N_∞-operads and associahedra

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Abstract

We provide a new combinatorial approach to studying the collection of N_∞-operads in G-equivariant homotopy theory for G a finite cyclic group. In particular, we show that for $G = C_{p^n}$ the natural order on the collection of N_∞-operads stands in bijection with the poset structure of the $(n+1)$-associahedron. We further provide a lower bound for the number of possible N_∞-operads for any finite cyclic group G.

1. *Introduction*

The study of homotopy commutative objects has always been compelling to homotopy theorists. Usually, homotopy commutative ring objects are encoded in terms of E_∞-operads. In the equivariant setting, constucting an appropriate version of E_∞-operads has its difficulties. For example, the naive version of an equivariant E_∞-algebra does not have any non-trivial norm maps, which leads to the phenomenon that weakly equivalent E_∞-operads do not necessarily possess Quillen equivalent categories of algebras. Thus, the definition of N_∞-operads has been developed, which governs the correct notions of homotopy commutativity for G-spectra.

Recent work by Blumberg and Hill [**1**] led to the conjecture, soon verified by [**2**, **4**, **7**], that for a group G, the data of an N_∞-operad is equivalent to a certain "indexing system". We show that this again is equivalent a set of norm maps $X = \{N_H^K\}$ for some subgroups $1 \leqslant H < K \leqslant G$ satisfying two specific rules. This implies that an N_∞-operad can be depicted by a graph whose vertices are conjugacy classes of subgroups, and an edge between subgroups exists if $N_H^K \in X$. This opens the door to a more combinatorial approach to studying those operads for a fixed group G.

We start with the case of G being a cyclic group C_{p^n}. A constructive approach leads to our first result that there are $\mathsf{Cat}(n+1)$ many N_∞-operads for C_{p^n}, where $\mathsf{Cat}(n)$ denotes the n^{th} Catalan number. In particular, there are as many N_∞-operads for C_{p^n} as there are binary trees with $n+2$ leaves.

The relation does not just stop there, though. Binary trees are one way of encoding associahedra (also known as *Tamari lattices* or *Stasheff polytopes*), where a binary tree corresponds to a vertex, and two vertices are related by a directed edge if one tree can be obtained from another by moving one branch to the right. On the other side, the set of all N_∞-operads for C_{p^n} can be ordered by inclusion of the corresponding graphs. We prove that these two posets are in fact isomorphic as posets, i.e. the bijection between N_∞-operads and binary trees is order-preserving.

When moving to a general cyclic group, unfortunately one will quickly find the combinatorics of the N_∞-operads unmanageable. This is due to the fact that in the corresponding graph diagram of an N_∞-operad for $C_{p_1^{n_1}\cdots p_k^{n_k}}$, the edges not induced from the C_{p^i} become hard to describe. We explain this phenomenon by developing the terms of *pure* and *mixed* N_∞-operads and give a non-trivial lower bound for the number of N_∞-operads for an arbitrary finite cyclic group G.

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This new approach of N_∞-operads as graph diagrams therefore sheds some light on the theory of equivariant homotopy commutativity.

2. *A brief tour of the theory of N_∞-operads*

We shall assume that the reader is somewhat familiar with G-equivariant homotopy theory in the sense of May [**6**]. We shall always assume that G is a compact Lie group. Our objects of interest, N_∞-operads, are a special class of G-operad, whence we begin our exposition.

Definition 1. A *G-operad* $\mathcal{O}$ is a symmetric operad in G-spaces. That is, we have a sequence of $(G \times \Sigma_n)$-spaces $\mathcal{O}_n$, $n \geqslant 0$, such that
(i) there is a G-fixed identity element $1 \in \mathcal{O}_1$,
(ii) there are G-equivariant composition maps

$$\mathcal{O}_k \times \mathcal{O}_{n_1} \times \cdots \times \mathcal{O}_{n_k} \to \mathcal{O}_{n_1+\cdots+n_k}$$

which satisfy the usual compatibility conditions with each other and the symmetric group actions.

A certain subclass of G-operads, known as N_∞-operads, are used to describe different levels of commutativity in genuine G-equivariant stable homotopy theory, see Blumberg and Hill [**1**]. That is, they are the correct analogue of E_∞-operads in the equivariant setting. Recall that for a group G a *family* $\mathcal{F}$ is a collection of subgroups which is closed under passage to subgroups and conjugacy. A *universal space* for a family $\mathcal{F}$ is a G-space $E\mathcal{F}$ such that for all subgroups H we have

$$(E\mathcal{F})^H \simeq \left\{ \begin{array}{cc} * & H \in \mathcal{F} \\ \emptyset & H \notin \mathcal{F} \end{array} \right\}.$$

Definition 2. An *N_∞-operad* is a G-operad $\mathcal{O}$ such that
(i) the space $\mathcal{O}$ is G-contractible,
(ii) the action of Σ_n on $\mathcal{O}_n$ is free,
(iii) $\mathcal{O}_n$ is a universal space for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times \{1\}$ for $H \leqslant G$.

We will denote by $N_\infty(G)$ the collection of all N_∞-operads for a given group G.

Although Definition 2 is perfectly good for theoretical purposes, we shall choose to work with a more computationally exploitable definition of N_∞-operads, which utilises the theory of *norm maps*. Denote by $\mathbf{Sp}_G$ the ∞-category of genuine G-equivariant spectra. Then for $H \leqslant G$, the *Hill-Hopkins-Ravanel norm* is a monoidal functor

$$N_H^G \colon \mathbf{Sp}_H \to \mathbf{Sp}_G$$

satisfying many desirable properties as given by Hill, Hopkins and Ravanel [**5**]. To give an equivalent formulation of the structure of an N_∞-operad, we first introduce an intermediary notion of indexing systems.

Definition 3. A *categorical coefficient system* is a contravariant functor $\underline{\mathcal{C}} \colon O_G^{op} \to \mathbf{Cat}$ from the orbit category of G to the category of small categories. Such a coefficient system is called *symmetric monoidal* if it takes values in symmetric monoidal categories and strong monoidal functors. We are particularly interested in the coefficient system $\mathcal{S}et$ with disjoint union which sends a subgroup H to the category $\mathbf{Set}^H$ of H-sets. A sub-symmetric coefficient $\underline{\mathcal{C}}$ of $\mathcal{S}et$ is said to be an *indexing system* if it is closed under direct products, subobjects and self-induction (i.e., $T \in \underline{\mathcal{C}}(K)$ and $H/K \in \underline{\mathcal{C}}(H)$ implies that $H \times_K T \in \underline{\mathcal{C}}(H)$).

The following result was first conjectured in Blumberg and Hill [**1**] and has subsequently been proven to hold in three independent articles by Bonventre and Pereira; Gutiérrez and White; and Rubin.

PROPOSITION 1 [**2**, **4**, **7**]. *The homotopy category of N_∞-operads is equivalent to the poset category of indexing systems.*

We now compare the notion of indexing systems to something that we choose to refer to as *norm systems*.

LEMMA 1. *An indexing system determines, and is determined by, a set $\mathcal{F}_H$ for each $H \leqslant G$ consisting of subgroups K of H, written as H/K, satisfying*
(Identity) $H/H \in \mathcal{F}_H$.
(Conjugation) $\mathcal{F}_H$ is closed under conjugation.
(Restriction) $H/K \in \mathcal{F}_H$ implies $M/(M \cap K) \in \mathcal{F}_M$ for all $M \leqslant H$.
(Composition) $H/K \in \mathcal{F}_H$ and $K/L \in \mathcal{F}_K$ implies $H/L \in \mathcal{F}_H$.
We call this data a norm system.

Proof. Given an indexing system $\underline{\mathcal{C}}$ we construct a norm system $\mathcal{F}$. The category $\underline{\mathcal{C}}(G/H)$ is a subcategory of H-sets closed under coproducts, products, sub-objects and must contain the one-point set. The categories $\underline{\mathcal{C}}(G/H)$ for varying H must be related by restriction and self-induction as we will explain later.

Since the category $\underline{\mathcal{C}}(G/H)$ is closed under subobjects and coproducts, it is determined by a collection of orbit H/K for certain subgroups K. This collection must be closed under conjugation and contain H. We let $\mathcal{F}_H$ be this set of orbits.

An inclusion of subgroups $i_M \colon M \leqslant H$ induces the restriction functor

$$i_M^* \colon \underline{\mathcal{C}}(G/H) \to \underline{\mathcal{C}}(G/M)$$

which sends H/K to $i_M^*(H/K)$. As an M-set this is a coproduct of M-sets conjugate to $M/(M \cap K)$.

Self-induction says that given $T \in \underline{\mathcal{C}}(G/K)$ and $H/K \in \underline{\mathcal{C}}(G/H)$ the H-set $H \times_K T$ is in $\underline{\mathcal{C}}(G/H)$. When $T = K/L$, this corresponds to the last point of a norm system.

For the converse, we start with a norm system $\mathcal{F}$ and construct an indexing system. Given $\mathcal{F}$ we can construct the category $\underline{\mathcal{C}}(G/H)$ of all H-sets isomorphic to those made from $\mathcal{F}_H$ under coproducts and sub-objects.

We claim that $\underline{\mathcal{C}}(G/H)$ is closed under products. The set $H/K \times H/L$ is a coproduct of orbits of type $H/(K \cap L)$ (up to suitable conjugations). Restricting $H/L \in \mathcal{F}_H$ to $\mathcal{F}_K$ gives the orbit $K/(K \cap L)$. The composition of H/K with $K/(K \cap L)$ gives $H/(K \cap L) \in \mathcal{F}_H$.

The restriction property for norm systems implies that the restriction functor (forgetful functor) of H-sets to K-sets passes to

$$\underline{\mathcal{C}}(G/H) \to \underline{\mathcal{C}}(G/K).$$

Similarly, the self-induction property for $\underline{\mathcal{C}}$ follows from composition for $\mathcal{F}$. □

COROLLARY 1. *Let G be a finite group. Up to homotopy, an N_∞-operad for G is the data of a set of norm maps $X = \{N_H^K\}_{1 \leqslant L < K \leqslant G}$ satisfying the following rules (and all conjugates thereof).*
(Restriction) If $N_K^H \in X$ and $M < H$, then $N_{K \cap M}^M \in X$.
(Composition) If $N_L^K \in X$ and $N_K^H \in X$, then $N_L^H \in X$.
In particular, N_∞-operads can be described as certain subgraphs of the lattice of (conjugacy classes of) subgroups of G.

Proof. Recall the relation between norm systems and norm maps: if $H/K \in \mathcal{F}_H$ then any corresponding N_∞-operad will have a norm map N_K^H. Since the norm map N_H^H is the identity, the identity condition of a norm system has no effect.

Given an N_∞-operad we have a norm system $\mathcal{F}$. We know that $H/K \in \mathcal{F}_H$ implies

$$M/(M \cap K) \in \mathcal{F}_M.$$

In terms of norms this is precisely the statement of the second form of restriction. The second axiom of a norm system says that $H/K \in \mathcal{F}_H$ and $K/L \in \mathcal{F}_K$ implies $H/L \in \mathcal{F}_H$. In terms of norms this is precisely the composition rule.

The converse is similar. □

This results leads to the following corollary, which motivates the results in this paper, namely, that for a finite group G, it makes sense to attempt to enumerate the number of N_∞-operad structures, and to understand the associated poset structure.

COROLLARY 2. *Let G be a finite group. Then the number of N_∞-operad structures G is finite. Moreover, the set $N_\infty(G)$ admits a canonical poset structure given by inclusions of sets of the corresponding norm systems.*

3. *The case $G = C_{p^n}$*

We will begin with the case of cyclic groups of the form C_{p^n}. We note that the choice of p here is arbitrary as the subgroup lattices of C_{p^n} and C_{q^n} are isomorphic for different primes p and q, indeed, they are isomorphic to the poset $\underline{n} = \{0 < 1 < \cdots < n\}$. To ease the notion we shall denote by N_i^j the norm map $N_{C_i}^{C_j}$ for $i \leqslant j$.

Before we continue to the theoretics, let us manually compute the first handful of values of $|N_\infty(C_{p^n})|$. The purpose of this is two-fold. Firstly it will give the reader an idea of how such computations are done, and second, for the avid integer sequence fan, these examples will suggest the general form for the sequence $\{|N_\infty(C_{p^n})|\}_{n \in \mathbb{N}}$. Note that we will not write the identity norm maps N_i^i, and shall only consider the non–trivial norm maps.

EXAMPLE 1. The case of $G = C_{p^0}$ is trivial. That is, there are no choices of non-trivial norms to make, and therefore $|N_\infty(C_{p^0})| = 1$. This is exactly the fact that for non-equivariant stable homotopy theory, there is only a single notion of commutativity as one may expect. We will write the single norm structure as $\{\emptyset\}$ to indicate that there are no non-trivial norm maps.

EXAMPLE 2. The situation for $G = C_p$ is only marginally more involved than the trivial case. Here we have a subgroup lattice $\{C_{p^0} < C_{p^1}\}$. Therefore the only choice to make is if we wish to include the only possible non-trivial norm N_0^1 or not. Therefore there are two norm structures, namely $\{\emptyset\}$ and $\{N_0^1\}$.

EXAMPLE 3. We shall now look at $G = C_{p^2}$. This is the first case where we need to take care of the rules appearing in Corollary 1. As always, we have the trivial N_∞-operad $\{\emptyset\}$ which we shall write diagrammatically as

$$\left(\begin{tikzcd} C_{p^0} & C_{p^1} & C_{p^2} \end{tikzcd}\right) = \{\emptyset\} .$$

At the other extreme, we could add in all of the norm maps. One can easily check the conditions to see that this will always be a valid N_∞-operad. We shall draw this N_∞-operad as

$$\left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) = \{N_0^1, N_1^2, N_0^2\}$$

where an arrow from C_{p^i} to C_{p^j} indicates the existence of the norm map N_i^j for $i < j$.

The technical part then, of course, is to identify what other N_∞-operads can appear in-between these two extremes. There are 2^3 different possibilities to try (indeed, there are three different norm maps which we much choose whether to include or not). Instead of investigating all of the remaining cases, we shall just show the failure of the ones that do not have an N_∞-operad structure. Figures 1, 2 and 3 give the invalid diagrams.

✗ $$\left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) = \{N_0^1, N_1^2\}$$

FIGURE 1. *This diagram is not valid as it violates the composition rule of Corollary 1. If we were to "complete" this diagram to get a valid N_∞-operad then we would need to add in the norm map N_0^2, and we get the operad above.*

✗ $$\left(\begin{tikzcd} C_{p^0} \arrow[rr, bend left] & C_{p^1} & C_{p^2} \end{tikzcd}\right) = \{N_0^2\}$$

FIGURE 2. *This diagram is not valid as it does not satisfy the restriction rules. To satisfy the rule we would need to also have the norm map N_0^1, and then all of the rules would be satisfied. The resulting operad would be different from the above two, and will appear in the list at the end of the this example.*

✗ $$\left(\begin{tikzcd} C_{p^0} \arrow[rr, bend left] & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) = \{N_1^2, N_0^2\}$$

FIGURE 3. *This is the final invalid diagram, which suffers from the same deficiency as the one above, that is, it does not satisfy the restriction rules.*

Consequently, we can write down the elements of $N_\infty(C_{p^2})$. Note that in particular, $|N_\infty(C_{p^2})| = 5$. We implore the reader to check these for themselves to gain confidence with the rules of Corollary 1 in preparation for the further sections. The valid N_∞-operad structures are as follows.

$$\checkmark \qquad \left(\begin{tikzcd} C_{p^0} & C_{p^1} & C_{p^2} \end{tikzcd}\right) = \{\emptyset\}$$

$$\checkmark \qquad \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} & C_{p^2} \end{tikzcd}\right) = \{N_0^1\}$$

$$\checkmark \qquad \left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] & C_{p^1} & C_{p^2} \end{tikzcd}\right) = \{N_0^1, N_0^2\}$$

$$\checkmark \qquad \left(\begin{tikzcd} C_{p^0} & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) = \{N_1^2\}$$

$$\checkmark \qquad \left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) = \{N_0^1, N_1^2, N_0^2\}$$

From our first analysis, we have obtained the integer sequence $1, 2, 5$ counting the number of N_∞-operads for C_{p^0}, C_{p^1} and C_{p^2} respectively. If one were to take the time to check the possibilities for C_{p^3}, they would see that there are 14 possibilities. Therefore the examples suggest a relation to the Catalan numbers. The next section will be devoted to recalling the necessary results regarding the Catalan numbers before we prove the first main result, Theorem 1 which says that $|N_\infty(C_{p^n})|$ coincides with the $(n+1)$-st Catalan number.

3.1. *A recollection of the Catalan numbers*

The Catalan numbers are a sequence of numbers which regularly appears in enumeration problems. The n^{th} Catalan number, which we denote $\mathsf{Cat}(n)$, is given as

$$\mathsf{Cat}(n) = \frac{(2n)!}{(n+1)!n!} .$$

The first few terms of the sequence are therefore $\mathsf{Cat}(0) = 1$, $\mathsf{Cat}(1) = 1$, $\mathsf{Cat}(2) = 2$, $\mathsf{Cat}(3) = 5$ and $\mathsf{Cat}(4) = 14$. There are many surprising ways to define the Catalan numbers, let us recall a few:

- $\mathsf{Cat}(n)$ is the number of valid expressions containing n-pairs of parentheses.

– $\mathsf{Cat}(n)$ is the number of triangulations of a regular $(n+2)$-gon.
– $\mathsf{Cat}(n)$ is the number of rooted binary trees with $n+1$ leaves.

This is but a few of a multitude of descriptions given in Stanley [**8**]. The last interpretation involving binary trees will be our canonical representation. Figure 4 gives the corresponding binary trees in the case of $n = 2$.

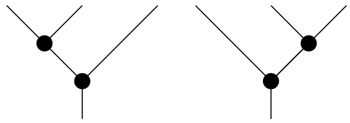


FIGURE 4. *The two binary trees giving the* 2^{nd} *Catalan number.*

The following well-known recurrence relation will be fundamental to the proof of the main result in this section.

LEMMA 2. *The Catalan numbers satisfy, and are completely determined by, the recurrence relation*

$$\mathsf{Cat}(0) = 1,$$
$$\mathsf{Cat}(n+1) = \sum_{i=0}^{n} \mathsf{Cat}(i)\mathsf{Cat}(n-i) \text{ for } n \geqslant 0.$$

3.2. *An operation on N_∞-operads*

To facilitate the proof of Theorem 1, we first introduce a function

$$\odot \colon N_\infty(C_{p^i}) \times N_\infty(C_{p^j}) \to N_\infty(C_{p^{i+j+2}}).$$

To be able to define this function explicitly, we need some auxiliary notation. We consider an $X \in N_\infty(C_{p^i})$ as being described by its finite set of norm maps. Secondly, we will undertake the assumption that $N_\infty(C_{p^{-1}})$ is defined to be the empty set (*not the set containing the empty set!*).

For a norm map $N_{k_1}^{k_2}$ in X, we define a shift operation

$$\Sigma^n N_{k_1}^{k_2} := N_{k_1+n}^{k_2+n} .$$

We write $\Sigma^n X$ to denote X where Σ^n has been applied to each norm map.

For $X \in N_\infty(C_{p^i})$ and $Y \in N_\infty(C_{p^j})$, we now define $X \odot Y \in N_\infty(C_{p^{i+j+2}})$ to be the N_∞-operad described by the set of norm maps

$$X \odot Y := X \coprod \Sigma^{i+2} Y \coprod \{N_{i+1}^{k}\}_{i+1<k<i+j+2}.$$

Figures 5, 6 and 7 give a pictorial presentation of $X \odot Y$. We exclude the norm maps for X and Y from the diagrams for clarity.

Note, that in particular, we can see that this operation is not commutative. Let us give some explicit examples of this construction before we prove that the resulting set of norms does indeed give an N_∞-operad as we have claimed.

EXAMPLE 4. Let

$$X = \left(C_{p^0} \qquad\qquad C_{p^1} \right) \in N_\infty(C_{p^1}),$$

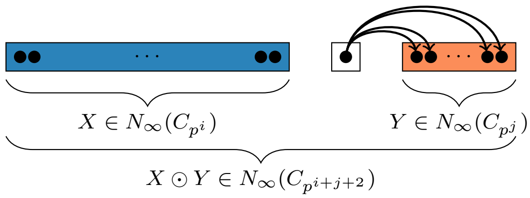


FIGURE 5. *The general picture for the operation $X \odot Y$.*

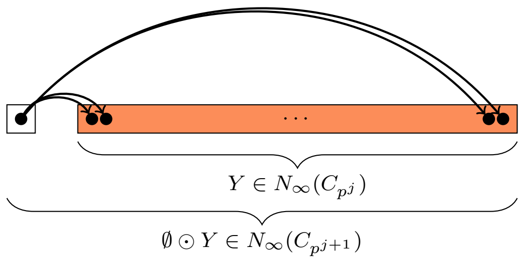


FIGURE 6. *The general picture for the operation $\emptyset \odot Y$.*

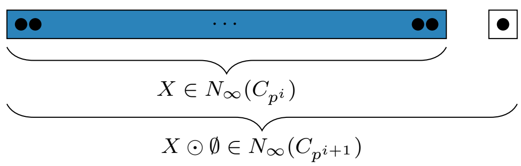


FIGURE 7. *The general picture for the operation $X \odot \emptyset$.*

$$Y = \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} \end{tikzcd} \right) \in N_\infty(C_{p^1}).$$

Then $X \odot Y = \left(\begin{tikzcd} C_{p^0} & C_{p^1} & C_{p^2} \arrow[r] \arrow[rr, bend left] & C_{p^3} \arrow[r] & C_{p^4} \end{tikzcd} \right) \in N_\infty(C_{p^4}),$

and $Y \odot X = \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} & C_{p^2} \arrow[r] \arrow[rr, bend left] & C_{p^3} & C_{p^4} \end{tikzcd} \right) \in N_\infty(C_{p^4}).$

EXAMPLE 5. Let

$$X = \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} & C_{p^2} \arrow[r] & C_{p^3} \end{tikzcd} \right) \in N_\infty(C_{p^3})$$

$$Y = N_\infty(C_{p^{-1}})$$

$$\text{Then } X \odot Y = \left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} & C_{p^2} \arrow[r] & C_{p^3} & C_{p^4} \end{tikzcd} \right) \in N_\infty(C_{p^4})$$

$$\text{and } Y \odot X = \left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] \arrow[rrr, bend left] \arrow[rrrr, bend left] & C_{p^1} \arrow[r] & C_{p^2} & C_{p^3} \arrow[r] & C_{p^4} \end{tikzcd} \right) \in N_\infty(C_{p^4}).$$

Proposition 2. *For $X \in N_\infty(C_{p^i})$ and $Y \in N_\infty(C_{p^j})$, $X \odot Y$ indeed does satisfy the rules of Corollary 1, and therefore is a valid object in $N_\infty(C_{p^{i+j+2}})$ for $-1 \leqslant i, j$. Moreover, the converse is true, that is, if $X \odot Y \in N_\infty(C_{p^{i+j+2}})$, then it follows that X and Y are both valid N_∞-operads for their respective groups.*

Proof. We must check that the collection of N_∞-operads satisfies the restriction and composition conditions. The simplest way to do this is to appeal to Figure 5. First of all, note that the the norms coming from X are disjoint from the rest of the structure, and as we have assumed that X is a valid N_∞-operad for $G = C_{p^i}$, this part does not need further consideration.

The restriction rule for the remaining norm maps is clear. This rule is satisfied due to the addition of the norm maps $\{N_{i+1}^k\}_{i+1<k<i+j+2}$. The composition rule will be satisfied because Y was chosen to be in $N_\infty(C_{p^j})$, and suspending it to its new position will not affect this.

To see the converse of the statement, take two lattices X and Y of size i and j respectively such that $X \odot Y \in N_\infty(C_{p^{i+j+2}})$. We first of all note that X must be an object of $N_\infty(C_{p^i})$. Clearly if Y was not an object in $N_\infty(C_{p^j})$, then neither would its shift. Therefore it only remains to show that the addition of the norm maps $\{N_{i+1}^k\}_{i+1<k<i+j+2}$ has no possibility of validating Y. As mentioned above, adding these maps only serves to ensure the restriction rule is satisfied for the additional point, hence they cannot turn Y into a valid diagram. □

3.3. *Computing the cardinality of $N_\infty(C_{p^n})$*

We now come to the first main result of this paper which gives the link between the set of N_∞-operads for C_{p^n} and the Catalan numbers. We shall prove that the cardinalities of these sets satisfy the defining recurrence relation for the Catalan numbers, and then we show how to construct a bijection between these N_∞-operads and binary trees.

Theorem 1. *The cardinalities $|N_\infty(C_{p^n})|$ satisfy the recurrence relation*

$$|N_\infty(C_{p^{-1}})| = 1,$$
$$|N_\infty(C_{p^n})| = \sum_{i=0}^{n} |N_\infty(C_{p^{i-1}})||N_\infty(C_{p^{n-i-1}})| \text{ for } n \geqslant 0.$$

In particular we have that $|N_\infty(C_{p^n})| = \mathsf{Cat}(n+1)$.

Proof. To prove this we shall show that every N_∞-operad in $Z \in N_\infty(C_{p^n})$ can be written in the form $X \odot Y$ for $X \in N_\infty(C_{p^{i-1}})$ and $Y \in N_\infty(C_{p^{n-i-1}})$. This fact, along with Proposition 2 completes the argument.

Suppose that $Z \in N_\infty(C_{p^n})$. We let $k \in \mathbb{Z}$ be the minimum integer such that the norm map N_k^n is in Z. We have three cases to deal with here, either $k = 0$, $0 < k < n$ or $k = n$ (i.e., there is no such norm map). We start with the two extreme cases before dealing with the intermediate one.

- When $k = 0$, we construct Z as $X \odot Y$ for $X = \emptyset \in N_\infty(C_{p^{-1}})$, and Y an N_∞-operad for $G = C_{p^{n-1}}$ as in Figure 6.
- When $k = n$, we construct Z as $X \odot Y$ for $Y = \emptyset \in N_\infty(C_{p^{-1}}))$, and X an N_∞-operad for $G = C_{p^{n-1}}$ as in Figure 7.
- When $0 < k < n$, we observe that we have two disjoint parts to Z. Namely we are able to split off the subgroups C_{p^i} for $0 \leqslant i < k$. Let us denote this part as X (which lives in $N_\infty(C_{p^{k-1}})$, and the remaining part Z'. The crucial observation to make now is that Z' looks like $\emptyset \odot Y$ for some $Y \in N_\infty(C_{p^{n-k-1}})$. We therefore conclude that $Z = X \odot Y$ as required.

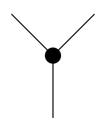
□

Corollary 3. *Every N_∞-operad Z for $G = C_{p^n}$ can be decomposed as $Z = X \odot Y$ for some N_∞-operads X and Y.*

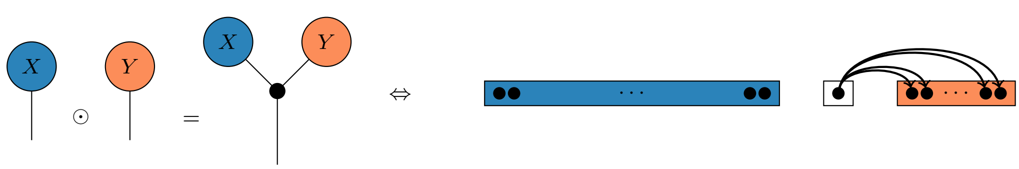
Corollary 4. *There is a bijection of sets*

$$\{N_\infty(C_{p^n})\} \Leftrightarrow \{\textit{rooted binary trees with } (n{+}2) \textit{ leaves}\}.$$

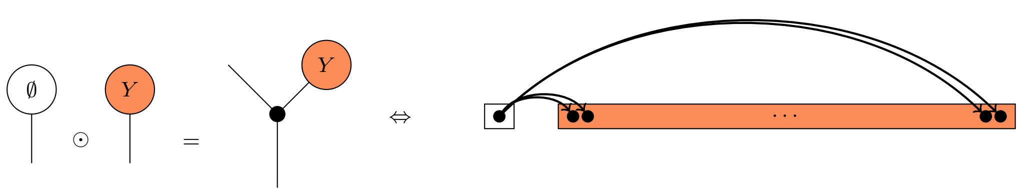
Proof. This follows immediately from Theorem 1 and the discussion in §3.1, however, it will be beneficial to the next section to spell out exactly how the correspondence works inductively. To the trivial for $G = C_{p^0}$ we assign the binary tree



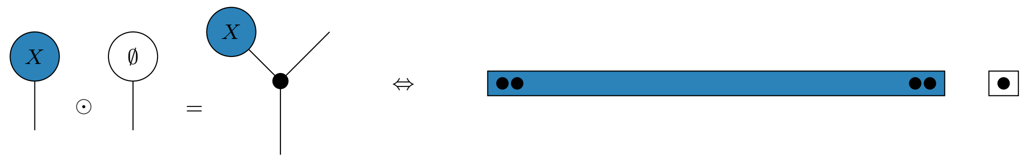
We will make the convention that $\emptyset$ is the empty tree. Assume that $n > 0$, we know from the above theorem that any N_∞-operad is of the form $X \odot Y$. We then have a binary tree associated to X and a binary tree associated to Y, and we can form the binary tree associated to $X \odot Y$ in the following way:



Following the convention of the empty diagram, we see that



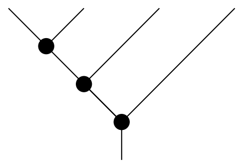
and



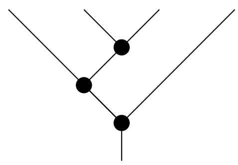
□

Example 6. One may use the above algorithm to compute the binary trees associated to the objects of $N_\infty(C_{p^2})$ as follows.

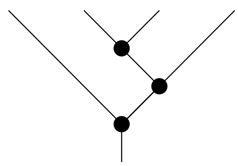
$$\left(\begin{tikzcd} C_{p^0} & C_{p^1} & C_{p^2} \end{tikzcd}\right) \Leftrightarrow$$



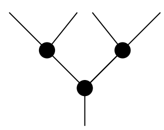
$$\left(\begin{tikzcd} C_{p^0} \arrow[r] & C_{p^1} & C_{p^2} \end{tikzcd}\right) \Leftrightarrow$$



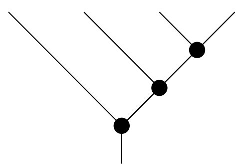
$$\left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] & C_{p^1} & C_{p^2} \end{tikzcd}\right) \Leftrightarrow$$



$$\left(\begin{tikzcd} C_{p^0} & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) \Leftrightarrow$$



$$\left(\begin{tikzcd} C_{p^0} \arrow[r] \arrow[rr, bend left] & C_{p^1} \arrow[r] & C_{p^2} \end{tikzcd}\right) \Leftrightarrow$$



3.4. *The relation to the associahedron*

We shall now see that the relationship between $N_\infty(C_{p^n})$ and the Catalan numbers runs deeper than just the result of Theorem 1. Recall that we can put an order on binary trees. Indeed, let X and Y be binary trees with $n+1$ edges. Then we say that $X < Y$ if Y can be obtained from X by clockwise *tree rotation operations, i.e., by moving a branch from left to right.*

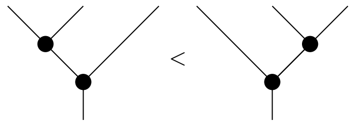


FIGURE 8. *An example of on order relation between two binary trees.*

*The poset structure on the set of binary trees with $n+1$ edges is known as the n-associaheadron, see Stasheff [**9**]. We shall denote this poset structure as $\mathcal{A}_n$.*

We can also implement a poset structure on $N_\infty(C_{p^n})$ by fixing that $X < Y$ if Y can be obtained from X via the addition of norm maps, for example we have the following.

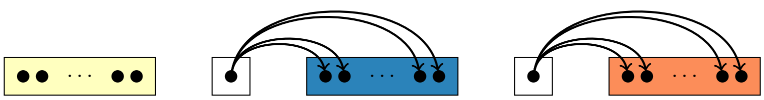
$$\left(C_{p^0} \qquad\qquad C_{p^1} \right) < \left(C_{p^0} \longrightarrow C_{p^1} \right)$$

Therefore, in our depiction, $X < Y$ for norm diagrams X and Y if Y can be obtained from X by adding edges.

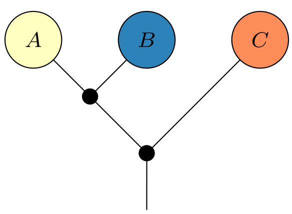
THEOREM 2. *There is an order preserving bijection of posets*

$$\{N_\infty(C_{p^n})\} \Leftrightarrow \{\textit{rooted binary trees with (n+2) leaves}\} \Leftrightarrow \mathcal{A}_{n+1}.$$

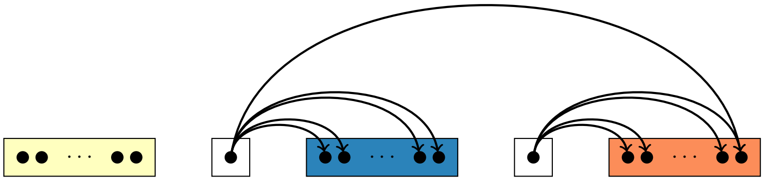
Proof. Let us begin by showing that a clockwise tree rotation corresponds to the addition of an edge in the corresponding N_∞-diagram, or more specifically, the addition of a norm map. We shall do this by appealing to the diagrammatic representations, as it provides the cleanest proof. We consider the norm diagram



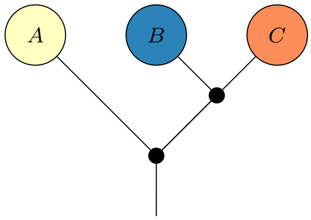
which corresponds to a tree of the following form.



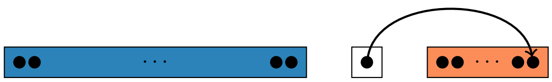
We then compare this to the diagram below, where restrictions of the largest arrow are omitted for clarity.



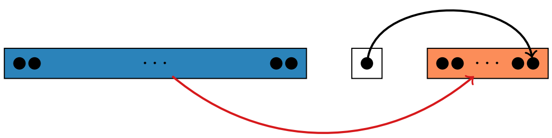
which arises from the following branch move.



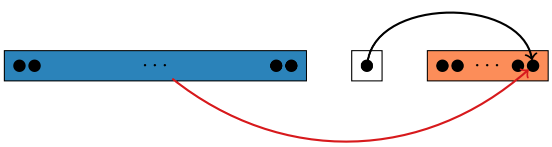
We now show that adding an edge in a norm diagram induces a clockwise tree rotation in the corresponding binary trees. We shall do this by induction on n. Note that the base case can be easily checked, see Example 6. Suppose that we begin with an arbitrary norm diagram



to which we add a non-trivial new edge as below.



Notice that from the composition and restriction rule, we can without loss of generality assume that the new edge has the following form,



that is, it goes up to the final vertex. We now have three different cases to consider based on where the new edge begins. We know that we can split up the left hand block into a diagram of the form $X \odot Y$ for some smaller diagrams X and Y. These situations are summarised in Diagram 9. In particular, we could land in Y, giving Case 1, we could land in X, giving Case 3, or the final option is that the new edge begins at vertex arising from the $\odot$ operation.

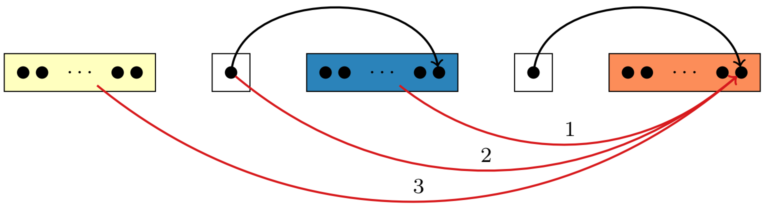
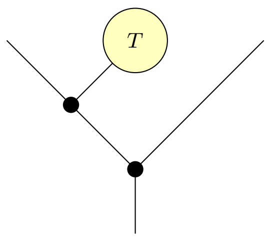
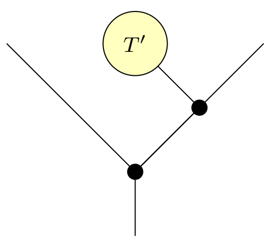


FIGURE 9. *The three cases for adding a non-trivial norm map*

Cases 1 and 2 can be verified using the induction hypothesis as we do not need to consider the leftmost block. Therefore the only non-trivial case is the third one. If the edge does not begin at the leftmost node, then we can repeat the process above and get the result once again by induction. Therefore we assume without loss of generality that we have added the norm map N_0^n. One could consider the original diagram corresponding to this tree.



The addition of the new edge then forces this to become



where T' is a tree which can be obtained from T via clockwise tree rotation moves (see Figure 8). This is therefore exactly a clockwise tree rotation. $\square$

4. *Generalising to other cyclic groups*

We would like to have a closed formula for the cardinality of $N_\infty(G)$ for all finite cyclic G. We shall explore the obstructions to obtaining such a result in this section. The main result is the construction of a lower bound of the number of such operads.

Trying to manually enumerate the norms for $G = C_{p^n q^m}$, $p \neq q$ or even just $C_{p^3 q}$ shows that the situation is already intangibly complicated. Indeed, we have computationally verified that there are 544 such N_∞-operads for $C_{p^3 q}$.

Let us highlight the style of norms that we must deal with in this circumstance. Figure 10 gives the 10 possible N_∞-operads for $G = C_{pq}$.

$$\begin{tikzcd} C_1 & C_p \\ C_q & C_{pq} \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r] & C_p \\ C_q & C_{pq} \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[d] & C_p \\ C_q & C_{pq} \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r] & C_p \\ C_q \arrow[r] & C_{pq} \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[d] & C_p \arrow[d] \\ C_q & C_{pq} \end{tikzcd}$$

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_p \\ C_q & C_{pq} \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \\ C_q \arrow[r] & C_{pq} \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \arrow[d] \\ C_q & C_{pq} \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \\ C_q & C_{pq} \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \arrow[d] \\ C_q \arrow[r] & C_{pq} \end{tikzcd}$$

FIGURE 10. *The 10 possible N_∞-operad structures for $G = C_{pq}$.*

A key observation to make is that there is an "odd one out" among these diagrams. In particular, consider the following.

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \\ C_q & C_{pq} \end{tikzcd}$$

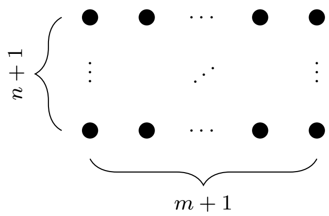
This norm is different from the other nine because it is the only one where the diagonal is not forced by the composition and restriction rules of Corollary 1. That is, if we were to remove the norm N_1^{pq}, then the resulting diagram is still a valid N_∞-operad. It follows that this N_∞-operad cannot be formed by just combining those for $G = C_p$ and $G = C_q$. We will call such an operad mixed. *If it can be obtained from the component groups, then we will call it* pure.

The main result of this section will be to give a closed expression for the number of pure N_∞-operads for $G = C_{p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}}$, which provides a non-trivial lower bound for the total number of N_∞-operads for G.

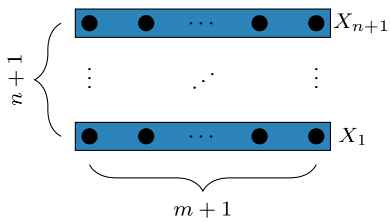
4.1. *Enumerating pure operads*

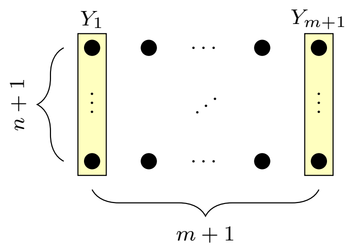
We begin with a more formal definition of "pure" and "mixed".

Let Z be an N_∞-operad for $G = C_{p^n q^m}$. That it, Z is an N_∞-diagram on the lattice



Then we can consider the rows and columns of these diagrams, to obtain a family of diagrams for $G = C_{p^m}$, namely $\{X_i\}_{1 \leqslant i \leqslant n+1}$ and a family of diagrams for $G = C_{p^n}$, namely $\{Y_i\}_{1 \leqslant i \leqslant m+1}$. Note that these are indeed valid diagrams as can be seen from observing the restriction and composition rules.





We shall say that an N_∞-operad is pure *if it is completely determined by the systems $\{X_i\}$ and $\{Y_j\}$. If an operad is not pure, then we will say that it is* mixed*. Note that an operad is mixed if after removing all norms of the form $N_{p^i}^{p^j q}$, $j \neq i$, and completing the set of norms according to the rules of Corollary 1, one does not recover the original operad one started with.*

Example 7. The following operad is pure as it has no diagonals, that is, no norms of the form $N_{p^i}^{p^j q}$. Therefore there is no condition to check

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] & C_p \\ C_q & C_{pq} \end{tikzcd}$$

The following is also pure, as when we remove the diagonal (highlighted in red) then the composition rule of Corollary 1 is violated. Completeing the set of norms according to the rules forces the diagonal, and we recover the original operad that we started with.

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d] \arrow[dr] & C_p \\ C_q \arrow[r] & C_{pq} \end{tikzcd}$$

By using the restriction rules, we see that there is a natural ordering on the systems $\{X_i\}$ and $\{Y_j\}$. Indeed, $X_1 \leqslant X_2 \leqslant \cdots \leqslant X_{n+1}$ and $Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_{m+1}$.

Definition 4. We will denote by $\mathcal{P}(n, r)$ the number of length r paths in the n-Tamari lattice $\mathcal{A}_n$. For example, $\mathcal{P}(n, 2)$ gives the sequence $1, 1, 3, 13, 68, 399, 2530, 16965, \ldots$ (starting at $n = 0$). In Châtel and Pons [**3**] this is given the closed form

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Theorem 3. *The number of pure N_∞-operads for $G = C_{p^n}C_{q^m}$ is given as*

$$\mathcal{P}(n+1, m)\mathcal{P}(m+1, n)$$

In general, for $G = C_{p_1^{n_1}} \cdots C_{p_k^{n_k}}$ the number of pure operads is

$$\prod_{j,i=1}^{k} \mathcal{P}(n_i + 1, n_j).$$

Proof. This is an exercise in counting using the orderings $X_1 \leqslant X_2 \leqslant \cdots \leqslant X_{n+1}$ and $Y_1 \leqslant Y_2 \leqslant \cdots \leqslant Y_{m+1}$. Once we have picked X_1, we must take a (possibly stationary) path of length n through the Tamari lattice $\mathcal{A}_{m+1}$ to pick the other entries. Therefore, there are $\mathcal{P}(m+1, n)$ such options for the X_i. We then have the choices for the Y_j giving us total of $\mathcal{P}(n+1, m)$ options via a similar argument. Combining these, we get the required total of $\mathcal{P}(n+1, m)\mathcal{P}(m+1, n)$.

The proof for the general case follows similarly. □

Example 8. One can compute the first few values for the sequence appearing in Theorem 3 (starting at $n = 0$ for $m = 1$) to be $1, 9, 52, 340, 2394, 17710, \ldots$. This sequence does not appear on the `OEIS` at the time of writing.

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