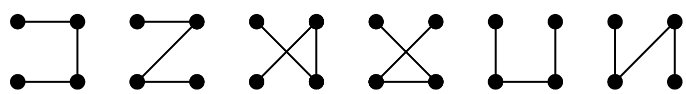
ORBITS OF HAMILTONIAN PATHS AND CYCLES IN COMPLETE GRAPHS

SAMUEL HERMAN AND EIRINI POIMENIDOU

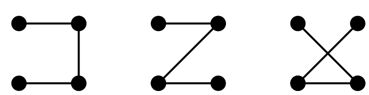
Abstract. We apply Burnside's Lemma to enumerate certain classes of undirected Hamiltonian paths and cycles in the complete graph defined on the vertices of a regular polygon.

1. Introduction

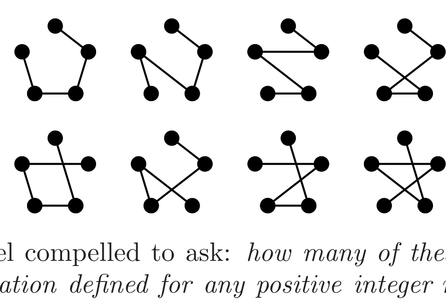
We begin with a motivating observation concerning the "shape" of undirected Hamiltonian paths in the complete graph defined on the vertices of a square. To see it for yourself, consider the Hamiltonian paths which have an endpoint at the top left vertex:



The observation in question is as follows: *each of these paths forms one of only three distinct shapes.* That is, all undirected Hamiltonian paths in the complete graph defined on these vertices form a shape which may be obtained from some composition of reflections and rotations of one (and only one!) of the three shapes illustrated below.



A natural follow-up to this observation is to consider the analogous situation in the complete graph defined on the vertices of a pentagon. If one were to carry out the (painful) process of drawing all possible Hamiltonian paths in this graph and classifying them by hand, one would find that there are eight of these "basic" shapes:

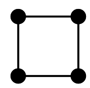
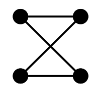


Here we should feel compelled to ask: *how many of these "basic" shapes exist in the analogous situation defined for any positive integer n?* The present paper exists in part to answer this question. Also answered in this paper is the analogous question regarding Hamiltonian *cycles* in complete graphs defined on the vertices of

Set	Paths or Cycles?	Similarity or Equivalence?
$\Gamma(n, P, S)$	Paths	Similarity
$\Gamma(n, P, E)$	Paths	Equivalence
$\Gamma(n, C, S)$	Cycles	Similarity
$\Gamma(n, C, E)$	Cycles	Equivalence

Table 1. Classification of Γ-paths and Γ-cycles.

regular n-gons. Said analogous question is indeed valid—for instance, the "basic" shapes formed by Hamiltonian cycles in the case $n = 4$ are

 and .

For convenience, we will state our results up front. However, before we may do so, we define some standard notation and a general framework for the problems addressed here.

Definition 1. Let $n \geq 3$ be an integer, and consider the complete graph K_n defined on the vertices of a regular n-gon. A Γ-*path* or Γ-*cycle* is an undirected Hamiltonian path or cycle in K_n, respectively. We say that two Γ-paths (cycles) which are obtainable from one another by some combination of reflections and rotations are *similar*, and that two Γ-paths (cycles) which are obtainable from one another by rotations only are *equivalent*. Otherwise, we say that they are *non-similar* or *inequivalent*, respectively.

We may partition the set of Γ-paths and Γ-cycles on n vertices into equivalence classes according to these considerations as follows. Denote a set of classes of Γ-paths or Γ-cycles on n vertices by $\Gamma(n, \alpha_1, \alpha_2)$, where

(1) $\alpha_1 = P$ if we are considering hamiltonian paths, and $\alpha_1 = C$ if we are considering hamiltonian cycles;
(2) $\alpha_2 = S$ if we are considering similarity, and $\alpha_2 = E$ if we are considering equivalence.

This language allows us to restate the questions we posed previously:

Question 1. *How many non-similar or inequivalent* Γ*-paths are there on* n *vertices? That is, what are the sizes of the sets* $\Gamma(n, P, S)$ *and* $\Gamma(n, P, E)$*, respectively?*

Question 2. *How many non-similar or inequivalent* Γ*-cycles are there on* n *vertices? That is, what are the sizes of the sets* $\Gamma(n, C, S)$ *and* $\Gamma(n, C, E)$*, respectively?*

Notation. Throughout the paper we use $n!!$ to denote the product of n with every number of the same parity as n which is less than or equal to n. That is,

$$n!! = \begin{cases} n(n-2)\cdots(2)(1) & \text{if } n \text{ is even,} \\ n(n-2)\cdots(3)(1) & \text{if } n \text{ is odd.} \end{cases}$$

The answers to these questions are as follows.

n	$\Gamma(n,P,S)$	$\Gamma(n,P,E)$	$\Gamma(n,C,S)$	$\Gamma(n,C,E)$
3	1	1	1	1
4	3	4	2	2
5	8	12	4	4
6	38	64	12	14
7	192	360	39	54
8	1320	2544	202	332
9	10176	20160	1219	2246
10	91296	181632	9468	18264

TABLE 2. Table of values for $3 \leq n \leq 10$.

$$|\Gamma(n,P,S)| = \begin{cases} \frac{1}{4}[(n-1)! + (\frac{n}{2}+1)(n-2)!!] & \text{if } n \text{ is even,} \\ \frac{1}{4}[(n-1)! + (n-1)!!] & \text{if } n \text{ is odd.} \end{cases}$$

$$|\Gamma(n,P,E)| = \begin{cases} \frac{1}{2}[(n-1)! + (n-2)!!] & \text{if } n \text{ is even,} \\ \frac{1}{2}(n-1)! & \text{if } n \text{ is odd.} \end{cases}$$

$$|\Gamma(n,C,S)| = \frac{1}{4n^2}\left[\sum_{d|n}\left(\left(\phi\left(\frac{n}{d}\right)\right)^2\left(\frac{n}{d}\right)^d d!\right) + \begin{cases} n!!\frac{n(n+6)}{4} & \text{if } n \text{ is even,} \\ n^2(n-1)!! & \text{if } n \text{ is odd.} \end{cases}\right]$$

$$|\Gamma(n,C,E)| = \frac{1}{2n^2}\left[\sum_{d|n}\left(\left(\phi\left(\frac{n}{d}\right)\right)^2\left(\frac{n}{d}\right)^d d!\right) + \begin{cases} \frac{n}{2}n!! & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}\right]$$

These formulae are proved in Theorems 7, 8, 16, and 17 respectively. Further, illustrations of $\Gamma(n,P,S)$ and $\Gamma(n,C,S)$ for $3 \leq n \leq 6$ may be found in Figures 3 and 4, respectively. Finally, Table 2 gives the size of each of these sets for $3 \leq n \leq 10$.

We obtain these answers by converting the original problem into one of enumerating the orbits of a specific group action. These orbits are enumerated by way of Burnside's lemma, which is a standard tool in the theory of finite group actions.

Burnside's lemma. *Consider a group G acting on a set A. For each $g \in G$, let $\mathsf{fix}(g)$ denote the set of elements of A which are fixed by g. That is,*

$$\mathsf{fix}(g) = \{a \in A \mid g \cdot a = a\}.$$

Let A/G denote the set of orbits of this action. Then the number of orbits in the action of G on A is given by

$$|A/G| = \frac{1}{|G|}\sum_{g\in G}|\mathsf{fix}(g)|.$$

2. ENUMERATING Γ-PATHS

We will begin by enumerating non-similar Γ-paths on n vertices, as this case will form the basis from which we approach the others. To accomplish this, we first associate each Γ-path with a string which encodes it. We then construct a group which will act on this set of strings. The number of orbits in this action will be the number of non-similar Γ-paths.

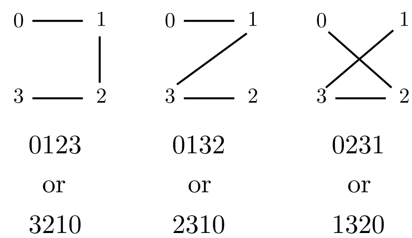


FIGURE 1. String representations of Γ-paths.

We associate each Γ-path with a string as follows. Let n be given, and label the vertices of the underlying graph with $\{0, 1, \ldots, n-1\}$. Then each Γ-path is associated with a string $x_1 x_2 \cdots x_n$ such that each entry x_i is the label of the ith vertex visited (as illustrated in Figure 1). Let X_n denote this set of strings, and note that X_n has $n!$ elements.

Next we construct a group to act on X_n such that any two similar Γ-paths are in the same orbit. First, all strings which represent symmetry-preserving rotations and reflections of a Γ-path should be contained in the same orbit. Further, since Γ-paths are undirected, two strings which are reversals of one another should also be contained in the same orbit. As such, it is natural that the group we seek will be isomorphic to the direct product $D_{2n} \times \mathbb{Z}_2$ of the dihedral group of order $2n$ with the cyclic group of order 2. Let $\mathcal{G}_p(n)$ denote this group, and consider its presentation as

$$\mathcal{G}_p(n) = \langle r, s, v \mid r^n = s^2 = v^2 = e,\ vs = sv,\ vr = rv,\ srs = r^{-1} \rangle$$

where e denotes the identity. Note also that $\mathcal{G}_p(n)$ has order $4n$. Here r represents a (clockwise) rotation, s a reflection over the axis through the vertex labelled by 0, and v a reversal of a string. The action of $\mathcal{G}_p(n)$ on X_n is given by

$$\begin{aligned} r \cdot\ x_1 x_2 \cdots x_n &= (x_1+1)(x_2+1)\cdots(x_n+1), \\ s \cdot\ x_1 x_2 \cdots x_n &= (-x_1)(-x_2)\cdots(-x_n), \\ v \cdot\ x_1 x_2 \cdots x_n &= x_n x_{n-1} \cdots x_1, \end{aligned} \tag{1}$$

where all arithmetical operations are considered modulo n. Figure 2 illustrates how the actions of elements of $\mathcal{G}_p(n)$ on strings are associated to geometric actions on Γ-paths.

With this in hand, we may proceed according to Burnside's lemma. The number of non-similar Γ-paths will be the number of orbits in the action of $\mathcal{G}_p(n)$ on X_n. The set of orbits in this action is denoted by $X_n/\mathcal{G}_p(n)$, and the size of this set is given by

$$|X_n/\mathcal{G}_p(n)| = \frac{1}{4n} \sum_{g \in \mathcal{G}_p(n)} |\mathsf{fix}(g)|.$$

We will now determine the number of strings in X_n which are fixed by each $g \in \mathcal{G}_p(n)$. Since v commutes with all elements of $\mathcal{G}_p(n)$, all elements of $\mathcal{G}_p(n)$ may be written as one of the forms r^k, sr^k, $r^k v$, and $sr^k v$, where k is an integer and $0 \leq k < n$. Considering this fact in the context of Burnside's lemma results in the following Proposition.[1]

[1]Note that the actions of these elements are composed *right to left*.

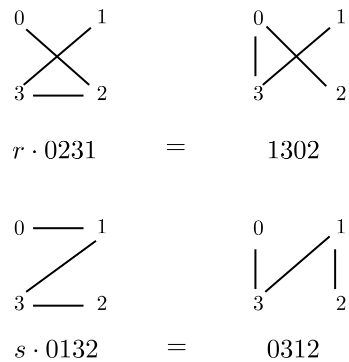


FIGURE 2. Illustration of the action of r, s on strings in X_n.

Proposition 2. *The number of orbits in the action of $\mathcal{G}_p(n)$ on X_n is given by*

$$|X_n/\mathcal{G}_p(n)| = \frac{1}{4n} \sum_{0 \le k < n} \left(|\mathsf{fix}(r^k)| + |\mathsf{fix}(sr^k)| + |\mathsf{fix}(r^k v)| + |\mathsf{fix}(sr^k v)| \right).$$

Lemma 3. *The number of strings in X_n fixed by all elements of $\mathcal{G}_p(n)$ of the form r^k for $0 \le k < n$ are given by*

$$\sum_{0 \le k < n} |\mathsf{fix}(r^k)| = |\mathsf{fix}(e)| = n!.$$

Proof. Clearly r^k will fix $x_1 x_2 \cdots x_n$ only when $k = 0$. That is, only the identity will fix $n!$ strings. □

Lemma 4. *The number of strings in X_n fixed by all elements of $\mathcal{G}_p(n)$ of the form sr^k for $0 \le k < n$ is given by*

$$\sum_{0 \le k < n} |\mathsf{fix}(sr^k)| = 0.$$

Proof. Notice that sr^k will fix $x_1 x_2 \cdots x_n$ if and only if $x_i = -x_i - k$ for all $1 \le i \le n$, and so $2x_i + k = 0$. But since there is some x_j such that $x_j = 0$, it follows that $2x_j + k = k = 0$ and so necessarily $sr^k = s$, which will clearly fix no strings. □

Lemma 5. *The number of strings in X_n fixed by all elements of $\mathcal{G}_p(n)$ of the form $r^k v$ for $0 \le k < n$ is given by*

$$\sum_{0 \le k < n} |\mathsf{fix}(r^k v)| = \begin{cases} n!! & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Notice that $r^k v$ will fix $x_1 x_2 \cdots x_n$ if and only if $x_i = x_{1-i} + k$ and $x_{1-i} = x_i + k$ for all $1 \le i \le n$. This implies that $x_i = x_i + 2k$ and thus that $2k = 0 = n$. Therefore n must be even and k must equal $n/2$. Then, observe that $x_i = x_{1-i} + n/2$ implies that $x_i - x_{1-i} = n/2$. Thus there are n choices for x_1, each of which determines x_n and leaves $n - 2$ choices for x_2, each of which determines x_{n-1}, and so on. □

Lemma 6. *The number of strings in X_n fixed by all elements of $\mathcal{G}_p(n)$ of the form $sr^k v$ for $0 \leq k < n$ is given by*

$$\sum_{0\leq k<n} |\mathsf{fix}(sr^k v)| = \begin{cases} (\frac{n}{2})n!! & \text{if } n \text{ is even,} \\ n(n-1)!! & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Notice that $sr^k v$ will fix $x_1 x_2 \cdots x_n$ if and only if $x_i = -(x_{1-i} + k)$ for all $1 \leq i \leq n$. There are the following cases.

If both n and k are even there must be some entry x_j such that $x_j = -k/2$, and so

$$\begin{aligned} x_j = -k/2 = -x_{1-j} - k &\Rightarrow k/2 = -x_{1-j} \\ &\Rightarrow -x_j = -x_{1-j} \\ &\Rightarrow x_j = x_{1-j} \\ &\Rightarrow n \text{ is odd.} \end{aligned}$$

Since we supposed that n is even, this is a contradiction and so k must be odd. Thus there are $n/2$ possible values of k, and for each of these there are n choices for x_1, each of which will determine x_n and leave $n-2$ choices for x_2, each of which will determine x_{n-1}, and so on.

If n is odd, then the action of v does not move the central entry of the string. It follows that if $sr^k v$ fixes $x_1 x_2 \cdots x_n$, then each of n choices of k will determine the central entry. Thus there are $n-1$ choices for x_1, each of which will determine x_n and leave $n-3$ choices for x_2, each of which will determine x_{n-1}, and so on. □

Theorem 7. *Let $n \geq 3$ be a integer. Then the number of non-similar Γ-paths on n vertices, denoted by $|\Gamma(n, P, S)|$, is equal to*

$$|\Gamma(n, P, S)| = \begin{cases} \frac{1}{4}[(n-1)! + (\frac{n}{2}+1)(n-2)!!] & \text{if } n \text{ is even,} \\ \frac{1}{4}[(n-1)! + (n-1)!!] & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The theorem follows from the application of Lemmas 3, 4, 5, and 6 to Proposition 2. □

Theorem 8. *The number of inequivalent Γ-paths on n vertices, denoted by $|\Gamma(n, P, E)|$, is equal to*

$$|\Gamma(n, P, E)| = \begin{cases} \frac{1}{2}[(n-1)! + (n-2)!!] & \text{if } n \text{ is even,} \\ \frac{1}{2}(n-1)! & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Consider a subgroup of $\mathcal{G}_p(n)$ isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_2$ acting on the set of string-representations X_n. Denote this acting subgroup by $\mathcal{G}'_p(n)$, and consider its presentation as

$$\mathcal{G}'_p(n) = \langle r, v \mid r^n = v^2 = e, rv = vr \rangle$$

where e denotes the identity and the action of r, v are defined as in (1). Every element of $\mathcal{G}'_p(n)$ can be written as either r^k or $r^k v$ for some $0 \leq k < n$. The theorem follows immediately from this fact and Lemmas 3 and 5. □

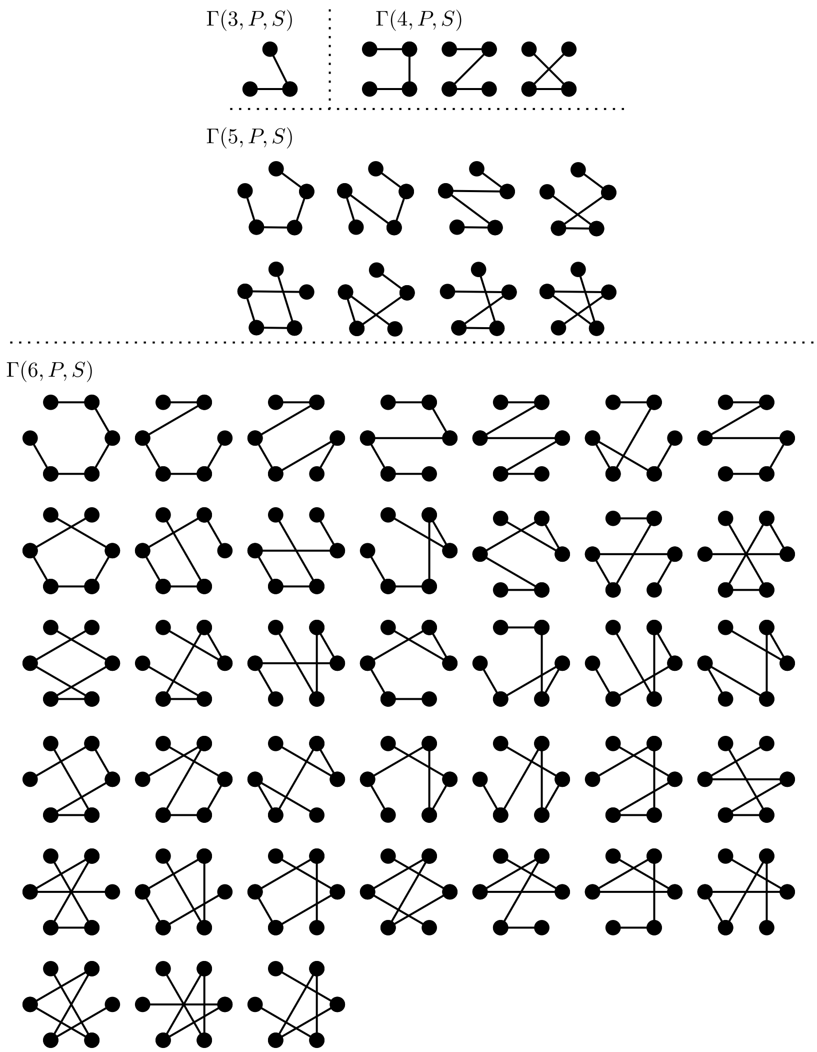


FIGURE 3. Illustration of $\Gamma(n, P, S)$ for $n = 3, 4, 5, 6$.

3. ENUMERATING Γ-CYCLES

We now turn to the more difficult problem of enumerating Γ-cycles. The case of non-similar Γ-cycles is illustrated in Figure 4. We will begin by enumerating non-similar Γ-cycles, and the enumeration of inequivalent Γ-cycles will follow as a corollary.

As before, we will proceed by applying Burnside's lemma to the action of a particular group on a set of strings which encode Γ-cycles. We represent cycles using the previously defined set X_n, but with the caveat that there is a slight change in their interpretation. Namely, we consider there to be an additional edge traversed between the vertices labelled by the first and last entries of the string. This means that we must also consider any cyclic permutation of a string to represent

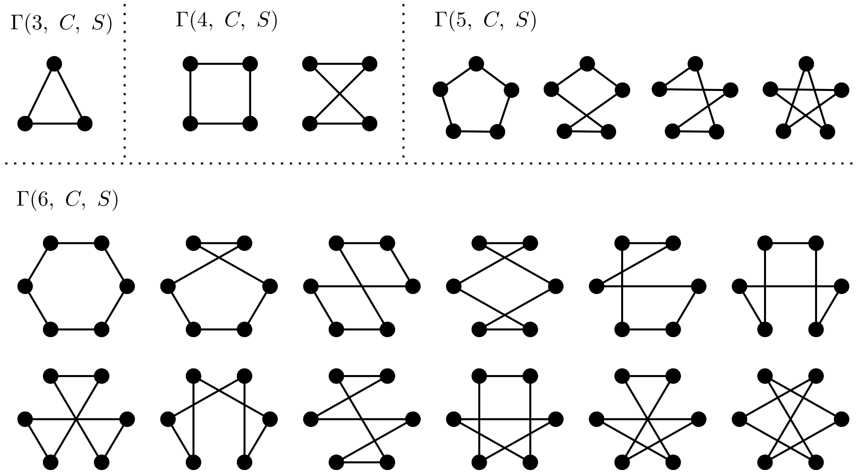
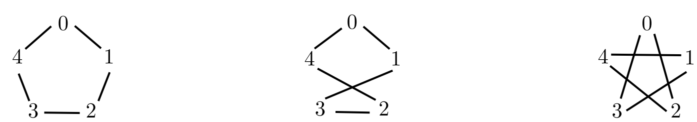


FIGURE 4. Illustration of $\Gamma(n, C, S)$ for $n = 3, 4, 5, 6$.



01234 or 43210	01324 or 42310	02413 or 31420
or	or	or
12340 or 04321	13240 or 04231	24130 or 03142
or	or	or
23401 or 10432	32401 or 10423	41302 or 20314
$\vdots$	$\vdots$	$\vdots$

FIGURE 5. String representations of Γ-cycles.

the same cycle, as there are no longer any distinguished start or end points. This interpretation is illustrated by Figure 5.

We will now construct a group to act on X_n with respect to the above considerations. Let $\mathcal{G}_c(n)$ denote this group, and note that $\mathcal{G}_c(n)$ is isomorphic to $D_{2n} \times D_{2n}$. Consider the presentation of $\mathcal{G}_c(n)$ as

$$\mathcal{G}_c(n) = \langle r, s, v, c \mid r^n = s^2 = c^n = v^2 = e,\ vs = sv,\ vr = rv,\\ cr = rc,\ srs = r^{-1},\ vcv = c^{-1} \rangle$$

where e denotes the identity. Note that $\mathcal{G}_c(n)$ has order $4n^2$. The elements r, s, v act on strings $x_1 x_2 \cdots x_n \in X_n$ as defined in (1), and c acts by a cyclic permutation of the string:

$$c \cdot\ x_1 x_2 \cdots x_n \ = \ x_2 x_3 \cdots x_n x_1. \tag{2}$$

As before, the number of non-similar Γ-paths will be the number of orbits in the action of $\mathcal{G}_c(n)$ on X_n. The set of these orbits is denoted by $X_n/\mathcal{G}_c(n)$, and the size of this set is given by

$$|X_n/\mathcal{G}_c(n)| = \frac{1}{4n^2}\sum_{g\in\mathcal{G}_c(n)} |\mathsf{fix}(g)|.$$

We will now consider the number of elements of X_n fixed by each $g \in \mathcal{G}_c(n)$. Since all elements of $\mathcal{G}_c(n)$ can be written as one of $c^m r^k$, $c^m r^k v$, $c^m s r^k$, or $c^m s r^k v$ for some $0 \le k, m < n$, we have the following proposition.

Proposition 9. *The number of orbits in the action of $\mathcal{G}_c(n)$ on X_n is given by*

$$|X_n/\mathcal{G}_c(n)| = \sum_{0\le k,m<n}\left(|\mathsf{fix}(c^m r^k)| + |\mathsf{fix}(c^m r^k v)| + |\mathsf{fix}(c^m s r^k)| + |\mathsf{fix}(c^m s r^k v)|\right).$$

Notation. It is convenient from this point onward to let $\bar{x}$ denote the string $x_1 x_2 \cdots x_n \in X_n$.

First, we enumerate strings fixed by elements of the form $c^m r^k$ for all $0 \le k, m < n$. To do this, we will adapt an argument from [Mos90].

For each $0 \le k, m < n$, notice that r^k is of order $\frac{n}{\gcd(k,n)}$ and c^m is of order $\frac{n}{\gcd(m,n)}$. Since $c^{-m} r^k \cdot \bar{x} = \bar{x}$ if and only if $r^k \cdot \bar{x} = c^m \cdot \bar{x}$, it follows that

$$\begin{aligned}\sum_{0\le k,m<n} |\mathsf{fix}(c^m r^k)| &= \sum_{0\le k,m<n} \left|\left\{\bar{x} \in X_n \mid c^{-m} r^k \cdot \bar{x} = \bar{x}\right\}\right|,\\ &= \sum_{0\le k,m<n} \left|\left\{\bar{x} \in X_n \mid r^k \cdot \bar{x} = c^m \cdot \bar{x}\right\}\right|.\end{aligned}$$

Let $\mathcal{F}(k,m) = \left\{\bar{x} \in X_n \mid r^k \cdot \bar{x} = c^m \cdot \bar{x}\right\}$. Thus we have

$$\sum_{0\le k,m<n} |\mathsf{fix}(c^m r^k)| = \sum_{0\le k,m<n} |\mathsf{fix}(c^{-m} r^k)| = \sum_{0\le k,m<n} |\mathcal{F}(k,m)|. \tag{3}$$

The sizes of $\mathcal{F}(k,m)$ for each choice of k, m are determined in the following Lemmas.

Lemma 10. *If $r^k \cdot \bar{x} = c^m \cdot \bar{x}$, then $\gcd(k,n) = \gcd(m,n)$, and therefore*

$$\sum_{0\le k,m<n} |\mathsf{fix}(c^m r^k)| = \sum_{d\mid n} \sum_{\substack{\gcd(k,n)=d\\ \gcd(m,n)=d}} |\mathcal{F}(k,m)| \tag{4}$$

Proof. To show the first assertion, notice that the action $r^{kn/\gcd(k,n)} \cdot \bar{x}$ keeps each x_i fixed, while the action $c^{mn/\gcd(k,n)} \cdot \bar{x}$ takes each x_i to $x_{i+mn/\gcd(k,n)}$. It follows that

$$\begin{aligned} r^{kn/\gcd(k,n)} \cdot \bar{x} = c^{mn/\gcd(k,n)} \cdot \bar{x} &\Rightarrow x_i = x_{i+mn/\gcd(k,n)} \quad \text{for all } x_i,\\ &\Rightarrow n \mid \frac{mn}{\gcd(k,n)},\\ &\Rightarrow \gcd(k,n) \mid m.\end{aligned}$$

Since $\gcd(k,n)$ divides n, it follows that $\gcd(k,n)$ divides $\gcd(m,n)$.

Similarly, the action $r^{kn/\gcd(m,n)} \cdot \bar{x}$ takes each x_i to $(x_i + \frac{kn}{\gcd(m,n)})$ while the action $c^{mn/\gcd(m,n)} \cdot \bar{x}$ keeps each x_i fixed. It follows that

$$\begin{aligned} r^{kn/\gcd(m,n)} \cdot \bar{x} = c^{mn/\gcd(m,n)} \cdot \bar{x} &\Rightarrow x_i = (x_i + \frac{kn}{\gcd(m,n)}) \quad \text{for all } x_i, \\ &\Rightarrow n \mid \frac{kn}{\gcd(m,n)}, \\ &\Rightarrow \gcd(m,n) \mid k. \end{aligned}$$

Since $\gcd(m,n)$ divides n, it follows that $\gcd(m,n)$ divides $\gcd(k,n)$, and so $\gcd(k,n) = \gcd(m,n)$. This proves the first assertion. The second assertion follows immediately from the application of the first assertion to (3). □

Lemma 11. *Let k, m, and d be given such that $d \mid n$ and $\gcd(k,n) = \gcd(m,n) = d$. Then $|\mathcal{F}(k,m)| = (n/d)^d\, d!$.*

Proof. Suppose $r^k \cdot \bar{x} = c^m \cdot \bar{x}$. It follows (in particular) that $r^{tk} \cdot \bar{x} = c^{tm} \cdot \bar{x}$ for all $1 \leq t \leq n/d$, and so

$$x_{i+tm} = (x_i + tk) \quad \text{for all} \quad 1 \leq t \leq \frac{n}{d} \quad \text{and} \quad 1 \leq i \leq d.$$

It follows that the entries of $\bar{x}$ are entirely dependent on $x_1 x_2 \cdots x_d$. Further, since r^k has order n/d, it follows that $x_1, x_2, \ldots, x_d$ must be pairwise incongruent modulo k. Therefore $x_1 x_2 \cdots x_d$ must be a permutation of the string $y_1 y_2 \cdots y_d$ with entries

$$\begin{aligned} y_1 &\in \left\{1, 1+k, 1+2k, \ldots, 1 + \left(\frac{n}{d} - 1\right) k\right\}, \\ y_2 &\in \left\{2, 2+k, 2+2k, \ldots, 2 + \left(\frac{n}{d} - 1\right) k\right\}, \\ &\vdots \\ y_d &\in \left\{d, d+k, d+2k, \ldots, d + \left(\frac{n}{d} - 1\right) k\right\}. \end{aligned}$$

Thus since there are n/d choices for each of d entries of $y_1 y_2 \cdots y_d$, and each permutation leads to $d!$ possible strings $x_1 x_2 \cdots x_d$, it follows that for each fixed k, m, d we have $|\mathcal{F}(k,m)| = (n/d)^d\, d!$. □

Lemma 12. *The number of strings in X_n fixed by all elements of $\mathcal{G}_c(n)$ of the form $c^m r^k$ for all $0 \leq k, m < n$ is given by*

$$\sum_{0 \leq k,m < n} |\mathsf{fix}(c^m r^k)| = \sum_{d \mid n} \left(\phi\left(\frac{n}{d}\right) \cdot \phi\left(\frac{n}{d}\right) \cdot \left(\frac{n}{d}\right)^d \cdot d! \right),$$

where ϕ denotes the Euler totient function.

Proof. A straightforward application of Lemma 11 to (4) shows that

$$\begin{aligned}\sum_{0\le k,m<n} |\mathsf{fix}(c^m r^k)| &= \sum_{d|n} \sum_{\substack{\gcd(k,n)=d\\ \gcd(m,n)=d}} |\mathcal{F}(k,m)|,\\ &= \sum_{d|n} \left(\frac{n}{d}\right)^d d! \sum_{\gcd(k,n)=d} \sum_{\gcd(m,n)=d} 1,\\ &= \sum_{d|n} \left(\phi\left(\frac{n}{d}\right)\cdot \phi\left(\frac{n}{d}\right)\cdot \left(\frac{n}{d}\right)^d \cdot d!\right). \qquad \square\end{aligned}$$

Lemma 13. *The number of strings X_n fixed by all elements of $\mathcal{G}_c(n)$ of the form $c^m sr^k$ for all $0 \le k, m < n$ is given by*

$$\sum_{0\le k,m<n} |\mathsf{fix}(c^m sr^k)| = \begin{cases} \frac{n}{2} n!! & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Notice that $c^m sr^k \cdot \bar{x} = \bar{x}$ if and only if $sr^k \cdot \bar{x} = c^{-m} \cdot \bar{x}$. This is the case if and only if $-x_i - k = x_{i-m}$ for all x_i; or, equivalently, $-x_{i-m} - k = x_{i-2m}$ for all x_i. It follows that $x_{i-2m} = x_i$ for all x_i, and so either $m = n/2$ or $m = 0$. But if $m = 0$ then by Lemma 4 no strings are fixed. Thus $m = n/2$ and so n must be even.

Then since $-x_i - k = x_{i-n/2}$ and $-x_{i-n/2} - k = x_i$ for all x_i, it follows that $x_i + x_{i-n/2} = -k$ for all x_i and therefore k must be odd. Thus for each of $n/2$ odd choices of k, $c^m sr^k$ will fix $n!!$ strings. □

Lemma 14. *The number of strings X_n fixed by all elements of $\mathcal{G}_c(n)$ of the form $c^m r^k v$ for all $0 \le k, m < n$ is given by*

$$\sum_{0\le k,m<n} |\mathsf{fix}(c^m r^k v)| = \begin{cases} \frac{n}{2} n!! & \text{if } n \text{ is even,}\\ 0 & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Notice that $c^m r^k v \cdot \bar{x} = \bar{x}$ if and only if $r^k v \cdot \bar{x} = c^{-m} \cdot \bar{x}$. This is the case if and only if $x_{1-i} + k = x_{i-m}$ for all x_i.

If n is odd, then for all values of m there will be exactly one entry x_a such that $x_{a-m} = x_{1-a}$. It follows that $x_{a-m} + k = x_{a-m}$ and so $k = 0$. But since all other entries are moved, no odd-length strings will satisfy this condition.

If n is even, there are two cases. If m is odd, then there exist precisely 2 entries x_a and x_b such that $x_{a-m} = x_{1-a}$ and $x_{b-m} = x_{1-b}$. It follows that $x_{a-m} + k = x_{a-m}$ and $x_{b-m} + k = x_{b-m}$, and thus that $k = 0$. But for the same reason as above, no even-length strings will satisfy this condition.

If m is even then each of $n/2$ even choices of m fully determines the value of k. Thus for each of $n/2$ valid choices of m, $c^m r^k v$ will fix $n!!$ strings. □

Lemma 15. *The number of strings X_n fixed by all elements of $\mathcal{G}_c(n)$ of the form $c^m sr^k v$ for $0 \le k < n$ and $0 \le m < n$ is given by*

$$\sum_{\substack{0\le k<n\\ 0\le m<n}} |\mathsf{fix}(c^m sr^k v)| = \begin{cases} (\frac{n}{2}+1)\frac{n}{2} n!! & \text{if } n \text{ is even,}\\ n^2 (n-1)!! & \text{if } n \text{ is odd.}\end{cases}$$

Proof. Notice that $c^m sr^k v \cdot \bar{x} = \bar{x}$ if and only if $sr^k v \cdot \bar{x} = c^{-m} \cdot \bar{x}$. This is the case if and only if $-x_{1-i} - k = x_{i-m}$ for all x_i.

If n is odd, then the argument for Lemma 6 applies for all n choices of m. Thus there are $n^2(n-1)!!$ fixed odd-length strings in total.

If n is even, there are two cases. If m is even, the argument for Lemma 6 applies for all $n/2$ even choices of m. Thus each choice of m leaves $n/2$ possible odd choices of k, for each of which $c^m sr^k v$ will fix $n!!$ elements.

If m is odd, there will be exactly two entries x_a, x_b such that $2a \equiv_n 2b \equiv_n m+1$. It follows that $2x_{a-m} \equiv_n 2x_{b-m} \equiv_n -k$, and so k must be even. Therefore for each of $n/2$ odd choices of m, there are $n/2$ even choices of k, each of which allows 2 choices for these particular entries x_a and x_b, both of which leave $(n-2)!!$ choices in total for the remaining entries of the string. Thus there are $(\frac{n}{2}+1)\frac{n}{2}n!!$ fixed even-length strings in total. ☐

Theorem 16. *The number of non-similar* Γ*-cycles on* n *vertices, denoted by* $\Gamma(n, C, S)$*, is equal to*

$$|\Gamma(n,C,S)| = \frac{1}{4n^2}\left[\sum_{d|n}\left(\left(\phi\left(\frac{n}{d}\right)\right)^2\left(\frac{n}{d}\right)^d d!\right) + \begin{cases} n!!\frac{n(n+6)}{4} & \text{if } n \text{ is even,} \\ n^2(n-1)!! & \text{if } n \text{ is odd.}\end{cases}\right]$$

Proof. The theorem follows from the application of Proposition 9 to Lemmas 12, 13, 14, and 15. ☐

Theorem 17. *The number of inequivalent* Γ*-cycles on* n *vertices, denoted by* $\Gamma(n, C, E)$*, is equal to*

$$|\Gamma(n,C,E)| = \frac{1}{2n^2}\left[\sum_{d|n}\left(\left(\phi\left(\frac{n}{d}\right)\right)^2\left(\frac{n}{d}\right)^d d!\right) + \begin{cases} \frac{n}{2}n!! & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.}\end{cases}\right]$$

Proof. Consider the action of a subgroup of $\mathcal{G}_c(n)$ isomorphic to $\mathbb{Z}_n \times D_{2n}$ on X_n. Let $\mathcal{G}'_c(n)$ denote this subgroup, and consider its presentation as

$$\mathcal{G}'_c(n) = \langle r, v, c \mid r^n = v^2 = c^n = e,\ rv = vr,\ rc = cr,\ vcv = c^{-1}\rangle,$$

where e denotes the identity, and the action of r, v, c on X_n are defined as in (1) and (2). Note that $\mathcal{G}'_c(n)$ is of order $2n^2$. The theorem follows immediately from the application of Burnside's lemma to Lemmas 12 and 14. ☐

Finally, it is worth noting a small corollary to the above theorems. Since $\phi(p) = (p-1)$ for any prime p, we have the following.

Corollary 18. *Let* p *be an odd prime. Then*

$$|\Gamma(p,C,S)| = \frac{1}{4p}\left[(p-1)^2 + p(p-1)!! + (p-1)!\right],$$

and

$$|\Gamma(p,C,E)| = \frac{1}{2p}[(p-1)^2 + (p-1)!].$$

4. Further Remarks

Here we note some interesting connections which the authors noticed over the course of writing this paper. After completing the enumeration of non-similar Γ-paths, we discovered that there are exactly as many of them as there are *tone rows* in n-tone music—the enumeration of which may be found in [Rei85]. The corresponding OEIS sequence is sequence A099030—which, as has been noted, is identical to sequence A089066.

Further, for reasons which should be clear, there are exactly as many non-similar Γ-cycles as there are classes of similar n-gons (that is, classes of n-gons which are equivalent up to rotations and reflections). These classes—as well as the analogous case of n-gons equivalent up to rotations only—were enumerated by Golomb and Welch in [GW60]. As such, this paper provides an alternative proof of their result. The corresponding OEIS sequences are A000940 and A000939, respectively. It should also be noted that an adaptation of an argument found in [Mos90] is vital to the proof of Lemma 12. In particular, the sum of Euler ϕ terms which makes an appearance in this paper as well as [GW60] is the same as that which appears in the case of $a = 1$ in [Mos90]. This connection is (as far as the authors are aware) not yet noted anywhere.

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Department of Mathematics, New College of Florida, Sarasota, FL 34243
E-mail address: samuel.herman18@ncf.edu

Division of Natural Sciences, New College of Florida, Sarasota, FL 34243
E-mail address: poimenidou@ncf.edu