

Real moduli space of stable rational curves revisited

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Abstract

We give a description of the operad formed by the real locus of the moduli space of stable genus zero curves with marked points $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ in terms of a homotopy quotient of an operad of associative algebras. We use this model to find different Hopf models of the algebraic operad of Chains and homologies of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$. In particular, we show that the operad $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is not formal. The manifolds $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ are known to be Eilenberg-MacLane spaces for the so called pure Cacti groups. As an application of the operadic constructions we prove that for each n the cohomology ring $H^*(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}), \mathbb{Q})$ is a Koszul algebra and that the manifold $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is not formal but is a rational $K(\pi, 1)$ space. We give the description of the Lie algebras associated with the lower central series filtration of the pure Cacti groups.

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0 Introduction

The Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,n}$ of the moduli space of genus zero algebraic curves with n marked points is a smooth algebraic variety defined over \mathbb{Q} ([6]). The natural stratification of the space $\overline{\mathcal{M}}_{0,n}$ by the number of double points on a curve defines a structure of a cyclic operad on $\overline{\mathcal{M}}_{0,n}$. The cohomology ring of $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ was found by Keel ([24]) and the description of an algebraic operad formed by the union $\cup_{n \geq 3} H_*(\overline{\mathcal{M}}_{0,n}(\mathbb{C}); \mathbb{Q})$ was presented by Kontsevich and Manin ([33]) and by Getzler([15]). Each complex projective smooth variety $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ is formal. Moreover, the operad $\cup \overline{\mathcal{M}}_{0,n}(\mathbb{C})$ is formal ([17]). Surprisingly, the detailed description of the (rational) homotopy groups of $\overline{\mathcal{M}}_{0,n}(\mathbb{C})$ as well as the description of the corresponding operad $\cup_{n \geq 3} \pi_*(\overline{\mathcal{M}}_{0,n}(\mathbb{C})) \otimes \mathbb{Q}$ in the category of L_∞ -algebras is still unknown.

On the other hand, the homotopy type of the real points of this variety (denoted by $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$) was found before its cohomology. Davis-Januszkiewicz-Scott ([4]) proved that the real manifold $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is aspherical and found a presentation of its fundamental group which we call *Pure Cacti group* and denote it by \mathcal{PCacti}_n (see Section 1.3). Kapranov ([22]) and Devadoss([7]) realized that $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ form an operad and Etingof-Henriques-Kamnitzer-Rains described the structure of the cohomology ring $H^*(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}))$ and found a presentation of the algebraic operad $\cup_{n \geq 2} H^*(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}))$. The (pure) cacti groups \mathcal{PCacti}_n has a lot of common properties with (pure) braid groups and are of particular interest for representation theory (e.g. [18]). The (pure) cacti groups play the same role in coboundary categories (introduced by Drinfeld [10]) as the (pure) braid groups in braided tensor categories (e.g. [23]). Various conjectures were stated about the lower central series and Malcev completions of the pure cacti groups in [12]. These conjectures streamline the comparisons of (pure) cacti groups and (pure) braid groups (Section 3 of [12]).

The main purpose of the present paper is to clarify the structure of the topological operad $\cup_{n \geq 2} \overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$, present nice algebraic models of this operad and prove almost all conjectures stated in [12]. In particular, we prove the following.

- The Koszul dual operad to the operad of $\text{Chains}_{\mathbb{Q}}(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}))$ coincides with \mathbb{Z}_2 -invariants of the associative operad (Theorem 2.2.3) where the generator of \mathbb{Z}_2 interchanges an algebra and its opposite.
- We find a presentation of the latter operad of \mathbb{Z}_2 -invariants of the associative operad (Theorem 3.4.1);
- This crucially simplifies the computation of the Poincaré polynomial and the description of the homology cooperad $\cup_{n \geq 2} H^*(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}), \mathbb{Q})$ (Corollary 3.4.8) compare to the one suggested in [12].
- The Koszul dual statement leads the homotopy equivalence of the operad $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ and the following homotopy quotient (Corollary 2.3.1):

$$\cup_{n \geq 2} \overline{\mathcal{M}}_{0,n+1}(\mathbb{R}) \simeq \frac{E_1 \rtimes \mathbb{Z}_2}{\mathbb{Z}_2} \simeq \frac{\text{As} \rtimes \mathbb{Z}_2}{\mathbb{Z}_2}$$

- We adopt certain Hopf models known for little balls operads E_d and for its cohomology operads Pois_d to the case of Hopf operads $\frac{E_d \rtimes \mathbb{Z}_2}{\mathbb{Z}_2}$ and their homology operads called $\text{Pois}_d^{\text{odd}}$ (Chapter §5).

- This description is used to prove the main conjectures stated in [12] concerning (pure) cacti groups (our results are stated in Section 1.7).
- In particular, we proved that the quadratic algebras $H^*(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}); \mathbb{Q})$ are Koszul (Theorem 6.1.4) and, consequently, that $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ are rational $K(\pi, 1)$ spaces (Corollary 6.2.1).

Moreover, we state that the technique used for the computations of Kontsevich graph complexes ([57, 58]) can be reasonably adopted to give a description of the deformation theory of the operads $\text{Pois}_d^{\text{odd}}$, $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ and the maps of operads $\frac{E_d \times \mathbb{Z}_2}{\mathbb{Z}_2} \rightarrow E_{d+1}$ that is presented by Drinfeld unitarization trick for $d = 1$. In particular, the deformation complex of the operad $\text{Pois}_d^{\text{odd}}$ is supposed to be quasi-isomorphic to the subspace of the Kontsevich graph complex GC_d spanned by graphs with odd Euler characteristic and that the operad $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ has no nontrivial deformations.

0.1 Structure of the paper

The first chapter §1 is an extended introduction and the advertisement of the material for those who do not want to deal seriously with operads. We first recollect the known material related to $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ (§1.1, §1.2) and the pure cacti groups (§1.3). We recall the notion of the Coboundary category and its relation with Braided Tensor categories suggested by Drinfeld in §1.6. We finish with the announcement of the corollaries of this paper for the pure cacti groups in §1.7.

§2 contains the operadic description of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ that clarifies the cell decomposition of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ discovered by Devadoss and Kapranov.

We collect all computations of the operads of \mathbb{Z}_2 -invariants on the operads of commutative, associative and Poisson algebras in Chapter §3.

Chapter §4 contains an outline of different known combinatorial dgca models of the little balls operad E_d that are generalized to the case of the mosaic operad in Chapter §5.

Chapter §6 contains the proof of the main Theorems 6.1.4 which is the most technical and complicated part of this paper.

The results on deformation of the Mosaic operad and its relations with the little discs operad are outlined in Chapter §7.

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1 Moduli space of stable rational curves and Cacti groups

1.1 Complex stable curves and $\overline{\mathcal{M}}_{0,n}$

Recall [6] that a point of $\overline{\mathcal{M}}_{0,n}$ (the Deligne-Mumford compactification of the space of genus zero curves with n marked points) is a stable curve. A stable curve of genus 0 with n labeled points is a finite union C of projective lines C_1, \dots, C_p , together with labeled distinct points $z_1, \dots, z_n \in C$ such that the following conditions are satisfied

1. Each marked point z_i belongs to a unique C_j .
2. The intersection of projective lines $C_i \cap C_j$ is either empty or consists of one point, and in the latter case the intersection is transversal.
3. The graph of components (whose vertices are the lines C_i and whose edges correspond to pairs of intersecting lines) is a tree.
4. The total number of special points (i.e. marked points or intersection points) that belong to a given component C_i is at least 3.

An equivalence between two stable curves $C = (C_1, \dots, C_p, z_1, \dots, z_n)$ and $C' = (C'_1, \dots, C'_p, z'_1, \dots, z'_n)$ is an isomorphism of algebraic curves $f : C \rightarrow C'$ which maps z_i to z'_i for each i .

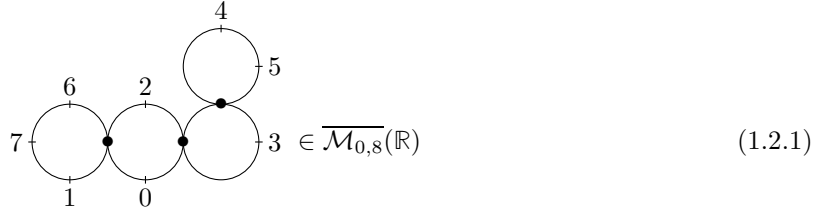
The gluing of two stable curves through marked points defines a collection of maps

$$\overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1} \rightarrow \overline{\mathcal{M}}_{0,m+n} \quad (1.1.1)$$

that assembles a structure of a cyclic operad on $\{\overline{\mathcal{M}}_{0,n+1}\}$. We will be interested in this paper only in the structure of symmetric operad. Thus we mark one of the points of a stable curve by 0 and call it an output.

1.2 The moduli space $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ (=the mosaic operad)

The real locus $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ of the Deligne-Mumford compactification consists of equivalence classes of stable curves of genus 0 with n labeled points defined over \mathbb{R} . The projective line over real numbers is a circle, thus pictorially a stable curve is a ‘‘cactus-like’’ structure – a tree of circles with labeled points on them:



Example 1.2.2. 1. $\overline{\mathcal{M}}_{0,3}(\mathbb{R})$ is a point.

2. $\overline{\mathcal{M}}_{0,4}(\mathbb{R})$ is a circle. The cross-ratio map defines an isomorphism $\overline{\mathcal{M}}_{0,4}(\mathbb{R}) \rightarrow \mathbb{RP}^1$.

3. $\overline{\mathcal{M}}_{0,5}(\mathbb{R})$ is a compact connected non-orientable surface with Poincaré polynomial equals to $1 + 4t$ (see [7]).

Let us summarize some of the known results about $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$.

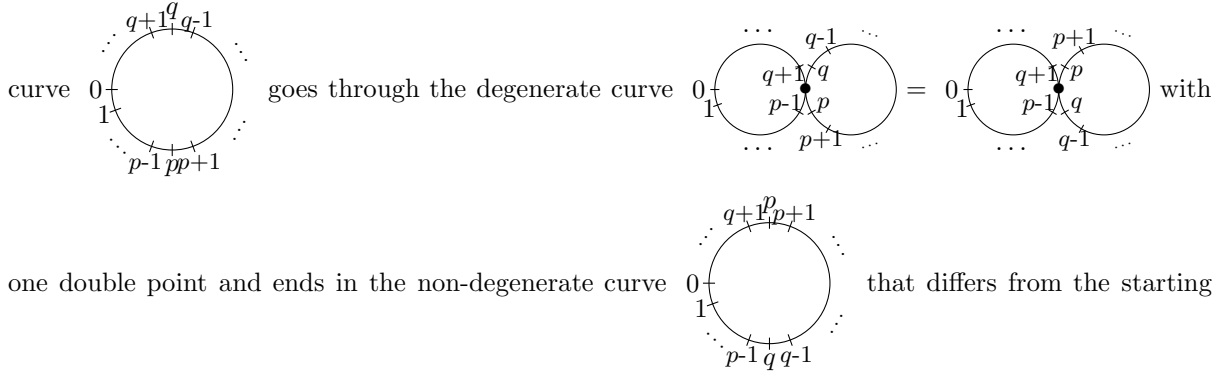
1. ([7]) The $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ are connected, compact, smooth manifolds of dimension $n - 2$ and the gluing maps (1.1.1) define an operad structure on their union. This operad is called the *mosaic operad* after S.Devadoss([7]). We will come back to the detailed description of this operad in Section 2.
2. ([4]) $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ is a $K(\pi, 1)$ -space. The fundamental group $\pi_1(\overline{\mathcal{M}}_{0,n}(\mathbb{R}))$ is called pure cacti group. We recall the definition of cacti groups and pure cacti groups in the next subsection 1.3.
3. ([12]) The rational cohomology $H^*(\overline{\mathcal{M}}_{0,n}(\mathbb{R}); \mathbb{Q})$ is a quadratic algebra. We recall these algebras in Section 5.1.
4. ([12]) The algebraic operad of homology groups $H_*(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}); \mathbb{Q})$ is a quadratic Koszul operad called the operad of *2-Gerstenhaber* algebras in [12]. We will denote this operad by $\text{Pois}_1^{\text{odd}}$. The precise definition, the simple proof of the coincidence of the operads $H_*(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}); \mathbb{Q})$ and $\text{Pois}_1^{\text{odd}}$ and, finally, the reason for this name will be explained in Section 3 and 2.

1.3 (Pure) Cacti groups

The symmetric group S_n act on the space $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ while permuting the labels of the marked points keeping untouched the label 0. Unfortunately this action has orbits of different size, thus the space $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})/S_n$ is an orbifold rather than a manifold. The orbifold fundamental group of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})/S_n$ is easy to compute ([7],[4],[18]) out of the natural cell decomposition of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$. We call the corresponding group *cacti group* and denote it by Cacti_n motivated by Picture (1.2.1). The group Cacti_n has the following presentation via generators and relations([4]):

$$\text{Cacti}_n := \left\langle \begin{array}{l} s_{pq}, \\ 1 \leq p < q \leq n \end{array} \left| \begin{array}{l} s_{pq}^2 = 1; \\ s_{pq}s_{kl} = s_{kl}s_{pq}, \text{ if } [pq] \cap [kl] = \emptyset, \\ s_{pq}s_{kl} = s_{p+q-l, p+q-k}s_{pq}, \text{ if } [pq] \supset [kl] \end{array} \right. \right\rangle$$

The generator s_{pq} can be presented by a path on $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ that starts in a marked non-degenerate



The fundamental group of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is called the *Pure Cacti group* (notation \mathcal{PCacti}_n) and coincides with the kernel of the surjection

$$\begin{array}{ccc} s_{pq} & \mapsto & \begin{pmatrix} 1 & \dots & p-1 & p & p+1 & \dots & q & q+1 & \dots & n \\ 1 & \dots & p-1 & q & q-1 & \dots & p & q+1 & \dots & n \end{pmatrix} \\ \mathcal{Cacti}_n & \twoheadrightarrow & S_n \end{array} \quad (1.3.1)$$

Unfortunately, we do not know any simple presentation via generators and relations of the group \mathcal{PCacti}_n . However, let us report some results on this group out of [12].

- Remark 1.3.2.**
1. The group \mathcal{PCacti}_n is a finitely presented torsion-free group, because the corresponding Eilenberg-MacLane space is a finite-dimensional compact manifold.
 2. ([12] Theorem 3.8) The abelianization $\mathcal{PCacti}_n/(\mathcal{PCacti}_n, \mathcal{PCacti}_n)$ is isomorphic to $\mathbb{Z}^r \oplus E$ where E is a vector space over \mathbb{F}_2 .

Note that (pure) cacti groups have a lot of common properties with the (pure) braid groups. For example, there is a simple generalization of cacti groups for other Dynkin diagrams ([4] Theorem 4.7.2). See also [46] and [37] for applications and references therein.

1.4 Lie algebras associated with completions of Pure Cacti groups

There are three family of Lie algebras associated with the groups \mathcal{PCacti}_n discussed in details in [12] (Section 3). We index the Lie algebras defined over \mathbb{Z} (over \mathbb{Q}) by a special superscript \mathbb{Z} (resp. \mathbb{Q}) if there is no special superscript we work with ordinary Lie algebras defined over \mathbb{Q} or over any other field of zero characteristics.

($\mathcal{L}_n^{\mathbb{Z}}$) The quadratic dual Lie algebra (defined over the integers) to the quadratic supercommutative algebra $H^*(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}); \mathbb{Z})$ is

$$\mathcal{L}_n^{\mathbb{Z}} := \text{Lie}^{\mathbb{Z}} \left(\begin{array}{l} \nu_{ijk}, 1 \leq i, j, k \leq n \\ \nu_{ijk} = (-1)^\sigma \nu_{\sigma(i)\sigma(j)\sigma(k)} \\ \deg(\nu_{ijk}) = 0 \end{array} \middle| \begin{array}{l} [\nu_{ijk}, \nu_{pqi} + \nu_{pqj} + \nu_{pqk}] = 0 \\ [\nu_{ijk}, \nu_{pqr}] = 0 \\ \text{for } \#\{i, j, k, p, q, r\} = 6. \end{array} \right)$$

($\mathcal{L}_n^{\mathbb{Z}}$) The commutator in a group \mathcal{PCacti}_n defines a \mathbb{Z} -Lie algebra structure on the associated graded space to the lower central series filtration:

$$\mathcal{PCacti}_n \supset \mathcal{PCacti}_n^2 := (\mathcal{PCacti}_n, \mathcal{PCacti}_n) \supset \dots \supset \mathcal{PCacti}_n^p := (\mathcal{PCacti}_n, \mathcal{PCacti}_n^{p-1}) \supset \dots$$

that we denote by $\mathcal{L}_n^{\mathbb{Z}}$. The quotient of $\mathcal{L}_n^{\mathbb{Z}}$ by the 2-torsion is known to be the \mathbb{Z} -Lie algebra $\mathcal{L}_n^{\mathbb{Z}}$ with the same set of generators as $\mathcal{L}_n^{\mathbb{Z}}$. Theorem 3.9 of [12] states that there exists a natural S_n -equivariant surjective map

$$\psi_n : \mathcal{L}_n^{\mathbb{Z}} \twoheadrightarrow \mathcal{L}_n^{\mathbb{Z}}.$$

(\mathfrak{L}_n) Let $\widehat{\mathcal{PCacti}}_n$ be the pronipotent (=Malcev) completion of \mathcal{PCacti}_n over \mathbb{Q} . Let $\mathfrak{L}_n := \text{Lie}(\widehat{\mathcal{PCacti}}_n)$ be the \mathbb{Q} -Lie algebra associated to $\widehat{\mathcal{PCacti}}_n$. That is \mathfrak{L}_n is the set of primitive elements of the complete

Hopf algebra $\mathbb{Q}[\widehat{\mathcal{PCacti}_n}]$. The rational homotopy theory ([45]) predicts that the associated graded to the lower central series filtration on $\text{Lie}(\widehat{\mathcal{PCacti}})$ is isomorphic to the rationalization of \mathcal{L}'_n :

$$\text{gr}(\mathcal{L}_n) := \text{gr}(\text{Lie}(\widehat{\mathcal{PCacti}_n})) \simeq \mathcal{L}'_n{}^{\mathbb{Z}} \otimes \mathbb{Q} = \mathcal{L}_n{}^{\mathbb{Z}} \otimes \mathbb{Q} =: \mathcal{L}_n{}^{\mathbb{Q}}$$

Let us also denote by $\widehat{\mathcal{Cacti}_n}$ the proalgebraic group

$$\mathcal{Cacti}_n \times_{\mathcal{PCacti}_n} \widehat{\mathcal{PCacti}_n} \quad (1.4.1)$$

that corresponds to the prounipotent completion of the subgroup of pure cacti elements.

1.5 (Pure) braid group and completions

Let us recall (after [30],[10],[11]) the analogous known results on the (pure) braid group $(\mathcal{P})\mathfrak{B}_n$ on n strands:

$$\mathfrak{B}_n := \left\langle \begin{array}{c} r_{i,i+1} \\ i = 1, \dots, n-1 \end{array} \left| \begin{array}{l} r_{i-1,i} r_{i,i+1} r_{i-1,i} = r_{i,i+1} r_{i-1,i} r_{i,i+1} \\ r_{i,i+1} r_{j,j+1} = r_{j,j+1} r_{i,i+1}, \text{ if } |i-j| > 1 \end{array} \right. \right\rangle$$

The pure braid group $\mathcal{P}\mathfrak{B}_n$ is the kernel of the surjection $\mathfrak{B}_n \xrightarrow{r_{i,i+1} \mapsto (i+1)}$ S_n . The configuration space of n (numbered) points on $\mathbb{C} = \mathbb{R}^2$ is the Eilenberg-MacLane space of \mathfrak{B}_n (respectively $\mathcal{P}\mathfrak{B}_n$).

The prounipotent completion of the pure braid group $\mathcal{P}\mathfrak{B}_n$ coincides with the rationalization of the lower central series completion and is denoted by $\widehat{\mathcal{P}\mathfrak{B}_n}$. The corresponding Lie algebra is called Drinfeld-Kohno Lie algebra (denoted by $\mathfrak{t}(n)^{\mathbb{Z}}$ and $\mathfrak{t}(n)^{\mathbb{Q}}$ respectively) and has the following presentation by generators and quadratic relations:

$$\mathfrak{t}(n) := \text{Lie} \left(\begin{array}{c} t_{ij}, 1 \leq i \neq j \leq n \\ t_{ij} = t_{ji} \end{array} \left| \begin{array}{l} [t_{ij}, t_{ik} + t_{jk}] = 0 \\ [t_{ij}, t_{kl}] = 0, \\ \text{for } \#\{i, j, k, l\} = 4. \end{array} \right. \right)$$

The kernel of the map $\mathfrak{t}(n) \rightarrow \mathfrak{t}(n-1)$ is known to be isomorphic to the free Lie algebra generated by the set $\{t_{in} | 1 \leq i \leq n-1\}$. In particular, $\mathfrak{t}(n)^{\mathbb{Z}}$ is the free \mathbb{Z} -module. Analogously to (1.4.1) we have:

$$\widehat{\mathfrak{B}}_n := \mathfrak{B}_n \times_{\mathcal{P}\mathfrak{B}_n} \widehat{\mathcal{P}\mathfrak{B}_n}$$

Example 1.5.1. For any given collection of objects X_1, \dots, X_n in a braided tensor category $(\mathcal{C}, \otimes, R, \Phi)$ the operators

$$r_{i,i+1}^{\mathcal{C}} := R_{X_i, X_{i+1}} : X_1 \otimes \dots \otimes X_i \otimes X_{i+1} \otimes \dots \otimes X_n \longrightarrow X_1 \otimes \dots \otimes X_{i+1} \otimes X_i \otimes \dots \otimes X_n$$

defines the action of the Braid groupoid on tensor products $\{X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)} | \sigma \in S_n\}$. Respectively, the pure braid group $\mathcal{P}\mathfrak{B}_n$ admits a natural representation on $X_1 \otimes \dots \otimes X_n$.

1.6 Coboundary categories and Drinfeld's Unitarization trick

Cacti and Pure cacti groups naturally arise in the theory of coboundary monoidal categories introduced by Drinfeld ([10]). Following [18] we recall that a *coboundary monoidal category* is a monoidal category \mathcal{C} together with a (functorial in $X, Y \in \text{Ob}(\mathcal{C})$) commutator morphism:

$$c_{X,Y} : X \otimes Y \rightarrow Y \otimes X \text{ such that } c_{X,Y} \circ c_{Y,X} = \text{Id}_{X \otimes Y},$$

and yielding the following relations for all $X, Y, Z \in \text{Ob}(\mathcal{C})$:

$$\begin{array}{ccccc} & & (X \otimes Y) \otimes Z & \xrightarrow{\Phi_{X,Y,Z}} & X \otimes (Y \otimes Z) \\ & \swarrow c_{X,Y} \otimes \text{Id}_Z & & & \searrow \text{Id}_X \otimes c_{Y,Z} \\ (Y \otimes X) \otimes Z & & & & X \otimes (Z \otimes Y) \\ & \searrow c_{Y \otimes X, Z} & & & \swarrow c_{X, Z \otimes Y} \\ & & Z \otimes (Y \otimes X) & \xleftarrow{\Phi_{Z,Y,X}} & (Z \otimes Y) \otimes X \end{array}$$

where $\Phi_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$ is the associativity isomorphism.

For each tensor product $X_1 \otimes X_2 \otimes \dots \otimes X_n$ denote by s_{pq}^C the following composition of commutators (and associators):

$$c_{(X_{q-1} \otimes \dots \otimes X_p), X_q} \circ \dots \circ c_{(X_{p+1} \otimes X_p), X_{p+2}} \circ c_{X_p, X_{p+1}} \quad (1.6.1)$$

that reverse the order of the subsequence of tensor multiples $X_p \otimes X_{p+1} \otimes \dots \otimes X_q$ inside $X_1 \otimes \dots \otimes X_n$:

$$s_{pq}^C : X_1 \otimes \dots \otimes X_p \otimes \dots \otimes X_q \otimes \dots \otimes X_n \longrightarrow X_1 \otimes \dots \otimes X_q \otimes \dots \otimes X_p \otimes \dots \otimes X_n$$

We omit identity operators, bracketings and associators in presentation (1.6.1) for simplicity.

Proposition. *For any given collection of objects $X_1, \dots, X_n \in \mathcal{C}$ the operators s_{pq}^C defines an action of the Cacti groupoid on tensor products $\{X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)} \mid \sigma \in S_n\}$. Respectively, the pure cacti group \mathcal{PCacti}_n admits a natural representation on $X_1 \otimes \dots \otimes X_n$.*

Proof. Theorem 7 (Section 3) of [18]. □

Drinfeld's motivation of coboundary categories comes from the following unitarization trick:

Example 1.6.2. ([10]) *Let $(\mathcal{C}, \otimes, R, \Phi)$ be a braided monoidal category. Then the operator*

$$c_{X,Y} := R_{X,Y} \circ (R_{Y,X} \circ R_{X,Y})^{-\frac{1}{2}} \in \widehat{\mathfrak{B}}_2$$

is well defined in the completion of the braid group and defines a commutator and a structure of the coboundary category on \mathcal{C} .

In particular, one can define a commutator for the category of finite-dimensional representations of the quantum group $U_q(\mathfrak{g})$ for generic q . It was shown in [20] that the commutator make sense for the case q tends to 0 and one has a nice example of a coboundary strict monoidal category with combinatorially defined commutator.

Example 1.6.3. *The category of crystals of finite-dimensional representations of a given simple Lie algebra \mathfrak{g} form a coboundary strict monoidal category. There are several different definitions of the commutator [18, 19, 20]. In spite of different definition they all happen to be the same what is explained via deformation theory of operads in Section 7.*

Drinfeld's unitarization trick 1.6.2 defines a map of prounipotent completions of nonpure groups and corresponding Lie algebras of prounipotent completions of pure groups.

$$\bar{\xi}_n : \widehat{\mathcal{Cacti}}_n \rightarrow \widehat{\mathfrak{B}}_n \quad \Rightarrow \quad \tilde{\xi}_n^{\mathbb{Q}} : \mathfrak{L}_n^{\mathbb{Q}} \rightarrow \mathfrak{t}(n)^{\mathbb{Q}}$$

The associated graded morphism with respect to the lower central series filtration is easy to compute on the level of generators of the corresponding Lie algebras and it happens to be defined over integers (see Section 3.11 of [12]):

$$\xi_n : \mathfrak{L}_n^{\mathbb{Z}} \xrightarrow{\nu_{ijk} \mapsto \nu_{ijk}} \mathfrak{L}_n^{\mathbb{Z}} \xrightarrow{\nu_{ijk} \mapsto [t_{ij}, t_{jk}]} \mathfrak{t}(n)^{\mathbb{Z}} \quad (1.6.4)$$

1.7 Overview of results (without operads)

In this paper we use a lot the language of operads while formulating and proving the results. However, we decided to state the outline of the applications of our results to the Pure Cacti groups and the associated Lie algebras that does not involve so far the word operad. We prove the following results for all n

1. The Lie algebras $\mathfrak{L}_n^{\mathbb{Q}}$ are Koszul (Theorem 6.1.4);
2. The morphisms $\xi_n^{\mathbb{Q}} : \mathfrak{L}_n^{\mathbb{Q}} \rightarrow \mathfrak{t}(n)^{\mathbb{Q}}$ and $\tilde{\xi}_n^{\mathbb{Q}} : \mathfrak{L}_n^{\mathbb{Q}} \rightarrow \mathfrak{t}(n)^{\mathbb{Q}}$ are embeddings of Lie algebras (Theorem 6.1.4 and Corollary 6.2.2 respectively);
3. $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ is a rational $K(\pi, 1)$ space (Corollary 6.2.1) meaning that its \mathbb{Q} -completion does not have higher homotopy groups;

4. The spaces $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ are not formal for all $n \geq 6$ as shown in [12] (Proposition 3.13). This means, in particular, that

$$\mathfrak{L}_n \not\simeq \text{gr}(\mathfrak{L}_n) \simeq \mathfrak{L}_n^{\mathbb{Q}}$$

We found a huge but purely combinatorial dg-model of the space $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ given by certain graphs and highly nontrivial differential (Theorem 5.3.4).

As it is mentioned in Section 3 of [12] these results imply in addition the following conclusions (that were stated as conjectures in [12]):

Corollary 1.7.1. 1. *The kernel of the projection $\pi_n : \mathfrak{L}_n^{\mathbb{Q}} \rightarrow \mathfrak{L}_{n-1}^{\mathbb{Q}}$ is a free Lie algebra on infinitely many generators;*

2. *$\mathfrak{L}_n^{\mathbb{Z}}$ and its universal enveloping algebra $U(\mathfrak{L}_n^{\mathbb{Z}})$ may have only 2-torsion.*

3. *$\psi_n : \mathfrak{L}_n^{\mathbb{Z}[\frac{1}{2}]} \rightarrow \mathfrak{L}_n^{\mathbb{Z}[\frac{1}{2}]}$ is an isomorphism. Thus, the only torsion of the lower central series completion \mathfrak{L}'_n of \mathcal{PCacti}_n is the 2-torsion.*

Proof. We have the following diagram of maps of Lie algebras defined over the integers:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker \pi_n & \longleftarrow & \mathfrak{L}_n^{\mathbb{Z}} & \xrightarrow{\pi_n} & \mathfrak{L}_{n-1}^{\mathbb{Z}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \xi_n & & \downarrow \xi_{n-1} & & \\ 0 & \longrightarrow & \text{Lie}^{\mathbb{Z}}(t_{1n}, \dots, t_{n-1,n}) & \longleftarrow & \mathfrak{t}(n)^{\mathbb{Z}} & \longrightarrow & \mathfrak{t}(n-1)^{\mathbb{Z}} & \longrightarrow & 0 \end{array}$$

Theorem 6.1.4 predicts that all vertical arrows tensored with \mathbb{Q} are known to be embeddings of \mathbb{Q} -Lie algebras. Therefore, the kernel of the projection π_n tensored with \mathbb{Q} is a Lie subalgebra of the free Lie algebra on $n-1$ generators. The Shirshov-Witt Theorem then states that any Lie subalgebra of the free Lie algebra is also free (see e.g. [50],[61]), thus proving the first assertion above.

In order to prove the second claim we recall that one of the main results of [12] states that the quadratic (and Koszul) dual algebra to $\mathfrak{L}_n^{\mathbb{Z}[\frac{1}{2}]}$ is equal to the cohomology ring $H^{\bullet}(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}), \mathbb{Z}[\frac{1}{2}])$ and, moreover, that the integral cohomology of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ has only 2-torsion. In particular, this implies that $H^2(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}), \mathbb{Z}[\frac{1}{2}])$ is a free finitely generated $\mathbb{Z}[\frac{1}{2}]$ -module which we denote by S . Let us denote by S^{\perp} the $\mathbb{Z}[\frac{1}{2}]$ -module spanned by quadratic relations in the cohomology ring and by the commutativity relations:

$$\nu_{ijk} \wedge \nu_{pqr} = -\nu_{pqr} \wedge \nu_{ijk}.$$

In particular, S is the $\mathbb{Z}[\frac{1}{2}]$ -span of quadratic relations in the Lie algebra \mathfrak{L}_n and in its universal enveloping algebra $U(\mathfrak{L}_n)$. Moreover, we have the following isomorphism of free $\mathbb{Z}[\frac{1}{2}]$ -modules:

$$\mathbb{Z}[\frac{1}{2}]\{\nu_{ijk}\} \otimes_{\mathbb{Z}[\frac{1}{2}]} \mathbb{Z}[\frac{1}{2}]\{\nu_{ijk}\} \simeq S \oplus S^{\perp}$$

The description of the ideal generated by quadratic algebras predicts the following isomorphisms for the graded component of the universal enveloping algebra:

$$U(\mathfrak{L}_n^{\mathbb{Z}[\frac{1}{2}]})_m \simeq \frac{\mathbb{Z}[\frac{1}{2}]\{\nu_{ijk}\}^{\otimes m}}{+\sum_{i=1}^{n-1} \mathbb{Z}[\frac{1}{2}]\{\nu_{ijk}\}^{\otimes i-1} \otimes S \otimes \mathbb{Z}[\frac{1}{2}]\{\nu_{ijk}\}^{\otimes m-i-1}} \simeq \bigcap_{i=1}^{n-1} \mathbb{Z}[\frac{1}{2}]\{\nu_{ijk}\}^{\otimes i-1} \otimes S^{\perp} \otimes \mathbb{Z}[\frac{1}{2}]\{\nu_{ijk}\}^{\otimes m-i-1}$$

In particular, since the intersection of free $\mathbb{Z}[\frac{1}{2}]$ modules is free we get the second item of Corollary 1.7.1.

Finally, the map $\xi_n^{\mathbb{Z}[\frac{1}{2}]} : \mathfrak{L}_n^{\mathbb{Z}[\frac{1}{2}]} \rightarrow \mathfrak{t}(n)^{\mathbb{Z}[\frac{1}{2}]}$ is the map of free $\mathbb{Z}[\frac{1}{2}]$ -modules such that tensored with \mathbb{Q} it is an embedding. Consequently, $\xi_n^{\mathbb{Z}[\frac{1}{2}]}$ is also an embedding.

As mentioned in (1.6.4) the map $\xi_n : \mathfrak{L}_n \hookrightarrow \mathfrak{t}(n)$ factors through the surjection $\psi_n : \mathfrak{L}_n \twoheadrightarrow \mathfrak{L}_n$ and, therefore, $\psi_n^{\mathbb{Z}[\frac{1}{2}]}$ is an isomorphism. \square

2 Conceptual algebraic model of the mosaic operad

This chapter is devoted to the understanding of the cell decomposition of the mosaic operad that is compatible with the operad structure. We do not recall the notion of an operad and Koszul duality for operads and refer to the original paper of Ginzburg and Kapranov [16] and the modern book on algebraic operads [36] and references therein.

2.1 \mathbb{Z}_2 -action on the associative operad

Recall that the symmetric associative operad \mathbf{As} is a quadratic Koszul operad whose space of n -ary operations is spanned by operations $\{x_{\sigma(1)} \dots x_{\sigma(n)} \mid \sigma \in S_n\}$. The associative operad \mathbf{As} has an automorphism $\tau^!$ of order 2 that flips the order of the multiplication:

$$\tau^! : x_1 x_2 \rightarrow x_2 x_1, \quad \tau^! : x_{\sigma(1)} \dots x_{\sigma(n)} \rightarrow x_{\sigma(n)} \dots x_{\sigma(1)}. \quad (2.1.1)$$

The associative operad is Koszul self-dual $\mathbf{As}^! = \mathbf{As}$ and the Koszul dual automorphism τ of order 2 has the following presentation:

$$\tau : x_1 x_2 \rightarrow -x_2 x_1, \quad \tau : x_{\sigma(1)} \dots x_{\sigma(n)} \rightarrow (-1)^{n-1} x_{\sigma(n)} \dots x_{\sigma(1)} \quad (2.1.2)$$

The subspace of invariants of the corresponding \mathbb{Z}_2 -action forms an operad which we denote by $[\mathbf{As}]^{\mathbb{Z}_2}$. In particular, we clearly have

$$\dim([\mathbf{As}]^{\mathbb{Z}_2}(n)) = \frac{n!}{2} \quad \forall n \geq 2. \quad (2.1.3)$$

We will find the presentation of this operad in terms of generators and relations later in Theorem 3.4.1. Let us denote by $[\mathbf{As}]_{\mathbb{Z}_2}^{\vee}$ the corresponding (linear dual) cooperad in the category of vector spaces. Note that $[\mathbf{As}]_{\mathbb{Z}_2}^{\vee}$ is the set of \mathbb{Z}_2 -coinvariants of the cooperad \mathbf{As}^{\vee} .

2.2 The Mosaic operad is bar-dual to $[\mathbf{As}]^{\mathbb{Z}_2}$

Let us come back to the geometry of the moduli spaces of curves. The space $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ admits a stratification by the graphs of irreducible components of a stable curve. The codimension one open strata consist of degenerate curves with two components with at least 3 points on each component. Thus, the codimension one closed strata are isomorphic to $\overline{\mathcal{M}}_{0,|I|+1}(\mathbb{R}) \times \overline{\mathcal{M}}_{0,|J|+1}(\mathbb{R})$ and are numbered by decompositions $[0n] = I \sqcup J$ with $|I|, |J| \geq 2$. The stratification defines a cyclic operad structure on the union $\cup_{n \geq 2} \overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$. This operad was introduced by S.Devadoss in [7] and called the *mosaic operad* and by Kapranov in [22]. We will use the slightly shorter notation $\overline{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}$ for the Mosaic operad. All open strata are contractible and the standard stratification considered by Devadoss defines a cell decomposition of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ compatible with the operadic compositions.

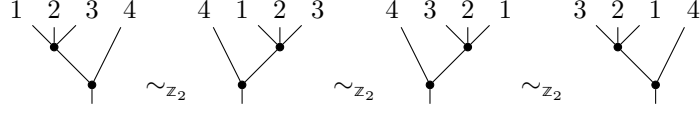
The underlying combinatorics of the latter cell decomposition are very similar to those of the cell decomposition of the Stasheff polytopes. Since we want to use the operadic structure and we want to have analogous constructions in higher dimensions we will use the notation $\mathbf{FM}_1(n)$ for the Fulton-McPherson compactification of the configuration space of points on a line. We recall the description of \mathbf{FM}_d as a model of E_d in 4.2 and refer to [34] for detailed discussions.

Let T_n be the set of all planar rooted trees with n leaves that are numbered from 1 to n . In particular, $|T_n| = n! a_n$ where a_n are the generalized Catalan numbers, also called the little Schröder numbers (see e.g. [44] and see Sloanes OEIS number A001003 [52]). The cells in the standard cell decomposition of the Stasheff polytope $\mathbf{FM}_1(n)$ are indexed by elements of T_n . The vertex splitting operation on trees defines the combinatorics of the boundary maps in the cell decomposition. In particular, the codimension of a cell equals the number of inner vertices (see [35] and references therein).

Let $\tau : T_n \rightarrow T_n$ be the reflection of a planar tree around the y -axes.¹ For each vertex $v \in \mathbf{Vert}(T)$ of a tree T we define the corresponding reflection τ_v of a maximal subtree of T whose root coincides with v . We denote by \sim_{z_2} the minimal equivalence relation generated by reflections in vertices. In other

¹The line around which the reflection is made does not really matter. We just have to change the orientation of the plane.

words, we say that planar leaf-labeled trees $T_1, T_2 \in T_n$ are equivalent if they are connected by a finite composition of reflections $\tau_{v_1} \circ \dots \circ \tau_{v_k}$. For example, we have



The description of the cell decomposition suggested in [7, 4] can be summarized as follows.

- Proposition 2.2.1.** 1. The cells of the Devadoss's cell decomposition of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ are indexed by elements of T_n/\sim_{z_2} ;
 2. There is a surjective cellular map of topological operads:

$$p : \text{FM}_1 \rightarrow \overline{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}. \quad (2.2.2)$$

such that for all $T \in T_n$ the restriction of the map p_n on the face (cell) U_T of the Stasheff polytope $\text{FM}_1(n)$ is a diffeomorphism between U_T and the cell $U_{[T]} \subset \overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$. Where $U_{[T]}$ is the cell of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ assigned to the class of T in T_n/\sim_{z_2} .

Proof. The proof is contained in [7] but is stated in a bit different form. Let us sketch the main idea. The open (top-dimensional) cells of $\text{FM}_1(n)$ are numbered by trees from T_n with the unique inner vertex. In other words, they are numbered by permutations $\sigma \in S_n$ (or cyclic structures on $n+1$ letters) given by the labels on leaves. Respectively, the open cells of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ are numbered by dihedral structures on $n+1$ letters. Consequently, one identifies the cell associated with σ and σ^{op} . The factorization property of the boundary defines by induction an equivalence \sim_{z_2} on the set of leaf-labeled planar trees T_n . \square

While applying the functor of chains to the cell decomposition of $\overline{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}$ compatible with the operadic structure we end up with the following simple but curious observation.

Theorem 2.2.3. The chains of the mosaic operad given by the cell decomposition of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ due to Devadoss assemble into a free symmetric operad that is a cobar construction of the cooperad $[\text{As}]_{\mathbb{Z}_2}^{\vee}$ (defined in the previous Section §2.1) :

$$\cup_{n \geq 1} \text{Chains}(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})) \simeq \Omega([\text{As}]_{\mathbb{Z}_2}^{\vee})$$

and the map $p : \text{FM}_1 \rightarrow \overline{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}$ is Koszul dual to the embedding $i : [\text{As}]^{\mathbb{Z}_2} \rightarrow \text{As}$.

Proof. Thanks to Proposition 2.2.1 we have a collection of isomorphism of vector spaces $\Omega([\text{As}]_{\mathbb{Z}_2}^{\vee})(n)$ and $\text{Chains}(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}))$ compatible with the operadic structure. What remains is to show that the differentials in these complexes coincide as well. The latter follows from the description of codimension one strata in stratification of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ that was mentioned before and the careful visualization of orientations of cells that is enough to work out for small n thanks to the operadic induction. \square

2.3 Mosaic operad as a homotopy quotient

Let us also mention another conceptual homotopical description of the mosaic operad.

Corollary 2.3.1. The mosaic operad $\overline{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}$ is homotopy equivalent to the homotopy quotient $\frac{E_1 \rtimes_{\mathbb{Z}_2}}{\mathbb{Z}_2}$. I.e. we have the following homotopy pushout square of topological operads and the corresponding algebraic models:

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,\bullet}^{\mathbb{R}} & \longleftarrow & pt \\ p \uparrow & \ulcorner & \uparrow \\ E_1 \rtimes_{\tau^!} \mathbb{Z}_2 & \longleftarrow & \mathbb{Z}_2 = O(1) \end{array} \quad \begin{array}{ccc} \text{Chains}(\overline{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}) & \longleftarrow & \mathbb{Q} \\ \uparrow & \ulcorner & \uparrow \tau^! \mapsto 1 \\ A_{\infty} \rtimes_{\tau^!} \mathbb{Q}[\mathbb{Z}_2] & \longleftarrow & \mathbb{Q}[\mathbb{Z}_2] \end{array}$$

where $pt, \mathbb{Z}_2, \mathbb{Q}$ and the group ring $\mathbb{Q}[\mathbb{Z}_2]$ are considered as operads with only unary operations.

Proof. It is enough to explain everything on the level of chains since the \mathbb{Z}_2 action is compatible with the standard cell decomposition of E_1 given by Stasheff polytopes. Moreover, this will help us to avoid discussions of bar-cobar constructions for topological operads.

The proof repeats the proof of the similar statement known for the complex moduli space (see e.g. [27]). One may also find a more topological approach to this statement presented in [55] in a big generality.

The operad $[\text{As}]^{\mathbb{Z}_2}$ is a suboperad of As and, moreover, has a right action of the associative operad As :

$$[\text{As}]^{\mathbb{Z}_2} \circ \text{As} \simeq \text{Hom}_{\mathbb{Z}_2}(\mathbb{Q}, \text{As}) \circ \text{As} \rightarrow \text{Hom}_{\mathbb{Z}_2}(\mathbb{Q}, \text{As} \circ \text{As}) \rightarrow \text{Hom}_{\mathbb{Z}_2}(\mathbb{Q}, \text{As}) \simeq [\text{As}]^{\mathbb{Z}_2}$$

and since the action of the group \mathbb{Z}_2 on the space of n -ary operations of the associative operad is free for $n \geq 2$ we have an isomorphism of right operadic modules over As :

$$\bigoplus_{n \geq 2} \mathbb{Q}[\mathbb{Z}_2] \otimes [\text{As}]^{\mathbb{Z}_2}(n) \simeq \bigoplus_{n \geq 2} \text{As}(n) \quad (2.3.2)$$

This observation is enough to see that there is an isomorphism of vector spaces that is compatible with the operadic structures, but that is not compatible with the differentials:

$$\Omega(\text{As}^\vee) \times \mathbb{Q}[\mathbb{Z}_2] \simeq \Omega(\mathbb{Q}[\mathbb{Z}_2] \otimes [\text{As}]_{\mathbb{Z}_2}^\vee) \times \mathbb{Q}[\mathbb{Z}_2] \simeq \Omega([\text{As}]_{\mathbb{Z}_2}^\vee) * \mathbb{Q}[\mathbb{Z}_2] \stackrel{\text{Thm. 2.2.3}}{\simeq} \text{Chains}(\bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}) * \mathbb{Q}[\mathbb{Z}_2] \quad (2.3.3)$$

where by $\mathcal{P} * \mathcal{Q}$ we mean the free product of operads. Combinatorially $\mathcal{P} * \mathbb{Q}[\mathbb{Z}_2]$ is given by the summation of operadic trees whose at least trivalent vertices are marked by elements of \mathcal{P} and bivalent vertices are marked by τ . In other words we can consider operadic trees (with all vertices at least trivalent) whose internal vertices are labelled by elements of \mathcal{P} and certain edges are labelled some edges by τ . Note that the (augmented) cobar construction $\Omega(\mathcal{P}^\vee)$ is already given by the summation of operadic trees, therefore, pictorially $\Omega(\mathbb{Q}[\mathbb{Z}_2] \otimes \mathcal{P}^\vee) = \Omega(\mathcal{P}^\vee \oplus \tau \otimes \mathcal{P}^\vee)$ is given by the summation of operadic trees such that certain outgoing edges are marked with τ . It remains to allow to mark the leaves with τ and get the isomorphism (2.3.3). Moreover, one can associate a filtration on $\Omega(\text{As}^\vee) \times \mathbb{Q}[\mathbb{Z}_2]$ such that the associated graded is isomorphic to $\Omega([\text{As}]_{\mathbb{Z}_2}^\vee) * \mathbb{Q}[\mathbb{Z}_2]$ as a dg-operad. Thus, it just remains to pass to the homotopy quotient:

$$\text{Chains}(\bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}) \simeq \frac{\text{Chains}(\bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}) * \mathbb{Q}[\mathbb{Z}_2]}{\mathbb{Q}[\mathbb{Z}_2]} \simeq \frac{\Omega(\text{As}^\vee) \times \mathbb{Q}[\mathbb{Z}_2]}{\mathbb{Q}[\mathbb{Z}_2]} \xrightarrow{\text{quies}} \frac{\text{As} \times \mathbb{Q}[\mathbb{Z}_2]}{\mathbb{Q}[\mathbb{Z}_2]}.$$

□

Recall that each configuration of intervals in E_1 can be considered as the intersection with the equator of a certain configuration of discs in E_2 . We require that intersections of each disc of a configuration with the equator coincides with the diameter of this disc. We call the corresponding embedding of operads $E_1 \rightarrow E_2$ by an *equator embedding*.

Corollary 2.3.4. *There exists a map of operads $\iota : \text{Chains}(\bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}) \rightarrow \text{Chains}(E_2)$ such that the standard equator embedding $\text{Chains}(E_1) \rightarrow \text{Chains}(E_2)$ factorises through the composition of the map $p : \text{Chains}(E_1) \rightarrow \text{Chains}(\bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}})$ (considered in (2.2.2)) and ι :*

$$\begin{array}{ccc} \text{Chains}(E_1) & \xrightarrow{\text{equator}} & \text{Chains}(E_2) \\ & \searrow p & \nearrow \exists \iota \\ & & \text{Chains}(\bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}) \end{array} \quad (2.3.5)$$

Proof. Consider the automorphism $\bar{\tau}$ of order 2 of the little discs operad E_2 given by the clockwise rotation over its center by 180° . Note that $\bar{\tau}$ is a homotopically trivial automorphism since one has a family of automorphisms given by rotations over its center by $0^\circ \leq t \leq 180^\circ$. Thus, there exists a model for $\text{Chains}(E_2)$ such that $\bar{\tau}$ acts trivially on $\text{Chains}(E_2)$. Therefore, there is a homotopy equivalence between the semidirect product and ordinary product of operads:

$$\text{Chains}(E_2) \times_{\bar{\tau}} \mathbb{Z}_2 \simeq \text{Chains}(E_2) \otimes \mathbb{Z}_2.$$

It is clear, that the automorphism $\bar{\tau}$ of E_2 restricts to the automorphism $\tau^!$ of E_1 . Consequently we have the following maps of (semi)direct products,

$$\text{Chains}(E_1 \rtimes_{\tau^!} \mathbb{Z}_2) \rightarrow \text{Chains}(E_2 \rtimes_{\bar{\tau}} \mathbb{Z}_2) \simeq \text{Chains}(E_2 \times \mathbb{Z}_2),$$

whose homotopy quotient gives the desired factorization:

$$\text{Chains}(E_1) \xrightarrow{p} \text{Chains}(\widehat{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}) \simeq \text{Chains}\left(\frac{E_1 \rtimes \mathbb{Z}_2}{\mathbb{Z}_2}\right) \longrightarrow \text{Chains}\left(\frac{E_2 \rtimes_{\bar{\tau}} \mathbb{Z}_2}{\mathbb{Z}_2}\right) \simeq \text{Chains}\left(\frac{E_2 \times \mathbb{Z}_2}{\mathbb{Z}_2}\right) \simeq \text{Chains}(E_2).$$

□

Remark 2.3.6. *The Drinfeld unitarization trick (Example (1.6.2)) defines a map of prounipotent completions of the groups $\widehat{\mathcal{P}\text{Cacti}}_n \rightarrow \widehat{\mathcal{P}\mathfrak{B}}_n$. An algebraic model of the latter map $\iota : \text{Chains}(\widehat{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}) \rightarrow \text{Chains}(\text{FM}_2)$ can be derived out of deformation theory of the operads under consideration (see Section 7 below). One can also explain why this map has no deformations. In other words, there exists a unique (in a proper sense) upto homotopy map of operad of prounipotent completions $\widehat{\mathcal{P}\text{Cacti}}_n \rightarrow \widehat{\mathcal{P}\mathfrak{B}}_n$. In particular, this implies that the Drinfeld map coincides with the one obtained in Corollary 2.3.4.*

3 Computing \mathbb{Z}_2 -invariants of operads Comm, Pois and As

In this section we describe several algebraic operads coming from the action of the finite group on the well known operads and find out the algebraic description of the operad $[\text{As}]^{\mathbb{Z}_2}$.

3.1 Convention on shifts in Koszul duality for operads

We deal a lot with Koszul duality for operads and would like to fix certain conventions. Let \mathcal{P} be an algebraic operad. We say that a structure of an algebra over a homologically shifted (k -suspended) operad $\mathcal{P}\{k\}$ on a chain complex V^\bullet is in one-to-one correspondence with the structure of a \mathcal{P} -algebra on a shifted complex $V^\bullet[k]$. In particular, the homological shift (also called the suspension) increases the homological degree of the space of n -ary operations $\mathcal{P}(n)$ by $(n - 1)$ and multiplies with a sign representation Sgn_n :

$$\mathcal{P}\{1\}(n) := \mathcal{P}(n)[1 - n] \otimes \text{Sgn}_n$$

Following the same ideology the homological shift of a cooperad shifts the degrees of cogenerators in the other direction. In particular, the cooperad $\text{Comm}^\vee\{k\}$ is cogenerated by a single element of degree $-k$ which is skew-symmetric for k odd.

In order to preserve the standard conventions suggested by [16] that predict the Koszul duality between Comm and Lie operads we pose the following conventions:

Let \mathcal{P} be an augmented algebraic operad, \mathcal{P}^\vee be the corresponding coaugmented cooperad and $\bar{\mathcal{P}}^\vee$ the coaugmentation coideal. Then as a chain complex the cobar construction $\Omega(\mathcal{P}^\vee)$ is isomorphic to the homological shift of the free operad generated by the shifted symmetric collection $\bar{\mathcal{P}}^\vee[-1] := \cup \bar{\mathcal{P}}^\vee(n)[-1]$:

$$\Omega(\mathcal{P}^\vee) := \mathcal{F}(\bar{\mathcal{P}}^\vee[-1])\{-1\}$$

Respectively, we use the following degree conventions for the bar-construction:

$$\mathcal{B}(\mathcal{P}) := \mathcal{F}^c(\bar{\mathcal{P}}[1])\{1\}$$

and we say that a Koszul dual operad $\mathcal{P}^!$ is the homology of the cobar construction $\Omega(\mathcal{P}^\vee)$. Note, that if the Koszul operad \mathcal{P} is generated by a single ternary element of degree 0 then the operad $\mathcal{P}^!$ is generated by a single ternary element of degree $1 - 2 = -1$.

3.2 \mathbb{Z}_2 -invariants of the operad of commutative algebras

Recall that the operad Comm of commutative algebras is a simplest quadratic operad generated by one binary symmetric generator $\mu_2(-, -)$ satisfying the associativity relation $\mu_2(\mu_2(x_1, x_2), x_3) = \mu_2(\mu_2(x_2, x_3), x_1) = \mu_2(\mu_2(x_3, x_1), x_2)$. This operad admits an automorphism τ of order 2 such that

$$\tau(\mu_2) = -\mu_2.$$

Lemma 3.2.1. *The \mathbb{Z}_2 -invariants $[\mathbf{Comm}]^{\mathbb{Z}_2}$ form a quadratic operad generated by the ternary symmetric operation $\mu_3(x_1, x_2, x_3) := \mu_2(x_1, \mu_2(x_2, x_3))$ subject to the following generalized associativity relation.*

$$\forall \sigma \in S_5 \quad \mu_3(\mu_3(x_1, x_2, x_4), x_3, x_5) = \mu_3(\mu_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}), x_{\sigma(4)}, x_{\sigma(5)}) \quad (3.2.2)$$

Concretely, this relation states that all quadratic monomials in $[\mathbf{Comm}]^{\mathbb{Z}_2}$ are the same.

The relations (3.2.2) form a quadratic Gröbner basis with respect to any compatible ordering of monomials suggested in [8] and, in particular, this operad is Koszul.

Proof. The space of n -ary operations $\mathbf{Comm}(n)$ is one-dimensional and is spanned by $\mu^{\circ n-1}$. Hence $\tau(\mu^{\circ n-1}) = (-1)^{n-1} \mu^{\circ n-1}$ and $[\mathbf{Comm}]^{\mathbb{Z}_2} = \bigoplus_{k \geq 1} \mathbf{Comm}(2k-1)$.

In order to show the Gröbner basis property it is enough to notice that quotient of the shuffle operad by the ideal generated by the leading monomials of relations (3.2.2) has the same size as $[\mathbf{Comm}]^{\mathbb{Z}_2}$. \square

We denote by $\mathbf{Lie}^{\text{odd}}$ the operad that is Koszul dual to the operad $[\mathbf{Comm}]^{\mathbb{Z}_2}$. (The notation refers to the fact that $\mathbf{Lie}^{\text{odd}}$ has nontrivial operations only in odd arities.) The latter operad $\mathbf{Lie}^{\text{odd}}$ is generated by a ternary skewsymmetric generator ν_3 of degree -1 subject to the so called generalized Jacobi identity:

$$\sum_{\sigma \in S_5} (-1)^\sigma \nu_3(x_{\sigma(1)}, x_{\sigma(2)}, \nu_3(x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)})) = 0 \quad (3.2.3)$$

3.3 \mathbb{Z}_2 -invariants of the Poisson operad and its Koszul-dual operad

The Poisson algebras are algebras over the Poisson operad \mathbf{Pois}_1 . The (graded) Poisson operad \mathbf{Pois}_d is the quadratic operad generated by two binary operations: μ_2 – an associative commutative multiplication and ν_2 – the Lie bracket of degree $1-d$, such that the Lie bracket with a given element is a derivation of the multiplication:

$$\mathbf{Pois}_d := \mathcal{F} \left(\begin{array}{l} \mu_2(x_1, x_2) = \mu_2(x_2, x_1), \\ \nu_2(x_1, x_2) = (-1)^d \nu_2(x_2, x_1) \end{array} \left| \begin{array}{l} \mu_2(\mu_2(x_1, x_2), x_3) = \mu_2(x_1, \mu_2(x_2, x_3)) \\ \nu_2(\nu_2(x_1, x_2), x_3) + \nu_2(\nu_2(x_2, x_3), x_1) + \nu_2(\nu_2(x_3, x_1), x_2) = 0 \\ \nu_2(\mu_2(x_1, x_2), x_3) = \mu_2(x_1, \nu_2(x_2, x_3)) + \mu_2(\nu_2(x_1, x_3), x_2). \end{array} \right. \right) \quad (3.3.1)$$

Remark 3.3.2. • *The Poisson operad \mathbf{Pois}_d is Koszul, admits a quadratic Gröbner basis and its Koszul dual coincides with itself upto a homological shift:*

$$(\mathbf{Pois}_d)^{\perp} = \mathbf{Pois}_d\{1-d\}, \quad \mu_2 \overset{\perp}{\leftrightarrow} \nu_2 \quad (3.3.3)$$

- For $d \geq 2$ the Poisson operad \mathbf{Pois}_d coincides with the homology of the little discs operad E_d which is known to be formal ([53],[34]).
- For $d = 1$ the little discs operad E_1 is also formal, but the homology coincides with the associative operad \mathbf{As} which admits a filtration by commutators, such that the associated graded operad is isomorphic to \mathbf{Pois}_1 .

The orthogonal group $O(d)$ acts on the operad E_d and, in particular, there exists an automorphism τ of order 2 that changes the orientation of the unit disc. Respectively, on the level of homology the Poisson operad has an automorphism τ^{\perp} of order 2:

$$\tau^{\perp}(\mu_2) = \mu_2, \quad \tau^{\perp}(\nu_2) = -\nu_2.$$

The Koszul dual automorphism τ of order 2 also defines an automorphism of the Poisson operad

$$\tau(\mu_2) = -\mu_2, \quad \tau(\nu_2) = \nu_2,$$

which we shall consider in the following lemma.

Lemma 3.3.4. 1. *The τ -invariants $[\mathbf{Pois}_d]^{\mathbb{Z}_2}$ form the quadratic operad generated by (skew)-symmetric binary Lie bracket $\nu_2(x_1, x_2)$ and by totally symmetric (commutative) ternary associative multiplication $\mu_3 := \mu_2(\mu_2(x_1, x_2), x_3)$ satisfying relation (3.2.2) and the analogue of the Leibniz identity*

$$\nu_2(\mu_3(x_1, x_2, x_3), x_4) = \mu_3(x_1, x_2, \nu_2(x_3, x_4)) + \mu_3(x_1, \nu_2(x_2, x_4), x_3) + \mu_3(\nu_2(x_1, x_4), x_2, x_3) \quad (3.3.5)$$

2. For $n \geq 2$ we have $\dim([\text{Pois}_d]^{\mathbb{Z}_2}) = \frac{n!}{2}$.

3. The aforementioned relations form a quadratic Gröbner basis for the operad of τ -invariants $[\text{Pois}_d]^{\mathbb{Z}_2}$ with respect to the path-lexicographic ordering and convention $\nu_2 > \mu_3$. In particular, the analogue of the Leibniz relation (3.3.5) defines a distributive law in the sense of [38].

Proof. The automorphism τ does not interact with the Lie bracket ν_2 and thus the Leibniz rule in the Poisson operad predicts that we have the following isomorphisms of τ -invariants:

$$[\text{Pois}_d]^{\mathbb{Z}_2} = [\text{Comm} \circ \text{Lie}\{1 - d\}]^{\mathbb{Z}_2} = [\text{Comm}]^{\mathbb{Z}_2} \circ \text{Lie}\{1 - d\}$$

This implies the following computation of the generating series of dimensions of $[\text{Pois}_d]^{\mathbb{Z}_2}$:

$$\begin{aligned} f_{[\text{Pois}_d]^{\mathbb{Z}_2}}(t) &= f_{[\text{Comm}]^{\mathbb{Z}_2}}(t) \circ f_{\text{Lie}}(t) = \left(\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \right) \circ (-\ln(1-t)) = \\ &= \frac{e^t - e^{-t}}{2} \circ (-\ln(1-t)) = \frac{1}{2} \left(\frac{1}{1-t} - (1-t) \right) = t + \sum_{n \geq 2} \frac{t^n}{2} \end{aligned} \quad (3.3.6)$$

Consequently, $\dim([\text{Pois}_d]^{\mathbb{Z}_2}(1)) = 1$ and $\dim([\text{Pois}_d]^{\mathbb{Z}_2}(n)) = \frac{n!}{2}$ for $n \geq 2$. We proved item 2 of Lemma 3.3.4. The relations (3.3.5) and (3.2.2) are obviously satisfied in $[\text{Pois}_d]^{\mathbb{Z}_2}$.

Recall that the defining relations for the operad Pois_d form a quadratic Gröbner basis with respect to any compatible ordering discussed in [8] whenever the generator ν_2 is considered to be greater than μ_2 . The automorphism τ is defined in the free operad generated by μ_2 and ν_2 and multiply a monomial in ν_2 and μ_2 either by 1 or by -1 . Therefore τ is compatible with the ordering of monomials. Let us consider the same ordering of monomials for the suboperad $[\text{Pois}_d]^{\mathbb{Z}_2}$. It is easy to see that thanks to the relation (3.3.5) the set of normal words for the operad defined by binary and ternary generators subject to the other defining relations coincides with the set of normal words of $[\text{Comm}]^{\mathbb{Z}_2} \circ \text{Lie}\{1 - d\}$, thus, has the same dimension $\frac{n!}{2}$ for all $n \geq 2$. Consequently, the defining relations of the operad $[\text{Pois}_d]^{\mathbb{Z}_2}$ form a quadratic Gröbner basis with $\nu_2 > \mu_2$. \square

Corollary 3.3.7. *The operad $[\text{Pois}_d]^{\mathbb{Z}_2}$ is Koszul. The Cobar-construction $\Omega([\text{Pois}_d^{\vee}]^{\mathbb{Z}_2})$ considered as a quasi-free dg-operad generated by the dual cooperad $[\text{Pois}_d^{\vee}]^{\mathbb{Z}_2}$ homologically shifted by 1 is quasiisomorphic to the quadratic operad generated by the ternary operation $\bar{\mu}_3$ of degree -1 and the binary operation $\bar{\nu}_2$ of degree $d - 1$.*

Notation 3.3.8. *We denote the shifted Koszul dual operad $H(\Omega([\text{Pois}_d^{\vee}]^{\mathbb{Z}_2}))\{1 - d\}$ by $\text{Pois}_d^{\text{odd}}$.*

We will use the same duality in notations between μ and ν as well as the operadic homological shift $\{1 - d\}$ as one has for the Koszul duality for the Poisson operad (3.3.3).

Corollary 3.3.9. *The operad $\text{Pois}_d^{\text{odd}}$ is generated by a binary operation of degree 0 (denoted $\mu_2 := \bar{\nu}_2\{d\}$) and a ternary operation $\nu_3 := \bar{\mu}_3\{d\}$ of homological degree $1 - 2d$ subject to the following symmetry conditions:*

$$\begin{aligned} \mu_2(x_1, x_2) &= \mu_2(x_2, x_1), \\ \nu_3(x_1, x_2, x_3) &= (-1)^{d|\sigma|} \nu_3(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) \end{aligned}$$

and the following quadratic relations:

$$\mu_2(\mu_2(x_1, x_2), x_3) = \mu_2(x_1, \mu_2(x_2, x_3)) \quad (3.3.10)$$

$$\nu_3(\mu_2(x_1, x_2), x_3, x_4) = \mu_2(x_1, \nu_3(x_2, x_3, x_4)) + \mu_2(\nu_3(x_1, x_3, x_4), x_2). \quad (3.3.11)$$

$$\sum_{\substack{\sigma \in S_5/S_2 \times S_3, \\ \sigma \text{ is even}}} \nu_3(x_{\sigma(1)}, x_{\sigma(2)}, \nu_3(x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)})) = 0 \quad (3.3.12)$$

In other words the operad $\text{Pois}_d^{\text{odd}}$ is generated by commutative associative multiplication μ_2 of degree 0 and a ternary operation ν_3 of homological degree $1 - 2d$ obeying the Leibniz rule (3.3.11) and the generalized Jacobi identity (3.3.12).

Remark 3.3.13. *It is worth mentioning that the operads Pois_d as well as the operads $\text{Pois}_d^{\text{odd}}$ are Hopf operads, i. e., operads in the category of (counital) commutative coalgebras. In particular the space of n -ary operations of the cooperad $(\text{Pois}_d^{\text{odd}})^\vee$ is a commutative graded algebra with unit. This observation will be extensively used later (Sections 4,5).*

Remark 3.3.14. *For $d = 1$ the operad $\text{Pois}_1^{\text{odd}}$ coincides with the operad called operad of 2-Gerstenhaber algebras in [12].*

Corollary 3.3.15. *The Leibniz rule (3.3.11) defines a distributive law in the sense of [38] between the operad of commutative algebras generated by μ_2 and the suboperad generated by the ternary bracket ν_3 called Lie^{odd} . In particular, we have an isomorphism of symmetric collections:*

$$\text{Pois}_d^{\text{odd}} \simeq \text{Comm} \circ \text{Lie}^{\text{odd}}\{1 - d\} \text{ where } \text{Lie}^{\text{odd}} := ([\text{Comm}]^{\mathbb{Z}_2})^!$$

Note, that the operad $[\text{Comm}]^{\mathbb{Z}_2}$ is generated by a ternary operation of homological degree 0. Respectively, the Koszul dual operad Lie^{odd} is generated by a ternary operation of homological degree -1 . The standard relation for generating series for Koszul dual operads (even for non binary generated) leads to the equation

$$f_{[\text{Comm}]^{\mathbb{Z}_2}}(-t) \circ f_{\text{Lie}^{\text{odd}}}(-t) = t$$

As mentioned earlier

$$f_{[\text{Comm}]^{\mathbb{Z}_2}}(t) = \sum_{n \geq 1} \frac{t^{2n-1}}{(2n-1)!} = \sinh(t)$$

Consequently

$$f_{\text{Lie}^{\text{odd}}}(t) := (-f_{[\text{Comm}]^{\mathbb{Z}_2}}(-t))^{\circ-1} = \text{arcsinh}(t) = \sum_{n \geq 0} \frac{(2n-1)!!}{(2n)!!} \frac{t^{2n+1}}{2n+1} = \sum_{n \geq 0} \frac{\binom{2n}{n}}{4^n(2n+1)} t^{2n+1}.$$

The space of n -ary operation $\text{Lie}^{\text{odd}}(n)$ differs from zero only for n odd and has homological degree $\frac{1-n}{2}$. Hence, we can write a generating series with additional parameter z that corresponds to the grading:

$$f_{\text{Lie}^{\text{odd}}}(t, z) := \sum_{n \geq 0} \frac{\dim \text{Lie}^{\text{odd}}(2n+1) t^{2n+1} z^n}{(2n+1)!} = \frac{\arcsin(t\sqrt{z})}{\sqrt{z}}$$

The isomorphism of graded symmetric collections $\text{Pois}_d^{\text{odd}} \simeq \text{Comm} \circ \text{Lie}^{\text{odd}}\{1 - d\}$ leads to the following presentation of the generating series of $\text{Pois}_d^{\text{odd}}$:

$$f_{\text{Pois}_d^{\text{odd}}}(t, z) := f_{\text{Comm}}(t) \circ f_{\text{Lie}^{\text{odd}}}(t, z) = \exp\left(\frac{\arcsin(t\sqrt{z})}{\sqrt{z}}\right).$$

Here the parameter z corresponds to the grading of a ternary operation. If we multiply the coefficient of t^n by $n!$ we obtain the generating series of the space of n -ary operations $\text{Pois}_d^{\text{odd}}(n)$

$$\prod_{1 \leq k < \frac{n}{2}} (1 + (n - 2k)^2 z). \quad (3.3.16)$$

The homological degree of z is equal to $1 - 2d$.

Remark 3.3.17. *A little bit more advanced computations with the generating series of symmetric functions given by S_n -characters on the space of n -ary operations for Comm and $[\text{Comm}]^{\mathbb{Z}_2}$ leads to the computation of S_n characters of $\text{Lie}^{\text{odd}}(n), \text{Pois}_d^{\text{odd}}(n)$. The details for the main case $d = 1$ can be found in [47].*

3.4 \mathbb{Z}_2 -invariants of the associative operad

We already discussed the \mathbb{Z}_2 -action and its Koszul dual action on the operad As of associative algebras in (2.1.1) and (2.1.2) given by flipping the order of the multiplication and changing the sign in the

Koszul dual case. The aim of this section is to give the algebraic presentation of the suboperad of τ -invariants $[\mathbf{As}]^{\mathbb{Z}_2}$ in terms of generators and relations. The key observation is that τ preserves filtration by commutators. Consequently, the space of τ -invariants of the associative operad and the space of τ -invariants of the associated graded operad (which coincides with the Poisson operad) have the same dimension.

Theorem 3.4.1. *The operad $[\mathbf{As}]^{\mathbb{Z}_2}$ is generated by*

- the binary generator $\nu_2 := x_1x_2 - x_2x_1$,
- the ternary generator $\mu_3 := \sum_{\sigma \in S_3} x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$, subject to the following relations²:

$$\nu_2(\nu_2(x_1, x_2), x_3) + \nu_2(\nu_2(x_2, x_3), x_1) + \nu_2(\nu_2(x_3, x_1), x_2) = 0 \quad (3.4.2)$$

$$\nu_2(\mu_3(x_1, x_2, x_3), x_4) = \mu_3(x_1, x_2, \nu_2(x_3, x_4)) + \mu_3(x_1, \nu_2(x_2, x_4), x_3) + \mu_3(\nu_2(x_1, x_4), x_2, x_3) \quad (3.4.3)$$

$$\begin{aligned} \mu_3(\mu_3(x_1, x_2, x_4), x_3, x_5) - \mu_3(\mu_3(x_1, x_2, x_3), x_4, x_5) &= B(x_1, x_2, x_3, x_4, x_5) - B(x_1, x_2, x_4, x_3, x_5) \\ \text{with } B(a, a, a, b, b) &:= 3\mu_3(\nu_2(a, b), \nu_2(a, b), a) + 6\mu_3(\nu_2(a, \nu_2(a, b)), a, b) + \\ &+ \frac{6}{5}\nu_2(b, \nu_2(a, \nu_2(a, \nu_2(a, b)))) + \frac{18}{5}\nu_2(a, \nu_2(b, \nu_2(a, \nu_2(b, a)))) \end{aligned} \quad (3.4.4)$$

These relations form a Gröbner basis with respect to the degree-path-lexicographic ordering of monomials and convention $\nu_2 > \mu_3$.

Proof. Let us explain that the aforementioned relations are indeed satisfied. First, notice that the commutator ν_2 is a Lie bracket in any associative algebra, hence the Jacobi identity (3.4.2) follows. Second, the Lie bracket is a derivation of the associative multiplication

$$[x_1x_2, x_3] = x_1x_2x_3 - x_3x_1x_2 = x_1x_2x_3 - x_1x_3x_2 + x_1x_3x_2 - x_3x_1x_2 = x_1[x_2, x_3] + [x_1, x_3]x_2,$$

which implies the Leibniz rule (3.4.3) for the iterated symmetrized multiplication μ_3 . Note that if we denote by μ_2 the symmetrization of multiplication $x_1x_2 + x_2x_1$, then

$$\mu_3(x_1, x_2, x_3) = \mu_2(x_1, \mu_2(x_2, x_3)) + \mu_2(x_2, \mu_2(x_3, x_1)) + \mu_2(x_3, \mu_2(x_1, x_2)) \quad (3.4.5)$$

We have the following relation together with the Leibniz rule:

$$\mu_2(x_1, \mu_2(x_2, x_3)) - \mu_2(x_2, \mu_2(x_3, x_1)) = \nu_2(\nu_2(x_1, x_2), x_3) \quad (3.4.6)$$

The remaining relation (3.4.4) was found and checked first using the computer and we do not provide these computations. However, one can derive it out of presentation (3.4.5) and relation (3.4.6).

In fact the precise form of the right-hand side of relation (3.4.4) will not be relevant, and that such a relation exists may be shown without direct computations. Indeed, the associative operad \mathbf{As} admits a filtration by commutators and the associated graded operad coincides with the Poisson operad \mathbf{Pois}_1 . The automorphism τ preserves the filtration by commutators and the corresponding automorphism of the associate graded operad coincides with the automorphism τ considered in the previous Section 3.3. Consequently, the operad $[\mathbf{As}]^{\mathbb{Z}_2}$ also admits a filtration by commutators and the associated graded coincides with the Koszul operad $[\mathbf{Pois}_1]^{\mathbb{Z}_2}$. The operations μ_3 and ν_2 generate the associated graded operad and, consequently, their preimages generate the initial operad $[\mathbf{As}]^{\mathbb{Z}_2}$. As we already mentioned the Jacobi identity for ν_2 as well as the Leibniz identity (3.3.5) for generators of $[\mathbf{Pois}_1]^{\mathbb{Z}_2}$ remain valid in $[\mathbf{As}]^{\mathbb{Z}_2}$ (relations (3.4.2) and (3.4.3)). However, the right-hand side of relation (3.2.2) has to be replaced by an expression in μ_3 and ν_2 that contains at least one ν_2 . Notice, that an operadic monomial of arity 5 in the binary operation ν_2 and the ternary operation μ_3 should be at least cubic in generators if the degree in ν_2 is greater than zero. Thus, the quadratic Gröbner basis of relations for $[\mathbf{Pois}_1]^{\mathbb{Z}_2}$ is replaced by a nonhomogeneous Gröbner basis of the relations for $[\mathbf{As}]^{\mathbb{Z}_2}$ but with the same homogeneous component of degree 2. \square

Corollary 3.4.7. *The homology of the cobar construction of the cooperad $[\mathbf{As}]_{\mathbb{Z}_2}^{\vee}$ dual to the operad $[\mathbf{As}]^{\mathbb{Z}_2}$ is equal to the operad $\mathbf{Pois}_1^{\text{odd}}$. However, the cobar construction is not formal and the nonhomogeneous relation (3.4.4) yields the existence of nontrivial ∞ -products on $\mathbf{Pois}_1^{\text{odd}}$ that make it equivalent to the mosaic operad $\mathcal{M}_{0, \bullet}^{\mathbb{R}}$.*

²not all the relations are invariant under the action of symmetric group, so first one has enlarge the space of quadratic relations using permutations of inputs

Proof. Consider, the Anick-type resolution of a (shuffle) operad (discovered in [9]) whose generators are constructed out of intersections of leading monomials of the given Gröbner basis of an operad. This resolution is minimal if the leading monomials are quadratic. Therefore, the space of generators for the minimal resolutions for the operad $[\text{As}]^{\mathbb{Z}_2}$ and its associated graded are the same:

$$\Omega((\text{Pois}_1^{\text{odd}})^\vee) := \mathcal{F}(s(\text{Pois}_1^{\text{odd}})^\vee, d) \xrightarrow{\text{quis}} [\text{Pois}_1]^{\mathbb{Z}_2} \quad \mathcal{F}(s(\text{Pois}_1^{\text{odd}})^\vee, d + d_{\geq 3}) \xrightarrow{\text{quis}} [\text{As}]^{\mathbb{Z}_2}$$

The differential in the Anick-type resolution replaces leading monomials by the corresponding relations in the Gröbner basis. Therefore, the differential can be split into part of homogeneity 2 and the part of homogeneity greater than 2. That is, the derivation $d_{\geq 3}$ maps $(\text{Pois}_1^{\text{odd}})^\vee$ to $\oplus_{k \geq 3} \underbrace{(\text{Pois}_1^{\text{odd}})^\vee \circ \dots \circ (\text{Pois}_1^{\text{odd}})^\vee}_k$.

The double bar construction explains that the generators of the minimal resolution coincide with the cohomology of the (co)bar construction. Consequently, the cohomology of the cobar constructions of $[\text{As}]_{\mathbb{Z}_2}^\vee$ and of its associated graded are the same vector spaces, meaning that the corresponding spectral sequence degenerates in the first term.

To see the non-formality, we first note that operads $[\text{As}]^{\mathbb{Z}_2}$ and $[\text{Pois}_1]^{\mathbb{Z}_2}$ are not isomorphic. This is because the generators are singled out uniquely, up to scale, by their symmetries, and they satisfy different relations. But then the operads $[\text{As}]^{\mathbb{Z}_2}$ and $[\text{Pois}_1]^{\mathbb{Z}_2}$ are also not quasi-isomorphic, since they have no differential, and any quasi-isomorphism would in particular induce an isomorphism on the level of the cohomology operads. Finally it follows that the cobar construction of $[\text{As}]_{\mathbb{Z}_2}^\vee$ cannot be formal. Otherwise, the bar-cobar construction of $[\text{As}]_{\mathbb{Z}_2}^\vee$ would be quasi-isomorphic to the bar construction of $\text{Pois}_1^{\text{odd}}$, i.e., to $[\text{Pois}_1]_{\mathbb{Z}_2}^\vee$. Of course, bar-cobar construction of $[\text{As}]_{\mathbb{Z}_2}^\vee$ is also quasi-isomorphic to $[\text{As}]_{\mathbb{Z}_2}^\vee$, and hence, dualizing, $[\text{Pois}_1]^{\mathbb{Z}_2}$ and $[\text{As}]^{\mathbb{Z}_2}$ were quasi-isomorphic, a contradiction. \square

As an easy consequence of Theorem 2.2.3 and Corollary 3.4.7 we recover the results of [12] on the rational cohomology of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$:

Corollary 3.4.8. 1. The \mathbb{Q} -homology of the mosaic operad is equal to $\text{Pois}_1^{\text{odd}}$.

2. The mosaic operad is not formal.

3. The Poincaré polynomial $\sum_{k \geq 0} \dim H^k(\overline{\mathcal{M}}_{0,n+1}(\mathbb{R}); \mathbb{Q}) \frac{t^k}{k!}$ equals $\prod_{1 \leq k < \frac{n}{2}} (1 + z(n - 2k)^2)$

Proof. Items 1 and 2 follows from Corollary 3.4.7 and the Poincaré polynomial was computed for $\text{Pois}_d^{\text{odd}}(n)$ in (3.3.16). \square

Remark 3.4.9. In order to cover the information of the mosaic operad over integers and, in particular, the description of the torsion homology groups of $\overline{\mathcal{M}}_{0,n}(\mathbb{R})$ one has to deal with the presentation of the cooperad $[\text{As}]_{\mathbb{Z}_2}^\vee$ over integers in terms of (co)generators and (co)relations. Unfortunately, we do not know a simple description of the latter cooperad over integers.

4 Hopf models for Pois_d and E_d

This section does not contain any new material. We give short overview of certain known combinatorial algebraic models of the little discs operad and the Poisson operad considered as Hopf operads in terms of graphs. We refer to e.g. [34],[57],[58] for more detailed expositions to these operads. The most complicated model of oriented graphs is upgraded in Section 5 in order to produce a model of $\text{Pois}_d^{\text{odd}}$. We also add some remarks about the deformations of the operad (of chains on) E_d . These remarks will be used in describing the deformations of $\text{Pois}_d^{\text{odd}}$ and for a description of an algebraic (Hopf) model of the mosaic operad $\mathcal{M}_{0,\bullet}^{\mathbb{R}}$.

By a Hopf operad we simply mean an operad in the category of differential graded cocommutative coalgebras. The multiplication on the space of n -ary operations of a Hopf operad \mathcal{P} will be denoted by $\Delta_{\mathcal{P}} : \mathcal{P} \rightarrow \mathcal{P} \otimes \mathcal{P}$.

4.1 Simplest Hopf models for Pois_d

The standard presentation (3.3.1) of the operad Pois_d admits the following very simple formula for the Hopf coproduct:

$$\Delta_{\text{Pois}_d}(\mu_2) = \mu_2 \otimes \mu_2, \quad \Delta_{\text{Pois}_d}(\nu_2) = \mu_2 \otimes \nu_2 + \nu_2 \otimes \mu_2 \text{ with } \nu_2, \mu_2 \in \text{Pois}_d(2). \quad (4.1.1)$$

In particular, the corresponding spaces of n -ary cooperations of the cooperad $\mathbf{Pois}_d^\vee(n) := H^\bullet(E_d(n)) = H^\bullet(\mathit{Conf}(n, \mathbb{R}^d))$ is known to be a quadratic Koszul algebra called Orlik-Solomon algebra (see e.g. [1],[41]):

$$OS_d(n) := H^\bullet(E_d(n); \mathbb{Q}) \simeq \mathbb{Q} \left[\begin{array}{l} \nu_{ij}, 1 \leq i \neq j \leq n \\ \nu_{ij} = (-1)^d \nu_{ji} \\ \deg(\nu_{ij}) = d - 1 \end{array} \middle| \begin{array}{l} \nu_{ij} \nu_{ij}, \\ \nu_{ij} \nu_{jk} + \nu_{jk} \nu_{ki} + \nu_{ki} \nu_{ij} \end{array} \right].$$

The Lie bracket $\nu_2 \in \mathbf{Pois}_d(2)$ generates a Koszul suboperad $\mathit{Lie}\{1-d\}$ of Lie algebras (with a Lie bracket of degree $(1-d)$). The Koszul dual operad $\mathit{Lie}\{1-d\}^\dagger$ is an operad of shifted commutative algebras and one has a simple resolution $L_\infty\{1-d\}$ of $\mathit{Lie}\{1-d\}$. This replacement defines another simple Hopf operad that is obviously quasiisomorphic to \mathbf{Pois}_d :

$$\mathit{hoPois}_d := \mathcal{F} \left(\begin{array}{l} \mu_2 \in \mathit{hoPois}_d(2), \\ \nu_k \in \mathit{hoPois}_d(k), k \geq 1 \\ \deg(\nu_k) = 1 - ks \end{array} \middle| \begin{array}{l} \mu_2(\mu_2(x_1, x_2), x_3) = \mu_2(x_1, \mu_2(x_2, x_3)) \\ d(\nu_k) = \sum_{i+j=k+1} \sum_{\sigma \in S_{k+1}} (-1)^\sigma \sigma \cdot \nu_i \circ_1 \nu_j \end{array} \right) \quad (4.1.2)$$

$$\Delta(\mu_2) = \mu_2 \otimes \mu_2, \quad \Delta(\nu_k) = \underbrace{\mu_2 \circ \mu_2 \circ \dots \circ \mu_2}_k \otimes \nu_k + \nu_k \otimes \underbrace{\mu_2 \circ \mu_2 \circ \dots \circ \mu_2}_k.$$

The case $d = 1$ is a bit exceptional meaning that the operad E_1 is equivalent to the operad \mathbf{As} of associative algebras. However, the operad hoPois_1 is also well defined and is a model for the operad \mathbf{Pois}_1 of Poisson algebras. There exist a standard filtration by the number of commutators on the operad of associative algebras whose associated graded is \mathbf{Pois}_1 . Respectively, the operad \mathbf{As} is considered as a deformation of the operad \mathbf{Pois}_1 and we will specify a particular element in the deformation complex of \mathbf{Pois}_1 that is responsible for this deformation in Section 4.6.

4.2 Fulton McPherson operad FM_d

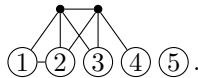
In [14] Fulton and McPherson defined a compactification of the space of configurations of points in a variety X given as a consecutive composition of blowups of all the diagonals in X^n . For $X = \mathbb{R}^d$ one has to consider real (spherical) blowups (bordifications). It is well known that the corresponding compactifications $FM_d(n)$ together with their natural stratifications define manifolds with corners which assemble into a model of the little d -dimensional discs operad E_d . In particular, $FM_d(2) \simeq S^{d-1}$ and the forgetful maps $FM_d(n) \rightarrow FM_d(2)$ forgetting all but i -th and j -th points is denoted by π_{ij} .

These operads were used by Kontsevich to obtain fibrant dgca models of \mathbf{Pois}_d (respectively FM_d) given by certain class of differential forms on FM_d . The drawback of the dgca models of the operad \mathbf{Pois}_d presented in the previous section 4.1 is that they are not fibrant, (essentially) due to the fact that the corresponding dg coalgebras are not cofree.

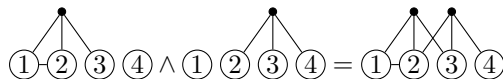
4.3 Kontsevich's operads \mathbf{Graphs}_d of graphs

In [32] M. Kontsevich defined the combinatorial model \mathbf{Graphs}_d for the little cubes operad $E_d(n)$. Let us only sketch the construction, a more detailed exposition is presented in [34].

First, one define spaces $\mathbf{Graphs}_d^\vee(n)$ whose elements are linear combinations of isomorphism classes of undirected "admissible" graphs with vertices of two sorts: there are n numbered *external* vertices and arbitrary (finite) number of unlabeled *internal* vertices of valence at least 3. One furthermore requires that each connected component of a graph contains at least one external vertex. Here is an example for $n = 5$



The cohomological degree of an inner vertex is $-d$ and the cohomological degree of an edge is $d - 1$. The spaces $\mathbf{Graphs}_d(n)^\vee$ assemble into a Hopf cooperad, i.e., a cooperad object in the category of dg commutative algebras. Combinatorially the cocomposition is given by subgraph contraction and the differential is given by edge contraction. The commutative multiplication is defined by fusing two graphs at the external vertices, e.g.,



We also consider the operad \mathbf{Graphs}_d dual to \mathbf{Graphs}_d^\vee . (Mind that we slightly abuse the notation \vee here.) Concretely, elements of \mathbf{Graphs}_d^\vee are finite linear combinations of graphs, while elements of \mathbf{Graphs}_d can be identified with series of graphs. The operadic composition in the operad \mathbf{Graphs}_d is given by insertion at external vertices, and the differential is vertex splitting.

The dg operad \mathbf{Graphs}_d is generally not a dg Hopf cooperad because of completion issues. It is however a complete dg Hopf cooperad, in the sense that to define the cocomposition one uses a completed tensor product. (The completion issue is furthermore inexistent if $d \geq 3$, since then it is of finite type in each arity.)

Also note that the spaces $\mathbf{Graphs}_d(n)^\vee(n)$ spanned by all graphs is isomorphic to the free graded commutative algebra generated by the space of internally connected graphs, meaning that the graphs are connected if one erases all external vertices. The span of internally connected graphs defines an operad in the category of L_∞ -algebras denoted $\mathbf{ICG}_d(n)$ (see [49] for detailed combinatorial description of L_∞ structure on $\mathbf{ICG}_d(n)$).

For any given choice of a propagator $\omega \in \Omega_{PA}^{d-1}(\mathbf{FM}_d(2))$ we assign to each graph Γ with n external and k internal vertices the differential form given by the product of pullbacks of propagators:

$$\tilde{\omega}_\Gamma := \bigwedge_{e \in \text{Edges}(\Gamma)} \pi_{in(e)out(e)}^* \omega \in \Omega_{PA}^{(d-1)(\#edges)}(\mathbf{FM}_d(n+k))$$

the direct image of this differential form with respect to the map $\mathbf{FM}_d(n+k) \rightarrow \mathbf{FM}_d(n)$ of forgetting the points defines a map of cooperads:

$$\begin{array}{ccc} \Gamma & \mapsto & \omega_\Gamma := \int_{\mathbf{FM}_d(n+k) \rightarrow \mathbf{FM}_d(n)} \tilde{\omega}_\Gamma \\ \mathbf{Graphs}_d^\vee(n) & \longrightarrow & \Omega_{PA}(\mathbf{FM}_d(n)) \end{array} \quad (4.3.1)$$

whenever the Kontsevich's vanishing lemma holds:

$$c_\Gamma := \int_{\mathbf{FM}_d(n)} \tilde{\omega}_\Gamma = 0, \text{ for } n > 2 \quad (4.3.2)$$

See [31, 34] for the existence of such a propagator ω and the detailed proof.

Connected components of graphs with no external vertices are forbidden in \mathbf{Graphs}_d but are of particular interest in the deformation theory of the little balls operad E_d . It was shown in [57] that the dg Lie algebra of homotopy deformations of the map $\mathbf{Pois}_d \rightarrow \mathbf{Graphs}_d$ of operads in dgca's is essentially³ quasiisomorphic to \mathbf{GC}_d . Where dgla \mathbf{GC}_d is called Kontsevich graph complex and is spanned by connected graphs with black vertices, vertex splitting differential and vertex substitution⁴ as a pre-Lie algebra structure.

4.4 Operads $d\mathbf{Graphs}_d$ of directed graphs

One may also consider a directed version of the operad of graphs $d\mathbf{Graphs}_d$ spanned by the same set of admissible graphs but with additional orientation of the edges (cf. [57]). $d\mathbf{Graphs}_d$ gives an equivalent dgca model for the little cubes operad E_d . Similar to the model with nondirected graphs the dg Lie algebra of homotopy deformations of the map $\mathbf{Pois}_d \rightarrow d\mathbf{Graphs}$ is (essentially) quasiisomorphic to the directed version $d\mathbf{GC}$ of the Kontsevich graph complex \mathbf{GC} .

The Hopf (co)multiplication is once again given by the graph gluing through external vertices. The space $d\mathbf{Graphs}_d(n)$ is the free dgca cogenerated by internally connected directed graphs with n external vertices. We denote the corresponding operad in the category of L_∞ -algebras by $d\mathbf{ICG}_d$. Once again we restrict to the case when all inner vertices are at least trivalent.

The same rule $\Gamma \mapsto \omega_\Gamma \in \Omega_{PA}(\mathbf{FM}_d(n))$ defines a map of cooperads whenever we start from a propagator $\omega \in \Omega_{PA}^{d-1}(\mathbf{FM}_d(2))$ yielding the Kontsevich vanishing property (4.3.2).

4.5 Shoikhet cocycle

Consider a propagator $\omega^\downarrow \in \Omega_{PA}^{d-1}(S^{d-1})$ representing the same class in homology as ω but whose support belongs to the upper semisphere (points whose last coordinate is positive and we may think them sitting

³there are some extra known classes in the deformation complex of $\mathbf{Pois}_d \rightarrow \mathbf{Pois}_d$ given by simple loops of different size

⁴one graph has to be inserted in all possible ways into a vertex of another graph

over the equator). Thus, the differential form ω_Γ^\downarrow remembers the direction of arrows in a graph. In other words ω^\downarrow is zero at the point $(\bar{x}, \bar{y}) \in \text{FM}_d(2)$ if we have a negative equality for the last coordinates of points: $x_d < y_d$. The Kontsevich vanishing property (4.3.2) does not hold for the propagator ω^\downarrow in dimension $d = 2$ and, thus, the assignment $\Gamma \rightarrow \omega_\Gamma$ is not a map of complexes for $d = 2$. One avoids this defect shifting the differential in $\text{dGraphs}_2(n)$ with the following MC element (called Shoikhet's element after [51])

$$\gamma^\downarrow := \sum_{G \in \text{directed graphs}} c_G^\downarrow \cdot G \in \text{dGC}, \text{ with } c_G^\downarrow := \int_{\text{FM}_2(n)} \omega_G^\downarrow \quad (4.5.1)$$

The nonzero summands of $\gamma^\downarrow \in \text{dGC}_2$ of the smallest loop order contains graphs with 4 vertices and looks as follows:

$$\gamma_0^\downarrow := \begin{array}{c} \text{graph 1} \\ \pm \\ \text{graph 2} \\ \pm 2 \\ \text{graph 3} \end{array} \quad (4.5.2)$$

Proposition. ([51]) *The assignment $\Gamma \rightarrow \omega_\Gamma$ defines a quasiisomorphism of Hopf cooperads:*

$$\text{dGraphs}_2^{\gamma^\downarrow}(n) \rightarrow \Omega_{PA}(\text{FM}_2(n))$$

We do not know how to compute all weights c_G^\downarrow . However, one can choose the propagator $\omega^\downarrow \in \Omega_{DR}(S^1)$ with additional symmetry:

$$\omega^\downarrow(-\bar{z}) = \omega^\downarrow(\pi - \varphi) = -\omega^\downarrow(\varphi) = -\omega^\downarrow(z). \quad (4.5.3)$$

This implies certain vanishing of weights c_G^\downarrow discovered by Shoikhet:

Lemma 4.5.4. ([51] Lemma 2.2) *If a directed graph G contains a directed loop or its Euler characteristic is even then the corresponding integral $c_G^\downarrow := \int_{\text{FM}_2(n)} \omega_G^\downarrow = 0$. In other words, the Shoikhet element γ^\downarrow is concentrated in even loop orders.*

Proof. If the graph G with n vertices contains a directed loop then the corresponding form $\omega_G^\downarrow \in \Omega(\text{FM}_2(n))$ is already zero before integration.

Consider the reflection of the plane $\mathbb{C} = \mathbb{R}^2$ around the y -axis

$$\tau : z = x + iy \mapsto -\bar{z} = -x + iy.$$

Denote by the same letter the corresponding reflection of a configuration of n points in \mathbb{R}^2 . Since points in $\text{FM}_2(n)$ are considered modulo joint translations, thus we may suppose that the first point of a configuration coincides with the origin. Therefore, the determinant of the Jacobian of τ on FM_2 is equal to $(-1)^{n-1}$. On the other hand, the symmetry (4.5.3) of the propagator implies that for each graph G with n vertices we have

$$\begin{aligned} (-1)^{\#\text{Edges}(G)} c_G^\downarrow &= (-1)^{\#\text{Edges}} \int_{\text{FM}_2(n)} \bigwedge_{e \in \text{Edges}(G)} \omega_e^\downarrow = \int_{\text{FM}_2(n)} \bigwedge_{e \in \text{Edges}(G)} \tau(\omega_e^\downarrow) = \\ &= \int_{\tau(\text{FM}_2(n))} \bigwedge_{e \in \text{Edges}(G)} \omega_e^\downarrow = (-1)^{n-1} \int_{\text{FM}_2(n)} \bigwedge_{e \in \text{Edges}(G)} \omega_e^\downarrow = (-1)^{\#\text{Vert}(G)-1} c_G^\downarrow \end{aligned}$$

Consequently, if the number of edges and the number of vertices of a connected directed graph G are of the same parity then $c_G^\downarrow = 0$. \square

Remark 4.5.5. *The MC element γ^\downarrow is gauge trivial as an element of dGC_d since there exists a family of interpolating propagators between the one yielding Kontsevich property and the directed one ω^\downarrow . Thus the operad dGraphs_d and the twisted one $\text{dGraphs}_d^{\gamma^\downarrow}$ are both equivalent to E_d .*

4.6 Merkulov's operads $\text{Graphs}_d^\downarrow$ of oriented graphs

Suppose that $d \geq 2$. The operad dGraphs_d of directed graphs contains an acyclic ideal spanned by directed graphs with at least one *target* internal vertex (i.e. vertex without outgoing edges). Let $\text{dGraphs}_d^{\downarrow}$ be the corresponding quotient that also gives a model for the little balls operad E_d . In [58] it was introduced the suboperad $\text{fGraphs}_d^\downarrow \subset \text{dGraphs}_d^{\downarrow}$ spanned by directed graphs such that

- There are no directed cycles.
- There are no edges starting at the external vertices.

Unfortunately, in the oriented case the restriction to the suboperad whose inner vertices have to be trivalent is no more a quasiisomorphism and has to be weakened to the following assumption.

Lemma 4.6.1. *The suboperad $\text{Graphs}_d^\downarrow \subset \text{fGraphs}_d^\downarrow$ spanned by directed graphs such that there are no internal vertices of valency 1 (with a unique outgoing edge) and vertices of valency 2 (with 1 incoming and 1 outgoing edges) is quasiisomorphic to the full operad $\text{fGraphs}_d^\downarrow$.*

Proof. The proof repeats the one that was used for the equivalence of operads $\text{fGraphs}_d \hookrightarrow \text{Graphs}_d$. \square

Theorem 4.6.2. ([58]) *The following assignment of a graph to each generator of hoPois_d*

$$\text{hoPois}_d(2) \ni \mu_2 \mapsto \circ \quad \circ, \quad \text{hoPois}_d(n) \ni \nu_n \mapsto \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \textcircled{1} \quad \textcircled{2} \quad \cdots \quad \textcircled{n-1} \quad \textcircled{n} \end{array} \quad (4.6.3)$$

extends to a quasiisomorphism of Hopf operads $\text{hoPois}_d \rightarrow \text{Graphs}_{d+1}^\downarrow$.

We will comment on the nature of the map (4.6.3) from the point of view of differential forms on the FM compactifications. Indeed, to each oriented graph $\Gamma \in \text{Graphs}^\downarrow(n)$ with m internal vertices we assign a differential form given by a fibered integral

$$\omega_\Gamma^\downarrow := \int_{\text{SC}_{d+1}(m,n) \rightarrow \text{FM}_d(n)} \bigwedge_{e \in \Gamma} \pi_{\text{in}(e)\text{out}(e)}^* \omega_{d+1}^\downarrow \quad (4.6.4)$$

Here $\omega_{d+1}^\downarrow \in \Omega_{PA}^d(\text{FM}_{d+1}(2))$ is a propagator that was used for directed graphs in 4.5, and $\text{SC}_{d+1}(m,n)$ is the Fulton-McPherson compactification of the Swiss Cheese operad in dimension $d+1$ (see [59] for the details of this construction). Roughly, $\text{SC}_{d+1}(m,n)$ is a compactification of the space of configurations of m points in the half space $\mathbb{R}_{>0} \times \mathbb{R}^d$ and n points in the boundary hyperplane $\{0\} \times \mathbb{R}^d$.

Recall, that the case $d = 1$ is a bit exceptional, meaning that the operad E_1 is equivalent to the associative operad As , however the operad hoe_1 (as we defined in (4.1.2)) is equivalent to its associated graded operad of Poisson algebras Pois .

Lemma 4.6.5. *The assignment $G \mapsto \omega_G^\downarrow$ defines a map of cooperads*

$$\left((\text{Graphs}_2^\downarrow)^\vee, d + [\gamma^\downarrow, -] \right) \rightarrow \Omega_{PA}(\text{FM}_1)$$

Proof. Stokes' theorem. \square

In other words, the Shoikhet MC-element γ^\downarrow is responsible for the deformation from the operad of Poisson algebras to the operad of associative algebras in the category of Hopf operads.

It was also proved in [58] that the oriented graph complex computes (essentially) the full set of deformations of the Hopf operad Pois_d , and as a consequence is (essentially) quasi-isomorphic to the ordinary (non-oriented) graph complex

$$H(\text{GC}_{d+1}^\downarrow) = H(\text{GC}_d) + \bigoplus_{\substack{j \geq 1, \\ j \equiv 2d+1 \pmod{4}}} \mathbb{k}[d-j]. \quad (4.6.6)$$

See also [62] for a more direct proof of this statement. In particular, we have $\dim H^1(\text{GC}_2^\downarrow) = \dim H^1(\text{GC}_1) = 1$ and the corresponding generator in GC_2^\downarrow can be presented as cycle (4.5.2).

5 Recognizing the Hopf structure of $\text{Pois}_d^{\text{odd}}$ and of $\bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}$.

Note that the cell decomposition of the mosaic operad is not compatible with the diagonal embedding $\Delta_{\bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}} : \bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}} \rightarrow \bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}} \times \bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}$. Nevertheless, $\bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}$ is a topological operad and one should expect its model in the category of differential graded cocommutative coalgebras. First, we describe several Hopf models of the homology operad $H_*(\bar{\mathcal{M}}_{0,\bullet}^{\mathbb{R}})$ that thanks to Corollary (3.4.8) is equal to $\text{Pois}_1^{\text{odd}}$. Second, we describe the corresponding deformation complex and, third, give a presentation of the free dgca model for the mosaic operad as a deformation of the free model of $\text{Pois}_1^{\text{odd}}$ with a given MC (Shoikhet's) element of deformation complex.

5.1 Simplest Hopf models for $\text{Pois}_d^{\text{odd}}$

One can verify that the operad $\text{Pois}_d^{\text{odd}}$ is a Hopf operad whose comultiplication $\Delta_{\text{Pois}_d^{\text{odd}}} : \text{Pois}_d^{\text{odd}} \rightarrow \text{Pois}_d^{\text{odd}} \otimes \text{Pois}_d^{\text{odd}}$ is easily defined on the generators of the operad $\text{Pois}_d^{\text{odd}}$:

$$\Delta(\mu_2) = \mu_2 \otimes \mu_2, \quad \Delta(\nu_3(-, -, -)) = \mu_2(\mu_2(-, -), -) \otimes \nu_3(-, -, -) + \nu_3(-, -, -) \otimes \mu_2(\mu_2(-, -), -). \quad (5.1.1)$$

Note that the latter formulas (5.1.1) resemble the description (4.1.1) of the comultiplication for the Pois_d operad.

The latter presentation (5.1.1) leads to the following description of the graded commutative algebra structure on the dual space $(\text{Pois}_d^{\text{odd}})^\vee(n)$ discovered in [12] for $d = 1$:

$$(\text{Pois}_d^{\text{odd}})^\vee(n) \simeq \mathbb{Q} \left[\begin{array}{l|l} \nu_{ijk}, 1 \leq i, j, k \leq n & \nu_{ijk}\nu_{ijl} = 0, \\ \nu_{ijk} = (-1)^{d\sigma} \nu_{\sigma(i)\sigma(j)\sigma(k)} & \nu_{ijk}\nu_{klm} + \nu_{jkl}\nu_{lmi} + \nu_{klm}\nu_{mij} + \\ \deg(\nu_{ijk}) = 2d - 1 & + \nu_{lmi}\nu_{ijk} + \nu_{mij}\nu_{jkl} = 0 \end{array} \right] \quad (5.1.2)$$

Remark 5.1.3. In [12] combinatorics of certain posets was used to prove the description of algebras (5.1.2). We postpone the alternative proof of this result to the forthcoming paper [26].

Following the comparison with the models we discussed for operads Pois_d in Section 4.1 we may consider the following homotopy replacement of the operad $\text{Pois}_d^{\text{odd}}$:

$$\text{hoPois}_d^{\text{odd}} := \mathcal{F} \left(\begin{array}{l|l} \mu_2, \nu_{2k+1}, k \geq 1 & \mu_2(\mu_2(x_1, x_2), x_3) = \mu_2(x_1, \mu_2(x_2, x_3)) \\ \deg(\nu_{2k+1}) = 1 - 2kd & d(\nu_{2k+1}) = \sum_{i+j=k} \sum_{\sigma \in S_{2k+1}} (-1)^\sigma \sigma \cdot \nu_{2i+1} \circ_1 \nu_{2j+1} \end{array} \right)$$

$$\Delta(\mu_2) = \mu_2 \otimes \mu_2, \quad \Delta(\nu_{2k+1}) = \underbrace{\mu_2 \circ \mu_2 \circ \dots \circ \mu_2}_{2k} \otimes \nu_{2k+1} + \nu_{2k+1} \otimes \underbrace{\mu_2 \circ \mu_2 \circ \dots \circ \mu_2}_{2k}.$$

where the generators $\{\nu_{2k+1} | k \in \mathbb{N}\}$ generate the free resolution $\mathbf{L}_\infty^{\text{odd}} = \Omega([\text{Comm}]_{\mathbb{Z}_2}^\vee)$ of the suboperad $\text{Lie}^{\text{odd}} \subset \text{Pois}_d^{\text{odd}}$. The latter suboperad Lie^{odd} is the suboperad generated by the ternary operation ν_3 .

5.2 Fibrant dgca model for $\text{Pois}_d^{\text{odd}}$ via "odd" graphs

We say that a vertex in a directed graph is *odd* (respectively *even*) if the number of outgoing edges is odd (resp. even). An oriented graph from $\text{fGraphs}_d^\downarrow$ is called *odd* if for each internal vertex the number of outgoing edges is odd. Notice that the subset of oriented graphs containing at least one *even* internal vertex forms an operadic ideal in $\text{fGraphs}_d^\downarrow$ closed under the differential. We denote the quotient operad of $\text{fGraphs}_d^\downarrow$ by this ideal $\text{fGraphs}_d^{\downarrow, \text{odd}}$. The notation shall indicate that it is spanned by oriented graphs with only odd internal vertices.

Lemma 5.2.1. *The suboperad $\text{Graphs}_d^{\downarrow, \text{odd}} \subset \text{fGraphs}_d^{\downarrow, \text{odd}}$ spanned by odd graphs whose internal vertices are at least trivalent is quasiisomorphic to the full operad of odd graphs $\text{fGraphs}_d^{\downarrow, \text{odd}}$.*

Proof. Notice that there are only two possibility of an internal odd vertex v of an odd graph Γ not to be trivalent:

- v is of valency 1 and has exactly one outgoing edge;
- v is of valency 2 and has one incoming and one outgoing edge.

These two possibilities are covered by usual spectral sequence argument related to the filtration of $\mathbf{fGraphs}$ given by the length of *antenna's* (a directed path consisting of bivalent vertices). \square

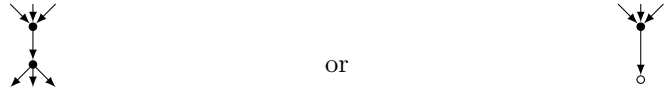
Theorem 5.2.2. *The following assignment of a graph to each generator of $\mathbf{hoPois}_d^{\text{odd}}$*

$$\mathbf{hoPois}_d^{\text{odd}}(2) \ni \mu_2 \mapsto \circ \quad \circ, \quad \mathbf{hoPois}_d^{\text{odd}}(2k+1) \ni \nu_{2k+1} \mapsto \begin{array}{c} \bullet \\ \swarrow \quad \downarrow \quad \searrow \\ \textcircled{1} \quad \textcircled{2} \quad \cdots \quad \textcircled{2k} \quad \textcircled{2k+1} \end{array} \quad (5.2.3)$$

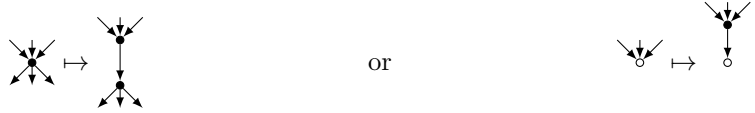
extends to a quasi-isomorphism of (complete) Hopf operads $F : \mathbf{hoPois}_d^{\text{odd}} \rightarrow \mathbf{Graphs}_{d+1}^{\downarrow, \text{odd}}$.

Proof. Let us compute the cohomology of $\mathbf{Graphs}_{d+1}^{\downarrow, \text{odd}}$. To this end we endow $\mathbf{Graphs}_{d+1}^{\downarrow, \text{odd}}$ with the descending complete filtration by the number of top vertices in graphs, i.e., the number of internal vertices with no inputs. We consider the associated graded complexes $\text{gr}^t \mathbf{Graphs}_{d+1}^{\downarrow, \text{odd}}$, where the "t" shall remind that this is via the filtration by the number of top vertices. The differential consists of just those splittings of vertices that do not increase the number of top vertices.

Next, we filter $\text{gr}^t \mathbf{Graphs}_{d+1}^{\downarrow, \text{odd}}$ again by the number of directed path between top vertices and external vertices. The differential on the path-associated graded $\text{gr}^p \text{gr}^t \mathbf{Graphs}_{d+1}^{\downarrow, \text{odd}}$ consists of only those splittings of vertices that do not decrease the number of paths. The cohomology can be evaluated as in [40]. To this end we take yet a third filtration as follows. We temporarily call an edge from a vertex v to a vertex w separating, such that the edge is the only outgoing edge at v , and the only incoming edge at w , pictorially



We filter by the number of non-separating edges, so that the differential on the associated graded complex $\text{gr}^{n,s} \text{gr}^p \text{gr}^t \mathbf{Graphs}_{d+1}^{\downarrow, \text{odd}}$ consist only of those terms creating a separating edge, while also not decreasing the number of path or creating top vertices. Pictorially it is given by the following operations



This differential has an obvious homotopy by contracting a separating edge. We can hence identify the cohomology of $\text{gr}^{n,s} \text{gr}^p \text{gr}^t \mathbf{Graphs}_{d+1}^{\downarrow, \text{odd}}$ with the subquotient consisting of graphs with no internal edge of valency $(\geq 2, \geq 2)$ and no external vertices of valency ≥ 2 , modulo all graphs with separating edges. Combinatorially in all nonzero such diagrams all vertices (internal and external) have either zero or one incoming edge. Now one quickly verifies that the space of such diagrams is identified precisely with $\mathbf{hoPois}_d^{\text{odd}}$, and the identification is done via our map F (i.e., the leading order terms with respect to our filtrations). Hence by standard spectral sequence arguments, the map F is a quasi-isomorphism. (Here we note that $\mathbf{Graphs}_{d+1}^{\downarrow, \text{odd}}$ splits into a direct product of subcomplexes according to the loop order of graphs, and each subcomplex is finite dimensional. Thus convergence of the spectral sequences considered is automatic.) \square

We remark that the exact same argument also gives a proof of Theorem 4.6.2, alternative to the one given in [58].

5.3 Hopf model of the mosaic operad $\overline{\mathcal{M}}_{0,\bullet}^{\mathbb{R}}$ via odd graphs

Let $\mathbf{GC}_d^{\downarrow, \text{odd}}$ be the Lie subalgebra of directed graphs whose loop order is even. In other words, the Euler characteristic is odd.

Proposition 5.3.1. *There is a natural action of the Lie algebra $\mathbf{GC}_d^{\downarrow, \text{odd}}$ on the operad $\mathbf{Graphs}_d^{\downarrow, \text{odd}}$ by operadic derivations. Pictorially the corresponding coaction of $(\mathbf{GC}_d^{\downarrow, \text{odd}})^{\vee}$ on $(\mathbf{Graphs}_d^{\downarrow, \text{odd}})^{\vee}$ is given by the contraction of a subgraph of odd Euler characteristic with at most one external vertex.*

Proof. See Section 7 for details. \square

Thanks to Lemma 4.6.5 the Shoikhet MC element γ^\downarrow recalled in (4.5.1) belongs to the subalgebra $\mathbf{GC}_2^{\downarrow, \text{odd}}$. In particular, one can consider the twisted (co)operad $(\mathbf{Graphs}_2^{\downarrow, \text{odd}})^{\gamma^\downarrow}$ twisted by the Shoikhet MC element γ^\downarrow . The twisting affects the differential, but does not change the operadic composition. The differential in the corresponding cooperad $(\mathbf{Graphs}_2^{\downarrow, \text{odd}})^{\gamma^\downarrow}$ is given by the following summation:

$$d^{\gamma^\downarrow}(\Gamma) = \sum_{e \in \text{Edges}(\Gamma)} \Gamma/e + \sum_{G \subset \Gamma} c_G^\downarrow \Gamma/G \quad \text{with} \quad c_G^\downarrow := \int_{\text{FM}_2(\text{Vert}(G))} \bigwedge_{e \in \text{Edges}G} \omega_e^\downarrow$$

where the second summation is given via subgraphs $G \subset \Gamma$ such that

- G has odd Euler characteristic and $2\#\text{vertices} - \#\text{edges} = 3$;
- G contains no more than one external vertex;
- the graph Γ/G obtained by contraction of a subgraph G inside Γ remains to be an oriented graph presenting an element of $\mathbf{Graphs}_2^{\downarrow, \text{odd}}$. That is, all inner vertices of Γ/G have an odd number of outputs, and there are no source vertices and no loops and double edges.

As mentioned earlier, the Stasheff polytope is homeomorphic to the Fulton-Macpherson compactification of points on the line and the latter coincides with the component $\text{SC}(0, n)$ of the Fulton-Macpherson compactification of the swiss cheese operad that does not contain the marked points outside of the boundary. Thus, the fibered integral assignment

$$\Gamma \mapsto \omega_\Gamma^\downarrow := \int_{\text{SC}(M, n) \rightarrow \text{SC}(0, n)} \bigwedge_{e \in \Gamma} \omega_e^\downarrow$$

defines a differential form on $\text{FM}_1(n)$ out of an odd directed graph Γ with n external vertices. Moreover, the preceding Lemma 5.3.2 explains why the differential form ω_Γ^\downarrow can be considered as a differential form on the manifold $\overline{\mathcal{M}}_{0, n+1}(\mathbb{R})$ and we will use the same notation ω_Γ^\downarrow for it.

Lemma 5.3.2. *Let $T_1 \sim_{z_2} T_2$ be a pair of leaf-labeled rooted trees from T_n that are equivalent under the equivalence relation introduced in Section 2.2. Let U_{T_1} (respectively U_{T_2}) be the corresponding cells in $\text{FM}_1(n)$. Then the pushforwards of the restrictions of the form ω_Γ^\downarrow on the corresponding cells coincides:*

$$\left(p_n \Big|_{U_{T_1}} \right)_* \left(\omega_\Gamma^\downarrow \Big|_{U_{T_1}} \right) = \left(p_n \Big|_{U_{T_2}} \right)_* \left(\omega_\Gamma^\downarrow \Big|_{U_{T_2}} \right)$$

Proof. The statement of the Lemma is obvious for the top-dimensional cells. For example, as predicted by the flip $\tau : \text{As} \rightarrow \text{As}$ defined in Section 3.4 the orientation of the cell U_σ and $U_{\sigma \circ p}$ differs by the sign $(-1)^{n-1}$. Consequently, the forms $p_{n*} \left(\omega_\Gamma^\downarrow \Big|_{U_\sigma} \right)$ and $p_{n*} \left(\omega_\Gamma^\downarrow \Big|_{U_{\sigma \circ p}} \right)$ coincides for the simplest directed

graph $\Gamma = \textcircled{1} \dots \textcircled{n}$ (with a unique inner vertex) if and only if n is odd.

The coincidence for general strata follows from the factorization property of the strata:

$$U_T := \bigtimes_{v \in \text{Vert}T} U_{\sigma_v}, \quad \text{with } \sigma_v \in \mathcal{S}_{|\text{in}(v)|}$$

and the factorization property of fibered integrals:

$$\omega_\Gamma^\downarrow \Big|_{U_T} = \bigotimes_{v \in \text{Vert}(T)} \omega_{\Gamma_v}^\downarrow \Big|_{U_{\sigma_v}}, \quad (5.3.3)$$

where the graphs Γ_v are uniquely determined by the cooperad structure and cooperadic cocomposition prescribed by T . \square

Finally, we are able to state one of the main results of this paper.

Theorem 5.3.4. *The fiber-integral assignment $\Gamma \mapsto \omega_\Gamma^\downarrow$ defines a quasi-isomorphism between the twisted cooperad $((\mathbf{Graphs}_2^{\downarrow, \text{odd}})^{\gamma^\downarrow})^\vee$ and the cooperad of differential forms on the Mosaic operad $\Omega_{DR}(\overline{\mathcal{M}}_{0, \bullet}^{\mathbb{R}})$.*

Equivalently, the twisted operad $(\mathbf{Graphs}_2^{\downarrow, \text{odd}}, d + [\gamma^\downarrow, -])$ defines a real algebraic model of the Mosaic operad in the category of (complete) dg-coalgebras.

Remark 5.3.5. *The functor of differential forms is not a comonoidal functor, thus the dgca's of differential forms of a topological operad is not really a cooperad but is very close to be. Following §3 of [28] and Definition 3.1 of [34] the existence of the following commutative diagrams for all pairs (m, n) :*

$$\begin{array}{ccc}
((\mathbf{Graphs}_2^{\downarrow, \text{odd}})^{\gamma^\downarrow})^\vee(m+n) & \xrightarrow{\quad\quad\quad} & \Omega_{DR}(\overline{\mathcal{M}}_{0, m+n+1}(\mathbb{R})) \\
\downarrow & & \downarrow \\
& & \Omega_{DR}(\overline{\mathcal{M}}_{0, m+2}(\mathbb{R})) \times \overline{\mathcal{M}}_{0, n+1}(\mathbb{R}) \\
& & \uparrow \\
((\mathbf{Graphs}_2^{\downarrow, \text{odd}})^{\gamma^\downarrow})^\vee(m+1) \otimes ((\mathbf{Graphs}_2^{\downarrow, \text{odd}})^{\gamma^\downarrow})^\vee(n) & \xrightarrow{\quad\quad\quad} & \Omega_{DR}(\overline{\mathcal{M}}_{0, m+2}(\mathbb{R})) \otimes \Omega(\overline{\mathcal{M}}_{0, n+1}(\mathbb{R}))
\end{array} \tag{5.3.6}$$

is enough to say that the twisted operad $(\mathbf{Graphs}_2^{\downarrow, \text{odd}})^{\gamma^\downarrow}$ is a rational model for the mosaic operad.

Proof. Let us explain why the fibered integral assignment is a quasiisomorphism of Hopf operads. In particular, we have to explain the following.

- The map is a map of operads. This is clear from the factorization property of forms ω_Γ that the diagram 5.3.6 is commutative.
- For each arity n the map $(\mathbf{Graphs}_2^{\downarrow, \text{odd}})^{\gamma^\downarrow}(n) \rightarrow \Omega_{DR}(\overline{\mathcal{M}}_{0, n+1}(\mathbb{R}))$ is a map of graded algebras. This follows from the definition of the fibered integral, because the algebra structure on the graphs is given by concatenation of graphs and gluing external vertices.
- The map is a map of complexes, i.e., compatible with the differentials. This is the core of Theorem 5.3.4. The proof is based on the Stokes formula applied to the fibered integrals on the Fulton-MacPherson compactification of the Swiss Cheese operad. The careful proof is a bit technical but completely repeats the proofs carefully written in [34] for the case of Little balls operad and the proof of the analogous statement with directed graphs in [51]. The only thing we have to mention is that we do the Stokes formula on the Swiss Cheese and on the Stasheff polytopes and use the fact of Lemma 5.3.2 that for odd graphs Γ the forms ω_Γ^\downarrow considered as a form on $\text{SC}(0, n)$ can be considered also as a form on $\overline{\mathcal{M}}_{0, n+1}(\mathbb{R})$.
- The map is a quasi-isomorphism. The loop order of a graph defines a filtration on $(\mathbf{Graphs}_2^{\downarrow, \text{odd}}, d + [\gamma^\downarrow, -])$. The corresponding associated graded is isomorphic to $\mathbf{Graphs}_2^{\downarrow, \text{odd}}$. One can define the filtration on \mathbf{As} and respectively on $[\mathbf{As}]_{\mathbb{Z}_2}$ that defines a filtration on $\Omega_{Op}([\mathbf{As}]_{\mathbb{Z}_2}^\vee)$ and on the quasiisomorphic complex $\Omega_{DR}(\overline{\mathcal{M}}_{0, n+1}(\mathbb{R}))$. We state that the desired morphism of operads is a map of filtered operads and Theorem 5.2.2 explains that the associated graded map is a quasi-isomorphism.

□

6 Rational homotopy type of $\overline{\mathcal{M}}_{0, n+1}(\mathbb{R})$

6.1 (Ho)Lie models of $\text{Pois}_d^{\text{odd}}$ and the rational homotopy type of $\text{Pois}_d^{\text{odd}}(n)$

We proved in Theorem 5.2.2 that the operad of oriented odd graphs $\mathbf{Graphs}_d^{\downarrow, \text{odd}}$ is a model of the Hopf operad $\text{Pois}_d^{\text{odd}}$. One of the main features of this model is that the space of n -ary operations $\mathbf{Graphs}_d^{\downarrow, \text{odd}}(n)$ is the free commutative algebra whose generators spanned by internally connected odd graphs. I.e. the graphs that remains to be connected after deleting all external vertices. We use the notation $\text{ICG}_d^{\downarrow, \text{odd}}$ for the L_∞ algebra spanned by these graphs following the notations invented in [49] for the case of the operad of Graphs. This space is endowed with the edge contracting differential and the L_∞ -structure that

can be read of the isomorphism between the space of all possible graphs and the Chevalley-Eilenberg complex of internally connected graphs

$$\mathbf{Graphs}_d^{\downarrow, \text{odd}}(n) = C_{\bullet}^{CE}(\mathbf{ICG}_d^{\downarrow, \text{odd}}) \quad (6.1.1)$$

The part of the vertex-splitting differential that decreases the number of internally connected components remembers the desired L_{∞} structure. The latter may happen only while splitting an external vertex.

In this section we compute the homology of the L_{∞} -algebra $\mathbf{ICG}_{d+1}^{\downarrow, \text{odd}}(n)$ for all n . In particular, we show that the cohomology of these L_{∞} -algebras are quadratic Koszul Lie algebras, which assemble into a model of $\mathbf{Pois}_d^{\text{odd}}$ in the category of Lie algebras.

The quadratic Lie algebra dual to $\mathbf{Pois}_d^{\text{odd}}(n)$ has the following presentation in terms of generators and relations:

$$\mathfrak{t}_d^{\text{odd}}(n) := \text{Lie} \left(\begin{array}{l|l} \nu_{ijk}, 1 \leq i, j, k \leq n & [\nu_{ijk}, \nu_{pqi} + \nu_{pqj} + \nu_{pqk}] = 0 \\ \nu_{ijk} = (-1)^{d\sigma} \nu_{\sigma(i)\sigma(j)\sigma(k)} & [\nu_{ijk}, \nu_{pqr}] = 0 \\ \deg(\nu_{ijk}) = 2 - 2d & \text{for } \#\{i, j, k, p, q, r\} = 6. \end{array} \right) \quad (6.1.2)$$

that generalizes the presentation found in [12] for the quadratic Lie algebra dual to the quadratic algebra $H^*(\mathcal{M}_{0, n+1}(\mathbb{R}))$. We denote this Lie algebra by $\mathfrak{t}^{\text{odd}}$ as an odd version of the Drinfeld-Kohno Lie algebra:⁵

$$\mathfrak{t}_d(n) := \text{Lie} \left(\begin{array}{l|l} t_{ij}, 1 \leq i \neq j \leq n & [t_{ij}, t_{ik} + t_{jk}] = 0 \\ t_{ij} = (-1)^d t_{ji} & [t_{ij}, t_{kl}] = 0, \\ \deg(t_{ij}) = 2 - d & \text{for } \#\{i, j, k, l\} = 4. \end{array} \right)$$

Lemma 6.1.3. *The assignment $\nu_{ijk} \mapsto \left[\begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ (i) \quad (j) \quad (k) \end{array} \right]$ extends to an embedding of Lie algebras*

$$\psi : \mathfrak{t}_d^{\text{odd}}(n) \rightarrow H^*(\mathbf{ICG}_{d+1}^{\downarrow, \text{odd}}(n)).$$

Proof. The existence of the embedding ψ follows from the general theory of Koszul duality for algebras ([43]). Recall that with each quadratic algebra A one assigns a Koszul-dual algebra $A^!$ (the cohomology of the cobar construction) and the quadratic-dual algebra $qA^!$ defined by the dual set of generators and the dual set of quadratic relations. The quadratic dual algebra $qA^!$ is isomorphic to the graded subalgebra of $A^!$ spanned by elements whose homological degree coincides with the inner degree. The algebra A is called Koszul if $qA^!$ and $A^!$ coincide.

The Lie algebra $\mathfrak{t}_d^{\text{odd}}(n)$ is the quadratic dual to the quadratic algebra $(\mathbf{Pois}_d^{\text{odd}}(n))^{\vee}$ and the L_{∞} algebra $\mathbf{ICG}_{d+1}^{\downarrow, \text{odd}}(n)$ is quasiisomorphic to the Harrison complex of $\mathbf{Graphs}_d^{\downarrow, \text{odd}}(n)$ and thanks to Theorem 5.2.2 we know that the latter dgca is equivalent to $(\mathbf{Pois}_d^{\text{odd}}(n))^{\vee}$. Consequently, $\mathbf{ICG}_{d+1}^{\downarrow, \text{odd}}(n)$ is equivalent to the Koszul dual Lie algebra of the commutative algebra $(\mathbf{Pois}_d^{\text{odd}}(n))^{\vee}$.

In order to ensure the reader that the map ψ is the desired map between quadratic dual and Koszul dual Lie algebras to the algebra $(\mathbf{Pois}_d^{\text{odd}}(n))^{\vee}$ let us check the defining relations (6.1.2) of the Lie algebra $\mathfrak{t}_d^{\text{odd}}(n)$.

Recall, that the Lie bracket (as well as all higher brackets) for internally connected graphs is given by gluing the graphs through external vertices and then splitting one of the external vertices. In particular, the component spanned by trees in the Lie bracket of the internally trivalent trees is also a sum of internally trivalent trees.

Notice that, if $|\{i, j, k, p, q, r\}| = 6$ then $\psi(\nu_{ijk})$ and $\psi(\nu_{pqr})$ commute, because the subsets of external vertices they are connected do not intersect. Moreover,

$$\begin{aligned} d \left(\begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ (i) \quad (j) \quad (k) \quad (p) \quad (q) \end{array} \right) &= \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ (i) \quad (j) \quad (k) \quad (p) \quad (q) \end{array} + \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ (j) \quad (k) \quad (i) \quad (p) \quad (q) \end{array} + \begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ (k) \quad (i) \quad (j) \quad (p) \quad (q) \end{array} = \\ &= [\psi(\nu_{ijk}), \psi(\nu_{kpq})] + [\psi(\nu_{ijk}), \psi(\nu_{ipq})] + [\psi(\nu_{ijk}), \psi(\nu_{jpq})]. \end{aligned}$$

The relations (6.1.2) are verified on the level of homology $H^*(\mathbf{ICG}_{d+1}^{\downarrow, \text{odd}}(n))$. \square

⁵The lower index d corresponds to the dimension of the unit ball in the little d -discs operad $E_d \simeq \text{FM}_d$. The classical Drinfeld-Kohno Lie algebra corresponds to the case $d = 2$ when $E_2(n)$ is an aspherical space with $\mathfrak{t}_2(n)$ equals the Malcev completion of the pure braid group.

Theorem 6.1.4. *The map $\psi : \mathfrak{t}_d^{\text{odd}}(n) \rightarrow H^\bullet(\text{ICG}_{d+1}^{\downarrow, \text{odd}}(n))$ is an isomorphism that fits into the following commutative diagram (whose vertical arrows are isomorphism and horizontal arrows are embeddings):*

$$\begin{array}{ccc}
\begin{array}{c} \nu_{ijk} \\ \downarrow \psi \\ \left[\begin{array}{c} \bullet \\ \swarrow \downarrow \searrow \\ \circ(i) \quad \circ(j) \quad \circ(k) \end{array} \right] \end{array} & \begin{array}{ccc} \mathfrak{t}_d^{\text{odd}}(n) & \xleftarrow{\xi_n: \nu_{ijk} \mapsto [t_{ij}, t_{jk}]} & \mathfrak{t}_{d+1}(n) \\ \downarrow \psi & & \downarrow \\ H(\text{ICG}_{d+1}^{\downarrow, \text{odd}}(n)) & \hookrightarrow & H(\text{dICG}_{d+1}(n)) \end{array} & \begin{array}{c} t_{ij} \\ \downarrow \\ \left[\circ(i) \rightarrow \circ(j) \right] + (-1)^d \left[\circ(i) \leftarrow \circ(j) \right] \end{array}
\end{array} \tag{6.1.5}$$

Before going into the somewhat cumbersome proof of the theorem let us mention the main corollary

Corollary 6.1.6. *The quadratic dual algebras $\text{Pois}_d^{\text{odd}}(n)^\vee$ and $\mathfrak{t}_d^{\text{odd}}(n)$ are Koszul for all n and the operad $\mathfrak{t}_d^{\text{odd}}$ is a model for the Hopf operad $\text{Pois}_d^{\text{odd}}$ in the category of Lie algebras.*

In particular, for $d = 1$ the quadratic skew-commutative algebra $H^\bullet(\overline{\mathcal{M}}_{0, n+1}(\mathbb{R}))$ and the Lie algebra $\mathfrak{t}_1^{\text{odd}}(n)$ are Koszul dual to each other.

Proof. It is enough to show that the operads $\mathfrak{t}_d^{\text{odd}}$ and $\text{ICG}_{d+1}^{\downarrow, \text{odd}}$ are homotopically equivalent operads in the category of L_∞ -algebras. We will state the proof in the case $d = 1$ for simplicity. The general case is covered by the same arguments but requires more careful comparison of homological and inner gradings in $\mathfrak{t}_d^{\text{odd}}$ that seems rather technical than conceptual.

Let us introduce an auxiliary operad in L_∞ algebras as a truncation of $\text{ICG}_2^{\downarrow, \text{odd}}$ in 0 homological degree:

$$\text{degree } s \text{ component } \left(\text{TICG}_2^{\downarrow, \text{odd}}(n) \right)^s := \begin{cases} \left(\text{ICG}_2^{\downarrow, \text{odd}} \right)^s, & \text{if } s < 0, \\ \left(\text{ICG}_2^{\downarrow, \text{odd}} \right)^0_{\text{closed}}, & \text{if } s = 0, \\ 0, & \text{if } s > 0, \end{cases} \tag{6.1.7}$$

Since, ψ is an isomorphism we know that the complexes $\text{ICG}_2^{\downarrow, \text{odd}}(n)$ have nontrivial homology only in 0 degree. Hence, both maps $\text{TICG}_2^{\downarrow, \text{odd}}(n) \rightarrow \mathfrak{t}_1^{\text{odd}}(n)$ and $\text{TICG}_2^{\downarrow, \text{odd}}(n) \rightarrow \text{ICG}_2^{\downarrow, \text{odd}}(n)$ are quasi-isomorphisms for all n that are compatible with the operadic structure. \square

6.1.1 Vertical arrows in Theorem 6.1.4

In this subsection we prove the vertical part of Theorem 6.1.4. The map $\mathfrak{t}_d(n) \rightarrow H(\text{dICG}_d(n))$ is known to be an isomorphism after [49]. Thus, this section is devoted to show that the map $\psi : \mathfrak{t}_d^{\text{odd}}(n) \rightarrow H(\text{ICG}_{d+1}^{\downarrow, \text{odd}}(n))$ is an isomorphism for all n .

Recall, that thanks to Lemma 5.2.1 we know that all internal vertices of an odd graph $\Gamma \in \text{ICG}_d^{\downarrow, \text{odd}}$ are at least trivalent. We know that the homological grading and the internal grading of the Lie algebra $\mathfrak{t}_d^{\text{odd}}(n)$ differ by linear transformation. Thanks to the Koszul duality theory and comparison between quadratic dual and Koszul dual algebras mentioned in the proof of Lemma 6.1.3 we know that any nontrivial higher Lie bracket on $H^\bullet(\text{ICG}_{d+1}^{\downarrow, \text{odd}})$ will be of wrong homological grading. Thus, in order to show that the map $\psi : \mathfrak{t}_d^{\text{odd}} \rightarrow H^\bullet(\text{ICG}_{d+1}^{\downarrow, \text{odd}})$ is an isomorphism it is enough to prove the following key result.


Lemma 6.1.8. *All non-trivial cohomology classes in $\text{ICG}_d^{\downarrow, \text{odd}}$ can be represented by linear combinations of internally trivalent trees.*

The proof is based on a collection of consecutive spectral sequence arguments such that the associated graded differential for each particular spectral sequence is the edge contraction that does not brake certain symmetries defined combinatorially in terms of graphs. Maschke's theorem is the key argument that helps in this type of computations.

Remark 6.1.9. *For each given pair of integers $n \geq 1$ and $l \geq 0$ the number of graphs in $\text{ICG}_d^{\downarrow, \text{odd}}(n)$ with a given loop order l is finite. Consequently all spectral sequences we are dealing with converge because all complexes are splitted to the direct sums of finite dimensional ones.*

Definition 6.1.10. For each subset $S \subset [1n]$ consider the subspace $\text{ICG}_{d,S}^{\downarrow,\text{odd}}(n) \subset \text{ICG}_d^{\downarrow,\text{odd}}(n)$ spanned by internally connected graphs Γ yielding the two following properties:

(i) for all $s \in S$ the graph Γ has a unique edge ending in the external vertex \textcircled{s} and the source vertex

of this edge has more than one outgoing edges: 

(ii) for all pairs of different elements $s, t \in S$ the length of the minimal (nondirected) path between \textcircled{s} and \textcircled{t} is greater than 2. In other words, we do not allow an internal vertex with two outgoing edges ending in external vertices from the subset S :



Lemma 6.1.11. The homology of the complex $\text{ICG}_{d,S}^{\downarrow,\text{odd}}(n)$ are spanned by internally trivalent trees. In particular, for each given loop order the homology is concentrated in a unique homological degree.

Note that Lemma 6.1.8 is a particular case of Lemma 6.1.11 when $S = \emptyset$.

Proof. The proof is the simultaneous (increasing) induction on the number n of externally connected vertices, (decreasing) induction on the cardinality of the subset S and (increasing) induction on the loop order of a graph.

For the base of induction ($|S| = n$) we notice that there are no oriented graphs (with no oriented loops) in $\text{ICG}_{d,[1n]}^{\downarrow,\text{odd}}(n)$. Indeed, each internal vertex of a graph $\Gamma \in \text{ICG}_d^{\downarrow,\text{odd}}$ has at least one outgoing edge and if, in addition, each external vertex of Γ is connected by a unique edge and the source of this edge has at least one edge ending in an internal vertex then one can construct a directed path of arbitrary length that avoids external vertices. However, the number of vertices is finite and hence this path contains a loop what is not allowed in $\text{ICG}_d^{\downarrow,\text{odd}}$.

(Induction step). Suppose $m \in [1n] \setminus S$ and thanks to the induction on n we may suppose that we are dealing with the ideal of graphs connected with \textcircled{m} . For each graph $\Gamma \in \text{ICG}_{d,S}^{\downarrow,\text{odd}}(n)$ we can define the maximal subtree $T_m := T_m(\Gamma) \subset \Gamma$ whose vertices are defined by the following property:

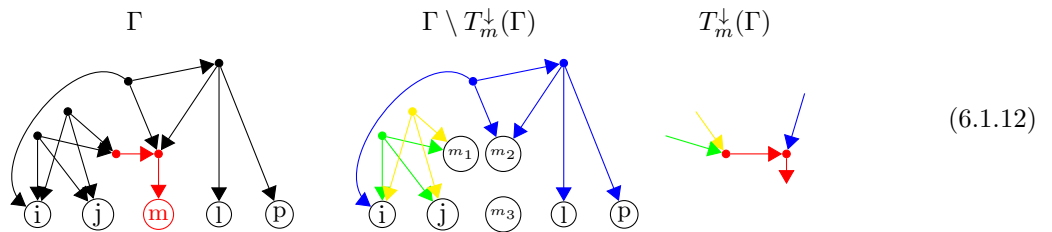
$w \in T_m \stackrel{\text{def}}{\iff}$ There exists a unique (nondirected) path (with no selfintersections) that starts in w and ends in \textcircled{m} .

Let T_m^\downarrow be the subtree of T_m spanned by those vertices v of T_m yielding the conditions:

(a1) the unique path starting at v and ending in \textcircled{m} is a directed path;

(a2) if the vertex v is internal (differs from \textcircled{m}) then it has a unique outgoing edge.


Let $\Gamma_1, \dots, \Gamma_k$ be the set of internally connected components of the complementary graph $\Gamma \setminus T_m^\downarrow$ (internal vertices of T_m^\downarrow are considered as external vertices of the complementary graph $\Gamma \setminus T_m^\downarrow$). Moreover, $\forall 1 \leq i \leq k$ there exists exactly one vertex $v_i \in T_m^\downarrow$ such that there exists not less than one edge ending in v_i that belongs to Γ_i . The uniqueness of v_i is governed by definition of $T_m \supset T_m^\downarrow$. Consider the following example of a graph Γ . The complementary graph $\Gamma \setminus T_m^\downarrow$ has three connected component drawn in different colors (green, yellow and blue). We draw the tree T_m^\downarrow together with the incoming edges that correspond to connected components that is designed by corresponding coloring:

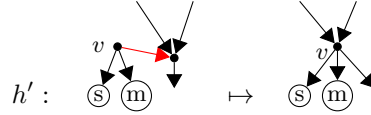


Consider the filtration of $\text{ICG}_{d,S}^{\downarrow,\text{odd}}(n)$ by the number k of internally connected components of the graph $\Gamma \setminus T_m^\downarrow(\Gamma)$. We will explain that the cohomology of the associated graded complex are represented by internally trivalent trees. Thanks to Mashke's theorem we can order the set of connected components and the complex is the tensor product of the same complexes with the unique connected component and the complex $L_\infty(k)$ that corresponds to the subtree T_m^\downarrow . The operad Lie is Koszul and therefore the

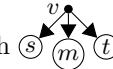
subtree T_m^\downarrow is represented by trivalent trees. Thus, it remains to show that the homology of the quotient complex $\mathrm{ICG}_{d,S}^{\downarrow,\mathrm{odd}}(n)^1$ spanned by graphs $\Gamma \in \mathrm{ICG}_{d,S}^{\downarrow,\mathrm{odd}}(n)$ with unique internally connected component Γ_1 of $\Gamma \setminus T_m^\downarrow(\Gamma)$ is spanned by internally trivalent trees.

Note that the graph $\Gamma \setminus T_m^\downarrow(\Gamma)$ may have a unique internally connected component if and only if $T_m^\downarrow(\Gamma)$ either is equal to (m) or consists of one edge $v \rightarrow (m)$. Therefore the complex $\mathrm{ICG}_{d,S}^{\downarrow,\mathrm{odd}}(n)^1$ admits a decomposition $\mathrm{ICG}_{d,S}^{\downarrow,\mathrm{odd}}(n)^1 = \mathrm{ICG}_{d,S}^{\downarrow,\mathrm{odd}}(n)_0^1 \oplus \mathrm{ICG}_{d,S}^{\downarrow,\mathrm{odd}}(n)_1^1$ where the additional rightmost lower index corresponds to the number of edges in T_m^\downarrow . Consider the homotopy $h : \mathrm{ICG}_{d,S}^{\downarrow,\mathrm{odd}}(n)_1^1 \rightarrow \mathrm{ICG}_{d,S}^{\downarrow,\mathrm{odd}}(n)_0^1$ to the first differential in the corresponding spectral sequence given by contraction of the edge $v \rightarrow (m)$ if allowed. The kernel of this surjection is spanned by graphs having more than one outgoing edge from the unique internal vertex $v \in T_m^\downarrow$. This kernel is decomposed into the sum $\mathrm{ICG}_{d,S \sqcup \{m\}}^{\downarrow,\mathrm{odd}}(n)$ and the subcomplex K spanned by graphs with at least one outgoing edge from the internal vertex v ending in an external vertex (s) with $s \in S$. Let us show that the subcomplex K is almost acyclic. Consider the filtration $K = \bigoplus_{s \in S} K_s$ by the minimal number of the external vertex (s) connected by an edge with v . The associated graded complex K_s admits additional twostep filtration $K_s = K_s^3 \oplus K_s^{>3}$, where K_s^3 is spanned by graphs with the vertex v trivalent. Note that if v is trivalent then v has 3 outgoing edges, one ending in (s) another ending in (m) and the latter connects v with the remaining part of the graph:

 Consider the homotopy h' to the associated graded differential given by contraction of the unique edge connecting v and the remaining part of the graph:



The homotopy h' defines a bijection between graphs spanning K_s^3 and K_s^3 except one particular case when the remaining part of the graph consists of one external vertex (t) and if $S = \{s\}$ or $S = \{s, t\}$

the cohomology of K_s is spanned by the simplest trivalent graph . The simultaneous induction finishes the proof that cohomology classes are represented by internally trivalent trees and therefore sits in one homological degree for each given loop order of a graph. \square

Let us denote by $\mathrm{LICG}_d^{\downarrow,\mathrm{odd}}(n)$ the L_∞ -ideal of $\mathrm{ICG}_d^{\downarrow,\mathrm{odd}}(n)$ spanned by graphs connected by an edge with the latter external vertex (n) . Following the aforementioned proof one can define a collection of subcomplexes $\mathrm{LICG}_{d,S}^{\downarrow,\mathrm{odd}} \subset \mathrm{LICG}_d^{\downarrow,\mathrm{odd}}$ and proof that their homology are represented by certain internally trivalent trees. Moreover, one easily conclude the following

Corollary 6.1.13. *Each homogeneous cocycle c of the loop order $l > 2$ representing a cohomological class of the complex $\mathrm{LICG}_{d,[1n-1]}^{\downarrow,\mathrm{odd}}(n)$ can be represented by a sum of internally trivalent trees with a unique edge connected with the latter external vertex (n) and no other outgoing edges for the source of the latter edges.*

6.1.2 Horizontal arrows in Theorem 6.1.4.

Lemma 6.1.8 implies that the algebras $\mathrm{Pois}_d^{\mathrm{odd}}(n)^\vee$ are Koszul and therefore the maps $\psi_n : \mathfrak{t}_d^{\mathrm{odd}}(n) \rightarrow H^*(\mathrm{ICG}_{d+1}^{\downarrow,\mathrm{odd}})$ is an isomorphism. On the other hand, we want to have a better description of the cohomology and we need an embedding to the ordinary Drinfeld-Kohno Lie algebra.

Note that the map $\xi_n : \mathfrak{t}_d^{\mathrm{odd}}(n) \xrightarrow{\nu_{ijk} \mapsto [t_{ij}, t_{jk}]} \mathfrak{t}_{d+1}(n)$ is well defined and fits into the following map of short exact sequences of Lie algebras (defined over \mathbb{Q}):

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \pi_n & \hookrightarrow & \mathfrak{t}_d^{\mathrm{odd}}(n) & \xrightarrow{\pi_n} & \mathfrak{t}_d^{\mathrm{odd}}(n-1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \xi_n & & \downarrow \xi_{n-1} & & \\ 0 & \longrightarrow & \mathrm{Lie}(t_{1n}, \dots, t_{n-1n}) & \hookrightarrow & \mathfrak{t}_{d+1}(n) & \twoheadrightarrow & \mathfrak{t}_{d+1}(n-1) & \longrightarrow & 0 \end{array}$$

Here the map π_n maps a generator ν_{ijk} to the corresponding generator ν_{ijk} whenever i, j, k are different from n , the generators ν_{ijn} are mapped to 0. The statement that ξ_n is a monomorphism will be proved by induction on n and it is enough to explain that the morphism $\xi_n : \ker \pi \rightarrow \text{Lie}(t_{1n}, \dots, t_{n-1,n})$ is an embedding. The latter will be explained below (Proposition 6.1.17) using the language of (internally connected) graphs.

Notice, that on the level of graphs the ideal $\ker \pi_n$ corresponds to the L_∞ -ideal of $\text{ICG}_{d+1}^{\downarrow, \text{odd}}$ spanned by graphs that are connected to the latter external vertex \textcircled{n} . We will call this ideal $\text{LICG}_{d+1}^{\downarrow, \text{odd}}$.

Let us define another L_∞ -algebra of graphs called $\text{LICG}_d^{\sharp, \text{odd}}(n)$ spanned by internally connected oriented odd graphs connected with the external vertex \textcircled{n} with a mild relief: *we allow outgoing edges from the latter external vertex \textcircled{n}* . All other external vertices may have only incoming edges. Let us define a combinatorial map:

$$\theta : \text{LICG}_d^{\sharp, \text{odd}}(n) \rightarrow \text{Lie}(\{t_{sn} | s = 1 \dots n-1\})$$

by (i) projecting onto internal trivalent trees with only one edge connecting to the external vertex \textcircled{n} , (ii) forgetting the directions of arrows, (iii) interpreting the tree as a Lie tree in t_{jn} : the vertex \textcircled{n} is considered to be a root, a leaf connected to the vertex \textcircled{j} corresponds to a copy of a t_{jn} and each inner vertex corresponds to a commutator. For example:

$$\begin{array}{ccc}
 \begin{array}{c} \textcircled{i} \quad \textcircled{j} \quad \textcircled{n} \quad \textcircled{k} \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \textcircled{i} \quad \textcircled{j} \quad \textcircled{n} \quad \textcircled{k} \end{array} & \mapsto & \begin{array}{c} \textcircled{j} \quad \textcircled{i} \quad \textcircled{k} \\ \downarrow \quad \downarrow \quad \downarrow \\ \textcircled{n} \end{array} & \mapsto & [[t_{jn}, t_{in}], [t_{in}, t_{kn}]], \\
 \\
 \begin{array}{c} \textcircled{i} \quad \textcircled{j} \quad \textcircled{k} \quad \textcircled{1} \\ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ \textcircled{i} \quad \textcircled{j} \quad \textcircled{k} \quad \textcircled{1} \end{array} & \mapsto & \begin{array}{c} \textcircled{i} \quad \textcircled{k} \quad \textcircled{j} \\ \downarrow \quad \downarrow \quad \downarrow \\ \textcircled{1} \end{array} & \mapsto & [t_{in}, [t_{kn}, [t_{in}, t_{jn}]]].
 \end{array} \tag{6.1.14}$$

Recall that the kernel of the map $\mathfrak{t}_d(n) \rightarrow \mathfrak{t}_d(n-1)$ is known to be isomorphic to the free Lie algebra on $n-1$ generator of degree $2-d$ which can be called for simplicity by $\{t_{sn} | s = 1 \dots n-1\}$.

Lemma 6.1.15. *The map $\theta : \text{LICG}_d^{\sharp, \text{odd}}(n) \rightarrow \text{Lie}(\{t_{sn} | s = 1 \dots n-1\})$ is a quasiisomorphism of graded L_∞ -algebras.*

Proof. Note that each higher Lie bracket with k arguments in $\text{ICG}_d^{\downarrow, \text{odd}}(n)$ produces a new vertex of valency $k+1$. Hence, the image of all higher Lie brackets is zero under θ . The ordinary Lie bracket in $\text{LICG}_d^{\sharp, \text{odd}}(n)$ maps trivalent trees to trivalent trees. Moreover, the Lie bracket of two internally trivalent trees Γ_1, Γ_2 with a unique edge connecting the last vertex n is a sum of trivalent trees, but there is only one summand with a unique edge ending at the last vertex n . This graph is glued out of Γ_1 and Γ_2 through the new internal trivalent vertex whose remaining edge is connected with the last external vertex n . The vertex splitting differential on an internally trivalent tree may differ from zero only while splitting an external vertex, however, the corresponding graph will have a nontrivial internal loop order. Thus, θ is a map from L_∞ algebra to an ordinary Lie algebra $\mathfrak{lt}(n)$ (compatible with the differential).

The proof that θ is a quasiisomorphism repeats the one given in [49] (Appendix B). Consider a two step filtration (a grading) of $\text{LICG}_d^{\sharp, \text{odd}}(n)$:

$$\text{LICG}_d^{\sharp, \text{odd}}(n) = \text{LICG}_d^{\sharp, \text{odd}}(n)^1 \oplus \text{LICG}_d^{\sharp, \text{odd}}(n)^{\geq 2}$$

with $\text{LICG}_d^{\sharp, \text{odd}}(n)^1$ consisting of graphs that have exactly one edge (either incoming or outgoing) connected to the vertex $\textcircled{1}$ and $\text{LICG}_d^{\sharp, \text{odd}}(n)^{\geq 2}$ contains at least two edges attached to $\textcircled{1}$. The first term of the corresponding spectral sequence is the subspace $\text{LICG}_d^{\sharp, \text{odd}}(n)_{disc}^1 \subset \text{LICG}_d^{\sharp, \text{odd}}(n)^1$ spanned by graphs that become disconnected after contracting the unique edge attached to $\textcircled{1}$. Let v be the vertex on the other end of the edge connecting to the vertex $\textcircled{1}$. If v is external the graph consists of one edge and we are done. If v is internal then by deleting v we get an isomorphism between $\text{LICG}_d^{\sharp, \text{odd}}(n)_{disc}^1$

and the truncated Chevalley-Eilenberg complex $C_{\geq 2}^{CE}(\mathrm{LICG}_d^{\sharp, \mathrm{odd}}(n))$. The cohomology computation of a truncated free Lie algebra:

$$H_{CE}^i(\mathrm{Lie}(V)) = \begin{cases} V, & \text{if } i = 1, \\ 0, & \text{if } i \neq 1 \end{cases} \Rightarrow H^i\left(C_{\geq 2}^{\bullet}(\mathrm{Lie}(V))\right) = \begin{cases} \mathrm{Lie}(V)_+ = \mathrm{Lie}(V)/V, & \text{if } i = 2, \\ 0, & \text{if } i > 2. \end{cases} \quad (6.1.16)$$

finishes the inductive prove of Lemma 6.1.15 □

Now we are ready to prove the key proposition of this section:

Proposition 6.1.17. *The morphism $\xi_n : \ker \pi \rightarrow \mathrm{Lie}(t_{1n}, \dots, t_{n-1,n})$ is monomorphic.*

Proof. We already see that $H^\bullet(\mathrm{LICG}d(n)) = \ker \pi_n$ and thanks to Lemma 6.1.15 we know that $H^\bullet(\mathrm{LICG}_d^{\sharp, \mathrm{odd}}(n)) = \mathrm{Lie}(t_{1n}, \dots, t_{n-1,n})$. Thus, it is enough to prove that the embedding of complexes $\mathrm{LICG}_d^{\downarrow, \mathrm{odd}}(n) \hookrightarrow \mathrm{LICG}_d^{\sharp, \mathrm{odd}}(n)$ remains to be an embedding on the level of cohomology. We state (without details) that the collection of consecutive filtrations and spectral sequences discusses in the core Lemma 6.1.11 can be adobted to these complexes.

Indeed, for each $S \subset [1n-1]$ one can define the subcomplexes $\mathrm{LICG}_{d,S}^{\sharp, \mathrm{odd}}(n) \subset \mathrm{LICG}_d^{\sharp, \mathrm{odd}}(n)$ following Definition 6.1.10. (The subcomplexes $\mathrm{LICG}_{d,S}^{\downarrow, \mathrm{odd}}(n) \subset \mathrm{ICG}_{d,S}^{\downarrow, \mathrm{odd}}$ are spanned by graphs connected to the last external vertex.) Formally, the corresponding complex for $S = [1n]$ has to be bigger since, in addition, it might contain graphs with a unique outgoing edge from the last external vertex \textcircled{n} but the same contradiction with absence of directed loops shows that $\mathrm{LICG}_{d,[1n]}^{\sharp, \mathrm{odd}}(n)$ is also empty and the induction base is satisfied. Following the lines of the proof of Lemma 6.1.11 one see the degeneration in the first term of the same spectral sequences for $\mathrm{LICG}_{d,S}^{\sharp, \mathrm{odd}}(n)$. Moreover, the same induction can be used in order to show the embedding of $H(\mathrm{LICG}_{d,S}^{\downarrow, \mathrm{odd}}(n))$ into $H(\mathrm{LICG}_{d,S}^{\sharp, \mathrm{odd}}(n))$. The first one is inductively

generated by *tripoids* $\textcircled{t} \leftarrow \textcircled{s} \rightarrow \textcircled{t}$ and the latter bigger complex has additional generators $\textcircled{s} \leftarrow \textcircled{n}$. However, all cohomologies are supported in one particular homological degree and the monomorphism property of ξ_n follows by induction. □

This finishes the inductive proof of the injectivity of horizontal arrows in Diagram (6.1.5) of Theorem 6.1.4.

6.2 Rational homotopy type of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ for individual n

Following [3] we say that the topological space X is a rational $K(\pi, 1)$ -space if its \mathbb{Q} -completion is a $K(\pi, 1)$ -space. Note that a $K(\pi, 1)$ -property does not imply the rational $K(\pi, 1)$ property. For example, the complement to the arrangement of the root system D_4 is a $K(\pi, 1)$ space but its \mathbb{Q} -completion is not.

Corollary 6.2.1. *The real locus of the moduli space $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is a rational $K(\pi, 1)$ -space.*

Proof. It was proved in [42] (Proposition 5.2) that if the rational cohomology of a connected topological space X is a finite-dimensional Koszul algebra generated by the first homology $H^1(X; \mathbb{Q})$ then X is a rational $K(\pi, 1)$ -space. So thanks to Corollary 6.1.6 we are done. □

However, in order to get a bit more feeling of the Lie algebra \mathfrak{L}_n which is the Lie algebra assigned to the pronipotent completion $\widehat{\mathcal{PCacti}}$ of the pure cacti group we want to give an independent proof of Corollary 6.2.1 adopted to our case.

Proof of Corollary 6.2.1. Recall that the dgca-model of the space of cochains on $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ for individual n discussed in Theorem 5.3.4 is given by the space of odd oriented graphs with n external vertices and the edge contracting differential twisted by γ^\downarrow . This model has a filtration by the loop order of a graph and the associated graded complex coincides with the nontwisted complex. Consider the corresponding filtration of the Harrison complex of the dgca of twisted graphs. The spectral sequence argument predicts that the cohomology of this Harrison complex differs from zero only in 0 cohomological degree and coincides with the quadratic Koszul Lie algebra $\mathfrak{t}_2^{\mathrm{dd}}(n)$. Hence there is no space for higher Lie brackets and the homotopy Lie algebra of the \mathbb{Q} -completion of $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ is an ordinary Lie algebra denoted by \mathfrak{L}_n with

all higher homotopy groups vanishing. Moreover, we know that there is a filtration on \mathfrak{L}_n such that the associated graded is isomorphic to $\mathfrak{t}_2^{\text{odd}}(n)$ that remembers the loop order of a graph. \square

Corollary 6.2.2. *The map $\tilde{\xi}_n^{\mathbb{Q}} : \mathfrak{L}_n^{\mathbb{Q}} \rightarrow \mathfrak{t}_2(n)^{\mathbb{Q}}$ is an embedding.*

Proof. Thanks to Theorem 6.1.4 we know that the associated graded map $\xi_n^{\mathbb{Q}} : \mathfrak{t}_1^{\text{odd}}(n)^{\mathbb{Q}} \rightarrow \mathfrak{t}_2(n)^{\mathbb{Q}}$ is an embedding. \square

It was checked by computer and announced in Section 3.9 of [12] that the spaces $\overline{\mathcal{M}}_{0,n+1}(\mathbb{R})$ are not formal for $n \geq 5$. The dgca models that we suggested lead to the following equivalent statement.

Corollary 6.2.3. *The dgca's $(\text{Graphs}_2^{\downarrow, \text{odd}}(n), d)$ and $(\text{Graphs}_2^{\downarrow, \text{odd}}(n), d + [\gamma^{\downarrow}, -])$ are not weakly equivalent. Equivalently, the Koszul-dual Lie algebra \mathfrak{L}_n is not isomorphic to its associated graded $\mathfrak{t}_1^{\text{odd}}(n)$.*

Unfortunately, we were not able to find a simple nontrivial Massey product. However, one can try to consider the filtration on the twisted complex of odd graphs given by the mod 4 internal loop order. The corresponding associated graded differential increases the loop order by at most 2 (since it should be an even number) is presented by the action of the leading terms (4.5.2) of the Shoikhet cocycle γ^{\downarrow} and what is important the associated graded differential does not break the connectivity of a graph, thus one can work directly with $\text{ICG}_2^{\downarrow, \text{odd}}(n)$ with a shifted differential. We do not present a nasty computation with graphs but we state that the cocycles of the twisted complex of odd graphs representing the generators $\nu_{ijk} \in \mathfrak{L}_n$ contain internally trivalent trees of loop order 4 what illustrates (but does not give a self contained proof of) the fact that the Lie algebra \mathfrak{L}_n is not isomorphic to its associated graded.

7 Around deformations of $\text{Pois}_d^{\text{odd}}$ and the mosaic operad

The deformation theory of the little n -disks operads E_n , and that of the natural maps $E_m \rightarrow E_n$ is well studied by now, at least rationally. Given our models for the odd Poisson and the mosaic operads, we may transcribe a list of results in the literature to extend to those operads. In this section we shall outline a few statements that are obtainable in this manner. We shall however defer a detailed discussion, and complete proofs to elsewhere, in order to avoid lengthy technical recollections.

We recall that the homotopy deformations and automorphisms of the operads (or Hopf operads) Pois_d are controlled by the graph complexes GC_d , see [57] or [13] (in the Hopf setting). Furthermore, the deformation theory of the natural operad maps $E_m \rightarrow E_n$ has similarly been studied. In short, the results are that the deformation theory of the operad maps $E_{n-1} \rightarrow E_n$ can be identified with that of the operad E_n , and that the deformations of $E_m \rightarrow E_n$ (for $n - m \geq 2$) are governed by hairy graph complexes [54, 13].

Our realization of the odd Poisson and mosaic operads as the cobar-duals to the \mathbb{Z}_2 -invariants of the Poisson and associative operads, and our graphical models $\text{Graphs}_{d+1}^{\downarrow, \text{odd}}$ allow us to extend most of the aforementioned results for the little disks operads to the odd Poisson and mosaic operads.

First, recall from section 5.3 that our graphical model $\text{Graphs}_{d+1}^{\downarrow, \text{odd}}$ of the odd Poisson operad $\text{Pois}_d^{\text{odd}}$ carries an action of the even-loop-order part of the oriented graph complex $\text{GC}_{d+1}^{\downarrow, \text{odd}}$. Hence we obtain a map of complexes

$$\mathbb{Q}L \times \text{GC}_{d+1}^{\downarrow, \text{odd}} \rightarrow \text{Def}(\Omega([\text{Pois}_d^{\vee}]^{\mathbb{Z}_2})\{1 - d\}) \rightarrow \text{Graphs}_{d+1}^{\downarrow, \text{odd}}[1] \quad (7.0.1)$$

into the the complex of operadic derivations of the natural quasi-isomorphism

$$\Omega([\text{Pois}_d^{\vee}]^{\mathbb{Z}_2})\{1 - d\} \rightarrow \text{hoPois}_d^{\text{odd}} \rightarrow \text{Graphs}_{d+1}^{\downarrow, \text{odd}}.$$

The element L on the left-hand side of (7.0.1) is the generator of the grading by loop order. The right-hand side of (7.0.1) can also be identified, up to a degree shift, with the homotopy derivations of the odd Poisson operad. Now the cohomology of that right-hand side can be computed along the lines of [58, Proposition 4]. Up to degree shifts, it is a symmetric product of its connected part $\text{Def}(\dots)_{\text{conn}}$, and one can show that the map (7.0.1) is a quasi-isomorphism onto that connected part. Noting the (loop order preserving) isomorphism (4.6.6), we can hence conclude that the deformation theory of the odd Poisson operads is controlled by the even-loop-order parts of the graph cohomology $H(\text{GC}_d)$.

Next, our model $(\mathbf{Graphs}_2^{\downarrow, \text{odd}})^{\gamma^\downarrow}$ of the mosaic operad of section 5.3 is obtained by twisting the above model $\mathbf{Graphs}_2^{\downarrow, \text{odd}}$ of the odd Poisson operad with the Shoikhet's Maurer-Cartan element $\gamma^\downarrow \in \mathbf{GC}_2^{\downarrow, \text{odd}}$. One can use this to show that the deformation theory of the mosaic operad is controlled by the twisted graph complex $(\mathbb{Q}L \rtimes \mathbf{GC}_2^{\downarrow, \text{odd}})^{\gamma^\downarrow}$. However, this complex is acyclic, as one can derive from the acyclicity result of [29]. In particular we can conclude that the deformation complex for the mosaic operad is acyclic and hence the mosaic operad is rigid.

Finally, one may study the space of maps from the mosaic operad to the chains operad of the little 2-disks operad. To this end, one can essentially re-use, with only small adaptations, the computation of the deformations of the operad maps $E_1 \rightarrow E_2$ from [54]. The end result is the following. Any operad map from the chains operad of the mosaic operad into the chains operad of the little 2-disks, which agrees up to arity three with the standard map, is obtained from the standard map by an action of the homotopy automorphisms of E_2 , i.e., by the Grothendieck-Teichmüller group.

In particular, note that any functorial construction of a coboundary category structure from a braided monoidal structure as in section 1.6 can be seen as an implicit description of a map from a model of the mosaic operad to a model of (chains of the) the little disks operad. Hence, such a construction is unique, up to the Grothendieck-Teichmüller group action, in the appropriate sense.

Now we know from the literature [19],[20],[21],[48] that there are many different definitions of a coboundary category on the set of crystals of a given quantum group $U_q(\mathfrak{g})$. We hope that one can use the deformation theory of the Mosaic operad in order to show that there exists a unique structure of coboundary category on the space of crystals of $U_q(\mathfrak{g})$.

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