [\ldots, \mathbf{a}, \mathbf{b}, \mathbf{a \pm 1}, \ldots]$$

2. *Separator of type II: $\sigma_{j_1}, \sigma_i, \sigma_{j_2}$ are subsequent numbers and $|j_1 - j_2| = 1$, i.e,*

$$\sigma = [\ldots, \mathbf{a}, \ldots, \mathbf{a \pm 1}, \mathbf{a \mp 1}, \ldots]$$

or

$$\sigma = [\ldots, \mathbf{a \pm 1}, \mathbf{a \mp 1}, \ldots, \mathbf{a}, \ldots].$$

Definition 3.6. *Let $Sep_I(\pi)$ and $Sep_{II}(\pi)$ be the sets of separators of π of types I and II respectively.*

Example 3.7. *Let $\sigma = [132465879]$. Then $Sep_I(\sigma) = \{3, 2, 6, 7\}$, and $Sep_{II}(\sigma) = \{3, 2, 5, 8\}$. Note that 7 is a separator of type I, even though 7 is a part of a $2-$block: 87, since by omitting 7 from σ we get a* **new** *$2-$block: 78.*

Remark 3.8. *Several comments are now in order:*

1. *Notice the significance of the word 'new' in Definition 3.5. For example, the identity permutation has plenty of* 2*-blocks even though it has no separators.*
2. *The numbers* 1 *and* n *can only be separators of type* I*,* σ_1 *and* σ_n *can only be separators of type* II*.*
3. *If* σ_i *is a separator of type* I *in* σ *then* i *is a separator of type* II *in* σ^{-1}*. Hence* $Sep_I(\sigma) = Sep_{II}(\sigma^{-1})$
4. $Sep_I(\sigma) = Sep_I(\sigma^r)$ *and* $Sep_{II}(\sigma) = Sep_{II}(\sigma^r)$ *where* σ^r *is the reverse of* σ*.*

In the following Theorem we present our first main result which claims that the permutations [2413] and [3142] serve as building blocks for the poset of king permutations.

Theorem 3.9. *For every* $\pi \in K_n$ $(n \geq 4)$*, either* $[2413] \preceq \pi$ *or* $[3142] \preceq \pi$*.*

Proof. By induction on n. For $n = 4$ this is trivial since $K_4 = \{[3142], [2413]\}$. Let $n > 4$ and assume to the contrary that $\pi \in K_n$ does not contain either [2413] or [3142]. Then $\sigma = \nabla_1^*(\pi)$ contains neither of them as well. By the induction hypotheses, $\sigma \notin K_{n-1}$, which implies, by 3.8, that 1 is a separator of type I. Hence we must have: $\pi = [\ldots, a, 1, a+1, \ldots]$ or $\pi = [\ldots, a+1, 1, a, \ldots]$. Without loss of generality, assume that $\pi = [\ldots, a, 1, a+1, \ldots]$.

Let us check the location of 2 in π. Obviously, $a = 2$ or $a + 1 = 2$ is impossible, so we may assume that $a > 2$. If 2 is located to the right of $a + 1$, then $\pi = [\ldots, a, 1, a + 1, \ldots, 2, \ldots]$ and so $a, 1, a+1, 2$ is a [3142] pattern, so we may assume that 2 is located to the left of a, thus $\pi = [\ldots, 2, \ldots, a, 1, a + 1, \ldots]$. Consider now $\tau = \nabla_{a+1}^*(\pi)$, the result of removing $a + 1$ from π. Since, according to the induction hypothesis, τ must contain a block, we have that $a + 1$ is a separator of type II. (Note that $a + 1$ can not be of type I since the digit 2 is far left.) Hence $\pi = [\ldots, 2, \ldots, a+2, a, 1, a+1, \ldots]$ and $2, a+2, 1, a+1$ forms a [2413] pattern.

□

The following lemma refers to the separators, and will be used later in some theorems related to the structure of the poset of the King permutations.

Lemma 3.10. *Let* $\sigma \in K_n$ *and let* $1 \leq j < i \leq n$*. If* $\nabla_j(\nabla_i(\sigma)) \in K_{n-2}$ *but* $\nabla_i(\sigma) \notin K_{n-1}$ *and* $\nabla_j(\sigma) \notin K_{n-1}$ *then* σ_i *separates* σ_j *from some digit and* σ_j *separates* σ_i *from some digit.*

Proof. If $\nabla_i(\sigma) \notin K_{n-1}$ then σ_i is a separator, i.e., there is a block in $\nabla_i(\sigma)$. Now, $\nabla_j(\nabla_i(\sigma)) \in K_{n-2}$, which can happen only if σ_j is a part of this block.

Recall that by Observation 3.2, $\nabla_j(\nabla_i(\sigma)) = \nabla_{i-1}(\nabla_j(\sigma))$ and the proof is complete by interchanging the rules of σ_i and σ_j.

□

Lemma 3.11. *Let* $\sigma \in K_n$*. Assume that* σ_i *separates* σ_j *from some digit and that* σ_j *separates* σ_i *from some digit. Then* σ_i *and* σ_j *are separators of the same type.*

Proof. Assume to the contrary that σ_i and σ_j are separators of different types. Without loss of generality, σ_i is a separator of type I, so that it separates σ_j from σ_k. Then j, i, k are subsequent numbers. In other words,

$$\sigma = \begin{bmatrix} 1 & \cdots & j & i & k & \cdots \\ \sigma_1 & \cdots & a & b & a \pm 1 & \cdots \end{bmatrix}.$$

Now, if σ_j is a separator of type II which separates σ_i from σ_m then we must have $m = k$ and $\sigma_k = b \pm 2$,and thus $\sigma_j = b \pm 1$, which contradicts the fact that $\sigma \in K_n$.

□

We observe now that if we omit a separator of type I, (separating a from $a+1$) from a permutation $\sigma \in S_n$ and then we omit one of the digits of the resulted new block, or we first omit one of the set $\{a, a+1\}$ and then we remove the separator, the result is the same permutation. The following picture depicts this situation where the digits written next to the edges are the ones we remove:

$$\begin{tikzcd}
 & {[742\mathbf{6}1\mathbf{5}3]} \arrow[dl, "1"'] \arrow[d, "5"] \arrow[dr, "6"] & \\
{[631542]} \arrow[dr, "4\text{ or }5"'] & {[642513]} \arrow[d, "1"] & {[642153]} \arrow[dl, "1"] \\
 & {[53142]} &
\end{tikzcd}$$

The separators of type II have a similar phenomenon which is depicted in the following picture:

$$\begin{tikzcd}
 & {[7426153]} \arrow[dl, "4"'] \arrow[d, "3"] \arrow[dr, "5"] & \\
{[625143]} \arrow[dr, "4\text{ or }3"'] & {[632514]} \arrow[d, "3"] & {[642513]} \arrow[dl, "4"] \\
 & {[52413]} &
\end{tikzcd}$$

Both claims are expressed formally in the following :

Observation 3.12. *Let $\sigma \in S_n$, and assume that σ_i separates σ_j from σ_k in σ. Then for $i > j$:*

$$\nabla_j(\nabla_i(\sigma)) = \nabla_i(\nabla_k(\sigma)) \ \text{ if } j < i < k$$

and

$$\nabla_j(\nabla_i(\sigma)) = \nabla_k(\nabla_i(\sigma)) \ \text{if } i > j > k$$

(By observation 3.2 it is sufficient to consider only the case $i > j$.)

In [3], it is proven that the poset of simple permutations is dense in the sense that for each two simple permutations $\sigma \prec \pi$ there exists a chain of simple permutations $\sigma^0 = \sigma \prec \sigma^1 \cdots \prec \sigma^k = \pi$ such that $|\sigma^i| - |\sigma^{i-1}| \in \{1, 2\}$. In our case, we prove in Theorem 3.14 that for each two king permutations such a chain exists with the stipulation that $|\sigma^i| - |\sigma^{i-1}| \in \{1,3\}$. The first step in this direction is the following:

Theorem 3.13. *Let $\sigma \in K_n$ with $n > 4$, and let $\pi \in K_{n-2}$ be such that $\pi \prec \sigma$. Then there exists $\tau \in K_{n-1}$ such that $\pi \prec \tau \prec \sigma$.*

Proof. Let $\sigma = [\sigma_1, \cdots, \sigma_n] \in K_n$, and $\pi \in K_{n-2}$ be such that $\pi \prec \sigma$.

Hence there are some $i > j$ such that $\pi = \nabla_j(\nabla_i(\sigma))$.

Let

$$G = \{(i,j) \mid \nabla_j(\nabla_i(\sigma)) = \pi, i > j\}.$$

For example, let $\sigma = [361425]$ and $\pi = [2413]$. Then $G = \{(4,3), (5,4), (6,5)\}$.

Assume to the contrary that there is no $\tau \in K_{n-1}$ such that $\pi \prec \tau \prec \sigma$.

According to our assumption, for each $(i,j) \in G$, both $\nabla_i(\sigma)$ and $\nabla_j(\sigma)$ are not in K_{n-1}, which means that each of them contains a block of order 2. Moreover, by Lemma 3.10, σ_i separates σ_j from some digit, and similarly σ_j separates σ_i from some digit and thus, by 3.11, both separators are of the same type.

Let $i_0 = \max\{i \mid (i,j) \in G\}$, and assume that σ_{i_0} separates σ_p from σ_q. We have two cases:

1. The separator σ_{i_0} is of type I. In this case we have w.l.o.g. that $p = i_0 - 1$ and $q = i_0 + 1$, and $|\sigma_p - \sigma_q| = 1$ so that

$$\sigma = \begin{bmatrix} 1 & \cdots & p & i_0 & q & \cdots \\ \sigma_1 & \cdots & a & b & a \pm 1 & \cdots \end{bmatrix},$$

and since the omitting of σ_{i_0} creates a block, which we must get rid of by removing σ_p or σ_q in the passage to $\pi \in K_{n-2}$ and $p < i_0 < q$, we must have by 3.12,

$$\nabla_{i_0}(\nabla_q(\sigma)) = \nabla_p(\nabla_{i_0}(\sigma)) = \pi.$$

This means that $(q, i_0) \in G$ which contradicts the maximality of i_0.

2. The separator σ_{i_0} is of type II. Let $\sigma_{i_0} = a + 1$, so that w.l.o.g. $\sigma_q = a + 2$ and $\sigma_p = a$, i.e.,

$$\sigma = \begin{bmatrix} 1 & \cdots & p & q & \cdots & i_0 & \cdots \\ \sigma_1 & \cdots & a & a+2 & \cdots & a+1 & \cdots \end{bmatrix}.$$

The removal of σ_{i_0} creates a block, which we must get rid of by removing σ_p or σ_q in the passage to $\pi \in K_{n-2}$, thus $\nabla_p(\nabla_{i_0}(\sigma)) \in K_{n-2}$ but $\nabla_{i_0}(\sigma) \notin K_{n-1}$ and $\nabla_p(\pi) \notin K_{n-1}$. As a result, by Lemma 3.10, we have that σ_{i_0} separates σ_p from some digit and σ_p separates σ_{i_0} from some digit, call it σ_k. In the same manner, σ_q separates σ_{i_0} from some σ_m.

Now, by 3.11, σ_p must be a separator of type II, so that $\sigma_k = a - 1$ and $k \in \{i_0 - 1, i_0 + 1\}$ and σ_q must be also a type II separator so that $\sigma_m = a + 3$ and $m \in \{i_0 - 1, i_0 + 1\}$ with $m \neq k$.

If $m = i_0 + 1$, then

$$\sigma = \begin{bmatrix} 1 & \cdots & p & q & \cdots & i_0 & i_0+1 & \cdots \\ \sigma_1 & \cdots & a & a+2 & \cdots & a+1 & a+3 & \cdots \end{bmatrix}$$

and by 3.12, $\nabla_q(\nabla_{i_0}(\sigma)) = \nabla_q(\nabla_{i_0+1}(\sigma)) = \pi$.

Similarly, if $k = i_0 + 1$, then $\nabla_p(\nabla_{i_0}(\sigma)) = \nabla_p(\nabla_{i_0+1}(\sigma)) = \pi$.

In any case, we contradicted the maximality of i_0.

$\square$

Theorem 3.14. *Let σ, π be king permutations such that $\pi \prec \sigma$ and $|\sigma| - |\pi| > 3$. Then there exists a king permutation τ such that $\pi \prec \tau \prec \sigma$ and $|\sigma| - |\tau| \in \{1, 3\}$.*

Proof. Let $\sigma = [\sigma_1, \ldots, \sigma_n]$. In order to pass from σ to π, we have to remove $d = |\sigma| - |\pi|$ different elements from σ. As there is more then one possibility to choose these elements, let F be the set of all sequences of d elements $(l_1, \ldots, l_d)$ such that the removal of them from σ achieves π.

Note that in this proof we chose to work with the digits of the permutation σ rather than with their locations as in previous proofs. This choice is justified by clarity considerations. Here is an example of using the sequence $(5, 6, 7, 8)$ for passing from $\sigma = [314296857]$ to $\pi = [31425]$. We start by omitting 5 from σ in order to get $\delta_1 = \nabla_5^*(\sigma) = [31428576]$. The second step will be to omit the digit 6 from σ, but since $6 > 5$, we actually omit the digit 5 (which fills in for 6) from δ_1 to get $\delta_2 = [3142765]$. Now, removing 7 from σ amounts to omitting 5 from δ_2, so we get $\delta_3 = [314265]$. The last step is to remove 8 from σ, and this is done by omitting 5 from δ_3 so we have $\delta_4 = [31425] = \pi$.

Here is another example: Again, let $\sigma = [314296857]$ and $\pi = [31425]$, but this time the procedure will be much easier since we use the descending sequence $(9, 8, 7, 6)$. We have $\delta_1 = [31426857]$, $\delta_2 = [3142657], \delta_3 = [314265]$ and $\delta_4 = [31425] = \pi$.

We are ready now to get to the proof. Let $f = (l_1, \ldots, l_d) \in F$ be chosen such that $\ell = l_1$ is maximal among all the elements appearing in the sequences of F. If ℓ is not a separator then by omitting it we get a king permutation τ such that $|\sigma| - |\tau| = 1$ and $\pi \prec \tau \prec \sigma$ and we are done. Otherwise, ℓ is a separator. We claim that it is a separator of type I but not of type II.

Indeed, if ℓ is of type II then the consecutive sub-sequence $\ell - 1, \ell + 1$ (or its reverse) appears in σ. After omitting ℓ, we get a block which must be removed in our way to π since π is a king permutation. Hence there is some $f \in F$ which contains $\ell + 1$ and this contradicts the maximality of ℓ.

Back to the proof, since ℓ is a separator of type I, there is some digit a such that σ contains the consecutive sequence $a, \ell, a+1$ or its reverse. By the maximality of ℓ and the fact that $\sigma \in K_n$, we get that $\ell > a + 2$. Let $\delta_1 = \nabla_l^*(\sigma)$ be the permutation obtained from σ by omitting the element ℓ. Without loss of generality, this permutation contains the block $a, a+1$. Now, let δ_2 be the permutation obtained from δ_1 by omitting the element $a + 1$. If δ_2 is a king permutation then by Theorem 3.13 there is some $\tau \in K_{n-1}$ such that $\pi \prec \delta_2 \prec \tau \prec \sigma$ and we are done. Otherwise, since π is a king permutation, $a + 1$ must be a separator in δ_1, and we have one of the following three options:

- The element $a+1$ is a separator of type I and type II in δ_1. In this case, we have the following situation:

$$\begin{array}{rl} \sigma = & \cdots \mathbf{a+2}, \mathbf{a}, \ell, \mathbf{a+1}, \mathbf{a-1} \cdots \\ & \downarrow \\ \delta_1 = \nabla^*_\ell(\sigma) = & \cdots \mathbf{a+2}, \mathbf{a}, \mathbf{a+1}, \mathbf{a-1} \cdots \end{array}$$

Now, $a-1$ is not a separator of type I in σ due to the location of $a+2$ and is not of type II since $\ell \neq a-2$. This means that $\nabla^*_{a-1}(\sigma) \in K_{n-1}$ and we are done.

- The element $a+1$ is a separator only of type I in $\delta_{\mathbf{1}}$. In this case, we have the following situation (note that $x \neq a+2$, since otherwise we are back in the former case):

$$\begin{array}{rl} \sigma = & \cdots x, \mathbf{a}, \ell, \mathbf{a+1}, \mathbf{a-1} \cdots \\ & \downarrow \\ \delta_1 = \nabla^*_\ell(\sigma) = & \cdots x, \mathbf{a}, \mathbf{a+1}, \mathbf{a-1} \cdots \end{array}$$

Now, it is clear that $a+1$ is not a separator in σ. This means that $\nabla^*_{a+1}(\sigma) \in K_{n-1}$ and we are done.

- The element $a+1$ is a separator only of type II in $\delta_{\mathbf{1}}$.

In this case, we have the following situation:

$$\begin{array}{rl} \sigma = & \cdots \mathbf{a+2}, \mathbf{a}, \ell, \mathbf{a}{+}1 \cdots \\ & \downarrow \\ \delta_1 = \nabla^*_\ell(\sigma) = & \cdots \mathbf{a+2}, \mathbf{a}, \mathbf{a+1} \cdots \end{array}$$

Now, if a is not a separator in σ, then $\nabla^*_a(\sigma) \in K_{n-1}$ and we are done. Otherwise, we have two cases:

1. The element a is a separator only of type I in σ. In this case we have

$$\sigma = \cdots x, \mathbf{a+2}, \mathbf{a}, \ell = \mathbf{a+3}, \mathbf{a+1}, y \cdots .$$

Note that the sub-sequence $a+2, a, \ell = a+3, a+1$ in σ is order isomorphic to 3142. Now, if $x = a+4$ or $y = a+4$ then omitting $a+3, a+2, a+1$ will force the omitting of $a+4$ in order to prevent a block. This is a contradiction to the maximality of $\ell = a+3$, and hence we can assume that $x, y \notin \{a+4\}$.

If x and y are both different from $a-1$ then by omitting $a+3$,$a+2$ and $a+1$ from σ, we get $\tau = \nabla^*_{a+1}\nabla^*_{a+2}\nabla^*_{a+3}(\sigma) \in K_{n-3}$ and we are done since $\pi \preceq \tau \prec \sigma$ and $|\sigma| - |\tau| = 3$. Otherwise, $x = a-1$ or $y = a-1$, in which case $\nabla^*_{a+1}\nabla^*_{a+2}\nabla^*_{a+3}(\sigma)$ contains the block $a-1, a$ or the block $a, a-1$ that we must get rid of in our way to π and thus $\pi \prec \nabla^*_{a-1}(\sigma) \prec \sigma$. It is easy to see that $a-1$ is not a separator in σ. This means that $\nabla^*_{a-1}(\sigma) \in K_{n-1}$ and again we are done.

2. The element a is a separator of type II (with or without being also a separator of type I) in σ. In this case we have: $\sigma = \cdots \mathbf{a+2}, \mathbf{a}, \ell, \mathbf{a+1}, \mathbf{a-1} \cdots$.

(Note that if $\ell = a+3$ then a is also of type I). Now, it is clear that $\pi \prec \nabla^*_{a-1}(\sigma) \prec \sigma$ and $a-1$ is not a separator in σ. This means that $\nabla^*_{a-1}(\sigma) \in K_{n-1}$ and we are done.

$\square$

As a corollary, we get one of our main results:

Corollary 3.15. *For each two king permutations $\pi \prec \sigma$ there exists a chain of king permutations $\pi = \pi^0 \prec \pi^1 \cdots \prec \pi^k = \sigma$ such that $|\pi^i| - |\pi^{i-1}| \in \{1, 3\}$.*

3.1 Kings without regents

Recall that a *regent* of a permutation $\sigma \in K_n$ is a king permutation $\tau \in K_{n-1}$ such that $\tau \prec \sigma$.

We can formulate now the following:

Lemma 3.16. *If for $\sigma \in K_n$ there is $\pi \in S_{n-1}$ such that $\pi \prec \sigma$ and π contains a single strict $2-$block then σ has a regent.*

Proof. Let $i \in \{1, \dots, n\}$ be such that $\pi = \nabla_i^*(\sigma)$ contains a single strict 2-block. Then, without loss of generality, σ contains the sub-sequence $k, k+1$, where $k-1$ and $k+2$ do not appear adjacent to k and $k+1$, so we can divide σ into 4 parts such that $\sigma = [\delta \quad k \quad k+1 \quad \epsilon]$ where both δ and ϵ are sequences that do not contain any $2-$block (and might be empty). If we now remove the digit $k+1$ from π, and standardize, we get $\tau = [\delta' \quad k \quad \epsilon'] \in S_{n-2}$ where $k-1$ and $k+1$ do not appear adjacent to k and δ', ϵ' do not contain any $2-$block. This means that $\tau = \nabla_{k+1}^* \nabla_i^*(\sigma) \in K_{n-2}$ and thus by Theorem 3.13, σ has a regent. $\square$

Example 3.17. *Let $\sigma = [3, 6, 1, 4, 7, 2, 9, 5, 8] \in K_9$. After omitting the digit 5 and normalizing we get $\nabla_5^*(\sigma) = \pi = [3, 5, 1, 4, 6, 2, \mathbf{8}, \mathbf{7}]$ which has the strict $2-$block $8, 7$. By removing from π the digit 8 (or the digit 7) and standardizing, we get $\nabla_8^*(\pi) = \tau = [3, 5, 1, 4, 6, 2, 7] \in K_7$. This implies the existence of a regent of σ (for example $[5, 1, 3, 6, 2, 8, 4, 7]$). The situation is depicted in the following:*

$$\begin{tikzcd}
\sigma = [3,6,1,4,7,2,9,5,8] \in K_9 \arrow[d, "\nabla_5^*(\sigma)"] & & \sigma = [3,6,1,4,7,2,9,5,8] \in K_9 \arrow[d, "\nabla_3^*(\sigma)"] \\
\pi = [3,5,1,4,6,2,\mathbf{8},\mathbf{7}] \in S_8 - K_8 \arrow[d, "\nabla_8^*(\pi)"] & \Longrightarrow & [5,1,3,6,2,8,4,7] \in K_8 \\
\tau = [3,5,1,4,6,2,7] \in K_7 & &
\end{tikzcd}$$

The permutations $\pi \in K_n$ which have no regents have a special structure. An example for such an element is $\pi = [7, 5, 8, 6, 2, 4, 1, 3, 10, 12, 9, 11] \in K_{12}$. Note that π has the following structure: $\pi = 213[3142, 2413, 2413]$. (Recall the definition of inflation from 2.6). Moreover, by removing any value from π we get a $3-$block. (by way of illustration, if we remove the digit 8 and standardise we get $[\mathbf{7}, \mathbf{5}, \mathbf{6}, 2, 4, 1, 3, 9, 11, 8, 10]$).

The following theorem shows that this is not coincidental.

Theorem 3.18. *The following conditions are equivalent for each $\pi \in K_n$ with $n \geq 4$.*

1. *There are $\pi^1, \dots, \pi^k \in \{[3142], [2413]\}$ and $\pi' \in S_k$ such that $\pi = \pi'[\pi^1, \dots, \pi^k]$.*

2. For each $i \in \{1, \ldots, n\}$, by removing i from π, we get a block of length 3.

3. π has no regents.

Proof. As the first two implications are trivial ((1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3)), we prove (3) $\Rightarrow$ (1). Assume that $\pi \in K_n$ has no regents. We prove that for every $k \equiv 3 \ (mod\ 4)$ such that $k+1 \leq n$: either $k, k-2, k+1, k-1$ or its reversal $k-1, k+1, k-2, k$ is a sub-sequence of π.

We start with $k = 3$:

Since π has no regents, removing any element of π will give us a 2-block. That block cannot be a single strict 2- block, otherwise by Lemma 3.16, π has a regent.

Hence, removing any element from π will give us either a 3-block or two strict 2-blocks. Note that the only way to get two strict 2-blocks is by removing a separator of both types, though it is not true that whenever we remove such a separator we get two strict 2-blocks.

We prove now that one of the sub-sequences 3142 and 2413 must appear in π. First, remove the digit 1 from π. Since 1 cannot be a separator of type II, its removal creates a 3-block, and hence π itself contains the sub-sequence $b, 1, b+1, b-1$ or its reversal: $b-1, b+1, 1, b$ for some $b \geq 3$. (The sub-sequence $b, 1, b-1, b+1$ and its reversal are impossible since each element of π is a separator, by the assumption that π has no regents, but had the sub-sequence been $b, 1, b-1, b+1$, the digit $b-1$ clearly would not have been a separator of type I. Furthermore, if $b-1$ was a separator of type II, π would contain the sub-sequence: $b-2, b, 1, b-1, b+1$ but then $b+1$ could not be a separator of any type). Now, we are done as soon as we show that $b = 3$. In order to do that, consider the removal of $b+1$ from the sub-sequence $b, 1, b+1, b-1$. If $b+1$ is a separator of type I then we must have $b = 3$. Otherwise, $b+1$ is a separator of type II and thus π contains the sub-sequence $b+2, b, 1, b+1, b-1$, but this cannot hold since in this case $b+2$ is not a separator of any type, which contradicts the assumption that π has no regents.

Now, for $k = 7$, the same argument holds, provided that we start by removing 5 instead of 1. Since 4 is confined between 1 and 2 in the case of 3142 or between 4 and 3 in the case of 2413, we have that 5 can not be a separator of type II, and we can continue as before. This procedure can be performed now sequentially for each k such that $k = 3(mod 4)$ and $k+1 \leq n$.

□

We can now get an exact enumeration of the elements of K_n which have no regents.

Corollary 3.19. *The number of permutations in K_n which have no regents is:*

$$\begin{cases} 2^k k! & n = 4k \\ 0 & O.W. \end{cases}$$

Actually, the entire downset of an element which has no regents can be detected. So, let $\sigma \in K_n$ be such that σ has no regents. By Theorem 3.18, $\sigma = \sigma'[\sigma^1, \ldots, \sigma^k]$, $\sigma^i \in \{[3142], [2143]\}$, $\sigma' \in S_k$. We have the following:

Corollary 3.20. *Let σ be as above. Then for each $\pi \in K_l$ $(l < n)$ such that $\pi \prec \sigma$ we have $\pi = \pi'[\pi^1, \ldots, \pi^m]$, $(m \leq k)$, where $\pi^i \in \{[3142], [2143], [1]\}$ and $\pi' \in S_m$ is such that $\pi' \prec \sigma'$.*

Proof. In order to get $\pi \prec \sigma$ such that π is a king, we must remove at least 3 digits from a single block π_i. □

The following corollary adds some more information about the structure of the posets of king permutations.

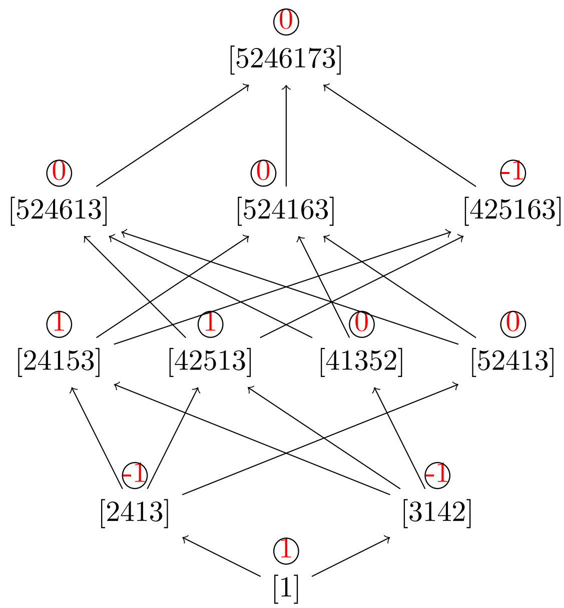
Theorem 3.21. *Let $n > 4$. For each $\sigma \in K_n$ there exists $\pi \in K_5$ s.t. $\pi \preceq \sigma$.*

Proof. We prove by induction on n, the case $n = 5$ being trivial. If σ has a regent then by the induction hypothesis we are done. Otherwise, by 3.18, σ is of the form $\sigma = \sigma'[\sigma^1, \dots, \sigma^k]$ with $\sigma^i \in \{[3142], [2413]\}$, $\sigma' \in S_k$. The first five digits of σ are order isomorphic to one of the permutations in the set $\{24135, 31425, 35241, 42531\} \subseteq K_5$.

$\square$

4 The Möbious function of the poset of kings permutations

In this section we present some results regarding the Möbious function of $\mathcal{K}$. We start with an example depicting the poset of the downset of the king permutation [5246173]. The red circled number above each permutation π is the value of $\mu(\pi)$.



We start with the following corner stone of our treatment of the Möbious function on $\mathcal{K}$.

Theorem 4.1. *Let $\pi \in K_n$, with $n > 4$. If $[2413] \not\preceq \pi$ or $[3142] \not\preceq \pi$ then $\mu(\pi) = 0$ in $\mathcal{K}$.*

Proof. We prove by induction on n. The basis is $n = 5$ which can be easily checked. Assume that the claim holds for each $\pi \in K_l$, $4 < l < n$. Let $\pi \in K_n$. By theorem 3.9, we may assume without loss of generality that π contains [2413] but not [3142].

Then

$$\mu(\pi) = -\sum_{\sigma \in [1,\pi)} \mu(\sigma) = -\sum_{\sigma \in [1,[2413]]} \mu(\sigma) - \sum_{\sigma \in ([2413],\pi)} \mu(\sigma).$$

The first summed is 0 since the interval is closed, while each element in the second one is 0 by the induction hypothesis. Hence we have $\mu(\pi) = 0$.

$\square$

For $\pi \in K_n$, define

$$C(\pi) = \{\sigma \in \bigcup_{l<n} K_l \mid \sigma \prec \pi, \text{ and there is no } \delta \text{ such that } \sigma \prec \delta \prec \pi\}.$$

Corollary 4.2. *Let $\pi \in K_n$ be such that there is only one $\sigma \in C(\pi)$ such that $[3142] \prec \sigma$ and $[2413] \prec \sigma$. Then $\mu(\pi) = 0$*

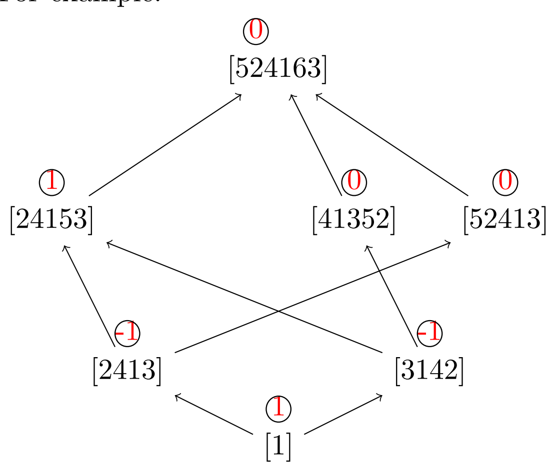
Proof. Let $\sigma \in C(\pi)$ be the single element which contains both [2413] and [3142].

$$\mu(\pi) = -\sum_{\tau \in [1,\pi)} \mu(\tau) = -\sum_{\tau \in [1,\sigma]} \mu(\tau) - \sum_{\tau \in [1,\pi)-[1,\sigma]} \mu(\tau).$$

The first summand vanishes since it runs through a closed interval, while the second summand vanishes by Theorem 4.1.

□

For example:



This result can be strengthen. In order to do that, we need the following:

Definition 4.3. *Let*

$$\mathbb{H} = \{[24153], [35142], [42513], [31524]\}.$$

It is easy to see that $\mathbb{H}$ consists of all the elements of K_5 which contain both [2413] *and* [3142].

Note that in K_5, $\mu(\pi) = 1$ if and only if $\pi \in \mathbb{H}$ (otherwise, by Theorem 4.1, $\mu(\pi) = 0$).

Theorem 4.4. *Let $\sigma \in K_n$ with $n > 5$ such that there is only one $\pi \prec \sigma$ such that $\pi \in \mathbb{H}$ and for each $\pi' \prec \sigma$ such that $\pi \not\prec \pi'$, we have either π' avoids* [3142]*, or π' avoids* [2413]*. Then $\mu(\sigma) = 0$.*

Proof. We prove by induction on n. First, for $n = 6$, let $\sigma \in K_6$ be such that the assumptions of the theorem are satisfied. Then, the permutation σ has only one regent $\pi \in \mathbb{H}$. The other regents avoid [2413] or avoid [3142]. Let $X = [1, \pi]$ and $Y = [1, \sigma) - X$. We have

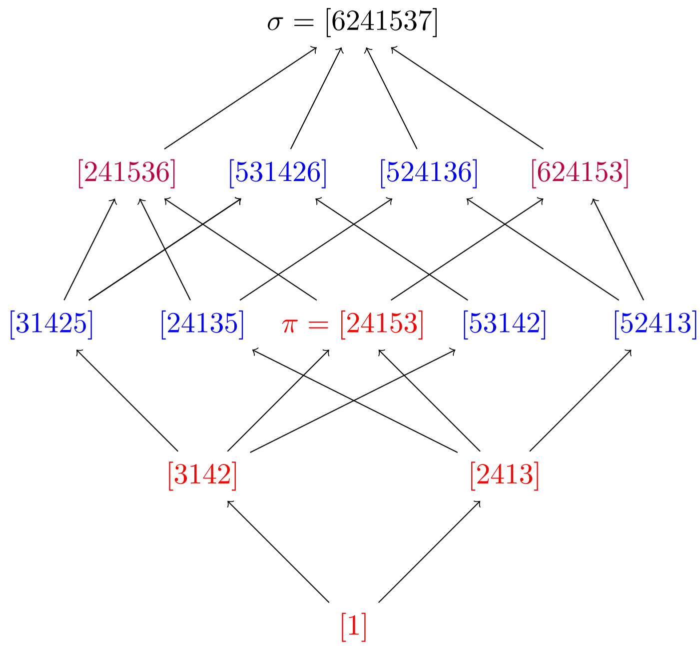
$$\mu(\sigma) = -\sum_{\tau \in [1,\sigma)} \mu(\tau) = -\sum_{\tau \in X} \mu(\tau) - \sum_{\tau \in Y} \mu(\tau).$$

This sum vanishes since the first summand is over a closed interval while the second one vanishes by Theorem 4.1.

Now, let $n \geq 6$ and assume the validity of the claim for $5 < k < n$. Let $\sigma \in K_n$ satisfy our assumptions. Then

$$\mu(\sigma) = -\sum_{\tau\in[1,\sigma)} \mu(\tau) = -\sum_{\tau\in(\pi,\sigma)} \mu(\tau) - \sum_{\tau\in[1,\pi]} \mu(\tau) - \sum_{\tau\in Z} \mu(\tau),$$

where $Z = [1,\sigma) - (\pi,\sigma) - [1,\pi]$. The elements of the first summand vanish by the induction hypothesis, the second one vanishes since it runs through a closed interval while the third one contains only elements which avoid [2413] or avoid [3142] and thus vanishes by Theorem 4.1. For example:



$\square$

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