Separators - a new statistic for permutations

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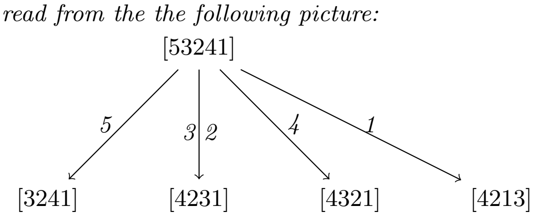
Abstract

A digit π_j in a permutation $\pi = [\pi_1, \ldots, \pi_n] \in S_n$ is defined to be a separator of π if by omitting it from π we get a new $2-$block. In this work we introduce a new statistic, the number of separators, on the symmetric group S_n and calculate its distribution over S_n. We also provide some enumerative and asymptotic results regarding this statistic.

1 Introduction

Let S_n be the Symmetric group of n elements. Let $\sigma, \pi \in \bigcup_{n\in\mathbb{N}} S_n$. We say that σ *contains* π if there is a sub-sequence of elements of σ that is order-isomorphic to π. As an example, the permutation $\sigma = [3624715]$ (written in one-line-notation) contains $\pi = [3142]$ as both the sub-sequences 6275 and 6475 testify. If π is contained in σ, then we write $\pi \preceq \sigma$. The set of all permutations $\cup_{n\in\mathbb{N}} S_n$ is a poset under the partial order given by containment. This is called the permutation pattern poset.

Example 1.1. *Let $\sigma = [53241] \in S_5$. In order to find the permutations in S_4 which are contained in σ, we shall remove each of the digits of π and standardize. If we remove $\sigma_1 = 5$, we get the permutation* [3241]*, while if we remove $\sigma_2 = 3$ or $\sigma_3 = 2$ we get the permutation* [4231]*. The removal of $\sigma_4 = 4$ produces the permutation* [4321]*, and the removal of $\sigma_4 = 1$ produces the permutation* [4213]*. The situation can also be read from the the following picture:*



In the example above, the removal of the digits 2 and 3 produced the same permutation. This is not coincidental. It happens since these two digits form a 2-block in π as we define below: (See also in [6], Definition 4)

Definition 1.2. *Let $\pi = [\pi_1, \ldots, \pi_n]$ be a permutation and let $i \in [n-1]$. We say that the pair (π_i, π_{i+1}) is a* 2-block *or a* bond *in π if $\pi_i - \pi_{i+1} = \pm 1$. We say that the sequence $(\pi_i, \pi_{i+1}, \ldots, \pi_{i+k-1})$ is a* run *of length $k > 1$ if, for $0 \le j \le k-2$, the pair (π_{i+j}, π_{i+j+1}) is a* bond*. We allow also (trivial) runs of length $k = 1$. Note that a run of a permutation might be ascending or descending. Occasionally, we omit the parentheses when we write blocks or runs.*

Example 1.3. *The permutation $\pi = [45187623]$ has* 45*,* 1*,* 876 *and* 23 *as its maximal runs.*

The distribution of the bonds has been examined previously in [4, 5, 6]. Each run of length $n \ge 1$ contains $n-1$ bonds. The number of bonds in a permutation $\sigma \in S_n$ affects the structure of the poset of all permutations contained in σ, the downset of σ. This happens since the number of permutations $\pi \in S_{n-1}$ such that $\pi \preceq \sigma$ is $n - \beta(\sigma)$, where $\beta(\sigma)$ is the number of bonds in σ. (See Theorem 6 in [6]).

To better understand the structure of the poset $\bigcup_{n\in\mathbb{N}} S_n$, we would like to get information not only about the number of bonds of a given $\sigma \in S_n$, but also about the number of bonds of the permutations contained in σ. Hence we introduce a new concept: A digit of a permutation, a removal of which produces a **new** bond, will be called a *separator*. (see the formal definition below).

Example 1.4. *In the permutation* $\pi = [567139482]$ *we can omit* $\pi_7 = 4$ *and after standardizing we get the permutation* $[45613872]$ *which has the* **new** *2-block* 87*, so* 4 *is a separator. The digit* 2 *is also a separator of* π*, since the removal of it creates the permutation* $[45612837]$ *containing the* ***new*** *bond* 12*. Note that if we remove* $\pi_2 = 6$ *from* π*, we get the permutation* $[56138472]$ *which contains the bond* 56 *that already exists in* π*, so* 6 *is not a separator in* π*.*

Formally:

Definition 1.5. *For* $\sigma = [\sigma_1, \dots, \sigma_n] \in S_n$ *we say that* $\sigma_{\mathbf{i}}$ separates σ_{j_1} *from* σ_{j_2} *in* σ *if by omitting* σ_i *from* σ *we get a* **new** $2-$*block. This happens if and only if one of the following cases holds:*

1. j_1, i, j_2 *are subsequent numbers and* $|\sigma_{j_1} - \sigma_{j_2}| = 1$*, i.e* $\sigma_i = b$ *and*

$$\sigma = [\dots, \mathbf{a}, \mathbf{b}, \mathbf{a \pm 1}, \dots]$$

   We call σ_i *a separator of type I or a vertical separator.*

2. $\sigma_{j_1}, \sigma_i, \sigma_{j_2}$ *are subsequent numbers and* $|j_1 - j_2| = 1$*, i.e* $\sigma_i = a$ *and*

$$\sigma = [\dots, \mathbf{a}, \dots, \mathbf{a \pm 1}, \mathbf{a \mp 1}, \dots]$$

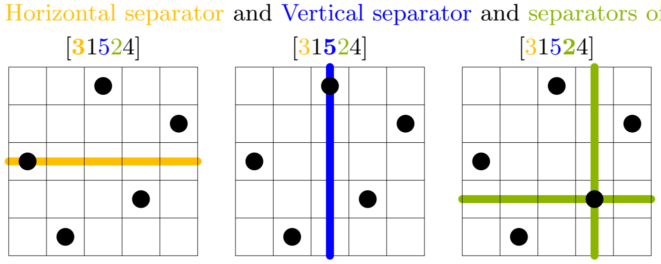
   or

$$\sigma = [\dots, \mathbf{a \pm 1}, \mathbf{a \mp 1}, \dots, \mathbf{a}, \dots].$$

   We call σ_i *a separator of type II or a horizontal separator.*

The choice of the names of the separators of types I and II is explained in the following picture in which $\sigma_1 = 3$ is a horizontal separator (which its omitting forms the 2-*block* 23), $\sigma_3 = 5$ is a vertical separator (which its omitting forms the 2-*block* 12) and $\sigma_4 = 2$ is both (which its omitting forms the two 2-*blocks* 21 and 43).

Horizontal separator and Vertical separator and separators of both types .



Definition 1.6. *Let* $Sep_V(\pi)$ *and* $Sep_H(\pi)$ *be the sets of vertical and horizontal separators of a permutation* π *respectively. Let* $Sep(\pi) = Sep_V(\pi) \cup Sep_H(\pi)$ *and* $sep(\pi) = |Sep(\pi)|$*.*

Example 1.7. *Let* $\sigma = [132465879]$*. Then* $Sep_V(\sigma) = \{3, 2, 6, 7\}$*, and* $Sep_H(\sigma) = \{3, 2, 5, 8\}$*. Note that* 7 *is a vertical separator, even though* 7 *is a part of a* $2-$*block:* 87*, since by omitting* 7 *from* σ *we get a* **new** $2-$*block:* 78*.*

Remark 1.8. *Several comments are now in order:*

1. *Notice the significance of the word 'new' in Definition 1.5. For example, the identity permutation has plenty of 2-blocks even though it has no separators.*
2. *The numbers* 1 *and* n *can only be vertical separators, while* σ_1 *and* σ_n *can only be horizontal separators.*
3. *If* σ_i *is a vertical separator in* σ *then* i *is a horizontal separator in* σ^{-1}*. Hence* $Sep_V(\sigma) = Sep_H(\sigma^{-1})$
4. $Sep_V(\sigma) = Sep_V(\sigma^r)$ *and* $Sep_H(\sigma) = Sep_H(\sigma^r)$ *where* σ^r *is the reverse of* σ*.*
5. *A separator can be of both types, vertical and horizontal. For instance, in example 1.7, the digits* 2, 3 *are separators of both types.*

Permutations of S_n which have no bonds are connected to the problem of placing n non-attacking kings in an $n \times n$ chess board. These permutations were counted in [2], and the structure of their containment poset is discussed in a recent paper by the authors of this note [1]. The set of such permutations will be denoted by K_n.

If $\sigma \in K_n$ then even though σ has no bonds, after omitting a digit from σ, the resulting permutation might have (at least) one. Recall that in this case the omitted digit is a separator of σ. The connection between the number of separators in σ and the number of $\pi \in K_{n-1}$ such that $\pi \preceq \sigma$ is given by the following:

Observation 1.9. *Let $\sigma \in K_n$. Then the number of $\pi \in K_{n-1}$ such that $\pi \preceq \sigma$ is $n - sep(\sigma)$ where $sep(\sigma)$ is the number of separators in σ.*

In [6], Homberger built a multivariate generating function which presented the distribution of the bonds throughout all permutations. He used the principal of inclusion-exclusion in its generating function version. The main result of this paper uses the same method for producing a multivariate generating function representing the distribution of the vertical separators (and thus also of horizontal separators, by Remark 1.8.3).

2 Permutations with no separators and permutations with maximal number of separators

The permutations in S_n that have no separators of any type, are counted by the sequence A137774 from OEIS. They correspond to the number of ways to place n non-attacking empresses (a chess piece which moves like a rook and a night) on an $n \times n$ chess board.

Theorem 2.7 below deals with the opposite case, i.e., the number of permutations, all the digits of which are separators. First, we have to include some definitions. A comprehensive survey of these concepts can be found in [3].

Definition 2.1. *Let $\pi = [\pi_1, \ldots, \pi_n] \in S_n$. A* block *(or* interval*) of π is a nonempty contiguous sequence of entries $\pi_i \pi_{i+1} \ldots \pi_{i+k}$ whose values also form a contiguous sequence of integers.*

Example 2.2. *If $\pi = [2647513]$ then 6475 is a block but 64751 is not.*

Each permutation can be decomposed into singleton blocks, and also forms a single block by itself; these are the *trivial blocks* of the permutation. All other blocks are called *proper*.

Definition 2.3. *A* block decomposition *of a permutation is a partition of it into disjoint blocks.*

For example, the permutation $\sigma = [67183524]$ can be decomposed as 67 1 8 3524. In this example, the relative order between the blocks forms the permutation [3142], i.e., if we take for each block one of its digits as a representative then the sequence of representatives is order-isomorphic to [3142]. Moreover, the block 67 is order-isomorphic to [12], and the block 3524 is order-isomorphic to [2413]. These are instances of the concept of *inflation*, defined as follows.

Definition 2.4. *Let $n_1, \ldots, n_k$ be positive integers with $n_1 + \cdots + n_k = n$. The* inflation *of a permutation $\pi \in S_k$ by the permutations $\alpha_i \in S_{n_i}$ $(1 \le i \le k)$ is the permutation $\pi[\alpha_1, \ldots, \alpha_k] \in S_n$ obtained by replacing the i-th entry of π by a block which is order-isomorphic to the permutation α_i on the numbers $\{s_i + 1, \ldots, s_i + n_i\}$ instead of $\{1, \ldots, n_i\}$, where $s_i = n_1 + \cdots + n_{i-1}$ $(1 \le i \le k)$.*

Example 2.5. *The inflation of [2413] by [213], [21], [132] and [1] is*

$$2413[213, 21, 132, 1] = [546\ 98\ 132\ 7].$$

We are interested in the structure of all permutations in S_n in which every digit is a separator. In Theorem 3.18 in [1] we proved that in a permutation $\sigma \in K_n$ each digit of σ is a separator if and only if $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$ where $\alpha_1, \ldots, \alpha_k \in \{[3142], [2413]\}$ and $\pi \in S_k$.

In the following Theorem, we extend this result and show that this structure holds for each permutation in S_n in which each one of its digits is a separator.

Theorem 2.6. *In a permutation $\sigma \in S_n$, each digit is a separator if and only if $n = 4k$, $k \in \mathbb{N}$ and there are $\alpha_1, \ldots, \alpha_k \in \{[3142], [2413]\}$ and $\pi \in S_k$ such that $\sigma = \pi[\alpha_1, \ldots, \alpha_k]$.*

Proof. The "only if" side is obvious, so we will prove only the "if" side. Let $\sigma \in S_n$ be a permutation such that each digit of σ is a separator. If we show that σ has no 2-block, i.e., $\sigma \in K_n$, then by Theorem 3.18 in [1], we are done. We assume to the contrary that σ contains a block of the form $a, a+1$ and show that not all the digits of σ are separators.

We divide in two different cases, according to the type of separation of $a+1$

- $a+1$ is a vertical separator: In this case, σ contains the sub-sequence $\cdots a, a+1, a-1 \cdots$. The digit $a-1$ is also a separator, so we distinguish between two cases according to the type of separation of $a-1$.
  1. If $a-1$ is a vertical separator then $\sigma = \cdots a, a+1, a-1, a+2 \cdots$. Hence $a+2$ must be a vertical separator so we have $\sigma = \cdots a, a+1, a-1, a+2, a-2 \cdots$. By the same argument, $a-2$ must be a vertical separator and so $\sigma = \cdots a, a+1, a-1, a+2, a-2, a+3 \cdots$. This process continues until we reach σ_n which can not be a horizontal separator but also can not be a vertical separator.
  2. If $a-1$ is a horizontal separator then $\sigma = \cdots a-2, a, a+1, a-1 \cdots$. Hence $a-2$ must be a horizontal separator so that $\sigma = \cdots a-2, a, a+1, a-1, a-3 \cdots$. By the same argument, $a-3$ must be a horizontal separator and so $\sigma = \cdots a-4, a-2, a, a+1, a-1, a-3 \cdots$. This process continues until we reach $a-k=1$ which can not be of a vertical separator but also can not be of a horizontal separator.
- $a+1$ ia a horizontal separator: In this case, we consider σ^{-1} in which $(\sigma^{-1})_{a+1}$ is a vertical separator. Since $a, a+1$ is a block in σ, σ^{-1} contains a block in locations $a, a+1$, which means that $(\sigma^{-1})_a = b, (\sigma^{-1})_{a+1} = b+1$ for some $b \in \{1, \ldots, n-1\}$. Now, by remark 1.8.3, we can apply the argument of the previous case in order to show that not all of the digits of σ^{-1} are separators. This implies that not all the digits of σ are separators and we are done.

We conclude that $\sigma \in K_n$ where K_n is the set of king permutations of order n. □

In Figure 1 we can see the structure of such permutations that each one of their digits is a separator, according to the above theorem.

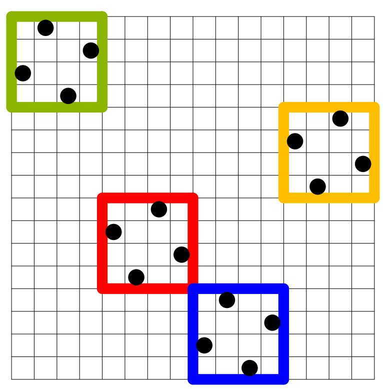


Figure 1: the plot of [14, 16, 13, 15, 7, 5, 8, 6, 2, 4, 1, 3, 11, 9, 12, 10]

According to Theorem 2.6, we can now enumerate those permutations.

Theorem 2.7. *The number of permutations in S_n which have exactly n different separators is:*

$$\begin{cases} 2^k k! & n = 4k \\ 0 & O.W. \end{cases}$$

3 A generating function for vertical separators

In this section we present a generating function for the number of vertical separators. For each $n, m \in \mathbb{N}$ let $s_{n,m}$ be the number of permutations $\pi \in S_n$ with exactly m vertical separators. We want to calculate the generating function $h(z,u) = \sum_{n\geq 0} \sum_{m=0}^{n} s_{n,m} z^n u^m$. According to remark 1.8.3, this generating function is the same for horizontal separators.

In order to construct the function $h(z,u)$, we will enlarge the set of elements we work with such that it will contain marked permutations. We then use the principle of inclusion-exclusion together with a method of splitting permutations into two parts to achieve the generating function for the number of vertical separators

3.1 Counting permutation with mark bonds

A *marked permutation* is a permutation in which each bond can be chosen to be marked or not. The marked bonds will be denoted by a bar above the corresponding part of the permutation. If several adjacent bonds are marked, then we put a long bar above the corresponding run. An entry that is not contained in a marked bond is considered to be a run of length 1.

Example 3.1. *Let* $\pi = [613452879]$*. Here are some permutations with marked bonds, made out of* π*:* $[61\overline{345}2\overline{87}9]$*,* $[61\overline{34}52879]$*,* $[613452\overline{87}9]$.

In order to count the marked permutations, we introduce another way to write them.

Recall that for $n \in \mathbb{N}$, a *composition* with m non-zero parts of n is a vector $(a_1, \ldots, a_m)$ such that $a_i \in \mathbb{N}$ and $\sum_{i=1}^{m} a_i = n$. We define an *arrowed composition* of n to be a composition in which after every part which is greater than 1 there exists one of the signs $\uparrow$ or $\downarrow$.

For example, $(2\uparrow, 1, 7\downarrow, 2\uparrow)$ is an arrowed composition of $n = 12$.

Now, each marked permutation $\pi \in S_n$ can be uniquely presented as an arrowed composition $(a_1, \ldots, a_m)$ of n, together with a permutation $\sigma \in S_m$. This idea will be best clarified by the following example.

Example 3.2. *Let* $\pi = [2\overline{45}61\overline{987}3]$*. We write* π *as a pair consisting of an arrowed composition of* $m = 6$ *parts* λ*, and a permutation* $\sigma \in S_6$*. First, write* π *as a sequence of runs:* $b_1 = 2, b_2 = \overline{45}, b_3 = 6, b_4 = 1, b_5 = \overline{987}, b_6 = 3$*. Each run contributes its length to the composition. Then for each part, we add the sign* $\uparrow$ *if the corresponding run is increasing, the sign* $\downarrow$ *if the run is decreasing and no arrow if the run is of length* 1*. In our case we get* $\lambda = (1, 2\uparrow, 1, 1, 3\downarrow, 1)$*. Now* $\sigma \in S_6$ *is the permutation induced by the order of the blocks. In our case we have:* $\sigma = [245163]$*. The marked permutation* π *is now uniquely defined by the pair* (λ, σ).

In other words, if we replace each $j\uparrow$ with the ascending permutation $[123\ldots j]$ and each $j\downarrow$ with the descending permutation $[j\ldots 321]$, we can see that this defines an inflation. In the previous example we can write $\pi = 245163[1, 12, 1, 1, 321, 1]$. For convenience, we denote this inflation by $\sigma[\lambda]$.

In [6] (during the proof of Theorem 10), the author calculated the generating function, counting the number of permutations having a specific number of bonds. This was done by calculating the generating function of marked bonds, and using the inclusion-exclusion principle. If we denote by $a_{n,m}$ the number of permutations of S_n with m marked bonds and put $A(z,u) = \sum_{n\geq 1} \sum_{m\geq 0} a_{n,m} z^n u^m$, then the identity permutation contributes z and for each $j \geq 2$, a run of order j can be either $[123\ldots j]$ or $[j\ldots 321]$, each of them contributes u^{n-1}, so the contribution is $2z^j u^{j-1}$. It is easy to see from the above that

$$A(z,u) = \sum_{m\geq 0} m!(z + 2z^2u + 2z^3u^2 + 2z^4u^3 + \cdots)^m = \sum_{m\geq 0} m!(z + \frac{2z^2u}{1-zu})^m.$$

(Here m denotes the number of runs).

3.2 Comb decomposition and marked separators

Coming back to our counting of permutations with respect to the number of vertical separators, we show now how to make a reduction of this problem to the problem of counting marked permutations with respect to the number of bonds. We start with the definition of what we call here *comb permutations* as follows:

Definition 3.3. *Let $\sigma = (\sigma_1, \ldots, \sigma_k), \tau = (\tau_1, \ldots, \tau_k)$ be two sequences such that $\{\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_k\} = \{1, 2, \ldots, 2k\}$. Define the comb permutation $\pi = \sigma \odot \tau$ by*

$$\pi = [\sigma_1, \tau_1, \sigma_2, \tau_2, \ldots, \sigma_k, \tau_k] \in S_{2k}.$$

Similarly, let $\sigma = (\sigma_1, \ldots, \sigma_{k+1}), \tau = (\tau_1, \ldots, \tau_k)$ be two sequences such that $\{\sigma_1, \ldots, \sigma_{k+1}, \tau_1, \ldots, \tau_k\} = \{1, 2, \ldots, 2k+1\}$. We define the comb permutation $\pi = \sigma \odot \tau$ by

$$\pi = [\sigma_1, \tau_1, \sigma_2, \tau_2, \ldots, \sigma_k, \tau_k, \sigma_{k+1}] \in S_{2k+1}.$$

When $\pi = \sigma \odot \tau$, we denote σ as π^{odd} and τ as π^{even}.

Now, let $\pi = \pi^{odd} \odot \pi^{even}$ where π^{odd} and π^{even} are sequences with marked bonds. Note that if $(\pi_i^{odd}\pi_{i+1}^{odd})$ is a marked bond of π^{odd} then the element of π^{even} which lies between π_i^{odd} and π_{i+1}^{odd} in π is a vertical separator, we call it a *marked separator* and denote it by putting a hat over it. For example, if $\pi^{odd} = (\overline{12}4)$ and $\pi^{even} = (53)$ then $\overline{12}$ is a marked bond in π^{odd} and thus 5 is a marked separator in $\pi^{odd} \odot \pi^{even} = 1\hat{5}234$. Similarly, define marked separators for marked bonds in π^{even}. Therefore, we have the following:

Observation 3.4. *Let π^{odd}, π^{even} be defined as in Definition 3.3 and let $\pi = \pi^{odd} \odot \pi^{even}$. Then the number of (marked) vertical separators in π is equal to the total number of (marked) bonds in π^{odd} and in π^{even}.*

Let $n = 2k$. Given two arrowed compositions of k: λ_o of size m_o and λ_e of size m_e, and given a permutation $\sigma \in S_{m_o+m_e}$, we take the inflation $\alpha = \sigma[\lambda_o, \lambda_e]$ (it is a permutation with marked bonds). We construct a permutation π as follows: denote the first k elements of α by π^{odd} and the last k elements of α by π^{even}. Now $\pi = \pi^{odd} \odot \pi^{even}$. The case $n = 2k+1$ is similar.

Example 3.5. *For $k = 4$, let $\lambda_o = (1, 3\downarrow)$, $\lambda_e = (1, 1, 2\uparrow)$ and let $\sigma = [34215]$. Then $\alpha = 34215[1, 321, 1, 1, 12] = [3\overline{654}21\overline{78}]$. Thus $\pi^{odd} = (3\overline{654})$ and $\pi^{even} = (21\overline{78})$ and therefore $\pi = (3\overline{654}) \odot (21\overline{78}) = [326\hat{1}5\hat{7}\hat{4}8]$.*

On the other side, given a permutation $\pi \in S_n$ with marked separators, let π^{odd} be the sequence of the odd entries of π and π^{even} be the sequence of the even entries of π, and mark the relevant bonds, i.e. if π_i is a marked separator in π, then $(\pi_{i-1}\pi_{i+1})$ is a marked bond in π^{odd} or in π^{even}. Denote by α the permutation with marked bonds obtained by π^{odd} followed by π^{even}. We know that α can be uniquely presented as an arrowed composition of n together with a permutation of the number of parts, m.

Example 3.6. $\pi = [27\hat{1}86\hat{3}549]$. *We use the sequences $\pi^{odd} = (21\overline{65}9)$ and $\pi^{even} = (\overline{78}34)$ to produce $\alpha = [21\overline{65}9\,\overline{78}34]$. This permutation can be presented as $\lambda = (1, 1, 2\downarrow, 1, 2\uparrow, 1, 1)$ with $\sigma = [2157634]$.*

3.3 Calculating the generating function for vertical separators

Recall that our goal is to find the function $h(z, u)$, which is the generating function for the number of vertical separators. In order to do that, we first calculate the generating function for the number of **marked** vertical separators. Denote by $b_{n,m}$ the number of permutations of S_n with m marked vertical separators, and let $g(z, u) = \sum_{n \geq 1} \sum_{m \geq 0} b_{n,m} z^n u^m$.

As we saw below, there is a correspondence between the number of marked bonds and the number of marked vertical separators, thus we construct the generating function $g(z, u)$ by calculating separately the generating functions for the marked bonds of the odd and the even parts of each permutation. The requirement that the odd part and the even part of a permutation must have (almost) the same size will be met by using the well known *Hadamard product* (element-wise) of polynomials and series.

Definition 3.7. *Let R be a ring and let $f(x) = \sum_{n \in \mathbb{N}} a_n x^n$, $g(x) = \sum_{n \in \mathbb{N}} b_n x^n \in R[[x]]$ be two power series in x. The Hadamard product of $f(x), g(x)$ is $f(x) * g(x) = \sum_{n=0}^{\infty} a_n b_n x^n$.*

Example 3.8. $(2 + 3x - 4x^2) * (5 + x + 7x^2) = 10 + 3x - 28x^2$.

In order to form the generating function of the marked separators, $g(z, u)$, let us have a look at the permutations $\pi \in S_n$ for a fixed n. We would like to find the monomial contributed by each $\pi = \pi^{odd} \odot \pi^{even} \in S_n$ using the monomial corresponding to the marked bonds of π^{odd} and π^{even}.

Let $n = 2k$, in this case, π^{odd}, π^{even} are sequences of order k, and therefore we use the Hadamard product to combine the two monomials. Each $\pi \in S_{2k}$ contributes to $g(z,u)$ a monomial of the form $u^m z^{2k}$ where the monomials corresponding to the marked bonds of π^{odd} and π^{even} are $f_o(z,u) = u^{m_1} z^k$ and $f_e(z,u) = u^{m_2} z^k$, respectively, where $m = m_1 + m_2$. If we look at those monomials as functions of the variable z, then we can easily see that the coefficient of z^{2k} in $g(z,u)$ should be the product of the coefficients of z^k of the odd and even parts. So we have to set z^2 instead of z in the monomoials of π^{odd} and π^{even} and get:

$$f_o(z^2,u) * f_e(z^2,u) = u^{m_1}(z^2)^k * u^{m_2}(z^2)^k = u^m z^{2k}.$$

Example 3.9. *Return to example 3.5. The monomial corresponding to the marked bonds of* $\pi^{odd} = (3\overline{654})$ *is* $f_o(z,u) = z^4u^2$. *In the same way, the monomial corresponding to the marked bonds of* $\pi^{even} = (21\overline{78})$ *is* $f_e(z,u) = z^4u$. *Thus* $f_o(z,u) * f_e(z,u) = z^4u^3$. *However, the monomial corresponding to the marked separators of* $\pi = (3\overline{654}) \odot (21\overline{78}) = [326\hat{1}5\hat{7}\hat{4}8]$ *is supposed to be* z^8u^3. *Note that if we take* $f_o(z^2,u) * f_e(z^2,u)$ *we get exactly what we need.*

Similarly, if $n = 2k+1$, then each $\pi = \pi^{odd} \odot \pi^{even} \in S_n$ contributes a monomial of the form $u^m z^{2k+1}$ where the monomial of π^{odd} is $f_o(z,u) = u^{m_1} z^{k+1}$ and the monomial of π^{even} is $f_e(z,u) = u^{m_2} z^k$, $m = m_1 + m_2$. Now,

$$\big[\frac{1}{z} f_o(z^2,u)\big] * \big[z\, f_e(z^2,u)\big] = \big[\frac{1}{z} u^{m_1}(z^2)^{k+1}\big] * \big[z\, u^{m_2}(z^2)^k\big] = u^m z^{2k+1}.$$

Example 3.10. *Let* $\pi = [2\hat{5}\hat{3}4176]$. *Then* $\pi^{odd} = (\overline{23}16)$ *and* $\pi^{even} = (\overline{54}7)$. *Then* $f_o(z,u) = uz^4$ *and* $f_e(z,u) = uz^3$. *So*

$$\big[\frac{1}{z} f_o(z^2,u)\big] * \big[z f_e(z^2,u)\big] = [\frac{1}{z} u(z^2)^4] * [zu(z^2)^3] = u^2 z^7$$

as required.

Using the above explanations, and the generating functions version to the inclusion-exclusion principle, we get the following calculation of the generating function of vertical separators, $h(z,u)$.

Theorem 3.11.

$$h(z,u) = \sum_{m_o, m_e \geq 0} (m_o + m_e)! \big[(z^2 + \frac{2z^4(u-1)}{1 - z^2(u-1)})^{m_o}\big] * \big[(z^2 + \frac{2z^4(u-1)}{1 - z^2(u-1)})^{m_e}\big]$$

$$+ \sum_{m_o, m_e \geq 0} (m_o + m_e)! \big[(z^2 + \frac{2z^4(u-1)}{1 - z^2(u-1)})^{m_o} \frac{1}{z}\big] * \big[(z^2 + \frac{2z^4(u-1)}{1 - z^2(u-1)})^{m_e} z\big]$$

where $*$ *is the Hadamard product in* $\mathbb{Q}[[u]][[z]]$.

Proof. Let us denote for each $m \in \mathbb{N}$:

$$p_m(z,v) = (z + 2z^2v + 2z^3v^2 + \cdots)^m = (z + \frac{2z^2v}{1 - zv})^m$$

Then $p_m(z,v)$ counts the number of ways to construct an arrowed composition λ of size m. We relate to n even and n odd separately. In order to construct a permutation of S_{2k} with marked separators, we have to choose two arrowed compositions of k: λ_o of size m_o, and λ_e of size m_e, we also choose a permutation $\sigma \in S_{m_o+m_e}$. It is easy to see that this contributes to our function $(m_o + m_e)! p_{m_o}(z^2,v) * p_{m_e}(z^2,v)$. For S_{2k+1} we have $(m_o + m_e)! \big[\frac{1}{z} p_{m_o}(z^2,v))\big] * \big[z\, p_{m_e}(z^2,v)\big]$. We go over all the values for m_o and m_e for both n even and n odd and we obtain the generating function of the marked vertical separators:

$$g(z,v) = \sum_{m_o, m_e \geq 0} (m_o + m_e)! \big[(z^2 + \frac{2z^4v}{1 - z^2v})^{m_o}\big] * \big[(z^2 + \frac{2z^4v}{1 - z^2v})^{m_e}\big]$$

$$+ \sum_{m_o, m_e \geq 0} (m_o + m_e)! \big[(z^2 + \frac{2z^4v}{1 - z^2v})^{m_o} \frac{1}{z}\big] * \big[(z^2 + \frac{2z^4v}{1 - z^2v})^{m_e} z\big]$$

Now, we can use this generating function to obtain $h(z,u)$. The variable v represents the **marked** vertical separators, while u is responsible for vertical separators. Since every vertical separator can either be marked or unmarked, it follows that by replacing $v+1$ by u we obtain that the generating function of the vertical separators is

$$h(z,u) = g(z, u-1).$$

□

4 The expectation of the number of separators

In this section we calculate the expectation of the number of separators in a randomly chosen permutation. In order to do that, let us first calculate the expectation of the number of vertical separators. Consider the sample space of all $n!$ permutations, with the uniform probability. For each $1 \le i \le n$, let X_i be the Bernuli random variable such that for each $\pi \in S_n$, $X_i = 1$ if the digit i is a vertical separator in π and $X_i = 0$ otherwise. Then the sum $X = \sum_{i=1}^{n} X_i$ counts the number of vertical separators for each $\pi \in S_n$.

In order to calculate $E[X] = \sum_{i=1}^{n} E[X_i]$, let us first calculate $E[X_i]$ for each $1 \le i \le n$. The digit i is a vertical separator in a permutation $\pi \in S_n$ if π contains a consecutive sequence of the form $a, i, a+1$ or its reverse. If $i \in \{1, n\}$, the digit a can be chosen in $n-2$ ways. After choosing a we have $(n-2)!$ ways to arrange the rest of the permutation, so that $E[X_1] = E[X_n] = \frac{2(n-2)(n-2)!}{n!}$. Now, for each $1 < i < n$, the same consideration applies, but now we have only $(n-3)$ ways to choose a. This gives us $E[X_i] = \frac{2(n-3)(n-2)!}{n!}$. We have now:

$$E[X] = \sum_{i=1}^{n} E[X_i] = 2E[X_1] + (n-2)E[X_2]$$

A simple calculation now yields the following:

Theorem 4.1. *The expectation of the number of vertical separators in a randomly chosen n-permutation is $\frac{2(n-2)}{n}$. The asymptotic value of the expectation is 2.*

We turn now to the calculation of the expectation of the number of separators that are both vertical and horizontal. Let Y be the random variable counting the number of digits of a permutation π which are separators of both types. Define, for each $1 < i < n$, Y_i to be the Bernuli random variable which for each $\pi \in S_n$ is equal to 1 if the digit i is both a vertical and a horizontal separator in π and is equal to 0 otherwise. Note that the digits 1 and n can not be separators of both types. Let us calculate $E[Y_i]$ for $1 < i < n$:

We have one of the following cases, depending on the structure of π. Either the digits of π which make i a separator of both types appear as one single part (like in $\pi = [624351]$, where 3 is a separator of both types due to the sequence 2435), or they appear in two different places (like in $\pi = [241536]$, where 24 makes 3 a horizontal separator and 5 and 6 make it a vertical one).

1. The first case occurs when $i = 2$ and i is a part of the consecutive sub-sequence 1**3**24 or its reverse. Similarly, when $i = n-1$ and i is a part of the consecutive sub-sequence $n-3, \mathbf{n-1}, n-2, n$ or its reverse. Also, for $2 < i < n-1$, i might be a part of one of the consecutive sub-sequences $i-2, \mathbf{i}, i-1, i+1$ or $i-1, i+1, \mathbf{i}, i+2$ or their reverses.

   For each one of those sub-cases, there are exactly $(n-3)!$ ways to arrange the rest of the permutation.
2. The permutation π contains the sub-sequences of the form $a, i, a+1$ and $i-1, i+1$ or their reverses. Again, we have to divide into two cases: $i \in \{2, n-1\}$ and $2 < i < n-1$.
   If $i \in \{2, n-1\}$ then there are $n-4$ ways to choose a, while if $2 < i < n-1$ then there are $n-5$ ways to choose a. Each choice of such two sub-sequences leaves $(n-3)!$ ways to arrange the rest of the permutation.

Hence, we have

$$E[Y_i] = \begin{cases} 0 & i = 1, n \\ \frac{2\cdot(n-3)!+2\cdot2\cdot(n-3)!(n-4)}{n!} & i = 2, n-1 \\ \frac{4\cdot(n-3)!+2\cdot2\cdot(n-3)!(n-5)}{n!} & 3 \le i \le n-2 \end{cases}$$

We have now:

$$E[Y] = \sum_{i=1}^{n} E[Y_i] = 2E[Y_2] + (n-4)E[Y_3].$$

A simple calculation now yields the following:

Theorem 4.2. *The expectation of the number of separators of both types: vertical and horizontal, in a randomly chosen n-permutation is*

$$\frac{4(n-3)^2}{n(n-1)(n-2)}$$

The asymptotic value of the expectation is 0.

Now, let Z be the random variable which counts the total number of separators (regardless of the type). By remark 1.8.3, the number of vertical separators has the same distribution as the number of horizontal separators, so $E(Z) = 2E(X) - E(Y)$. So, we have the following:

Theorem 4.3. *The expectation of the number of separators, in a randomly chosen n-permutation is*

$$\frac{4(n^3 - 6n^2 + 14n - 13)}{n(n-1)(n-2)}.$$

The asymptotic value of the expectation is 4.

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