ON THE $x-$COORDINATES OF PELL EQUATIONS WHICH ARE SUMS OF TWO PADOVAN NUMBERS

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Abstract. Let $\{P_n\}_{n\geq 0}$ be the sequence of Padovan numbers defined by $P_0 = 0$, $P_1 = P_2 = 1$ and $P_{n+3} = P_{n+1} + P_n$ for all $n \geq 0$. In this paper, we find all positive square-free integers d such that the Pell equations $x^2 - dy^2 = \pm 1$, $X^2 - dY^2 = \pm 4$ have at least two positive integer solutions (x, y) and (x', y'), (X, Y) and (X', Y'), respectively, such that each of $x,\ x',\ X,\ X'$ is a sum of two Padovan numbers.

1. Introduction

Let $\{P_n\}_{n\geq 0}$ be the sequence of Padovan numbers given by

$$P_0 = 0,\ P_1 = 1,\ P_2 = 1,\ \text{and } P_{n+3} = P_{n+1} + P_n \text{ for all } n \geq 0.$$

This is sequence $A000931$ on the Online Encyclopedia of Integer Sequences (OEIS). The first few terms of this sequence are

$$\{P_n\}_{n\geq 0} = 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, \ldots.$$

Let $d \geq 2$ be a positive integer which is not a square. It is well known that the Pell equations

$$x^2 - dy^2 = \pm 1, \tag{1}$$

and

$$X^2 - dY^2 = \pm 4, \tag{2}$$

have infinitely many positive integer solutions (x, y) and (X, Y), respectively. By putting (x_1, y_1) and (X_1, Y_1) for the smallest positive solutions to (1) and (2), respectively, all solutions are of the forms (x_k, y_k) and (X_k, Y_k) for some positive integer k, where

$$x_k + y_k\sqrt{d} = (x_1 + y_1\sqrt{d})^k \qquad \text{for all} \quad k \geq 1,$$

and

$$\frac{X_k + Y_k\sqrt{d}}{2} = \left(\frac{X_1 + Y_1\sqrt{d}}{2}\right)^k \qquad \text{for all} \quad k \geq 1.$$

Furthermore, the sequences $\{x_k\}_{k\geq 1}$ and $\{X_k\}_{k\geq 1}$ are binary recurrent. In fact, the following formulae

$$x_k = \frac{(x_1 + y_1\sqrt{d})^k + (x_1 - y_1\sqrt{d})^k}{2},$$

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and

$$X_k = \left(\frac{X_1 + Y_1\sqrt{d}}{2}\right)^k + \left(\frac{X_1 - Y_1\sqrt{d}}{2}\right)^k$$

hold for all positive integers k.

Recently, Bravo, Gómez-Ruiz and Luca [1] studied the Diophantine equation

$$x_l = T_m + T_n, \tag{3}$$

where x_l are the $x-$coordinates of the solutions of the Pell equation (1) for some positive integer l and $\{T_n\}_{n\geq 0}$ is the sequence of Tribonacci numbers given by $T_0 = 0,\ T_1 = 1 = T_2$ and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geq 0$. They proved that for each square free integer $d \geq 2$, there is at most one positive integer l such that x_l admits the representation (3) for some nonnegative integers $0 \leq m \leq n$, except for $d \in \{2, 3, 5, 15, 26\}$. Furthermore, they explicitly stated all the solutions for these exceptional cases.

In the same spirit, Rihane, Hernane and Togbé [14] studied the Diophantine equations

$$x_n = P_m \qquad \text{and} \qquad X_n = P_m, \tag{4}$$

where x_n and X_n are the $x-$coordinates of the solutions of the Pell equations (1) and (2), respectively, for some positive integers n and $\{P_m\}_{m\geq 0}$ is the sequence of Padovan numbers. They proved that for each square free integer $d \geq 2$, there is at most one positive integer x participating in the Pell equation (1) and one positive integer X participating in the Pell equation (2) that is a Padovan number with a few exceptions of d that they effectively computed. Furthermore, the exceptional cases were $d \in \{2, 3, 5, 6$ and $d \in \{5\}$ for the the first and second equations in (4), respectively. Several other related problems have been studied where x_l belongs to some interesting positive integer sequences. For example, see [3, 4, 6, 8, 9, 10, 11, 12].

2. Main Results

In this paper, we study a problem related to that of Bravo, Gómez-Ruiz and Luca [1] but with the Padovan sequence instead of the Tribonacci sequence. We also extend the results from the Pell equation (1) to the Pell equation (2). In both cases we find that there are only finitely many solutions that we effectively compute.

Since $P_1 = P_2 = P_3 = 1$, we discard the situations when $n = 1$ and $n = 2$ and just count the solutions for $n = 3$. Similarly, $P_4 = P_5 = 2$, we discard the situation when $n = 4$ and just count the solutions for $n = 5$. The main aim of this paper is to prove the following results.

Theorem 1. *For each integer $d \geq 2$ which is not a square, there is atmost one positive integer k such that x_k admits a representation as*

$$x_k = P_n + P_m \tag{5}$$

for some nonnegative integers $0 \leq m \leq n$, except when $d \in \{2, 3, 6, 15, 110, 483\}$ in the $+1$ case and $d \in \{2, 5, 10, 17\}$ in the -1 case.

Theorem 2. *For each integer $d \geq 2$ which is not a square, there is atmost one positive integer k such that X_k admits a representation as*

$$X_k = P_n + P_m \tag{6}$$

for some nonnegative integers $0 \leq m \leq n$*, except when* $d \in \{3, 5, 21\}$ *in the* $+4$ *case and* $d \in \{2, 5\}$ *in the* -4 *case.*

For the exceptional values of d listed in Theorem 1 and Theorem 2, all solutions (k, n, m) are listed at the end of the proof of each result. The main tools used in this paper are the lower bounds for linear forms in logarithms of algebraic numbers and the Baker-Davenport reduction procedure, as well as the elementary properties of Padovan numbers and solutions to Pell equations.

3. Preliminary results

3.1. **The Padovan sequence.** Here, we recall some important properties of the Padovan sequence $\{P_n\}_{n\geq 0}$. The characteristic equation

$$\Psi(x) := x^3 - x - 1 = 0$$

has roots $\alpha, \beta, \gamma = \bar{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \qquad \beta = \frac{-(r_1 + r_2) + \sqrt{-3}(r_1 - r_2)}{12} \tag{7}$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \quad \text{and} \quad r_2 = \sqrt[3]{108 - 12\sqrt{69}}. \tag{8}$$

Furthermore, the Binet formula is given by

$$P_n = a\alpha^n + b\beta^n + c\gamma^n \qquad \text{for all} \quad n \geq 0, \tag{9}$$

where

$$a = \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)}, \quad c = \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \bar{b}. \tag{10}$$

Numerically, the following estimates hold:

$$\begin{gathered} 1.32 < \alpha < 1.33 \\ 0.86 < |\beta| = |\gamma| = \alpha^{-\frac{1}{2}} < 0.87 \\ 0.72 < a < 0.73 \\ 0.24 < |b| = |c| < 0.25. \end{gathered} \tag{11}$$

From (7), (8) and (11), it is easy to see that the contribution the complex conjugate roots β and γ, to the right-hand side of (9), is very small. In particular, setting

$$e(n) := P_n - a\alpha^n = b\beta^n + c\gamma^n \quad \text{then} \quad |e(n)| < \frac{1}{\alpha^{n/2}} \tag{12}$$

holds for all $n \geq 1$. Furthermore, by induction, we can prove that

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \quad \text{holds for all} \quad n \geq 4. \tag{13}$$

3.2. **Linear forms in logarithms.** Let η be an algebraic number of degree D with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{D}(x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{D}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$\begin{aligned} h(\eta_1 \pm \eta_2) &\leq h(\eta) + h(\eta_1) + \log 2, \\ h(\eta_1\eta_2^{\pm 1}) &\leq h(\eta_1) + h(\eta_2), \\ h(\eta^s) &= |s|h(\eta) \ (s \in \mathbb{Z}). \end{aligned} \tag{14}$$

Theorem 3. *Let $\eta_1, \ldots, \eta_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K} \subset \mathbb{R}$ of degree $D_{\mathbb{K}}$, $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1, \tag{15}$$

is nonzero. Then

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D_{\mathbb{K}}^2(1 + \log D_{\mathbb{K}})(1 + \log B)A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{D_{\mathbb{K}}h(\eta_i), |\log\eta_i|, 0.16\}, \qquad \text{for all} \qquad i = 1, \ldots, t.$$

3.3. **Reduction procedure.** During the calculations, we get upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some results from the theory of continued fractions.

For the treatment of linear forms homogeneous in two integer variables, we use the well-known classical result in the theory of Diophantine approximation.

Lemma 1. *Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction of τ and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M) + 2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [5], Lemma 5a). For a real number X, we write $||X|| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 2. *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let further $\varepsilon := ||\mu q|| - M||\tau q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \textit{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

At various occasions, we need to find a lower bound for linear forms in logarithms with bounded integer coefficients in three and four variables. In this case we use the LLL-algorithm that we describe below. Let $\tau_1, \tau_2, \ldots \tau_t \in \mathbb{R}$ and the linear form

$$x_1\tau_1 + x_2\tau_2 + \cdots + x_t\tau_t \quad \text{with} \quad |x_i| \leq X_i. \tag{16}$$

We put $X := \max\{X_i\}$, $C > (tX)^t$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\tau_j \rceil \quad \text{for} \quad 1 \leq j \leq t-1 \quad \text{and} \quad \mathbf{b}_t := \lfloor C\tau_t \rceil \mathbf{e}_t,$$

where C is a sufficiently large positive constant.

Lemma 3. *Let $X_1, X_2, \ldots, X_t$ be positive integers such that $X := \max\{X_i\}$ and $C > (tX)^t$ is a fixed sufficiently large constant. With the above notation on the lattice Ω, we consider a reduced base $\{\boldsymbol{b}_i\}$ to Ω and its associated Gram-Schmidt orthogonalization base $\{\boldsymbol{b}_i^*\}$. We set*

$$c_1 := \max_{1 \leq i \leq t} \frac{||\boldsymbol{b}_1||}{||\boldsymbol{b}_i^*||}, \quad \theta := \frac{||\boldsymbol{b}_1||}{c_1}, \quad Q := \sum_{i=1}^{t-1} X_i^2 \quad \textit{and} \quad R := \left(1 + \sum_{i=1}^{t} X_i\right)/2.$$

If the integers x_i are such that $|x_i| \leq X_i$, for $1 \leq i \leq t$ and $\theta^2 \geq Q + R^2$, then we have

$$\left|\sum_{i=1}^{t} x_i\tau_i\right| \geq \frac{\sqrt{\theta^2 - Q} - R}{C}.$$

For the proof and further details, we refer the reader to the book of Cohen. (Proposition 2.3.20 in [[2], Pg. 58–63).

Finally, the following Lemma is also useful. It is Lemma 7 in [7].

Lemma 4. *If $r \geqslant 1$, $H > (4r^2)^r$ and $H > L/(\log L)^r$, then*

$$L < 2^r H(\log H)^r.$$

4. Proof of Theorem 1

Let (x_1, y_1) be the smallest positive integer solution to the Pell quation (1). We Put

$$\delta := x_1 + y_1\sqrt{d} \quad \text{and} \quad \sigma = x_1 - y_1\sqrt{d}. \tag{17}$$

From which we get that

$$\delta \cdot \sigma = x_1^2 - dy_1^2 =: \epsilon, \quad \text{where} \quad \epsilon \in \{\pm 1\}. \tag{18}$$

Then

$$x_k = \frac{1}{2}(\delta^k + \sigma^k). \tag{19}$$

Since $\delta \geq 1+\sqrt{2}$, it follows that the estimate

$$\frac{\delta^k}{\alpha^4} \leq x_k \leq \delta^k \quad \text{holds for all} \quad k \geq 1. \tag{20}$$

We assume that (k_1, n_1, m_1) and (k_2, n_2, m_2) are triples of integers such that

$$x_{k_1} = P_{n_1} + P_{m_1} \quad \text{and} \quad x_{k_2} = P_{n_2} + P_{m_2} \tag{21}$$

We asuume that $1 \leq k_1 < k_2$. We also assume that $3 \leq m_i < n_i$ for $i = 1, 2$. We set $(k, n, m) := (k_i, n_i, m_i)$, for $i = 1, 2$. Using the inequalities (13) and (20), we get from (21) that

$$\frac{\delta^k}{\alpha^4} \leq x_k = P_n + P_m \leq 2\alpha^{n-1} \quad \text{and} \quad \alpha^{n-2} \leq P_n + P_m = x_k \leq \delta^k.$$

The above inequalities give

$$(n-2)\log\alpha < k\log\delta < (n+3)\log\alpha + \log 2.$$

Dividing through by $\log\alpha$ and setting $c_2 := 1/\log\alpha$, we get that

$$-2 < c_2 k \log\delta - n < 3 + c_2\log 2,$$

and since $\alpha^3 > 2$, we get

$$|n - c_2 k \log\delta| < 6. \tag{22}$$

Furthermore, $k < n$, for if not, we would then get that

$$\delta^n \leq \delta^k < 2\alpha^{n+3}, \quad \text{implying} \quad \left(\frac{\delta}{\alpha}\right)^n < 2\alpha^3,$$

which is false since $\delta \geq 1+\sqrt{2}$, $1.32 < \alpha < 1.33$ (by (11)) and $n \geq 4$.

Besides, given that $k_1 < k_2$, we have by (13) and (21) that

$$\alpha^{n_1-2} \leq P_{n_1} \leq P_{n_1} + P_{m_1} = x_{k_1} < x_{k_2} = P_{n_2} + P_{m_2} \leq 2P_{n_2} < 2\alpha^{n_2-1}.$$

Thus, we get that

$$n_1 < n_2 + 4. \tag{23}$$

4.1. **An inequality for n and k (I).** Using the equations (9) and (19) and (21), we get

$$\frac{1}{2}(\delta^k + \sigma^k) = P_n + P_m = a\alpha^n + e(n) + a\alpha^m + e(m)$$

So,

$$\frac{1}{2}\delta^k - a(\alpha^n + \alpha^m) = -\frac{1}{2}\sigma^k + e(n) + e(m),$$

and by (12), we have

$$\begin{aligned} \left|\delta^k (2a)^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1} - 1\right| &\leq \frac{1}{2\delta^k a(\alpha^n+\alpha^m)} + \frac{2|b|}{\alpha^{n/2}a(\alpha^n+\alpha^m)} \\ &\quad + \frac{2|b|}{\alpha^{m/2}a(\alpha^n+\alpha^m)} \\ &\leq \frac{1}{a\alpha^n}\left(\frac{1}{2\delta^k} + \frac{2|b|}{\alpha^{n/2}} + \frac{2|b|}{\alpha^{m/2}}\right) < \frac{1.5}{\alpha^n}. \end{aligned}$$

Thus, we have

$$\left|\delta^k(2a)^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1}-1\right| \quad < \quad \frac{1.5}{\alpha^n}. \tag{24}$$

Put

$$\Lambda_1 := \delta^k(2a)^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1}-1$$

and

$$\Gamma_1 := k\log\delta - \log(2a) - n\log\alpha - \log(1+\alpha^{m-n}).$$

Since $|\Lambda_1| = |e^{\Gamma_1}-1| < \frac{1}{2}$ for $n \geq 4$ (because $1.5/\alpha^4 < 1/2$), since the inequality $|y| < 2|e^y-1|$ holds for all $y \in \left(-\frac{1}{2},\frac{1}{2}\right)$, it follows that $e^{|\Gamma_1|} < 2$ and so

$$|\Gamma_1| < e^{|\Gamma_1|}|e^{\Gamma_1}-1| < \frac{3}{\alpha^n}.$$

Thus, we get that

$$\left|k\log\delta - \log(2a) - n\log\alpha - \log(1+\alpha^{m-n})\right| < \frac{3}{\alpha^n}. \tag{25}$$

We apply Theorem 3 on the left-hand side of (24) with the data:

$$t := 4, \quad \eta_1 := \delta, \quad \eta_2 := 2a, \quad \eta_3 := \alpha, \eta_4 := 1+\alpha^{m-n},$$
$$b_1 := k, \quad b_2 := -1, \quad b_3 := -n, \quad b_4 := -1.$$

Furthermore, we take the number field $\mathbb{K} = \mathbb{Q}(\sqrt{d},\alpha)$ which has degree $D = 6$. Since $\max\{1,k,n\} \leq n$, we take $D_{\mathbb{K}} = n$. First we note that the left-hand side of (24) is non-zero, since otherwise,

$$\delta^k = 2a(\alpha^n + \alpha^m).$$

The left-hand side belongs to the quadratic field $\mathbb{Q}(\sqrt{d})$ while the right-hand side belongs to the cubic field $\mathbb{Q}(\alpha)$. These fields only intersect when both sides are rational numbers. Since δ^k is a positive algebraic integer and a unit, we get that to $\delta^k = 1$. Hence, $k = 0$, which is a contradiction. Thus, $\Lambda_1 \neq 0$ and we can apply Theorem 3.

We have $h(\eta_1) = h(\delta) = \frac{1}{2}\log\delta$ and $h(\eta_3) = h(\alpha) = \frac{1}{3}\log\alpha$. Further,

$$2a = \frac{2\alpha(\alpha+1)}{3\alpha^2-1},$$

the mimimal polynomial of $2a$ is $23x^3 - 46x^2 + 24x - 8$ and has roots $2a, 2b, 2c$. Since $2|b| = 2|c| < 1$ (by (11)), then

$$h(\eta_2) = h(2a) = \frac{1}{3}(\log 23 + \log(2a)).$$

On the other hand,

$$\begin{aligned} h(\eta_4) &= h(1+\alpha^{m-n}) \leq h(1) + h(\alpha^{m-n}) + \log 2 \\ &= (n-m)h(\alpha) + \log 2 = \frac{1}{3}(n-m)\log\alpha + \log 2. \end{aligned}$$

Thus, we can take $A_1 := 3\log\delta$,

$$A_2 := 2(\log 23 + \log(2a)), \quad A_3 := 2\log\alpha, \quad A_4 := 2(n-m)\log\alpha + 6\log 2.$$

Now, Theorem 3 tells us that

$$\begin{aligned}\log|\Lambda_1| &> -1.4\times 30^7\times 4^{4.5}\times 6^2(1+\log 6)(1+\log n)(3\log\delta)\\ &\quad\times(2(\log 23+\log(2a))(2\log\alpha)(2(n-m)\log\alpha+6\log 2)\\ &> -2.33\times 10^{17}(n-m)(\log n)(\log\delta).\end{aligned}$$

Comparing the above inequality with (24), we get

$$n\log\alpha-\log 1.5<2.33\times 10^{17}(n-m)(\log n)(\log\delta).$$

Hence, we get that

$$n<8.30\times 10^{17}(n-m)(\log n)(\log\delta). \tag{26}$$

We now return to the equation $x_k=P_n+P_m$ and rewrite it as

$$\frac{1}{2}\delta^k-a\alpha^n=-\frac{1}{2}\sigma^k+e(n)+P_m,$$

we obtain

$$\left|\delta^k(2a)^{-1}\alpha^{-n}-1\right|\le\frac{1}{a\alpha^{n-m}}\left(\frac{1}{\alpha}+\frac{1}{\alpha^{m+n/2}}+\frac{1}{2\delta^k\alpha^m}\right)<\frac{2.5}{\alpha^{n-m}}. \tag{27}$$

Put

$$\Lambda_2:=\delta^k(2a)^{-1}\alpha^{-n}-1,\quad \Gamma_2:=k\log\delta-\log(2a)-n\log\alpha.$$

We assume for technical reasons that $n-m\ge 10$. So $|e^{\Lambda_2}-1|<\frac{1}{2}$. It follows that

$$|k\log\delta-\log(2a)-n\log\alpha|=|\Gamma_2|<e^{|\Lambda_2|}|e^{\Lambda_2}-1|<\frac{5}{\alpha^{n-m}}. \tag{28}$$

Furthermore, $\Lambda_2\neq 0$ (so $\Gamma_2\neq 0$), since $\delta^k\in\mathbb{Q}(\alpha)$ by the previous argument.

We now apply Theorem 3 to the left-hand side of (27) with the data

$$t:=3,\quad \eta_1:=\delta,\quad \eta_2:=2a,\quad \eta_3:=\alpha,\quad b_1:=k,\quad b_2:=-1,\quad b_3:=-n.$$

Thus, we have the same $A_1,\ A_2,A_3$ as before. Then, by Theorem 3, we conclude that

$$\log|\Lambda|>-9.82\times 10^{14}(\log\delta)(\log n)(\log\alpha).$$

By comparing with (27), we get

$$n-m<9.84\times 10^{14}(\log\delta)(\log n). \tag{29}$$

This was obtained under the assumption that $n-m\ge 10$, but if $n-m<10$, then the inequality also holds as well. We replace the bound (29) on $n-m$ in (26) and use the fact that $\delta^k\le 2\alpha^{n+3}$, to obtain bounds on n and k in terms of $\log n$ and $\log\delta$. We now record what we have proved so far.

Lemma 5. *Let (k,n,m) be a solution to the equation $x_k=P_n+P_m$ with $3\le m<n$, then*

$$k<2.5\times 10^{32}(\log n)^2(\log\delta)\quad \textit{and}\quad n<8.2\times 10^{32}(\log n)^2(\log\delta)^2. \tag{30}$$

4.2. **Absolute bounds (I).** We recall that $(k,n,m)=(k_i,n_i,m_i)$, where $3\le m_i<n_i$, for $i=1,2$ and $1\le k_1<k_2$. Further, $n_i\ge 4$ for $i=1,2$. We return to (28) and write

$$\left|\Gamma_2^{(i)}\right| := |k_i\log\delta-\log(2a)-n_i\log\alpha| < \frac{5}{\alpha^{n_i-m_i}},\qquad \text{for}\qquad i=1,2.$$

We do a suitable cross product between $\Gamma_2^{(1)}$, $\Gamma_2^{(2)}$ and k_1,k_2 to eliminate the term involving $\log\delta$ in the above linear forms in logarithms:

$$\begin{aligned}|\Gamma_3| &:= |(k_1-k_2)\log(2a)+(k_1n_2-k_2n_1)\log\alpha| = |k_2\Gamma_2^{(1)}-k_1\Gamma_2^{(2)}|\\ &\le k_2|\Gamma_2^{(1)}|+k_1|\Gamma_2^{(2)}| \le \frac{5k_2}{\alpha^{n_1-m_1}}+\frac{5k_1}{\alpha^{n_2-m_2}} \le \frac{10n_2}{\alpha^{\lambda}},\end{aligned}\tag{31}$$

where

$$\lambda := \min_{1\le i\le 2}\{n_i-m_i\}.$$

We need to find an upper bound for λ. If $10n_2/\alpha^{\lambda}>1/2$, we then get

$$\lambda < \frac{\log(20n_2)}{\log\alpha} < 4\log(20n_2).\tag{32}$$

Otherwise, $|\Gamma_3|<\frac{1}{2}$, so

$$\left|e^{\Gamma_3}-1\right| = \left|(2a)^{k_1-k_2}\alpha^{k_1n_2-k_2n_1}-1\right| < 2|\Gamma_3| < \frac{20n_2}{\alpha^{\lambda}}.\tag{33}$$

We apply Theorem 3 with the data: $t:=2$, $\eta_1:=2a$, $\eta_2:=\alpha$, $b_1:=k_1-k_2$, $b_2:=k_1n_2-k_2n_1$. We take the number field $\mathbb{K}:=\mathbb{Q}(\alpha)$ and $D=3$. We begin by checking that $e^{\Gamma_3}-1\neq 0$ (so $\Gamma_3\neq 0$). This is true because α and $2a$ are multiplicatively independent, since α is a unit in the ring of integers $\mathbb{Q}(\alpha)$ while the norm of $2a$ is $8/23$.

We note that $|k_1-k_2|<k_2<n_2$. Further, from (31), we have

$$|k_2n_1-k_1n_2| < (k_2-k_1)\frac{|\log(2a)|}{\log\alpha}+\frac{10k_2}{\alpha^{\lambda}\log\alpha} < 11k_2 < 11n_2$$

given that $\lambda\ge 1$. So, we can take $B:=11n_2$. By Theorem 3, with the same $A_1:=\log 23$ and $A_2:=\log\alpha$, we have that

$$\log|e^{\Gamma_3}-1| > -1.55\times 10^{11}(\log n_2)(\log\alpha).$$

By comparing this with (33), we get

$$\lambda < 1.56\times 10^{11}\log n_2.\tag{34}$$

Note that (34) is better than (32), so (34) always holds. Without loss of generality, we can assume that $\lambda=n_i-m_i$, for $i=1,2$ fixed.

We set $\{i,j\}=\{1,2\}$ and return to (25) to replace $(k,n,m)=(k_i,n_i,m_i)$:

$$|\Gamma_1^{(i)}| = \left|k_i\log\delta-\log(2a)-n_i\log\alpha-\log(1+\alpha^{m_i-n_i})\right| < \frac{3}{\alpha^{n_i}},\tag{35}$$

and also return to (28), replacing with $(k,n,m)=(k_j,n_j,m_j)$:

$$|\Gamma_2^{(j)}| = |k_j\log\delta-\log(2a)-n_j\log\alpha| < \frac{5}{\alpha^{n_j-m_j}}.\tag{36}$$

We perform a cross product on (35) and (36) in order to eliminate the term on $\log\delta$:

$$
\begin{aligned}
|\Gamma_4| &:= \left|(k_j-k_i)\log(2a)+(k_jn_i-k_in_j)\log\alpha+k_j\log(1+\alpha^{m_i-n_i})\right|\\
&= \left|k_i\Gamma_2^{(j)}-k_j\Gamma_1^{(i)}\right| \le k_i\left|\Gamma_2^{(j)}\right|+k_j\left|\Gamma_1^{(i)}\right|\\
&< \frac{5k_i}{\alpha^{n_j-m_j}}+\frac{3k_j}{\alpha^{n_i}}<\frac{8n_2}{\alpha^{\nu}} \qquad (37)
\end{aligned}
$$

with $\nu := \min\{n_i, n_j - m_j\}$. As before, we need to find an upper bound on ν. If $8n_2/\alpha^{\nu} > 1/2$, then we get

$$\nu < \frac{\log(16n_2)}{\log\alpha} < 4\log(16n_2). \qquad (38)$$

Otherwise, $|\Gamma_4| < 1/2$, so we have

$$\left|e^{\Gamma_4}-1\right| \le 2|\Gamma_4| < \frac{16n_2}{\alpha^{\nu}}. \qquad (39)$$

In order to apply Theorem 3, first if $e^{\Gamma_4} = 1$, we obtain

$$(2a)^{k_i-k_j} = \alpha^{k_jn_i-k_in_j}(1+\alpha^{-\lambda})^{k_j}.$$

Since α is a unit, the right-hand side in above is an algebraic integer. This is a contradiction because $k_1 < k_2$ so $k_i - k_j \ne 0$, and neither $(2a)$ nor $(2a)^{-1}$ are algebraic intgers. Hence $e^{\Gamma_4} \ne 1$. By assuming that $\nu \ge 100$, we apply Theorem 3 with the data:

$$
\begin{aligned}
&t := 3, \quad \eta_1 := 2a, \quad \eta_2 := \alpha, \quad \eta_3 := 1+\alpha^{-\lambda},\\
&b_1 := k_j - k_i, \quad b_2 := k_jn_i - k_in_j, \quad b_3 := k_j,
\end{aligned}
$$

and the inequalities (34) and (39). We get

$$\nu = \min\{n_i, n_j - m_j\} < 1.14\times10^{14}\lambda\log n_2 < 1.78\times10^{25}(\log n_2)^2. \qquad (40)$$

The above inequality also holds when $\nu < 100$. Further, it also holds when the inequality (38) holds. So the above inequality holds in all cases. Note that the case $\{i, j\} = \{2, 1\}$ leads to $n_1 - m_1 \le n_1 \le n_2 + 4$ whereas $\{i, j\} = \{1, 2\}$ lead to $\nu = \min\{n_1, n_2 - m_2\}$. Hence, either the minimum is n_1, so

$$n_1 < 1.78\times10^{25}(\log n_2)^2, \qquad (41)$$

or the minimum is $n_j - m_j$ and from the inequality (34) we get that

$$\max_{1\le j\le 2}\{n_j - m_j\} < 1.78\times10^{25}(\log n_2)^2. \qquad (42)$$

Next, we assume that we are in the case (42). We evaluate (35) in $i = 1, 2$ and make a suitable cross product to eliminate the term involving $\log\delta$:

$$
\begin{aligned}
|\Gamma_5| &:= |(k_2-k_1)\log(2a)+(k_2n_1-k_1n_2)\log\alpha\\
&\quad +k_2\log(1+\alpha^{m_1-n_1})-k_1\log(1+\alpha^{m_2-n_2})|\\
&= \left|k_1\Gamma_1^{(2)}-k_2\Gamma_1^{(1)}\right| \le k_1\left|\Gamma_1^{(2)}\right|+k_2\left|\Gamma_1^{(1)}\right| < \frac{6n_2}{\alpha^{n_1}}. \qquad (43)
\end{aligned}
$$

In the above inequality we used the inequality (23)to conclude that $\min\{n_1, n_2\} \ge n_1 - 4$ as well as the fact that $n_i \ge 4$ for $i = 1.2$. Next, we apply a linear form in four logarithms to obtain an upper bound to n_1. As in the previous calculations, we pass from (43) to

$$\left|e^{\Gamma_5}-1\right| < \frac{12n_2}{\alpha^{n_1}}, \qquad (44)$$

which is implied by (43) except if n_1 is very small, say

$$n_1 \leq 4\log(12n_2). \tag{45}$$

Thus, we assume that (45) does not hold, therefore (44) holds. Then to apply Theorem 3, we fist justify that $e^{\Gamma_5} \neq 1$. Otherwise,

$$(2a)^{k_1-k_2} = \alpha^{k_2n_1-k_1n_2}(1+\alpha^{n_1-m_1})^{k_2}(1+\alpha^{n_2-m_2})^{-k_1},$$

By the fact that $k_1 < k_2$, the norm $\mathbf{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(2a) = \frac{8}{23}$ and that α is a unit, we have that 23 divides the norm $\mathbf{N}_{\mathbb{K}/\mathbb{Q}}(1+\alpha^{n_1-m_1})$. The factorization of the ideal generated by 23 in $\mathcal{O}_{\mathbb{Q}(\alpha)}$ is $(23) = \mathfrak{p}_1^2\mathfrak{p}_2$, where $\mathfrak{p}_1 = (23,\ \alpha+13)$ and $\mathfrak{p}_2 = (23,\ \alpha+20)$. Hence $\mathfrak{p}_2$ divides $\alpha^{n_1-m_1}+1$. Given that $\alpha \equiv -20 \pmod{\mathfrak{p}_2}$, then $(-20)^{n_1-m_1} \equiv -1 (\bmod\ \mathfrak{p}_2)$. Taking the norm $\mathbf{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}$, we obtain that $(-20)^{n_1-m_1} \equiv -1 \pmod{23}$. If $n_1 - m_1$ is even -1 is a quadratic residue modulo 23 and if $n_1 - m_1$ is odd then 20 is a quadratic residue modulo 23. But, neither -1 nor 20 are quadratic residues modulo 23. Thus, $e^{\Gamma_5} \neq 1$.

Then, we apply Theorem 3 on the left-hand side of the inequalities (44) with the data

$$t := 4, \quad \eta_1 := 2a, \quad \eta_2 := \alpha, \quad \eta_3 := 1+\alpha^{m_1-n_1}, \quad \eta_4 := 1+\alpha^{m_2-n_2},$$
$$b_1 := k_2-k_1, \quad b_2 := k_2n_1-k_1n_2, \quad b_3 := k_2, \quad b_4 := k_1.$$

Together with combining the right-hand side of (44) with the inequalities (34) and (42), Theorem 3 gives

$$\begin{aligned} n_1 &< 3.02\times 10^{16}(n_1-m_1)(n_2-m_2)(\log n_2) \\ &< 8.33\times 10^{52}(\log n_2)^4. \end{aligned} \tag{46}$$

In the above we used the facts that

$$\min_{1\leq i\leq 2}\{n_i-m_i\} < 1.56\times 10^{11}\log n_2 \quad \text{and} \quad \max_{1\leq i\leq 2}\{n_i-m_i\} < 1.78\times 10^{25}(\log n_2)^2.$$

This was obtained under the assumption that the inequality (45) does not hold. If (45) holds, then so does (46). Thus, we have that inequality (46) holds provided that inequality (42) holds. Otherwise, inequality (41) holds which is a better bound than (46). Hence, conclude that (46) holds in all posibble cases.

By the inequality (22),

$$\log\delta \leq k_1\log\delta \leq n_1\log\alpha + \log 6 < 2.38\times 10^{52}(\log n_2)^4.$$

By substituting this into (30) we get $n_2 < 4.64\times 10^{137}(\log n_2)^{10}$, and then, by Lemma 4, with the data $r := 10,\ H := 4.64\times 10^{137}$ and $L := n_2$, we get that $n_2 < 4.87\times 10^{165}$. This immediately gives that $n_1 < 1.76\times 10^{63}$.

We record what we have proved.

Lemma 6. *Let (k_i, n_i, m_i) be a solution to $x_{k_i} = P_{n_i} + P_{m_i}$, with $3 \leq m_i < n_i$ for $i \in \{1,2\}$ and $1 \leq k_1 < k_2$, then*

$$\max\{k_1, m_1\} < n_1 < 1.76\times 10^{63}, \quad \textit{and} \quad \max\{k_2, m_2\} < n_2 < 4.87\times 10^{165}.$$

5. Reducing the bounds for n_1 and n_2 (I)

In this section we reduce the bounds for n_1 and n_2 given in Lemma 6 to cases that can be computationally treated. For this, we return to the inequalities for Γ_3, Γ_4 and Γ_5.

5.1. **The first reduction (I).** We divide through both sides of the inequality (31) by $(k_2 - k_1)\log\alpha$. We get that

$$\left|\frac{\log(2a)}{\log\alpha} - \frac{k_2 n_1 - k_1 n_2}{k_2 - k_1}\right| < \frac{36 n_2}{\alpha^{\lambda}(k_2 - k_1)} \quad \text{with} \quad \lambda := \min_{1\le i\le 2}\{n_i - m_i\}. \tag{47}$$

We assume that $\lambda \geq 10$. Below we apply Lemma 1. We put $\tau := \frac{\log(2a)}{\log\alpha}$, which is irrational and compute its continued fraction

$$[a_0, a_1, a_2, \ldots] = [1, 3, 3, 1, 11, 1, 2, 1, 1, 1, 3, 1, 1, 1, 2, 5, 1, 15, 2, 19, 1, 1, 2, 2, \ldots]$$

and its convergents

$$\left[\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots\right] = \left[1, \frac{4}{3}, \frac{13}{10}, \frac{17}{13}, \frac{200}{153}, \frac{217}{166}, \frac{634}{485}, \frac{851}{651}, \frac{1485}{1136}, \frac{2336}{1787}, \frac{8493}{6497}, \ldots\right].$$

Furthermore, we note that taking $M := 4.87 \times 10^{165}$ (by Lemma 6), it follows that

$$q_{315} > M > n_2 > k_2 - k_1 \quad \text{and} \quad a(M) := \max\{a_i : 0 \le i \le 315\} = a_{282} = 2107.$$

Thus, by Lemma 1, we have that

$$\left|\tau - \frac{k_2 n_1 - k_1 n_2}{k_2 - k_1}\right| > \frac{1}{2109(k_2 - k_1)^2}. \tag{48}$$

Hence, combining the inequalities (47) and (48), we obtain

$$\alpha^{\lambda} < 75924 n_2 (k_2 - k_1) < 1.75 \times 10^{336},$$

so $\lambda \le 2714$. This was obtained under the assumption that $\lambda \ge 10$, Otherwise, $\lambda < 10 < 2714$ holds as well.

Now, for each $n_i - m_i = \lambda \in [1, 2714]$ we estimate a lower bound $|\Gamma_4|$, with

$$\Gamma_4 \;=\; (k_j - k_i)\log(2a) + (k_j n_i - k_i n_j)\log\alpha + k_j \log(1 + \alpha^{m_i - n_i}) \tag{49}$$

given in the inequality 37, via the procedure described in Subsection 3.3 (LLL-algorithm). We recall that $\Gamma_4 \neq 0$.

We apply Lemma 3 with the data:

$$t := 3, \quad \tau_1 := \log(2a), \quad \tau_2 := \log\alpha, \quad \tau_3 := \log(1 + \alpha^{-\lambda}),$$
$$x_1 := k_j - k_i, \quad x_2 := k_j n_i - k_i n_j, \quad x_3 := k_j.$$

We set $X := 5.4 \times 10^{166}$ as an upper bound to $|x_i| < 11 n_2$ for all $i = 1, 2, 3$, and $C := (20X)^5$. A computer in *Mathematica* search allows us to conclude, together with the inequality (37), that

$$2 \times 10^{-671} < \min_{1\le\lambda\le 2714} |\Gamma_4| < 8 n_2 \alpha^{-\nu}, \quad \text{with} \quad \nu := \min\{n_i, n_j - m_j\}$$

which leads to $\nu \le 6760$. As we have noted before, $\nu = n_1$ (so $n_1 \le 6760$) or $\nu = n_j - m_j$.

Next, we suppose that $n_j - m_j = \nu \le 6760$. Since $\lambda \le 2714$, we have

$$\lambda := \min_{1\le i\le 2}\{n_i - m_i\} \le 2714 \quad \text{and} \quad \chi := \max_{1\le i\le 2}\{n_i - m_i\} \le 6760.$$

Now, returning to the inequality (43) which involves

$$\begin{aligned}\Gamma_5 :\; &= (k_2 - k_1)\log(2a) + (k_2 n_1 - k_1 n_2)\log\alpha \\ &\quad + k_2\log(1 + \alpha^{m_1 - n_1}) - k_1 \log(1 + \alpha^{m_2 - n_2}) \neq 0,\end{aligned} \tag{50}$$

we use again the LLL-algorithm to estimate the lower bound for $|\Gamma_5|$ and thus, find a bound for n_1 that is better than the one given in Lemma 6.

We distinguish the cases $\lambda < \chi$ and $\lambda = \chi$.

5.2. **The case $\lambda < \chi$.** We take $\lambda \in [1, 2714]$ and $\chi \in [\lambda + 1, 6760]$ and apply Lemma 3 with the data:

$$t := 4, \quad \tau_1 := \log(2a), \quad \tau_2 := \log\alpha, \quad \tau_3 := \log(1 + \alpha^{m_1 - n_1}), \quad \tau_4 := \log(1 + \alpha^{m_2 - n_2}),$$
$$x_1 := k_2 - k_1, \quad x_2 := k_2 n_1 - k_1 n_2, \quad x_3 := k_2, \quad x_4 := -k_1.$$

We also put $X := 5.4 \times 10^{166}$ and $C := (20X)^9$. After a computer search in *Mathematica* together with the inequality 43, we can confirm that

$$8 \times 10^{-1342} < \min_{\substack{1 \le \lambda \le 2714 \\ \lambda + 1 \le \chi \le 6760}} |\Gamma_5| < 6 n_2 \alpha^{-n_1}. \tag{51}$$

This leads to the inequality

$$\alpha^{n_1} < 7.5 \times 10^{1341} n_2. \tag{52}$$

Subsitituting for the bound n_2 given in Lemma 6, we get that $n_1 \le 12172$.

5.3. **The case $\lambda = \chi$.** In this case, we have

$$\Lambda_5 := (k_2 - k_1)(\log(2a) + \log(1 + \alpha^{m_1 - n_1})) + (k_2 n_1 - k_1 n_2)\log\alpha \neq 0.$$

We divide through the inequality 43 by $(k_2 - k_1)\log\alpha$ to obtain

$$\left| \frac{\log(2a) + \log(1 + \alpha^{m_1 - n_1})}{\log\alpha} - \frac{k_2 n_1 - k_1 n_2}{k_2 - k_1} \right| < \frac{21 n_2}{\alpha^{n_1}(k_2 - k_1)} \tag{53}$$

We now put

$$\tau_\lambda := \frac{\log(2a) + \log(1 + \alpha^{-\lambda})}{\log\alpha}$$

and compute its continued fractions $[a_0^{(\lambda)}, a_1^{(\lambda)}, \ldots]$ and its convergents $[p_0^{(\lambda)}/q_0^{(\lambda)}, p_1^{(\lambda)}/q_1^{(\lambda)}, \ldots]$ for each $\lambda \in [1, 2714]$. Furthermore, for each case we find an integer t_λ such that $q_{t_\lambda}^{(\lambda)} > M := 4.87 \times 10^{165} > n_2 > k_2 - k_1$ and calculate

$$a(M) := \max_{1 \le \lambda \le 2714} \left\{ a_i^{(\lambda)} : 0 \le i \le t_\lambda \right\}.$$

A computer search in *Mathematica* reveals that for $\lambda = 321$, $t_\lambda = 330$ and $i = 263$, we have that $a(M) = a_{321}^{(330)} = 306269$. Hence, combining the conclusion of Lemma 1 and the inequality (53), we get

$$\alpha^{n_1} < 21 \times 306271 n_2 (k_2 - k_1) < 1.525 \times 10^{338},$$

so $n_1 \le 2730$. Hence, we obtain that $n_1 \le 12172$ holds in all cases ($\nu = n_1$, $\lambda < \chi$ or $\lambda = \chi$). By the inequality (22), we have that

$$\log\delta \le k_1 \log\delta \le n_1 \log\alpha + \log 6 < 3475.$$

By considering the second inequality in (30), we can conclude that $n_2 \le 9.9 \times 10^{39} (\log n_2)^2$, which immediately yields $n_2 < 3.36 \times 10^{44}$, by a simple application of Lemma 4. We summarise the first cycle of our reduction process as follows:

$$n_1 \le 12172 \quad \text{and} \quad n_2 \le 3.36 \times 10^{44}. \tag{54}$$

From the above, we note that the upper bound on n_2 represents a very good reduction of the bound given in Lemma 6. Hence, we expect that if we restart our reduction cycle with the new bound on n_2, then we get a better bound on n_1. Thus, we return to the inequality (47) and take $M := 3.36 \times 10^{44}$. A computer search in *Mathematica* reveals that

$$q_{88} > M > n_2 > k_2 - k_1 \quad \text{and} \quad a(M) := \max\{a_i : 0 \leq i \leq 88\} = a_{54} = 373,$$

from which it follows that $\lambda \leq 752$. We now return to (49) and we put $X := 3.36 \times 10^{44}$ and $C := (10X)^5$ and then apply the LLL-algorithm in Lemma 3 to $\lambda \in [1, 752]$. After a computer search, we get

$$5.33 \times 10^{-184} < \min_{1 \leq \lambda \leq 752} |\Gamma_4| < 8n_2\alpha^{-\nu},$$

then $\nu \leq 1846$. By continuing under the assumption that $n_j - m_j = \nu \leq 1846$, we return to (50) and put $X := 3.36 \times 10^{44}$, $C := (10X)^9$ and $M := 3.36 \times 10^{44}$ for the case $\lambda < \chi$ and $\lambda = \chi$. After a computer search, we confirm that

$$2 \times 10^{-366} < \min_{\substack{1 \leq \lambda \leq 752 \\ \lambda + 1 \leq \chi \leq 1846}} |\Gamma_5| < 6n_2\alpha^{-n_1},$$

gives $n_1 \leq 3318$, and $a(M) = a_{175}^{(205)} = 206961$, leads to $n_1 \leq 772$. Hence, in both cases $n_1 \leq 3318$ holds. This gives $n_2 \leq 5 \times 10^{42}$ by a similar procedure as before, and $k_1 \leq$.

We record what we have proved.

Lemma 7. *Let (k_i, n_i, m_i) be a solution to $X_i = P_{n_i} + P_{m_i}$, with $3 \leq m_i < n_i$ for $i = 1, 2$ and $1 \leq k_1 < k_2$, then*

$$m_1 < n_1 \leq 3318, \quad k_1 \leq 3125 \quad \textit{and} \quad n_2 \leq 5 \times 10^{42}.$$

5.4. **The final reduction (I).** Returning back to (17) and (19) and using the fact that (x_1, y_1) is the smallest positive solution to the Pell equation (1), we obtain

$$\begin{aligned} x_k &= \frac{1}{2}(\delta^k + \sigma^k) = \frac{1}{2}\left(\left(x_1 + y_1\sqrt{d}\right)^k + \left(x_1 - y_1\sqrt{d}\right)^k\right) \\ &= \frac{1}{2}\left(\left(x_1 + \sqrt{x_1^2 \mp 1}\right)^k + \left(x_1 - \sqrt{x_1^2 \mp 1}\right)^k\right) := Q_k^{\pm}(x_1). \end{aligned}$$

Thus, we return to the Diophantine equation $x_{k_1} = P_{n_1} + P_{m_1}$ and consider the equations

$$Q_{k_1}^{+}(x_1) = P_{n_1} + P_{m_1} \quad \text{and} \quad Q_{k_1}^{-}(x_1) = P_{n_1} + P_{m_1}, \tag{55}$$

with $k_1 \in [1, 3125]$, $m_1 \in [3, 3318]$ and $n_1 \in [m_1 + 1, 3318]$.

Besides the trivial case $k_1 = 1$, with the help of a computer search in *Mathematica* on the above equations in (55), we list the only nontrivial solutions in the tables below. We also note that $3 + 2\sqrt{2} = (1 + \sqrt{2})^2$, so these solutions come from the same Pell equation when $d = 2$.

$Q_{k_1}^{+}(x_1)$				
k_1	x_1	y_1	d	δ
2	2	1	3	$2+\sqrt{3}$
2	3	2	2	$3+2\sqrt{2}$
2	4	1	15	$4+\sqrt{15}$
2	5	2	6	$5+2\sqrt{6}$
2	21	2	110	$21+2\sqrt{110}$
2	22	1	483	$22+\sqrt{483}$
2	47	4	138	$47+4\sqrt{138}$

$Q_{k_1}^{-}(x_1)$				
k_1	x_1	y_1	d	δ
2	1	1	2	$1+\sqrt{2}$
2	2	1	5	$2+\sqrt{5}$
2	3	1	10	$3+\sqrt{10}$
2	4	1	17	$4+\sqrt{17}$
2	5	1	26	$5+\sqrt{26}$
2	9	1	82	$9+\sqrt{82}$
2	10	1	101	$10+\sqrt{101}$
2	17	1	290	$17+\sqrt{290}$
2	42	1	1765	$42+\sqrt{1765}$
2	47	1	2210	$47+\sqrt{2210}$
2	63	1	3970	$63+\sqrt{3970}$

From the above tables, we set each $\delta := \delta_t$ for $t = 1, 2, \ldots 17$. We then work on the linear forms in logarithms Γ_1 and Γ_2, in order to reduce the bound on n_2 given in Lemma 7. From the inequality (28), for $(k, n, m) := (k_2, n_2, m_2)$, we write

$$\left| k_2 \frac{\log \delta_t}{\log \alpha} - n_2 + \frac{\log(2a)}{\log(\alpha^{-1})} \right| < \left(\frac{5}{\log \alpha} \right) \alpha^{-(n_2 - m_2)}, \quad \text{for} \quad t = 1, 2, \ldots 17. \tag{56}$$

We put

$$\tau_t := \frac{\log \delta_t}{\log \alpha}, \qquad \mu_t := \frac{\log(2a)}{\log(\alpha^{-1})} \qquad \text{and} \quad (A_t, B_t) := \left(\frac{5}{\log \alpha}, \alpha \right).$$

We note that τ_t is transcendental by the Gelfond-Schneider's Theorem and thus, τ_t is irrational. We can rewrite the above inequality, 56 as

$$0 < |k_2 \tau_t - n_2 + \mu_t| < A_t B_t^{-(n_2 - m_2)}, \quad \text{for} \quad t = 1, 2, \ldots, 17. \tag{57}$$

We take $M := 5 \times 10^{42}$ which is the upper bound on n_2 according to Lemma 7 and apply Lemma 2 to the inequality (57). As before, for each τ_t with $t = 1, 2, \ldots, 17$, we compute its continued fraction $[a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, \ldots]$ and its convergents $p_0^{(t)}/q_0^{(t)}, p_1^{(t)}/q_1^{(t)}, p_2^{(t)}/q_2^{(t)}, \ldots$. For each case, by means of a computer search in *Mathematica*, we find and integer s_t such that

$$q_{s_t}^{(t)} > 3 \times 10^{43} = 6M \qquad \text{and} \qquad \epsilon_t := ||\mu_t q^{(t)}|| - M||\tau_t q^{(t)}| > 0.$$

We finally compute all the values of $b_t := \lfloor \log(A_t q_{s_t}^{(t)}/\epsilon_t)/\log B_t \rfloor$. The values of b_t correspond to the upper bounds on $n_2 - m_2$, for each $t = 1, 2, \ldots, 17$, according to Lemma 2. The results of the computation for each t are recorded in the table below.

t	δ_t	s_t	q_{s_t}	$\epsilon_t >$	b_t
1	$2+\sqrt{3}$	85	8.93366×10^{43}	0.3100	374
2	$4+\sqrt{15}$	90	3.90052×10^{43}	0.3124	371
3	$5+2\sqrt{6}$	80	3.16032×10^{43}	0.0122	382
4	$21+2\sqrt{110}$	88	6.33080×10^{43}	0.2200	374
5	$22+\sqrt{483}$	75	4.19689×10^{43}	0.2361	372
6	$47+4\sqrt{138}$	96	7.76442×10^{43}	0.3732	373
7	$1+\sqrt{2}$	78	1.46195×10^{44}	0.3328	375
8	$2+\sqrt{5}$	94	1.48837×10^{44}	0.2146	377
9	$3+\sqrt{10}$	88	4.21425×10^{43}	0.1347	374
10	$4+\sqrt{17}$	92	1.11753×10^{44}	0.2529	375
11	$5+\sqrt{26}$	98	3.23107×10^{43}	0.1043	374
12	$9+\sqrt{82}$	74	5.25207×10^{43}	0.2181	373
13	$10+\sqrt{101}$	94	1.86122×10^{44}	0.2672	377
14	$17+\sqrt{290}$	87	1.06422×10^{44}	0.0193	384
15	$42+\sqrt{1765}$	78	3.81406×10^{43}	0.1768	373
16	$47+\sqrt{2210}$	94	3.92482×10^{43}	0.4476	370
17	$63+\sqrt{3970}$	85	6.00550×10^{43}	0.4056	371

By replacing $(k,n,m) := (k_2,n_2,m_2)$ in the inequality (25), we can write

$$\left| k_2\frac{\log\delta_t}{\log\alpha} - n_2 + \frac{\log(2a(1+\alpha^{-(n_2-m_2)}))}{\log(\alpha^{-1})}\right| < \left(\frac{3}{\log\alpha}\right)\alpha^{-n_2}, \tag{58}$$

for $t = 1,2,\ldots,17$.

We now put

$$\tau_t := \frac{\log\delta_t}{\log\alpha}, \quad \mu_{t,n_2-m_2} := \frac{\log(2a(1+\alpha^{-(n_2-m_2)}))}{\log(\alpha^{-1})} \quad \text{and} \quad (A_t,B_t) := \left(\frac{3}{\log\alpha},\alpha\right).$$

With the above notations, we can rewrite (58) as

$$0 < |k_2\tau_t - n_2 + \mu_{t,n_2-m_2}| < A_t B_t^{-n_2}, \quad \text{for} \quad t = 1,2,\ldots 17. \tag{59}$$

We again apply Lemma 2 to the above inequality (59), for

$$t = 1,2,\ldots,17, \quad n_2 - m_2 = 1,2,\ldots,b_t, \quad \text{with} \quad M := 5\times 10^{43}.$$

We take

$$\epsilon_{t,n_2-m_2} := ||\mu_t q^{(t,n_2-m_2)}|| - M||\tau_t q^{(t,n_2-m_2)}|| > 0,$$

and

$$b_t = b_{t,n_2-m_2} := \lfloor \log(A_t q_{s_t}^{(t,n_2-m_2)}/\epsilon_{t,n_2-m_2})/\log B_t \rfloor.$$

With the help of Mathematica, we obtain that

t	1	2	3	4	5	6	7	8	9
b_{t,n_2-m_2}	388	389	394	394	393	394	396	392	392
t	10	11	12	13	14	15	16	17	
b_{t,n_2-m_2}	396	392	408	390	396	396	388	389	

Thus, $\max\{b_{t,n_2-m_2} : t = 1,2,\ldots,17 \quad \text{and} \quad n_2 - m_2 = 1,2,\ldots b_t\} \leq 408.$

Thus, by Lemma 2, we have that $n_2 \leq 408$, for all $t = 1, 2, \ldots, 17$, and by the inequality (23) we have that $n_1 \leq n_2 + 4$. From the fact that $\delta^k \leq 2\alpha^{n+3}$, we can conclude that $k_1 < k_2 \leq 133$. Collecting everything together, our problem is reduced to search for the solutions for (21) in the following range

$$1 \leq k_1 < k_2 \leq 133, \quad 0 \leq m_1 < n_1 \in [3, 408] \quad \text{and} \quad 0 \leq m_2 < n_2 \in [3, 408]. \tag{60}$$

After a computer search on the equation (21) on the above ranges, we obtained the following solutions, which are the only solutions for the exceptional d cases we have stated in Theorem 1:

For the $+1$ case:

$$\begin{aligned}
&(d = 2) && x_1 = 3 = P_6 + P_0 = P_5 + P_3, \quad x_2 = 17 = P_{12} + P_3;\\
&(d = 3) && x_1 = 2 = P_3 + P_0 = P_3 + P_3,\ x_2 = 7 = P_9 + P_0 = P_7 + P_6,\\
& && x_3 = 26 = P_{13} + P_8;\\
&(d = 6) && x_1 = 5 = P_8 + P_0 = P_7 + P_3 = P_6 + P_5,\\
& && x_2 = 49 = P_{16} + P_0 = P_{15} + P_{12} = P_{14} + P_{13};\\
&(d = 15) && x_1 = 4 = P_7 + P_0 = P_6 + P_3 = P_5 + P_5, \quad x_2 = 31 = P_14 + P_6;\\
&(d = 110) && x_1 = 21 = P_{13} + P_0 = P_{12} + P_8 = P_{11} + P_{10},\\
& && x_2 = 881 = P_{26} + P_{17} = P_{25} + P_{22};\\
&(d = 483) && x_1 = 22 = P_{13} + P_3, \quad x_2 = 967 = P_{26} + P_{20} = P_{25} + P_{23}.
\end{aligned}$$

For the -1 case:

$$\begin{aligned}
&(d = 2) && x_1 = 1 = P_3 + P_0, \quad x_2 = 7 = P_9 + P_0 = P_8 + P_5 = P_7 + P_6,\\
& && x_3 = 41 = P_{15} + P_7 = P_{14} + P_{10} = P_{13} + P_{12};\\
&(d = 5) && x_1 = 2 = P_5 + P_0 = P_3 + P_3, \quad x_2 = 38 = P_{15} + P_3;\\
&(d = 10) && x_1 = 3 = P_6 + P_0 = P_5 + P_3, \quad x_2 = 117 = P_{19} + P_6;\\
&(d = 17) && x_1 = 4 = P_7 + P_0 = P_6 + P_3 = P_5 + P_5, \quad x_2 = P_{22} + P_6.
\end{aligned}$$

This completes the proof of Theorem 1. □

6. Proof of Theorem 2

The proof of Theorem 2 will be similar to that of Theorem 1. We also give the details for the benefit of the reader. Further, for technical reasons in our proof, we assume that $d \geq 5$ and then treat the cases $d \in \{2, 3\}$ during the reduction procedure.

Let (X_1, Y_1) be the smallest positive integer solution to the Pell quation (2). We Put

$$\rho := \frac{X_1 + Y_1\sqrt{d}}{2} \quad \text{and} \quad \varrho = \frac{X_1 - Y_1\sqrt{d}}{2}. \tag{61}$$

From which we get that

$$\rho \cdot \varrho = \frac{X_1^2 - dY_1^2}{4} =: \epsilon, \quad \text{where} \quad \epsilon \in \{\pm 1\}. \tag{62}$$

Then

$$X_n = \rho^k + \varrho^k. \tag{63}$$

Since $\rho \geq \frac{1+\sqrt{5}}{2}$, it follows that the estimate

$$\frac{\rho^k}{\alpha^2} \leq X_k \leq 2\rho^k \quad \text{holds for all} \quad k \geq 1. \tag{64}$$

Similarly, as before, we assume that (k_1, n_1, m_1) and (k_2, n_2, m_2) are triples of integers such that

$$X_{k_1} = P_{n_1} + P_{m_1} \quad \text{and} \quad X_{k_2} = P_{n_2} + P_{m_2} \tag{65}$$

We asuume that $1 \leq k_1 < k_2$. We also assume that $4 \leq m_j < n_j$ for $j = 1, 2$. We set $(k, n, m) := (k_j, n_j, m_j)$, for $j = 1, 2$. Using the inequalities (12) and (64), we get from (65) that

$$\frac{\rho^k}{\alpha^2} \leq X_k = P_n + P_m \leq 2\alpha^{n-1} \quad \text{and} \quad \alpha^{n-2} \leq P_n + P_m = X_k \leq 2\rho^k.$$

The above inequalities give

$$(n-2)\log\alpha - \log 2 < k\log\rho < (n+1)\log\alpha + \log 2.$$

Dividing through by $\log\alpha$ and setting $c_1 := 1/\log\alpha$, as before, we get that

$$-2 - c_1\log 2 < c_1 k\log\rho - n < 1 + c_1\log 2,$$

and since $\alpha^3 > 2$, we get

$$|n - c_1\log\rho| < 5. \tag{66}$$

Furthermore, $k < n$, for if not, we would then get that

$$\rho^n \leq \rho^k < 2\alpha^{n+1}, \quad \text{implying} \quad \left(\frac{\rho}{\alpha}\right)^n < 2\alpha,$$

which is false since $\rho \leq \frac{1+\sqrt{5}}{2}$, $1.32 < \alpha < 1.33$ and $n \geq 5$.

Besides, given that $k_1 < k_2$, we have by (13) and (65) that

$$\alpha^{n_1-2} \leq P_{n_1} \leq P_{n_1} + P_{m_1} = X_{k_1} < X_{k_2} = P_{n_2} + P_{m_2} \leq 2P_{n_2} < 2\alpha^{n_2-1}.$$

Thus, as before, we get that

$$n_1 < n_2 + 4. \tag{67}$$

6.1. **An inequality for n and k (II).** Using the equations (9) and (61) and (65), we get

$$\rho^k + \varrho^k = P_n + P_m = a\alpha^n + e(n) + a\alpha^m + e(m)$$

So,

$$\rho^k - a(\alpha^n + \alpha^m) = -\varrho^k + e(n) + e(m),$$

and by (12), we have

$$\begin{aligned} \left|\rho^k a^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1} - 1\right| &\leq \frac{1}{\rho^k a(\alpha^n + \alpha^m)} + \frac{2|b|}{\alpha^{n/2}a(\alpha^n + \alpha^m)} \\ &\quad + \frac{2|b|}{\alpha^{m/2}a(\alpha^n + \alpha^m)} \\ &\leq \frac{1}{a\alpha^n}\left(\frac{1}{\rho^k} + \frac{2|b|}{\alpha^{n/2}} + \frac{2|b|}{\alpha^{m/2}}\right) < \frac{2.5}{\alpha^n}. \end{aligned}$$

Thus, we have

$$\left|\rho^k a^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1}-1\right| \quad < \quad \frac{2.5}{\alpha^n}. \tag{68}$$

Put

$$\Lambda_1^{'} := \rho^k a^{-1}\alpha^{-n}(1+\alpha^{m-n})^{-1}-1$$

and

$$\Gamma_1' := k\log\rho - \log a - n\log\alpha - \log(1+\alpha^{m-n}).$$

Since $|\Lambda_1'| = |e^{\Gamma_1'}-1| < 0.83$ for $n \geq 4$ (because $2.5/\alpha^4 < 0.83$), it follows that $e^{|\Gamma_1'|} < 4$ and so

$$|\Gamma_1'| < e^{|\Gamma_1'|}|e^{\Gamma_1'}-1| < \frac{10}{\alpha^n}.$$

Thus, we get that

$$\left|k\log\rho - \log a - n\log\alpha - \log(1+\alpha^{m-n})\right| < \frac{10}{\alpha^n}. \tag{69}$$

We apply Theorem 3 on the left-hand side of (68) with the data:

$$t := 4, \quad \eta_1 := \rho, \quad \eta_2 := a, \quad \eta_3 := \alpha, \quad \eta_4 := 1+\alpha^{m-n},$$
$$b_1 := k, \quad b_2 := -1, \quad b_3 := -n, \quad b_4 := -1.$$

Furthermore, we take same the number field as before, $\mathbb{K} = \mathbb{Q}(\sqrt{d},\alpha)$ with degree $D = 6$. We also take $D_{\mathbb{K}} = n$. First we note that the left-hand side of (24) is non-zero, since otherwise,

$$\rho^k = a(\alpha^n + \alpha^m).$$

By the same argument as before, we get a contradiction. Thus, $\Lambda_1' \neq 0$ and we can apply Theorem 3. Further,

$$a = \frac{\alpha(\alpha+1)}{3\alpha^2-1},$$

the mimimal polynomial of a is $23x^3 - 23x^2 + 6x - 1$ and has roots a, b, c. Since $\max\{a,b,c\} < 1$ (by (11)), then $h(\eta_2) = h(a) = \frac{1}{3}\log 23$. Thus, we can take $A_1 := 3\log\rho$, $A_2 := 2\log 23$, $A_3 := 2\log\alpha$, and $A_4 := 2(n-m)\log\alpha + 6\log 2$.

Now, Theorem 3 tells us that

$$\begin{aligned}\log|\Lambda_1'| \quad > \quad & -1.4\times 30^7 \times 4^{4.5}\times 6^2(1+\log 6)(1+\log n)(3\log\rho)\\ & \times(2\log 23)(2\log\alpha)(2(n-m)\log\alpha + 6\log 2)\\ > \quad & -2.08\times 10^{17}(n-m)(\log n)(\log\rho).\end{aligned}$$

Comparing the above inequality with (68), we get

$$n\log\alpha - \log 2.5 < 2.08\times 10^{17}(n-m)(\log n)(\log\rho).$$

Hence, we get that

$$n < 7.40\times 10^{17}(n-m)(\log n)(\log\rho). \tag{70}$$

We now return to the equation $X_k = P_n + P_m$ and rewrite it as

$$\rho^k - a\alpha^n = -\varrho^k + e(n) + P_m,$$

we obtain

$$\left|\rho^k a^{-1}\alpha^{-n} - 1\right| \leq \frac{1}{a\alpha^{n-m}}\left(\frac{1}{\alpha} + \frac{1}{\alpha^{m+n/2}} + \frac{1}{\rho^k\alpha^m}\right) < \frac{3}{\alpha^{n-m}}. \tag{71}$$

Put

$$\Lambda_2' := \rho^k a^{-1}\alpha^{-n} - 1, \quad \Gamma_2' := k\log\rho - \log a - n\log\alpha.$$

We assume for technical reasons that $n - m \geq 10$. So $|e^{\Lambda_2} - 1| < \frac{1}{2}$. It follows that

$$|k\log\rho - \log a - n\log\alpha| = |\Gamma_2'| < e^{|\Lambda_2'|}|e^{\Lambda_2'} - 1| < \frac{6}{\alpha^{n-m}}. \tag{72}$$

Furthermore, $\Lambda_2' \neq 0$ (so $\Gamma_2' \neq 0$), since $\rho^k \in \mathbb{Q}(\alpha)$ by the previous argument.

We now apply Theorem 3 to the left-hand side of (71) with the data

$$t := 3, \quad \eta_1 := \rho, \quad \eta_2 := a, \quad \eta_3 := \alpha, \quad b_1 := k, \quad b_2 := -1, \quad b_3 := -n.$$

Thus, we have the same $A_1,\ A_2, A_3$ as before. Then, by Theorem 3, we conclude that

$$\log|\Lambda| > -9.50 \times 10^{14}(\log\rho)(\log n)(\log\alpha).$$

By comparing with (71), we get

$$n - m < 9.52 \times 10^{14}(\log\rho)(\log n). \tag{73}$$

This was obtained under the assumption that $n - m \geq 10$, but if $n - m < 10$, then the inequality also holds as well. We replace the bound (73) on $n - m$ in (70) and use the fact that $\rho^k \leq 2\alpha^{n+1}$, to obtain bounds on n and k in terms of $\log n$ and $\log\rho$. We again record what we have proved.

Lemma 8. *Let* (k, n, m) *be a solution to the equation* $X_k = P_n + P_m$ *with* $3 \leq m < n$, *then*

$$k < 1.98 \times 10^{32}(\log n)^2(\log\rho) \quad \text{and} \quad n < 7.03 \times 10^{32}(\log n)^2(\log\rho)^2. \tag{74}$$

6.2. **Absolute bounds (II).** We recall that $(k, n, m) = (k_j, n_j, m_j)$, where $3 \leq m_j < n_j$, for $j = 1, 2$ and $1 \leq k_1 < k_2$. Further, $n_j \geq 4$ for $j = 1, 2$. We return to (72) and write

$$\left|\Gamma_2^{(j)'}\right| := |k_j\log\rho - \log a - n_j\log\alpha| < \frac{6}{\alpha^{n_j - m_j}}, \quad \text{for} \quad j = 1, 2.$$

We do a suitable cross product between $\Gamma_2^{(1)'}$, $\Gamma_2^{(2)'}$ and k_1, k_2 to eliminate the term involving $\log\rho$ in the above linear forms in logarithms:

$$\begin{aligned} |\Gamma_3'| &:= |(k_1 - k_2)\log a + (k_1n_2 - k_2n_1)\log\alpha| = |k_2\Gamma_2^{(1)'} - k_1\Gamma_2^{(2)'}| \\ &\leq k_2|\Gamma_2^{(1)'}| + k_1|\Gamma_2^{(2)'}| \leq \frac{6k_2}{\alpha^{n_1-m_1}} + \frac{6k_1}{\alpha^{n_2-m_2}} \leq \frac{12n_2}{\alpha^{\lambda'}}, \end{aligned} \tag{75}$$

where

$$\lambda' := \min_{1\leq j\leq 2}\{n_j - m_j\}$$

.

We need to find an upper bound for λ'. If $12n_2/\alpha^{\lambda'} > 1/2$, we then get

$$\lambda' < \frac{\log(24n_2)}{\log\alpha} < 4\log(24n_2). \tag{76}$$

Otherwise, $|\Gamma_3'| < \frac{1}{2}$, so

$$\left|e^{\Gamma_3'} - 1\right| = \left|a^{k_1-k_2}\alpha^{k_1n_2-k_2n_1} - 1\right| < 2|\Gamma_3'| < \frac{24n_2}{\alpha^{\lambda'}}. \tag{77}$$

We apply Theorem 3 with the data: $t := 2$, $\eta_1 := a$, $\eta_2 := \alpha$, $b_1 := k_1 - k_2$, $b_2 := k_1n_2 - k_2n_1$. We take the number field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $D = 3$. We begin by checking that $e^{\Gamma_3'} - 1 \neq 0$ (so $\Gamma_3' \neq 0$). This is true because α and a are multiplicatively independent, since α is a unit in the ring of integers $\mathbb{Q}(\alpha)$ while the norm of a is $1/23$.

We note that $|k_1 - k_2| < k_2 < n_2$. Further, from (75), we have

$$|k_2n_1 - k_1n_2| < (k_2 - k_1)\frac{|\log a|}{\log\alpha} + \frac{12k_2}{\alpha^\lambda \log\alpha} < 13k_2 < 13n_2$$

given that $\lambda \geq 1$. So, we can take $B := 13n_2$. By Theorem 3, with the same $A_1 := \log 23$ and $A_2 := \log\alpha$, we have that

$$\log|e^{\Gamma_3'} - 1| > -4.63 \times 10^{10}(\log n_2)(\log\alpha).$$

By comparing this with (77), we get

$$\lambda' < 1.62 \times 10^{11} \log n_2. \tag{78}$$

Note that (78) is better than (77), so (78) always holds. Without loss of generality, we can assume that $\lambda' = n_j - m_j$, for $j = 1, 2$ fixed.

We set $\{j, i\} = \{1, 2\}$ and return to (69) to replace $(k, n, m) = (k_i, n_i, m_i)$:

$$|\Gamma_1^{(i)'}| = \left|k_i \log\rho - \log a - n_i \log\alpha - \log(1 + \alpha^{m_i - n_i})\right| < \frac{10}{\alpha^{n_i}}, \tag{79}$$

and also return to (72), with $(k, n, m) = (k_j, n_j, m_j)$:

$$|\Gamma_2^{(j)'}| = |k_j \log\rho - \log a - n_j \log\alpha| < \frac{6}{\alpha^{n_j - m_j}}. \tag{80}$$

We perform a cross product on (79) and (80) in order to eliminate the term on $\log\rho$:

$$\begin{aligned} |\Gamma_4'| &:= \left|(k_j - k_i)\log a + (k_jn_i - k_in_j)\log\alpha + k_j \log(1 + \alpha^{m_i - n_i})\right| \\ &= \left|k_i\Gamma_2^{(j)'} - k_j\Gamma_1^{(i)'}\right| \leq k_i\left|\Gamma_2^{(j)'}\right| + k_j\left|\Gamma_1^{(i)'}\right| \\ &< \frac{6k_i}{\alpha^{n_j - m_j}} + \frac{10k_j}{\alpha^{n_i}} < \frac{16n_2}{\alpha^{\nu'}} \qquad (81) \end{aligned}$$

with $\nu' := \min\{n_i, n_j - m_j\}$. As before, we need to find an upper bound on ν'. If $16n_2/\alpha^{\nu'} > 1/2$, then we get

$$\nu' < \frac{\log(32n_2)}{\log\alpha} < 4\log(32n_2). \tag{82}$$

Otherwise, $|\Gamma_4'| < 1/2$, so we have

$$\left|e^{\Gamma_4'} - 1\right| \leq 2|\Gamma_4'| < \frac{32n_2}{\alpha^{\nu'}}. \tag{83}$$

In order to apply Theorem 3, first if $e^{\Gamma_4'} = 1$, we obtain

$$a^{k_i - k_j} = \alpha^{k_jn_i - k_in_j}(1 + \alpha^{-\lambda'})^{k_j}.$$

Since α is a unit, the right-hand side in above is an algebraic integer. This is a contradiction because $k_1 < k_2$ so $k_i - k_j \neq 0$, and neither a nor a^{-1} are algebraic intgers. Hence $e^{\Gamma_4'} \neq 1$. By assuming that $\nu' \geq 100$, we apply Theorem 3 with the data:

$$\begin{aligned} &t := 3, \quad \eta_1 := a, \quad \eta_2 := \alpha, \quad \eta_3 := 1 + \alpha^{-\lambda'}, \\ &b_1 := k_j - k_i, \quad b_2 := k_jn_i - k_in_j, \quad b_3 := k_j, \end{aligned}$$

and the inequalities (78) and (83). We get

$$\nu' = \min\{n_i, n_j - m_j\} < 1.85 \times 10^{13}\lambda' \log n_2 < 3 \times 10^{24}(\log n_2)^2. \tag{84}$$

The above inequality also holds when $\nu' < 100$. Further, it also holds when the inequality (82) holds. So the above inequality holds in all cases. Note that the case $\{i, j\} = \{2, 1\}$ leads to $n_1 - m_1 \le n_1 \le n_2 + 4$ whereas $\{i, j\} = \{1, 2\}$ lead to $\nu' = \min\{n_1, n_2 - m_2\}$. Hence, either the minimum is n_1, so

$$n_1 < 3 \times 10^{24}(\log n_2)^2, \tag{85}$$

or the minimum is $n_j - m_j$ and from the inequality (34) we get that

$$\max_{1\le j\le 2}\{n_j - m_j\} < 3 \times 10^{24}(\log n_2)^2. \tag{86}$$

Next, we assume that we are in the case (86). We evaluate (79) in $i = 1, 2$ and make a suitable cross product to eliminate the term involving $\log \rho$:

$$\begin{aligned} |\Gamma_5'| &:= \big|(k_2 - k_1)\log a + (k_2 n_1 - k_1 n_2)\log\alpha \\ &\quad + k_2 \log(1 + \alpha^{m_1 - n_1}) - k_1 \log(1 + \alpha^{m_2 - n_2})\big| \\ &= \left|k_1 \Gamma_1^{(2)} - k_2 \Gamma_1^{(1)}\right| \le k_1 \left|\Gamma_1^{(2)}\right| + k_2 \left|\Gamma_1^{(1)}\right| < \frac{20 n_2}{\alpha^{n_1}}. \end{aligned} \tag{87}$$

In the above inequality we used the inequality (23)to conclude that $\min\{n_1, n_2\} \ge n_1 - 4$ as well as the fact that $n_i \ge 4$ for $i = 1.2$. Next, we apply a linear form in four logarithms to obtain an upper bound to n_1. As in the previous calculations, we pass from (87) to

$$\left|e^{\Gamma_5'} - 1\right| < \frac{40 n_2}{\alpha^{n_1}}, \tag{88}$$

which is implied by (87) except if n_1 is very small, say

$$n_1 \le 4 \log(40 n_2). \tag{89}$$

Thus, we assume that (89) does not hold, therefore (88). Then to apply Theorem 3, we fist justify that $e^{\Gamma_5'} \ne 1$. Otherwise,

$$a^{k_1 - k_2} = \alpha^{k_2 n_1 - k_1 n_2}(1 + \alpha^{n_1 - m_1})^{k_2}(1 + \alpha^{n_2 - m_2})^{-k_1}.$$

By a similar argument as before, we get a contradiction. Thus, $e^{\Gamma_5'} \ne 1$.

Then, we apply Theorem 3 on the left-hand side of the inequalities (44) with the data

$$t := 4, \quad \eta_1 := a, \quad \eta_2 := \alpha, \quad \eta_3 := 1 + \alpha^{m_1 - n_1}, \quad \eta_4 := 1 + \alpha^{m_2 - n_2},$$
$$b_1 := k_2 - k_1, \quad b_2 := k_2 n_1 - k_1 n_2, \quad b_3 := k_2, \quad b_4 := k_1.$$

Together with combining the right-hand side of (88) with the inequalities (78) and (86), Theorem 3 gives

$$\begin{aligned} n_1 &< 4.99 \times 10^{15}(n_1 - m_1)(n_2 - m_2)(\log n_2) \\ &< 2.43 \times 10^{51}(\log n_2)^4. \end{aligned} \tag{90}$$

In the above we used the facts that

$$\min_{1\le i\le 2}\{n_i - m_i\} < 1.62 \times 10^{11} \log n_2 \quad \text{and} \quad \max_{1\le i\le 2}\{n_i - m_i\} < 3 \times 10^{24}(\log n_2)^2.$$

This was obtained under the assumption that the inequality (89) does not hold. If (89) holds, then so does (90). Thus, we have that inequality (90) holds provided that inequality

(86) holds. Otherwise, inequality (85) holds which is a better bound than (90). Hence, conclude that (90) holds in all posibble cases.

By the inequality (66),

$$\log\rho \leq k_1 \log\rho \leq n_1 \log\alpha + \log 5 < 6.92 \times 10^{50} (\log n_2)^4.$$

By substituting this into (74) we get $n_2 < 3.67 \times 10^{134} (\log n_2)^{10}$, and then, by Lemma 4, with the data $r := 10$, $P := 3.67 \times 10^{134}$, $L := n_2$, we get that $n_2 < 3.07 \times 10^{162}$. This immediately gives that $n_1 < 4.76 \times 10^{61}$.

We record what we have proved.

Lemma 9. *Let* (k_i, n_i, m_i) *be a solution to* $X_{k_i} = P_{n_i} + P_{m_i}$, *with* $3 \leq m_i < n_i$ *for* $i \in \{1, 2\}$ *and* $1 \leq k_1 < k_2$, *then*

$$\max\{k_1, m_1\} < n_1 < 4.76 \times 10^{61}, \quad \textit{and} \quad \max\{k_2, m_2\} < n_2 < 3.07 \times 10^{162}.$$

7. Reducing the bounds for n_1 and n_2 (II)

In this section we reduce the bounds for n_1 and n_2 given in Lemma 6 to cases that can be computationally treated. For this, we return to the inequalities for Γ_3', Γ_4' and Γ_5'.

7.1. **The first reduction (II).** We divide through both sides of the inequality (75) by $(k_2 - k_1)\log\alpha$. We get that

$$\left| \frac{|\log a|}{\log\alpha} - \frac{k_2 n_1 - k_1 n_2}{k_2 - k_1} \right| < \frac{42 n_2}{\alpha^{\lambda'} (k_2 - k_1)} \quad \text{with} \quad \lambda' := \min_{1\leq i \leq 2} \{n_i - m_i\}. \tag{91}$$

We assume that $\lambda' \geq 10$. Below we apply Lemma 1. We put $\tau' := \frac{|\log a|}{\log\alpha}$, which is irrational and compute its continued fraction

$$[a_0, a_1, a_2, \ldots] = [1, 6, 2, 1, 18, 166, 1, 2, 13, 1, 2, 5, 1, 5, 1, 2, 3, 1, 1, 31, 1, 3, 2, 3, \ldots]$$

and its convergents

$$\left[\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots\right] = \left[1, \frac{7}{6}, \frac{15}{13}, \frac{22}{19}, \frac{411}{355}, \frac{68248}{58949}, \frac{68659}{59304}, \frac{205566}{177557}, \frac{2741017}{2367545}, \frac{2946583}{2545102}, \ldots\right].$$

Furthermore, we note that taking $N := 3.07 \times 10^{162}$ (by Lemma 9), it follows that

$$q_{296} > N > n_2 > k_2 - k_1 \quad \text{and} \quad a(N) := \max\{a_j : 0 \leq j \leq 296\} = a_{189} = 1028.$$

Thus, by Lemma 1, we have that

$$\left| \tau' - \frac{k_2 n_1 - k_1 n_2}{k_2 - k_1} \right| > \frac{1}{1030 (k_2 - k_1)^2}. \tag{92}$$

Hence, combining the inequalities (91) and (92), we obtain

$$\alpha^{\lambda'} < 43260 n_2 (k_2 - k_1) < 4.08 \times 10^{329},$$

so $\lambda' \leq 2661$. This was obtained under the assumption that $\lambda' \geq 10$, Otherwise, $\lambda' < 10 < 2661$ holds as well.

Now, for each $n_i - m_i = \lambda' \in [1, 2661]$ we estimate a lower bound $|\Gamma_4'|$, with

$$\Gamma_4' \quad = \quad (k_j - k_i)\log a + (k_j n_i - k_i n_j)\log\alpha + k_j \log(1 + \alpha^{m_i - n_i}) \tag{93}$$

given in the inequality 81, via the same procedure described in Subsection 3.3 (LLL-algorithm). We recall that $\Gamma_4' \neq 0$.

We apply Lemma 3 with the data:

$$t:=3,\quad \tau_1:=\log a,\quad \tau_2:=\log\alpha,\quad \tau_3:=\log(1+\alpha^{-\lambda'}),$$
$$x_1:=k_j-k_i,\quad x_2:=k_jn_i-k_in_j,\quad x_3:=k_j.$$

We set $X:=3.99\times 10^{163}$ as an upper bound to $|x_i|<13n_2$ for all $i=1,2,3$, and $C:=(20X)^5$. A computer in *Mathematica* search allows us to conclude, together with the inequality (81), that

$$8\times 10^{-660}<\min_{1\leq\lambda\leq 2661}|\Gamma_4'|<16n_2\alpha^{-\nu'},\quad \text{with}\quad \nu':=\min\{n_i,n_j-m_j\}$$

which leads to $\nu'\leq 6643$. As we have noted before, $\nu'=n_1$ (so $n_1\leq 6643$) or $\nu'=n_j-m_j$.

Next, we suppose that $n_j-m_j=\nu'\leq 6643$. Since $\lambda'\leq 2661$, we have

$$\lambda':=\min_{1\leq i\leq 2}\{n_i-m_i\}\leq 2661\quad\text{and}\quad \chi':=\max_{1\leq i\leq 2}\{n_i-m_i\}\leq 6643.$$

Now, returning to the inequality (87) which involves

$$\begin{aligned}\Gamma_5' :&= (k_2-k_1)\log a+(k_2n_1-k_1n_2)\log\alpha\\ &\quad +k_2\log(1+\alpha^{m_1-n_1})-k_1\log(1+\alpha^{m_2-n_2})\neq 0,\end{aligned}\tag{94}$$

we use again the LLL-algorithm to estimate the lower bound for $|\Gamma_5'|$ and thus, find a bound for n_1 that is better than the one given in Lemma 9.

We distinguish the cases $\lambda'<\chi'$ and $\lambda'=\chi'$.

7.2. **The case $\lambda'<\chi'$.** We take $\lambda'\in[1,2661]$ and $\chi'\in[\lambda'+1,6643]$ and apply Lemma 3 with the data:

$$t:=4,\quad \tau_1:=\log a,\quad \tau_2:=\log\alpha,\quad \tau_3:=\log(1+\alpha^{m_1-n_1}),\quad \tau_4:=\log(1+\alpha^{m_2-n_2}),$$
$$x_1:=k_2-k_1,\quad x_2:=k_2n_1-k_1n_2,\quad x_3:=k_2,\quad x_4:=-k_1.$$

We also put $X:=3.99\times 10^{163}$ and $C:=(20X)^9$. As before, after a computer search in *Mathematica* together with the inequality 87, we can confirm that

$$9.9\times 10^{-1317}<\min_{\substack{1\leq\lambda\leq 2661\\ \lambda+1\leq\chi\leq 6643}}|\Gamma_5'|<20n_2\alpha^{-n_1}.\tag{95}$$

This leads to the inequality

$$\alpha^{n_1}<2.02\times 10^{1317}n_2.\tag{96}$$

Subsitituting for the bound n_2 given in Lemma 9, we get that $n_1\leq 11948$.

7.3. **The case$\lambda'=\chi'$.** In this case, we have

$$\Lambda_5':=(k_2-k_1)(\log a+\log(1+\alpha^{m_1-n_1}))+(k_2n_1-k_1n_2)\log\alpha\neq 0.$$

We divide through the inequality 87 by $(k_2-k_1)\log\alpha$ to obtain

$$\left|\frac{|\log a+\log(1+\alpha^{m_1-n_1})|}{\log\alpha}-\frac{k_2n_1-k_1n_2}{k_2-k_1}\right|<\frac{70n_2}{\alpha^{n_1}(k_2-k_1)}\tag{97}$$

We now put

$$\tau_{\lambda'}:=\frac{|\log a+\log(1+\alpha^{-\lambda'})|}{\log\alpha}$$

and compute its continued fractions $[a_0^{(\lambda')}, a_1^{(\lambda')}, \ldots]$ and its convergents $[p_0^{(\lambda)}/q_0^{(\lambda')}, p_1^{(\lambda')}/q_1^{(\lambda')}, \ldots]$ for each $\lambda' \in [1, 2661]$. Furthermore, for each case we find an integer $t_{\lambda'}$ such that $q_{t_{\lambda'}}^{(\lambda')} > N := 3.07 \times 10^{162} > n_2 > k_2 - k_1$ and calculate

$$a(N) := \max_{1 \le \lambda' \le 2661} \left\{ a_i^{(\lambda')} : 0 \le i \le t_{\lambda'} \right\}.$$

A computer search in *Mathematica* reveals that for $\lambda' = 2466$, $t_{\lambda'} = 298$ and $i = 295$, we have that $a(N) = a_{295}^{(2466)} = 2818130$. Hence, combining the conclusion of Lemma 1 and the inequality (97), we get

$$\alpha^{n_1} < 70 \times 2818132 n_2(k_2 - k_1) < 1.86 \times 10^{333},$$

so $n_1 \le 2690$. Hence, we obtain that $n_1 \le 11948$ holds in all cases ($\nu' = n_1$, $\lambda' < \chi'$ or $\lambda' = \chi'$). By the inequality (66), we have that

$$\log \rho \le k_1 \log \rho \le n_1 \log \alpha + \log 5 < 3410.$$

By considering the second inequality in (74), we can conclude that $n_2 \le 8.17 \times 10^{39} (\log n_2)^2$, which yields $n_2 < 2.76 \times 10^{44}$, by a simple application of Lemma 4 as before. Below, we summarise the first cycle of our reduction process:

$$n_1 \le 11948 \quad \text{and} \quad n_2 \le 2.76 \times 10^{44}. \tag{98}$$

As in the previous case, from the above, we note that the upper bound on n_2 represents a very good reduction of the bound given in Lemma 9. Hence, we expect that if we restart our reduction cycle with the new bound on n_2, then we get a better bound on n_1. Thus, we return to the inequality (48) and take $N := 2.76 \times 10^{44}$. A computer search in *Mathematica* reveals that

$$q_{88} > N > n_2 > k_2 - k_1 \quad \text{and} \quad a(N) := \max\{a_i : 0 \le i \le 88\} = a_{55} = 397,$$

from which it follows that $\lambda \le 738$. We now return to (93) and we put $X := 2.76 \times 10^{44}$ and $C := (10X)^5$ and then apply the LLL-algorithm in Lemma 3 to $\lambda \in [1, 738]$. After a computer search, we get

$$8.6 \times 10^{-183} < \min_{1 \le \lambda' \le 738} |\Gamma_4'| < 16 n_2 \alpha^{-\nu'},$$

then $\nu' \le 1838$. By continuing under the assumption that $n_j - m_j = \nu \le 1838$, we return to (94) and put $X := 2.76 \times 10^{44}$, $C := (10X)^9$ and $N := 2.76 \times 10^{44}$ for the case $\lambda' < \chi'$ and $\lambda' = \chi'$. After a computer search, we confirm that

$$8 \times 10^{-365} < \min_{\substack{1 \le \lambda \le 738 \\ \lambda + 1 \le \chi \le 1838}} |\Gamma_5'| < 6 n_2 \alpha^{-n_1},$$

gives $n_1 \le 3304$, and $a(N) = a_{125}^{(160)} = 155013$, leads to $n_1 \le 774$. Hence, in both cases $n_1 \le 3304$ holds. This gives $n_2 \le 4 \times 10^{42}$ by a similar procedure as before, and $k_1 \le$.

We record what we have proved.

Lemma 10. *Let (k_i, n_i, m_i) be a solution to $X_i = P_{n_i} + P_{m_i}$, with $3 \le m_i < n_i$ for $i = 1, 2$ and $1 \le k_1 < k_2$, then*

$$m_1 < n_1 \le 3304, \quad k_1 \le 3108 \quad \textit{and} \quad n_2 \le 4 \times 10^{42}.$$

7.4. **The final reduction (II).** Returning back to (61) and (63) and using the fact that (X_1, X_1) is the smallest positive solution to the Pell equation (2), we obtain

$$\begin{aligned} X_k &= \rho^k + \varrho^k = \left(\frac{(X_1 + Y_1\sqrt{d}}{2}\right)^k + \left(\frac{X_1 - Y_1\sqrt{d}}{2}\right)^k \\ &= \left(\frac{X_1 + \sqrt{X_1^2 \mp 4}}{2}\right)^k + \left(\frac{X_1 - \sqrt{X_1^2 \mp 4}}{2}\right)^k := R_k^{\pm}(X_1). \end{aligned}$$

Thus, we return to the Diophantine equation $X_{k_1} = P_{n_1} + P_{m_1}$ and consider the equations

$$R_{k_1}^{+}(X_1) = P_{n_1} + P_{m_1} \quad \text{and} \quad R_{k_1}^{-}(X_1) = P_{n_1} + P_{m_1}, \tag{99}$$

with $k_1 \in [1, 3108]$, $m_1 \in [3, 3304]$ and $n_1 \in [m_1 + 1, 3304]$.

A computer search in *Mathematica* on the above equations in (99) shows that there are only finitely many solutions that we list in the tables below. We note that

$$\frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2 \qquad \text{and} \qquad 2+\sqrt{5} = \left(\frac{1+\sqrt{5}}{2}\right)^3,$$

so these come from the same Pell equation with $d = 5$. Similarly,

$$\frac{11+\sqrt{13}}{2} = \left(\frac{3+\sqrt{13}}{2}\right)^2, \qquad \text{and} \qquad \frac{51+7\sqrt{53}}{2} = \left(\frac{7+\sqrt{53}}{2}\right)^2$$

these also come from the same Pell equation with $d = 13$ and $d = 53$, respectively.

$R_{k_1}^{+}(X_1)$				
k_1	X_1	Y_1	d	ρ
2	3	1	5	$(3+\sqrt{5})/2$
2	4	2	3	$2+\sqrt{3}$
2	5	1	21	$(5+\sqrt{21})/2$
3	9	1	77	$(9+\sqrt{77})/2$
2	10	4	6	$5+2\sqrt{6}$
2	11	3	13	$(11+3\sqrt{13})/2$
2	12	2	35	$6+\sqrt{35}$
2	13	1	165	$(13+2\sqrt{165})/2$
3	15	1	221	$(15+\sqrt{221})/2$
2	25	3	69	$(25+3\sqrt{69})/2$
2	44	2	483	$22+\sqrt{483}$
2	51	7	53	$(51+7\sqrt{53})/2$
2	88	6	215	$44+3\sqrt{215}$
2	2570	4	412806	$1285+2\sqrt{412806}$

$R_{k_1}^{-}(X_1)$				
k_1	X_1	Y_1	d	ρ
2	1	1	5	$(1+\sqrt{5})/2$
2	2	2	2	$1+\sqrt{2}$
2	3	1	13	$(3+\sqrt{13})/2$
2	4	2	5	$2+\sqrt{5}$
2	6	2	10	$3+\sqrt{10}$
2	7	1	53	$(7+\sqrt{53})/2$
2	8	2	17	$4+\sqrt{17}$
2	10	2	26	$5+\sqrt{26}$
2	11	5	5	$(11+5\sqrt{5})/2$
2	19	1	365	$(19+\sqrt{365})/2$
2	22	2	122	$11+\sqrt{122}$
2	30	2	226	$15+\sqrt{226}$
2	58	2	842	$29+\sqrt{842}$
2	88	2	1937	$44+\sqrt{1937}$
2	178	2	7922	$89+\sqrt{7922}$
2	3480	2	3027601	$1740+\sqrt{3027601}$

From the above tables, we set each $\rho := \rho_t$ for $t = 1, 2, \ldots 25$. We then work on the linear forms in logarithms Γ_1' and Γ_2', in order to reduce the bound on n_2 given in Lemma

10. From the inequality (72), for $(k,n,m) := (k_2,n_2,m_2)$, we write

$$(100)\quad \left|k_2\frac{\log\rho_t}{\log\alpha} - n_2 + \frac{\log a}{\log(\alpha^{-1})}\right| < \left(\frac{6}{\log\alpha}\right)\alpha^{-(n_2-m_2)}, \quad \text{for} \quad t = 1,2,\ldots 25.$$

We put

$$\tau_t := \frac{\log\rho_t}{\log\alpha}, \qquad \mu_t := \frac{\log a}{\log(\alpha^{-1})} \qquad \text{and} \quad (A_t,B_t) := \left(\frac{6}{\log\alpha},\alpha\right).$$

We note that τ_t is transcendental by the Gelfond-Schneider's Theorem and thus, τ_t is irrational. We can rewrite the above inequality, 100 as

$$(101)\qquad 0 < |k_2\tau_t - n_2 + \mu_t| < A_t B_t^{-(n_2-m_2)}, \quad \text{for} \quad t = 1,2,\ldots,25.$$

We take $N := 4\times 10^{42}$ which is the upper bound on n_2 according to Lemma 10 and apply Lemma 2 to the inequality (101). As before, for each τ_t with $t = 1,2,\ldots,26$, we compute its continued fraction $[a_0^{(t)}, a_1^{(t)}, a_2^{(t)},\ldots]$ and its convergents $p_0^{(t)}/q_0^{(t)}, p_1^{(t)}/q_1^{(t)}, p_2^{(t)}/q_2^{(t)},\ldots$. For each case, by means of a computer search in *Mathematica*, we find and integer s_t such that

$$q_{s_t}^{(t)} > 2.4\times 10^{43} = 6N \qquad \text{and} \qquad \epsilon_t := ||\mu_t q^{(t)}|| - N||\tau_t q^{(t)}| > 0.$$

We finally compute all the values of $b_t := \lfloor\log(A_t q_{s_t}^{(t)}/\epsilon_t)/\log B_t\rfloor$. The values of b_t correspond to the upper bounds on $n_2 - m_2$, for each $t = 1,2,\ldots,25$, according to Lemma 2. We record the results of the computations for each t in the following table.

t	ρ_t	s_t	q_{s_t}	$\epsilon_t >$	b_t
1	$1+\sqrt{2}$	78	1.46195×10^{44}	0.1578	379
2	$2+\sqrt{3}$	100	8.93366×10^{43}	0.3147	374
3	$(1+\sqrt{5})/2$	82	2.96985×10^{43}	0.4479	369
4	$5+2\sqrt{6}$	80	3.16032×10^{43}	0.1940	372
5	$3+\sqrt{10}$	88	4.21425×10^{43}	0.2358	373
6	$(3+\sqrt{13})/2$	91	6.62314×10^{43}	0.0666	379
7	$4+\sqrt{17}$	92	1.11753×10^{44}	0.2387	376
8	$(5+\sqrt{21})/2$	73	2.44965×10^{43}	0.0400	377
9	$5+\sqrt{26}$	98	3.23107×10^{43}	0.2333	372
10	$6+\sqrt{35}$	83	1.87425×10^{44}	0.1172	381
11	$(7+\sqrt{53})/2$	96	1.82440×10^{44}	0.3875	376
12	$(25+3\sqrt{69})/2$	80	2.40911×10^{43}	0.2013	371
13	$(9+\sqrt{77})/2$	82	2.54747×10^{43}	0.1470	373
14	$11+\sqrt{122}$	76	4.91937×10^{44}	0.4004	380
15	$(13+2\sqrt{165})/2$	86	2.61323×10^{43}	0.1664	372
16	$44+3\sqrt{215}$	80	3.14146×10^{43}	0.3298	371
17	$(15+\sqrt{221})/2$	75	5.70467×10^{43}	0.4661	371
18	$15+\sqrt{226}$	79	4.78438×10^{43}	0.4046	371
19	$(19+\sqrt{365})/2$	78	3.05270×10^{43}	0.1985	372
20	$22+\sqrt{483}$	75	4.19689×10^{43}	0.1559	374
21	$29+\sqrt{842}$	87	8.14707×10^{44}	0.2964	382
22	$44+\sqrt{1937}$	87	4.70884×10^{43}	0.1191	376
23	$89+\sqrt{7922}$	79	2.43413×10^{43}	0.4418	369
24	$1285+2\sqrt{412806}$	85	2.22078×10^{45}	0.4501	385
25	$1740+\sqrt{3027601}$	77	2.33761×10^{44}	0.3352	378

By replacing $(k,n,m) := (k_2,n_2,m_2)$ in the inequality (69), we can write

$$\left|k_2\frac{\log\delta_t}{\log\alpha} - n_2 + \frac{\log(a(1+\alpha^{-(n_2-m_2)}))}{\log(\alpha^{-1})}\right| < \left(\frac{10}{\log\alpha}\right)\alpha^{-n_2}, \tag{102}$$

for $t=1,2,\ldots,25$. We now put

$$\tau_t := \frac{\log\delta_t}{\log\alpha}, \quad \mu_{t,n_2-m_2} := \frac{\log(a(1+\alpha^{-(n_2-m_2)}))}{\log(\alpha^{-1})} \quad \text{and} \quad (A_t,B_t) := \left(\frac{10}{\log\alpha},\alpha\right).$$

With the above notations, we can rewrite (102) as

$$0 < |k_2\tau_t - n_2 + \mu_{t,n_2-m_2}| < A_tB_t^{-n_2}, \quad \text{for} \quad t=1,2,\ldots 25. \tag{103}$$

We again apply Lemma 2 to the above inequality (103), for

$$t=1,2,\ldots,25, \quad n_2-m_2=1,2,\ldots,b_t, \quad \text{with} \quad N := 4\times 10^{43}.$$

We take

$$\epsilon_{t,n_2-m_2} := ||\mu_t q^{(t,n_2-m_2)}|| - N||\tau_t q^{(t,n_2-m_2)}|| > 0,$$

and

$$b_{t,n_2-m_2} := \lfloor \log(A_t q_{s_t}^{(t,n_2-m_2)}/\epsilon_{t,n_2-m_2})/\log B_t\rfloor.$$

With the help of Mathematica, we obtain that

t	1	2	3	4	5	6	7	8	9	10	11	12	13
b_{t,n_2-m_2}	398	404	399	413	390	398	401	397	390	413	401	396	396
t	14	15	16	17	18	19	20	21	22	23	24	25	
b_{t,n_2-m_2}	402	393	395	392	401	396	392	400	401	392	414	395	

$$\max\{b_{t,n_2-m_2} : t = 1, 2, \ldots, 25 \quad \text{and} \quad n_2 - m_2 = 1, 2, \ldots d_t\} \leq 414.$$

Thus, by Lemma 2, we have that $n_2 \leq 414$, for all $t = 1, 2, \ldots, 25$, and by the inequality (67) we also have that $n_1 \leq n_2 + 4$. From the fact that $\rho^k \leq 2\alpha^{n+1}$, we can conclude that $k_1 < k_2 \leq 248$. Collecting everything together, our problem is reduced to search for the solutions for (65) in the following range

$$(104)\; 1 \leq k_1 < k_2 \leq 248, \quad 0 \leq m_1 < n_1 \in [3, 414] \quad \text{and} \quad 0 \leq m_2 < n_2 \in [3, 414].$$

After a computer search on the equation (65) on the above ranges, we obtained the following solutions, which are the only solutions for the exceptional d cases we have stated in Theorem 2:

For the $+4$ case:

$(d = 3)$ $X_1 = 4 = P_7 + P_0 = P_6 + P_3 = P_5 + P_5, \quad X_2 = 14 = P_{11} + P_5 = P_{10} + P_8,$
$X_3 = 52 = P_{16} + P_6;$

$(d = 5)$ $X_1 = 3 = P_6 + P_0 = P_5 + P_3, \quad X_2 = 7 = P_9 + P_0 = P_7 + P_6,$
$X_3 = 18 = P_{12} + P_5;$

$(d = 21)$ $X_1 = 5 = P_8 + P_0 = P_7 + P_3 = P_6 + P_5, \quad X_2 = 23 = P_{13} + P_5 = P_{12} + P_9,$
$X_3 = 2525 = P_{30} + P_{11}.$

For the -4 case:

$(d = 2)$ $X_1 = 2 = P_5 + P_0 = P_3 + P_3, \quad X_2 = 14 = P_{11} + P_5 = P_{10} + P_8;$

$(d = 5)$ $X_1 = 1 = P_3 + P_0, \quad X_2 = 4 = P_7 + P_0 = P_6 + P_3 = P_5 + P_5,$
$X_3 = 11 = P_{10} + P_5 = P_9 + P_7, \quad X_4 = 29 = P_{14} + P_3.$

This completes the proof of Theorem 2. □

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