PERMUTATIONS OF TYPE B WITH FIXED NUMBER OF DESCENTS AND MINUS SIGNS

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ABSTRACT. We study three dimensional array of numbers $B(n, k, j)$, $0 \le j, k \le n$, where $B(n, k, j)$ is the number of type B permutations of order n with k descents and j minus signs. We prove in particular, that $b(n, k, j) := B(n, k, j)/\binom{n}{j}$ is an integer and provide two combinatorial interpretations for these numbers.

INTRODUCTION

Let $B(n, k, j)$ denote the number of type B permutations $(0, \sigma_1, \ldots, \sigma_n)$ which have k descents and j minus signs. We study properties of the three-dimensional array $B(n, k, j)$, $0 \le j, k \le n$. Some of these properties appear in the work of Brenti [4]. In particular he computed the three-variable generating function and proved real rootedness of some linear combinations of the polynomials $P_{n,j}(x) := \sum_{k=0}^{n} B(n, k, j) x^k$ (Corollary 3.7 in [4], see also Corollary 6.9 in [2]). Here we will prove that the numbers $b(n, k, j) := B(n, k, j)/\binom{n}{j}$ are also integers. We provide two combinatorial interpretations of them.

For a subset $U \subseteq \{1, \ldots, n\}$ and $0 \le k \le n$ let $\mathcal{B}_{n,k,U}$ denote the family of all type B permutations $\sigma = (0, \sigma_1, \ldots, \sigma_n)$ that σ has k descents and satisfy: $\sigma_i < 0$ iff $|\sigma_i| \in U$. We will show (Theorem 9) that the cardinality of $\mathcal{B}_{n,k,U}$ is $b(n, k, |U|)$.

Conger [5, 6] defined the refined Eulerian number $\left\langle {n \atop k} \right\rangle_j$ as the cardinality of the set $\mathcal{A}_{n,k,j}$ of all type A permutations $\tau = (\tau_1, \ldots, \tau_n)$ such that $\tau_1 = j$ and τ has k descents. He proved many interesting properties of these numbers, like direct formula, asymptotic behavior, lexicographic unimodality, formula for the generating function and real rootedness of the corresponding polynomials. It turns out that for $0 \le j, k \le n$ we have $b(n, k, j) = \left\langle {n+1 \atop k} \right\rangle_{j+1}$. We will prove this equality providing a bijection $\mathcal{A}_{n+1,k,j+1} \to \mathcal{B}_{n,k,U}$, where $U = \{1, \ldots, j\}$ (Theorem 11). The array $b(n, k, 1)$, $1 \le k \le n$, appears in OEIS [8] as A120434. It also counts permutations $\sigma \in \mathcal{A}_n$ which have $k-1$ big descents, i.e. such descents $\sigma_i > \sigma_{i+1}$ that $\sigma_i - \sigma_{i+1} \ge 2$.

Conger proved that the polynomials $p_{n,j}(x) := \sum_{k=0}^{n} b(n, k, j) x^k$ have only real roots (Theorem 5 in [5]). Brändén [3] showed something stronger: for every $n \ge 1$ the sequence of polynomials $\{p_{n,j}(x)\}_{j=0}^{n}$ is interlacing, in particular for every $c_0, c_1, \ldots, c_n \ge 0$ the polynomial $c_0 p_{n,0}(x) + c_1 p_{n,1}(x) + \ldots + c_n p_{n,n}(x)$ has only real roots. Here we remark, that $P_{n,j}(x) = \binom{n}{j} p_{n,j}(x)$, so the polynomials $P_{n,j}(x)$ admit the same property, which is a generalization of Corollary 3.7 in [4] and of Corollary 6.9 in [2].

1. PRELIMINARIES

For a sequence $(a_0, \ldots, a_s)$, $a_i \in \mathbb{R}$, the *number of descents*, denoted $\mathrm{des}(a_0, \ldots, a_s)$, is defined as the cardinality of the set $\{i \in \{1, \ldots, s\} : a_{i-1} > a_i\}$. We will use the *Iverson bracket:* $[p] := 1$ if the statement p is true and $[p] := 0$ otherwise, see [7].

Denote by $\mathcal{A}_n$ the group of permutations of the set $\{1, \ldots, n\}$. We will identify $\sigma \in \mathcal{A}_n$ with the sequence $(\sigma_1, \ldots, \sigma_n)$ (we will usually write σ_k instead of $\sigma(k)$). For $0 \le k \le n$ we define $\mathcal{A}_{n,k}$ as the set of those $\sigma \in \mathcal{A}_n$ such that the sequence $(\sigma_1 \ldots, \sigma_n)$ has k descents. Then the classical *type A Eulerian number* $A(n, k)$ (see entry A123125 in OEIS) is defined as the cardinality of $\mathcal{A}_{n,k}$. We have the following recurrence relation:

$$(1) \qquad A(n, k) = (n - k)A(n - 1, k - 1) + (k + 1)A(n - 1, k)$$

for $0 < k < n$, with the boundary conditions: $A(n, 0) = 1$ for $n \ge 0$ and $A(n, n) = 0$ for $n \ge 1$. These numbers can be expressed as:

$$(2) \qquad A(n, k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n+1}{k-i} (i+1)^n.$$

For the Eulerian polynomials

$$P_n^{\mathrm{A}}(t) := \sum_{k=0}^{n} A(n, k) t^k$$

the exponential generating function is equal to

$$(3) \qquad f^{\mathrm{A}}(t, z) := \sum_{n=0}^{\infty} \frac{P_n^{\mathrm{A}}(t)}{n!} z^n = \frac{(1 - t)e^{(1-t)z}}{1 - te^{(1-t)z}}.$$

By $\mathcal{B}_n$ we will denote the group of such permutations σ of the set

$$\{-n, \ldots, -1, 0, 1, \ldots, n\}$$

such that σ is odd, i.e. $\sigma(-k) = -\sigma(k)$ for every $-n \le k \le n$. Then $|\mathcal{B}_n| = 2^n n!$. We will identify $\sigma \in \mathcal{B}_n$ with the sequence $(0, \sigma_1, \ldots, \sigma_n)$. For $\sigma \in \mathcal{B}_n$ we define $\mathrm{des}(\sigma)$ (resp. $\mathrm{neg}(\sigma)$) as the number of descents (resp. of negative numbers) in the sequence $(0, \sigma_1, \ldots, \sigma_n)$. For $0 \le k, j \le n$ we define sets

$$\mathcal{B}_{n,k} := \{\sigma \in \mathcal{B}_n : \mathrm{des}(\sigma) = k\},$$
$$\mathcal{B}_{n,k,j} := \{\sigma \in \mathcal{B}_n : \mathrm{des}(\sigma) = k, \mathrm{neg}(\sigma) = j\},$$

and the numbers $B(n, k) := |\mathcal{B}_{n,k}|$ (*type B Eulerian numbers*, see entry A060187 in OEIS), $B(n, k, j) := |\mathcal{B}_{n,k,j}|$. The numbers $B(n, k)$ satisfy the following recurrence relation:

$$(4) \qquad B(n, k) = (2n - 2k + 1)B(n - 1, k - 1) + (2k + 1)B(n - 1, k),$$

$0 < k < n$, with the boundary conditions $B(n, 0) = B(n, n) = 1$, and can be expressed as

$$(5) \qquad B(n, k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n+1}{k-i} (2i + 1)^n.$$

The type B Eulerian polynomials are defined by

$$P_n^{\mathrm{B}}(t) := \sum_{k=0}^{n} B(n,k) t^k,$$

and the corresponding exponential generating function is equal to

$$(6) \qquad f^{\mathrm{B}}(t,z) := \sum_{n=0}^{\infty} \frac{P_n^{\mathrm{B}}(t)}{n!} z^n = \frac{(1-t)e^{(1-t)z}}{1 - te^{2(1-t)z}}.$$

2. Descents and signs in type B permutations

This section is devoted to the numbers $B(n,k,j) := |\mathcal{B}_{n,k,j}|$. First we observe the following symmetry.

Proposition 1. *For $0 \le j, k \le n$ we have*

$$(7) \qquad B(n,k,j) = B(n, n-k, n-j).$$

Proof. It is sufficient to note that the map

$$(0, \sigma_1, \ldots, \sigma_n) \mapsto (0, -\sigma_1, \ldots, -\sigma_n)$$

is a bijection of $\mathcal{B}_{n,k,j}$ onto $\mathcal{B}_{n,n-k,n-j}$. $\qquad \square$

Now we provide two summation formulas.

Proposition 2.

$$(8) \qquad \sum_{j=0}^{n} B(n,k,j) = B(n,k),$$

$$(9) \qquad \sum_{k=0}^{n} B(n,k,j) = \binom{n}{j} n!.$$

Proof. The former sum counts all $\sigma \in \mathcal{B}_n$ which have k descents, while the latter counts all $\sigma \in \mathcal{B}_n$ which have j minus signs in the sequence $(\sigma_1, \ldots, \sigma_n)$. $\qquad \square$

From Corollary 4.4 in [1] we have also

$$(10) \qquad \sum_{\substack{j=0 \\ j \text{ even}}}^{n} B(n,k,j) = \frac{1}{2} B(n,k) + \frac{(-1)^k}{2} \binom{n}{k},$$

$$(11) \qquad \sum_{\substack{j=0 \\ j \text{ odd}}}^{n} B(n,k,j) = \frac{1}{2} B(n,k) - \frac{(-1)^k}{2} \binom{n}{k},$$

see A262226 and A262227 in OEIS.

Now we present the basic recurrence relations for the numbers $B(n,k,j)$.

Theorem 3. *The numbers $B(n,k,j)$ admit the following recurrence:*

$$(12) \qquad \begin{aligned} B(n,k,j) &= (k+1)B(n-1,k,j) + (n-k)B(n-1,k-1,j) \\ &\quad + kB(n-1,k,j-1) + (n-k+1)B(n-1,k-1,j-1) \end{aligned}$$

for $0 < k, j < n$, with boundary conditions:

(13) $\qquad B(n, 0, j) = [j = 0], \qquad\qquad B(n, n, j) = [j = n],$

(14) $\qquad B(n, k, 0) = A(n, k), \qquad\qquad B(n, k, n) = A(n, n - k)$

for $0 \le k, j \le n$.

Equality (12) remains true for $0 \le j, k \le n$ under convention that $B(n, k, j) = 0$ whenever $j \in \{-1, n + 1\}$ or $k \in \{-1, n + 1\}$.

Proof. For $(\sigma_0, \ldots, \sigma_n) \in \mathcal{B}_n$, $n \ge 1$, we define

$$\Lambda\sigma := (\sigma_0, \ldots, \widehat{\sigma}_i, \ldots, \sigma_n) \in \mathcal{B}_{n-1},$$

where i is such that $\sigma_i = \pm n$, and the symbol "$\widehat{\sigma}_i$" means, that the element σ_i has been removed from the sequence.

For given $\sigma \in \mathcal{B}_{n,k,j}$, $0 < k, j < n$, we have four possibilities:

- $\sigma_i = n$ and either $i = n$ or $\sigma_{i-1} > \sigma_{i+1}$, $1 \le i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k,j}$.
- $\sigma_i = n$ and $\sigma_{i-1} < \sigma_{i+1}$, $1 \le i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k-1,j}$.
- $\sigma_i = -n$ and $\sigma_{i-1} > \sigma_{i+1}$, $1 \le i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k,j-1}$.
- $\sigma_i = -n$ and either $i = n$ or $\sigma_{i-1} < \sigma_{i+1}$, $1 \le i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k-1,j-1}$.

Now, suppose we are given a fixed $\tau = (\tau_0, \ldots, \tau_{n-1})$ which belongs to one of the sets $\mathcal{B}_{n-1,k,j}$, $\mathcal{B}_{n-1,k-1,j}$, $\mathcal{B}_{n-1,k,j-1}$ or $\mathcal{B}_{n-1,k-1,j-1}$. We are going to count all $\sigma \in \mathcal{B}_{n,k,j}$ such that $\Lambda\sigma = \tau$.

If $\tau \in \mathcal{B}_{n-1,k,j}$ then we should either put n at the end of τ, or insert into a descent of τ, i.e. between τ_{i-1} and τ_i, where $1 \le i \le n - 1$, $\tau_{i-1} > \tau_i$, therefore we have $k + 1$ possibilities.

Similarly, if $\tau \in \mathcal{B}_{n-1,k-1,j}$ then we construct σ by inserting n between τ_{i-1} and τ_i, $1 \le i \le n - 1$, where $\tau_{i-1} < \tau_i$. For this we have $n - k$ possibilities.

Now assume that $\tau \in \mathcal{B}_{n-1,k,j-1}$. Then we should insert $-n$ between τ_{i-1} and τ_i, $1 \le i \le n - 1$, where $\tau_{i-1} > \tau_i$, for which we have k possibilities.

Finally, if $\tau \in \mathcal{B}_{n-1,k-1,j-1}$ then we put $-n$ either at the end of τ or between τ_{i-1} and τ_i, $1 \le i \le n - 1$, where $\tau_{i-1} < \tau_i$, for which we have $n - k + 1$ possibilities.

Therefore the number of $\sigma \in \mathcal{B}_{n,k,j}$ such that $\Lambda\sigma$ belongs to the set $\mathcal{B}_{n-1,k,j}$, $\mathcal{B}_{n-1,k-1,j}$, $\mathcal{B}_{n-1,k,j-1}$ or $\mathcal{B}_{n-1,k-1,j-1}$ is equal to $(k + 1)B(n - 1, k, j)$, $(n - k)B(n - 1, k - 1, j)$, $kB(n - 1, k, j - 1)$ or $(n - k + 1)B(n - 1, k - 1, j - 1)$ respectively. This proves (12).

For the boundary conditions it is clear that if $\text{neg}(\sigma) > 0$ then $\text{des}(\sigma) > 0$, which yields $B(n, 0, j) = [j = 0]$. We note that the map $(\sigma_0, \sigma_1, \ldots, \sigma_n) \mapsto (\sigma_1, \ldots, \sigma_n)$ is a bijection of $\mathcal{B}_{n,k,0}$ onto $\mathcal{A}_{n,k}$, consequently $B(n, k, 0) = A(n, k)$. For the two others we refer to (7). $\qquad\qquad\square$

Below we present tables for the numbers $B(n, k, j)$ for $n = 0, 1, 2, 3, 4, 5$:

$k \backslash j$	0
0	1

,

$k \backslash j$	0	1
0	1	0
1	0	1

,

$k \backslash j$	0	1	2
0	1	0	0
1	1	4	1
2	0	0	1

,

$k \backslash j$	0	1	2	3
0	1	0	0	0
1	4	12	6	1
2	1	6	12	4
3	0	0	0	1

,

$k \setminus j$	0	1	2	3	4
0	1	0	0	0	0
1	11	32	24	8	1
2	11	56	96	56	11
3	1	8	24	32	11
4	0	0	0	0	1

$k \setminus j$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	26	80	80	40	10	1
2	66	330	600	480	180	26
3	26	180	480	600	330	66
4	1	10	40	80	80	26
5	0	0	0	0	0	1

For example we have $B(n, 1, 0) = 2^n - n - 1$ and $B(n, 1, j) = \binom{n}{j} 2^{n-j}$ for $1 \le j \le n$ (cf. A038207 in OEIS). We will see that $B(n, k, j) / \binom{n}{j}$ is always an integer.

3. GENERATING FUNCTIONS

Now we define three families of polynomials corresponding to the numbers $B(n, k, j)$:

$$(15) \qquad P_{n,j}(x) := \sum_{k=0}^{n} B(n, k, j) x^k,$$

$$(16) \qquad Q_{n,k}(y) := \sum_{j=0}^{n} B(n, k, j) y^j,$$

$$(17) \qquad R_n(x, y) := \sum_{j,k=0}^{n} B(n, k, j) x^k y^j.$$

The polynomials $R_n(x, y)$ were studied by Brenti [4], who called them "q-Eulerian polynomials of type B".

The symmetry (7) implies:

$$(18) \qquad\qquad P_{n,j}(x) = x^n P_{n,n-j}(1/x),$$
$$(19) \qquad\qquad Q_{n,k}(y) = y^n Q_{n,n-k}(1/y),$$
$$(20) \qquad\qquad R_n(x, y) = x^n y^n R_n(1/x, 1/y).$$

Proposition 4. *The polynomials $P_{n,j}(x)$ satisfy the following recurrence:*

$$(21) \qquad P_{n,j}(x) = (1 + nx - x) P_{n-1,j}(x) + (x - x^2) P'_{n-1,j}(x)$$
$$+ nx P_{n-1,j-1}(x) + (x - x^2) P'_{n-1,j-1}(x),$$

with the initial conditions: $P_{n,0}(x) = P_n^A(x)$ for $n \ge 0$ and $P_{n,n}(x) = x P_n^A(x)$ for $n \ge 1$.

Proof. It is easy to verify that

$$\sum_{k=0}^{n} (k+1) B(n-1, k, j) x^k = P_{n-1,j}(x) + x P'_{n-1,j}(x),$$

$$\sum_{k=0}^{n} (n-k) B(n-1, k-1, j) x^k = nx P_{n-1,j}(x) - x P_{n-1,j}(x) - x^2 P'_{n-1,j}(x),$$

$$\sum_{k=0}^{n} k B(n-1, k, j-1) x^k = x P'_{n-1,j-1}(x),$$

and

$$\sum_{k=0}^{n}(n-k+1)B(n-1,k-1,j-1)x^k = nxP_{n-1,j-1}(x) - x^2P'_{n-1,j-1}(x).$$

Summing up and applying (12) we obtain (21).   □

Brändén [2], Corollary 6.9, proved that for every nonempty subset $S \subseteq \{1,\ldots,n\}$ the polynomial $\sum_{j \in S} P_{n,j}(x)$ has only real and simple roots. Combining (47) with Example 7.8.8 in [3] we will note (Theorem 20) that in fact every linear combination $c_0 P_{n,0}(x) + c_1 P_{n,1}(x) + \ldots + c_n P_{n,n}(x)$, with $c_0, c_1, \ldots, c_n \geq 0$, has only real roots. The cases when S is the set of even or odd numbers in $\{1,\ldots,n\}$ were studied in [1]. The Newton's inequality implies that if $0 \leq j \leq n$ then the sequence $\{B(n,k,j)\}_{k=0}^{n}$ satisfies a stronger version of log-concavity, namely

$$(22) \qquad B(n,k,j)^2 \geq B(n,k-1,j)B(n,k+1,j)\frac{(k+1)(n-k+1)}{k(n-k)}$$

for $0 < k < n$, in particular this sequence is unimodal.

For the polynomials $Q_{n,k}(y)$ we have the following, see (18) in [4]:

Proposition 5. *The polynomials $Q_{n,k}(y)$ satisfy the following recurrence:*

$$Q_{n,k}(y) = (k+1+ky)Q_{n-1,k}(y) + (n-k+(n-k+1)y)Q_{n-1,k-1}(y)$$

with the initial conditions: $Q_{n,0}(y) = 1$, $Q_{n,n}(y) = y^n$ for $n \geq 0$.

The polynomials $Q_{n,k}$ however do not have all roots real. They satisfy the following versions of Worpitzky identity:

$$(23) \qquad \sum_{k=0}^{n}\binom{u+n-k}{n}Q_{n,k}(y) = (u+1+uy)^n,$$

$$(24) \qquad \sum_{k=0}^{n}\binom{u+k}{n}Q_{n,k}(y) = (u+y+uy)^n.$$

The former is proved in [4], Theorem 3.4, the latter follows from the former and the symmetry (19).

Now we recall the recurrence relation for $R_n(x,y)$ (see Theorem 3.4 in [4]):

Proposition 6. *The polynomials $R_n(x,y)$ admit the following recurrence:*

$$R_n(x,y) = (1+nxy+nx-x)R_{n-1}(x,y) + (x-x^2)(1+y)\frac{\partial}{\partial x}R_{n-1}(x,y),$$

$n \geq 1$, *with initial condition $R_0(x,y) = 1$.*

Brenti [4] also found the generating function for the numbers $B(n,k,j)$:

$$(25) \qquad f(x,y,z) := \sum_{n=0}^{\infty}\frac{R_n(x,y)}{n!}z^n = \frac{(1-x)e^{(1-x)z}}{1-xe^{(1-x)(1+y)z}}.$$

Note that

$$(26) \qquad f(x,y,z) = f^A\big(x,(1+y)z\big)e^{(x-1)yz}.$$

4. Refined numbers

For $0 \leq k \leq n$ and a subset $U \subseteq \{1, 2, \ldots, n\}$ we define $\mathcal{B}_{n,k,U}$ as the set of those $\sigma \in \mathcal{B}_{n,k}$ which have minus sign at σ_i, $1 \leq i \leq n$, if and only if $|\sigma_i| \in U$. Therefore we have

$$(27) \qquad \bigcup_{\substack{U \subseteq \{1,\ldots,n\} \\ |U|=j}}^{\cdot} \mathcal{B}_{n,k,U} = \mathcal{B}_{n,k,j}.$$

The cardinality of $\mathcal{B}_{n,k,U}$ will be denoted $b(n, k, U)$. By convention we put $b(n, -1, U) = b(n, n+1, U) := 0$. It is quite easy to observe boundary conditions.

Proposition 7. *For $n \geq 1$, $0 \leq k \leq n$, $U \subseteq \{1, \ldots, n\}$ we have*

$$b(n, 0, U) = [U = \emptyset], \qquad\qquad b(n, n, U) = [U = \{1, \ldots, n\}],$$
$$b(n, k, \emptyset) = A(n, k), \qquad b(n, k, \{1, \ldots, n\}) = A(n, n-k).$$

Now we provide a recurrence relation.

Proposition 8. *For $0 \leq k \leq n$, $U \subseteq \{1, 2, \ldots, n\}$ we have*

$$(28) \qquad b(n, k, U) = (k+1) \cdot b(n-1, k, U) + (n-k) \cdot b(n-1, k-1, U)$$

if $n \notin U$ and

$$(29) \qquad b(n, k, U) = k \cdot b(n-1, k, U') + (n-k+1) \cdot b(n-1, k-1, U')$$

if $n \in U$, where $U' := U \setminus \{n\}$.

Proof. Both formulas are true when $k = 0$ or $k = n$. Assume that $0 < k < n$. We will apply the same map $\Lambda : \mathcal{B}_n \to \mathcal{B}_{n-1}$ as in the proof of Theorem 2.1. Fix $\sigma \in \mathcal{B}_{n,k,U}$ and assume that i is such that $\sigma_i = n$ (when $n \notin U$) or $\sigma_i = -n$ (when $n \in U$), $1 \leq i \leq n$. We have now four possibilities:

- $n \notin U$ and either $i = n$ or $\sigma_{i-1} > \sigma_{i+1}$, $1 \leq i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k,U}$.
- $n \notin U$ and $\sigma_{i-1} < \sigma_{i+1}$, $1 \leq i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k-1,U}$.
- $n \in U$ and $\sigma_{i-1} > \sigma_{i+1}$, $1 \leq i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k,U\setminus\{n\}}$.
- $n \in U$ and either $i = n$ or $\sigma_{i-1} < \sigma_{i+1}$, $1 \leq i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k-1,U\setminus\{n\}}$.

On the other hand, as in the proof of Theorem 3, we see that for a given τ in $\mathcal{B}_{n-1,k,U}$ (resp. in $\mathcal{B}_{n-1,k-1,U}$) there are $k+1$ (resp. $n-k$) such σ's in $\mathcal{B}_{n,k,U}$ that $\Lambda\sigma = \tau$. We simply insert n into a descent or at the end of τ (resp. into an ascent). Similarly, for a given τ in $\mathcal{B}_{n-1,k,V}$ (resp. in $\mathcal{B}_{n-1,k-1,V}$) there are k (resp. $n-k+1$) such σ's in $\mathcal{B}_{n,k,V\cup\{n\}}$ that $\Lambda\sigma = \tau$. $\qquad \square$

Now we will see that $b(n, k, U)$ depends only on n, k and the cardinality of U.

Theorem 9. *If $0 \leq k \leq n$, $U, V \subseteq \{1, \ldots, n\}$ and $|U| = |V|$ then*

$$b(n, k, U) = b(n, k, V).$$

Proof. Fix $U, V \subseteq \{1, \ldots, n\}$, with $|U| = |V|$ and define $\tau \in \mathcal{A}_n$ as the unique permutation of $\{1, \ldots, n\}$ such that: $\tau(U) = V$, $\tau|_U$ preserves the order and $\tau|_{\{1,\ldots,n\}\setminus U}$ preserves the order. We extend τ to an element of $\mathcal{B}_n$ by putting $\tau(-i) = -\tau(i)$. Now let $\sigma \in \mathcal{B}_{n,k,U}$. Then, by definition, $\tau(\sigma(i)) < 0$ if and only if $\sigma(i) < 0$, $-n \leq i \leq n$. Moreover, if $1 \leq i \leq n$ then $\tau(\sigma(i-1)) < \tau(\sigma(i))$ if and only if $\sigma(i-1) < \sigma(i)$. This is clear when $\sigma(i-1)$ and $\sigma(i)$ have different signs. If they have the same sign then

this is a consequence of the order preserving property of τ on U and on $\{1, \ldots, n\} \setminus U$. Consequently, the map $\sigma \mapsto \tau \circ \sigma$ is a bijection of $\mathcal{B}_{n,k,U}$ onto $\mathcal{B}_{n,k,V}$. $\qquad \square$

The theorem justifies the following definition: for $0 \le j, k \le n$ we put

$$b(n, k, j) := b(n, k, U),$$

where U is an arbitrary subset of $\{1, \ldots, n\}$ with $|U| = j$. In addition, if $j < 0$ or $k < 0$ or $n < j$ or $n < k$ then we put $b(n, k, j) = 0$. From (27) we obtain

Corollary 10. *For $0 \le j, k \le n$ we have*

$$(30) \qquad \binom{n}{j} b(n, k, j) = B(n, k, j).$$

5. CONNECTIONS WITH PERMUTATIONS OF TYPE A

For given $n \ge 0$ we define a map $F_n : \mathcal{A}_{n+1} \to \mathcal{B}_n$ in the following way: $F_n(\sigma) = \widetilde{\sigma}$, where for $1 \le i \le n$ we put

$$(31) \qquad \widetilde{\sigma}_i := \begin{cases} \sigma_{i+1} - \sigma_1 & \text{if } \sigma_{i+1} < \sigma_1, \\ \sigma_{i+1} - 1 & \text{if } \sigma_{i+1} > \sigma_1, \end{cases}$$

$\widetilde{\sigma}_{-i} := -\widetilde{\sigma}_i$ and $\widetilde{\sigma}_0 := 0$. Note that $\widetilde{\sigma}_{i-1} > \widetilde{\sigma}_i$ if and only if $\sigma_i > \sigma_{i+1}$ for $1 \le i \le n$, so the number of descents in $(0, \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n)$ is the same as in $(\sigma_1, \ldots, \sigma_{n+1})$. It is easy to see that F_n is one-to-one. Its image is the set of such $\tau \in \mathcal{B}_n$ which satisfy the following property: if $1 \le i_1, i_2 \le n$, $|\tau_{i_1}| < |\tau_{i_2}|$, $\tau_{i_2} < 0$ then $\tau_{i_1} < 0$. Denote

$$\mathcal{A}_{n,k,j} := \{\sigma \in \mathcal{A}_{n,k} : \sigma_1 = j\}.$$

The cardinalities of these sets were studied by Conger [5], who denoted $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle_j := |\mathcal{A}_{n,k,j}|$.

From our remarks we have

Theorem 11. *For $0 \le j, k \le n$ the function F_n maps $\mathcal{A}_{n+1,k}$ into $\mathcal{B}_{n,k}$ and is a bijection from $\mathcal{A}_{n+1,k,j+1}$ onto $\mathcal{B}_{n,k,\{1,\ldots,j\}}$. Consequently,*

$$(32) \qquad b(n, k, j) = |\mathcal{A}_{n+1,k,j+1}|.$$

In the rest of this section we briefly collect some properties of the numbers $b(n, k, j) = \left\langle \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right\rangle_{j+1}$, most of them are immediate consequences of the results of Conger [5, 6].

Proposition 12. *If $0 \le k, j \le n$ then*

$$(33) \qquad b(n, 0, j) = [j = 0],$$

$$(34) \qquad b(n, n, j) = [j = n],$$

$$(35) \qquad b(n, k, 0) = A(n, k),$$

$$(36) \qquad b(n, k, n) = A(n, n - k),$$

$$(37) \qquad b(n, k, j) = (k + 1)b(n - 1, k, j) + (n - k)b(n - 1, k - 1, j), \quad j < n,$$

$$(38) \qquad b(n, k, j) = kb(n - 1, k, j - 1) + (n - k + 1)b(n - 1, k - 1, j - 1), \quad j > 0,$$

$$(39) \qquad b(n, k, j) = b(n, n - k, n - j).$$

Proof. These formulas are consequences of Proposition 7, Proposition 8, (7) and (30) (see formulas (3) and (8) in [5]). Note that (38) is absent in [5]. $\qquad \square$

Applying (37), with $j - 1$ instead of j, and (38) we obtain (see (10) in [5])

Corollary 13. *For $1 \leq j, k \leq n$*

(40) $b(n, k, j-1) - b(n, k, j) = b(n-1, k, j-1) - b(n-1, k-1, j-1).$

Below we present tables for the numbers $b(n, k, j)$ for $n = 0, 1, 2, 3, 4, 5, 6$ (they also appear in Appendix A of [6]):

$k \backslash j$	0
0	1

$k \backslash j$	0	1
0	1	0
1	0	1

$k \backslash j$	0	1	2
0	1	0	0
1	1	2	1
2	0	0	1

$k \backslash j$	0	1	2	3
0	1	0	0	0
1	4	4	2	1
2	1	2	4	4
3	0	0	0	1

$k \backslash j$	0	1	2	3	4
0	1	0	0	0	0
1	11	8	4	2	1
2	11	14	16	14	11
3	1	2	4	8	11
4	0	0	0	0	1

$k \backslash j$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	26	16	8	4	2	1
2	66	66	60	48	36	26
3	26	36	48	60	66	66
4	1	2	4	8	16	26
5	0	0	0	0	0	1

$k \backslash j$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	57	32	16	8	4	2	1
2	302	262	212	160	116	82	57
3	302	342	372	384	372	342	302
4	57	82	116	160	212	262	302
5	1	2	4	8	16	32	57
6	0	0	0	0	0	0	1

From (30), (37) and (38) we can provide new recurrence formulas for the numbers $B(n, k, j)$:

Corollary 14. *For $0 \leq j, k \leq n$ we have*

$$B(n, k, j) = \frac{(k+1)n}{n-j} B(n-1, k, j) + \frac{(n-k)n}{n-j} B(n-1, k-1, j),$$

if $0 \leq j < n$ and

$$B(n, k, j) = \frac{kn}{j} B(n-1, k, j-1) + \frac{(n-k+1)n}{j} B(n-1, k-1, j-1),$$

if $0 < j \leq n$.

Now we introduce the following lexicographic order on the set $\{0, 1, \ldots, n\}^2$: $(k_1, j_1) \preceq (k_2, j_2)$ if and only if either $k_1 < k_2$ or $k_1 = k_2$, $j_1 \geq j_2$. This is a linear order, in which the successor of $(k, 0)$, with $0 \leq k < n$, is $(k+1, n)$, and for $1 \leq j \leq n$ the successor of (k, j) is $(k, j-1)$. It turns out that for every $n \geq 1$ the array $\left(b(n, k, j)\right)_{k,j=0}^{n}$ is lexicographically unimodal, cf. Theorem 7 in [5].

Proposition 15. *For every $n \geq 1$ we have the following:*
a) If either $0 \leq k < n/2$, $1 \leq j \leq n$ or $k = n/2$, $n/2 < j \leq n$ then

$$b(n, k, j-1) \geq b(n, k, j).$$

This inequality is sharp unless either $k = 0$, $2 \le j \le n$ or n is odd, $k = (n-1)/2$, $j = 1$.

b) If either $1 \le k \le n/2$, $0 \le j \le n$ or n is odd, $k = (n+1)/2$, $(n+1)/2 \le j \le n$ then

$$b(n, k-1, j) \le b(n, k, j)$$

and this inequality is sharp unless n is even, $k = n/2$, $j = 0$.

c) The array of numbers $b(n, k, j)$, $0 \le j, k \le n$, is unimodal with respect to the order "$\preceq$", with the maximal value $b(n, n/2, n/2)$ if n is even and

$$b(n, (n-1)/2, n) = b(n, (n+1)/2, 0)$$

if n is odd.

Proof. First we note that (a) implies (c) as a consequence of the symmetry (39) and the equality

$$b(n, k-1, 0) = A(n, k-1) = A(n, n-k) = b(n, k, n).$$

Similarly we get (b).

Now assume that the statement holds for $n-1$. If either $k < n/2$ or $k = n/2$, $n/2 < j$ then, due to (3), the right hand side of (40) is nonnegative which proves (a), (b) and consequently (c) for n. Moreover, it is positive unless $j = 1$, $n-1 = 2k$, as $A(2k, k-1) = A(2k, k)$. $\qquad\square$

Now we note two summation formulas (see (4) and (5) in [5]).

Proposition 16. *For $0 \le j, k \le n$ we have*

$$(41) \qquad\qquad \sum_{j=0}^{n} b(n, k, j) = A(n+1, k),$$

$$(42) \qquad\qquad \sum_{k=0}^{n} b(n, k, j) = n!.$$

Proof. For (41) we apply (32) to the following decomposition:

$$\mathcal{A}_{n+1,k,1} \dot\cup \mathcal{A}_{n+1,k,2} \dot\cup \ldots \dot\cup \mathcal{A}_{n+1,k,n+1} = \mathcal{A}_{n+1,k}.$$

The latter identity is a consequence of (9) and (30). $\qquad\square$

It turns out that (2) can be generalized to a formula which expresses the numbers $b(n, k, j)$, see Theorem 1 in [5].

Theorem 17. *For any $0 \le j, k \le n$ we have*

$$(43) \qquad\qquad b(n, k, j) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n+1}{k-i} i^j (i+1)^{n-j},$$

under convention that $0^0 = 1$.

Proof. It can be proved by induction by applying (2), (36) and (38). $\qquad\square$

From (43) and (30) we can derive a formula for the numbers $B(n, k, j)$.

Corollary 18. *For any $0 \leq j, k \leq n$ we have*

$$(44) \qquad B(n, k, j) = \binom{n}{j} \sum_{i=0}^{k} (-1)^{k-i} \binom{n+1}{k-i} i^j (i+1)^{n-j},$$

under convention that $0^0 = 1$.

Now we can prove Worpitzky type formula:

Proposition 19. *For $0 \leq j \leq n$ we have*

$$(45) \qquad \sum_{k=0}^{n} b(n, k, j) \binom{x+n-k}{n} = x^j (1+x)^{n-j}.$$

Proof. If $x \in \{0, 1, \ldots, n\}$ then

$$\sum_{k=0}^{n} (-1)^{k-i} \binom{n+1}{k-i} \binom{x+n-k}{n} = [x = i]$$

(see (5.25) in [7]). Applying (43) we see that (45) holds for $x \in \{0, 1, \ldots, n\}$ (see formula (4.18) in [6]). Since the left hand side is a polynomial of degree at most n, this implies that (45) is true for all $x \in \mathbb{R}$. $\qquad\square$

6. REAL ROOTEDNESS

For $0 \leq j \leq n$ denote

$$(46) \qquad p_{n,j}(x) := \sum_{k=0}^{n} b(n, k, j) x^k$$

so that

$$(47) \qquad P_{n,j}(x) = \binom{n}{j} p_{n,j}(x).$$

By Proposition 4 we have the following recurrence:

$$(48) \qquad p_{n,j}(x) = \frac{n-j}{n}(1 + xn - x)p_{n-1,j}(x) + \frac{n-j}{n}(x - x^2)p'_{n-1,j}(x)$$

$$+ jx p_{n-1,j-1}(x) + \frac{j}{n}(x - x^2)p'_{n-1,j-1}(x),$$

with the initial conditions: $p_{n,0}(x) = P_n^{\mathrm{A}}(x)$ for $n \geq 0$ and $p_{n,n}(x) = x P_n^{\mathrm{A}}(x)$ for $n \geq 1$. By (32) the polynomial $p_{n,j}(x)$ coincides with $A_{n+1,j+1}(x)$ considered by Brändén [3], Example 7.8.8. He noted that

$$(49) \qquad p_{n,j}(x) = \sum_{i=0}^{j-1} x p_{n-1,i}(x) + \sum_{i=j}^{n-1} p_{n-1,i}(x),$$

which is equivalent to

$$(50) \qquad b(n, k, j) = \sum_{i=0}^{j-1} b(n-1, k-1, i) + \sum_{i=j}^{n-1} b(n-1, k, i)$$

(see (9) in [5]). Note that if $0 \leq j < n$ then $\deg p_{n,j}(x) = n - 1$ and $\deg p_{n,n}(x) = n$. In fact, $p_{n,n}(x) = x p_{n,0}(x)$. Conger [5], Theorem 5, proved that all $p_{n,j}(x)$ have only real roots. It turns out that they admit a much stronger property.

Let $f, g \in \mathbb{R}[x]$ be real-rooted polynomials with positive leading coefficients. We say that f is an *interleaver* of g, which we denote $f \ll g$, if

$$\ldots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1,$$

where $\{\alpha_i\}_{i=1}^m$, $\{\beta_i\}_{i=1}^n$ are the roots of f and g respectively. A sequence $\{f_i\}_{i=0}^n$ of real-rooted polynomials is called *interlacing* if $f_i \ll f_j$ whenever $0 \leq i < j \leq n$.

From [3], Example 7.8.8 and (47) we have the following property of the polynomials $p_{n,j}(x)$ and $P_{n,j}(x)$:

Theorem 20. *For every $n \geq 1$ the sequence $\{p_{n,j}(x)\}_{j=0}^n$ is interlacing. Consequently, for any $c_0, c_1, \ldots, c_n \geq 0$ the polynomial*

$$c_0 p_{n,0}(x) + c_1 p_{n,1}(x) + \ldots + c_n p_{n,n}(x)$$

has only real roots.

The same statement holds for the polynomials $P_{n,j}(x)$.

Note that Theorem 20 generalizes Corollary 3.7 in [4] and Corollary 6.9 in [2].

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