Generating Functions for Domino Matchings in the $2 \times k$ Game of Memory

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Abstract

When all the elements of the multiset $\{1, 1, 2, 2, 3, 3, \ldots, k, k\}$ are placed in the cells of a $2 \times k$ rectangular array, in how many configurations are exactly v of the pairs directly over top one another, and exactly h directly beside one another — thus forming 1×2 dominoes? We consider the sum of matching numbers over the graphs obtained by deleting h horizontal and v vertical vertex pairs from the $2 \times k$ grid graph in all possible ways, providing a generating function for these aggregate matching polynomials. We use this result to derive a formal generating function enumerating the domino matchings, making connections with linear chord diagrams.

1 Introduction

The game of memory consists of the placement of a set of distinct pairs of cards in a rectangular array. The present author [1] considered the enumeration of the configurations in which exactly p of the pairs are placed directly beside, or over top of one another, thus forming 1×2 dominoes. In this paper we consider the case of $2 \times k$ arrays in more detail. In Figure 1 we show a configuration of the case $k = 4$ with $h = 1$ horizontal dominoes, and $v = 1$ vertical dominoes. We can think of the domino enumeration problem in different ways: as a Brauer diagram[1], as a chord diagram (cf. Krasko and Omelchenko [4]), or unfolded as a linear chord diagram (cf. Cameron and Killpatrick [5]). In the present paper we provide a generating function for the numbers $D_{k,v,h}$ which count, considering the pairs to be indistinguishable, the number of configurations with h horizontal dominoes and v vertical dominoes. Considering $D_{k,v,h}$ as matrices with $v \geq 0$ indexing rows, and $h \geq 0$ indexing columns, the first few values are as follows:

$$D_{0,v,h} = 1, \quad D_{1,v,h} = \begin{pmatrix} 0 & 0 \\ 1 & \end{pmatrix}, \quad D_{2,v,h} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & \\ 1 & & \end{pmatrix}, \quad D_{3,v,h} = \begin{pmatrix} 2 & 4 & 2 & 0 \\ 4 & 0 & 2 & \\ 0 & 0 & & \\ 1 & & & \end{pmatrix}.$$

[1]Terada [2] and Marsh and Martin [3] have considered Brauer diagrams in the context of combinatorics.

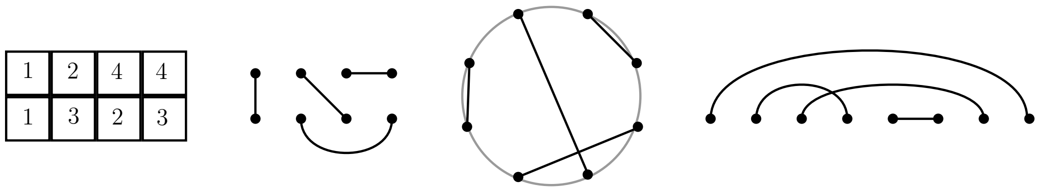

Figure 1: A configuration for the 2×4 array with one horizontal and one vertical domino is shown in four different representations. From left to right: a placement of paired cards in a game of memory, a Brauer diagram whose links correspond to the pairs, a chord diagram, and finally as a linear chord diagram resulting from breaking the circle in the chord diagram at the Westernmost point.

The matrices are naturally upper anti-triangular since $h + v \leq k$. The sum of the numbers on the anti-diagonal are the Fibonacci numbers, which count the domino tilings of the $2 \times k$ array. The sum of all numbers in the matrix is $(2k-1)!!$, which is the number of ways of placing the cards, modulo re-labelling of the pairs.

The strategy we will employ is to consider the matching numbers of the $2 \times k$ grid graph, see Figure 2, whose vertices represent the $2 \times k$ array of cards, and whose edges define the possible domino matchings. The present author [1] provided a method for computing the number of 0-domino configurations on any analogous graph G. Let ρ_j be the number of j-edge matchings on G. Then the number of 0-domino configurations is given by

$$\sum_{j=0}^{n} (-1)^j (2n-2j-1)!!\, \rho_j,$$

where n is the number of pairs. We may therefore compute the $D_{k,v,h}$ by computing the matching numbers for the graphs which arise from removing v vertical, and h horizontal vertex pairs (and their incident edges), from the $2 \times k$ grid graph in all possible ways.

2 Preliminaries

The *board* associated with the grid graph is defined in the usual way, see Figure 2. We color the vertices of the grid graph black and white, in a checkerboard pattern. The columns (rows) of the board represent the black (white) vertices. The cells enclosed by the outline represent the edges of the of graph. The vertical edges correspond to the cells on the central diagonal, while the horizontal edges correspond to the upper and lower diagonals. The rook or matching polynomial $r(x) = \sum r_j x^j$ encodes the number r_j of j-edge matchings on the graph and enjoys two important properties. One can *develop* the board using the property that the rook polynomial of a board B is equal to that of B with a given cell removed, plus x times the rook polynomial of B with the entire row and column containing that cell

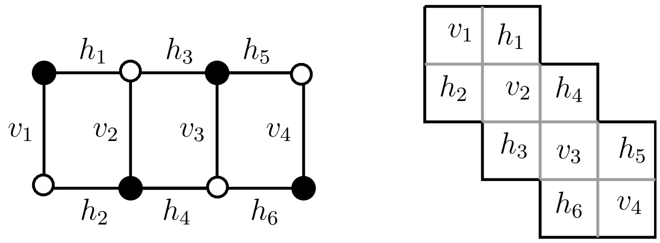

Figure 2: The 2×4 grid graph is shown on the left, while the corresponding board is shown on the right. The mapping of the edges of the graph to the cells of the board is also displayed.

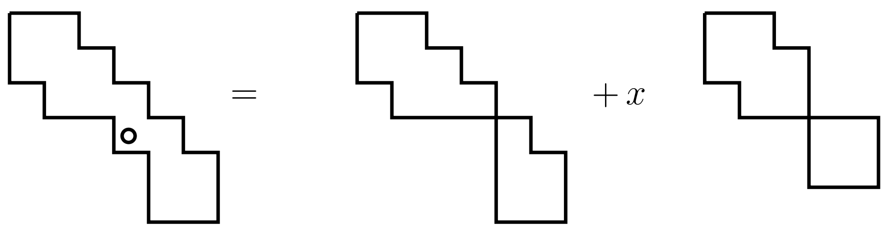

Figure 3: The board on the left is developed using the marked cell, according to the rules of the rook polynomial. The boards on the right hand side both factorize into the product of two rook polynomials.

removed, see Figure 3. The second property is that if a board can be separated into regions whose cells share no common row or column with another region (as in the right hand side of Figure 3), the rook polynomial factorizes into a product of the polynomials for the regions.

Riordan [6, 7, p. 230] (McQuistan and Lichtman [8] give connections to dimer models in Physics) provided the rook polynomials for the $2 \times k$ boards

$$T(x,y) = \frac{1-xy}{1-y-2xy-xy^2+x^3y^3} = \sum_{k=0}^{\infty} T_k(x) y^k,$$

where $T_k(x)$ are the rook polynomials. Riordan also provided similar generating functions for several related boards, shown in Figure 6,

$$\begin{aligned}
T(x,y) &= \frac{1-xy}{1-y-2xy-xy^2+x^3y^3}, \\
s(x,y) &= \frac{T(x,y)}{(1-xy)^2}, \quad r(x,y) = (1-xy)\, s(x,y), \quad R(x,y) = y\, r(x,y), \\
S(x,y) &= \left(1-2xy-xy^2+x^3y^3\right) s(x,y).
\end{aligned}$$

For example, the rook polynomial corresponding to the board on the left hand side of Figure 3 is $R_3(x)\, r_3(x) + x\, T_3(x)\, T_2(x)$.

3 Generating functions for matching numbers

We will require the sum of the matching numbers taken over the graphs obtained by removing h horizontal and v vertical vertex pairs from the $2 \times k$ grid graph in all possible ways. We will express this sum of matching numbers as a rook polynomial $\mathcal{T}_{k,v,h}(x)$, where the coefficient of x^q represents the number of q-edge matchings in the sum, see Figure 4.

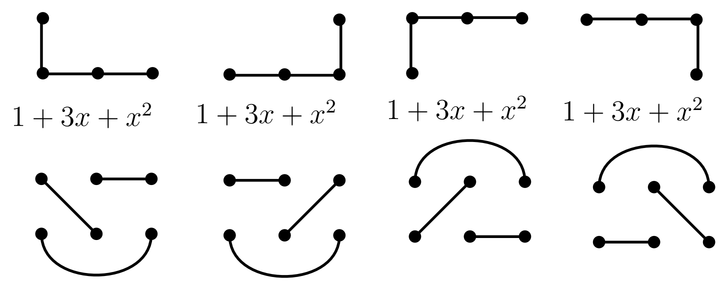

Figure 4: On the bottom row, the four configurations counted by $D_{3,0,1}$ are shown. On the top row, the graphs corresponding to the removal of the vertex pair corresponding to the domino, along with their matching polynomials, are shown. The sum of these polynomials yields $\mathcal{T}_{3,0,1}(x) = 4(1 + 3x + x^2)$.

Theorem 1. *The generating function for the $\mathcal{T}_{k,v,h}(x)$ is given by*[2]

$$\begin{aligned}\mathcal{T}(x, y, w, z) &= \sum_{k,h,v} \mathcal{T}_{k,v,h}(x)\, y^{k-v-h} w^v z^h \\ &= \frac{1 - xy - z}{1 - (1 + 2x)y - z - w(1 - xy - z) + (xy + z)(x^2y^2 - (1 - z)z - y(1 - 2xz))}. \end{aligned} \tag{1}$$

Proof. The removal of a horizontal vertex pair from the board results in the deletion of a cell on the lower or upper diagonal, together with its entire row and column. In Figure 5, boards resulting from the deletion of two horizontal vertex pairs are shown. When the boards are developed using the cells marked with a dot in Figure 5, various shapes arise; these are shown in Figure 6. For example, the configuration shown on the left in Figure 5 corresponds to

$$\begin{aligned} &R(x, y) \cdot S(x, y) \cdot r(x, y) + xy\, T(x, y) \cdot R(x, y) \cdot r(x, y) \\ &+ R(x, y) \cdot r(x, y) \cdot xy\, T(x, y) + x^2y^2\, T(x, y) \cdot T(x, y) \cdot T(x, y) \\ &= yr^2S + 2xy^2Tr^2 + x^2y^2T^3, \end{aligned} \tag{2}$$

[2]The power of y is taken to be $k - v - h$; this is a useful parameterization for computing the generating function for the $D_{k,v,h}$.

where the $\cdot$ denotes ordinary multiplication, and has been included to aid in the following explanation. The first term corresponds to the removal of both marked cells, the second (third) term to the additional removal of the row and column containing the first (second) marked cell, and the last term to the removal of the rows and columns containing both marked cells. The multiplication also accounts for the ordered sum over all possible positions of the removed horizontal vertex pairs[3]. When the row and column containing a marked cell is removed, the overall board is shortened, and hence earns an extra factor of y; this is why the expression xy appears for each such removal. It is straightforward to see that the corresponding expression for the board shown on the right in Figure 5 differs from the above only in the first term, which becomes $R(x, y) \cdot s(x, y) \cdot R(x, y) = y^2r^2s$, and hence is produced by the substitution $S \to ys$.

It now becomes clear how to generalize this argument to account for an arbitrary number of horizontal vertex pairs. We will require all words formed from the alphabet $\{T,\ yr^2S^m(ys)^l\}$, where $m, l \geq 0$. A q-letter word is proportional to $(xy)^{q-1}$, and is formed as follows:

$$(2xyz)^{q-1}\left(T + \frac{2yzr^2}{1 - z(S + ys)}\right)^q,$$

where we have introduced the variable z, so that the coefficient of z^h gives the expression corresponding to the removal of h non-coincident (see Footnote 3) horizontal vertex pairs. The factors of 2 arise as there are two choices from which to remove a horizontal vertex pair: from the upper or the lower diagonal of the board. For example, in Equation (2), the horizontal vertex pairs could have both been taken from the *lower* diagonal.

Taking the sum over $q > 0$, we obtain

$$\mathcal{H} = \frac{T + \frac{2yzr^2}{1-z(S+ys)}}{1 - 2xyz\left(T + \frac{2yzr^2}{1-z(S+ys)}\right)}.$$

It remains to account for the removal of vertical vertex pairs and coincident horizontal vertex pairs from the grid graph. The removal of a single pair of coincident horizontal vertex pairs is equivalent to the removal of two neighboring vertical vertex pairs. The effect of either of these removals on the board is to break it into two factors. Thus we find that the removal of coincident horizontal and vertical vertex pairs is accounted for as follows:

$$\sum_{j=0}^{\infty}(w + z^2)^j\mathcal{H}^{j+1} = \frac{T + \frac{2yzr^2}{1-z(S+ys)}}{1 - (2xyz + z^2 + w)\left(T + \frac{2yzr^2}{1-z(S+ys)}\right)},$$

where the coefficient of w^v corresponds to the removal of v vertical vertex pairs. Simplifying this expression, we obtain Equation (1). □

[3] The exception is when the two horizontal vertex pairs are directly over top of one another, these coincident configurations will be accounted for below.

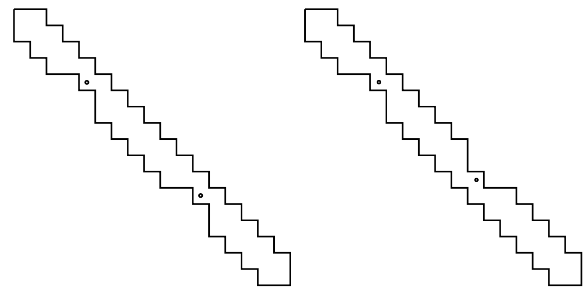

Figure 5: Boards corresponding to removing two horizontal vertex pairs. On the left both pairs have been removed from the upper diagonal. On the right one upper and one lower diagonal pair have been removed. The marked cells are used in further decomposing the boards using the calculus of the rook polynomial.

4 From matching numbers to domino-counting generating functions

We now use the result of Theorem 1 to compute the number of configurations with exactly h horizontal, and v vertical dominoes. Let the $\rho_j(k,v,h)$ be defined as follows:

$$\mathcal{T}_{k,v,h}(x) = \sum_{j=0}^{n} \rho_j(k,v,h)\, x^j,$$

where $n = k - h - v$. As mentioned in the Introduction, the number $D_{k,h,v}$ of configurations with exactly h horizontal, and v vertical dominoes is given by

$$D_{k,v,h} = \sum_{j=0}^{n} (-1)^j (2n-2j-1)!!\, \rho_j(k,v,h). \tag{3}$$

We define the corresponding generating function in the usual way,

$$D(y,w,z) = \sum_{k,h,v} D_{k,v,h}\, y^k w^v z^h.$$

We now translate Equation (3) into an operation on $\mathcal{T}(x,y,w,z)$.

Theorem 2. *The generating function $D(y,w,z)$ may be obtained using the following integral representation:*

$$D(y,w,z) = \int_0^\infty dt\, e^{-t} \frac{1}{2\pi i} \oint_{C_\epsilon} \frac{dx}{x\sqrt{1+2x}}\, \mathcal{T}\left(\frac{x}{t}, \frac{-yt}{x}, yw, yz\right),$$

where the contour integral with respect to x is taken around a small circle containing the origin.

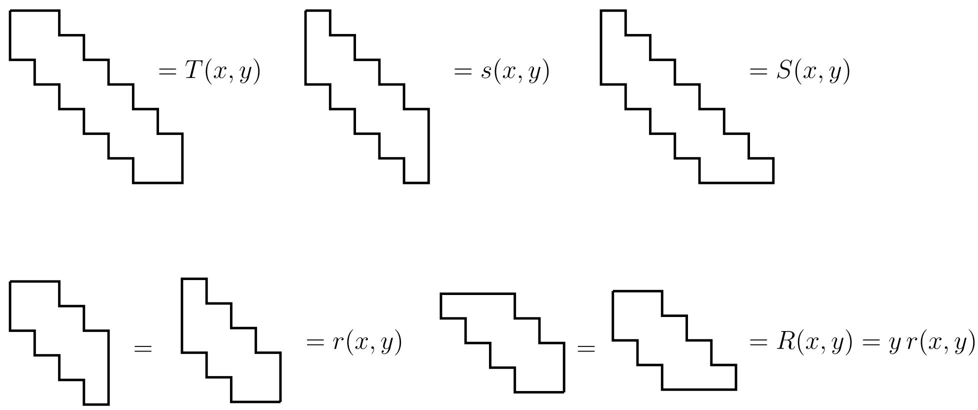

Figure 6: The boards which arise when the calculus of the rook polynomial is applied to the boards shown in Figure 5.

Proof. We consider the coefficient of y^n in the Taylor expansion of $\mathcal{T}(x, y, w, z)$

$$[y^n]\mathcal{T}(x, y, w, z) = \sum_{j=0}^{n} \rho_j(w, z)\, x^j.$$

Under the integration in t, the replacements $x \to xt^{-1}$ and $y \to yt$ dress this result by a factor of $(n - j)!$

$$[y^n] \int_0^\infty dt\, e^{-t}\, \mathcal{T}(xt^{-1}, yt, w, z) = \sum_{j=0}^{n} (n - j)!\, \rho_j(w, z)\, x^j.$$

We now consider the coefficient of x^n which results from multiplying this series by the series expansion of $(1 + 2x)^{-1/2}$

$$[x^n] \left(\sum_{q=0}^{\infty} (-1)^q \frac{(2q-1)!!}{q!}\, x^q \right) \left(\sum_{j=0}^{n} (n - j)!\, \rho_j(w, z)\, x^j \right)$$
$$= (-1)^n \sum_{j=0}^{n} (-1)^j (2n - 2j - 1)!!\, \rho_j(w, z).$$

We may, therefore, compute this quantity by further scaling $y \to y/x$ and taking the residue at the origin after an overall multiplication by x^{-1}. The factor of $(-1)^n$ is absorbed by a final replacement $y \to -y$. The variables w and z are also scaled by y as they correspond to removed vertex pairs, which hence shorten the original grid graph. □

Corollary 3. *The generating function $D(y, w, z)$ is given by*

$$D(y,w,z) = \int_0^\infty dt \frac{e^{-t}}{(1+(1-w)y-(1-z)^2y^2)\sqrt{1-\frac{2ty(1-(1-z)y)}{(1+(1-z)y)(1+(1-w)y-(1-z)^2y^2)}}}$$
$$= \sum_{j=0}^{\infty}(2j-1)!! \frac{y^j(1-(1-z)y)^j}{(1+(1-z)y)^j(1+(1-w)y-(1-z)^2y^2)^{j+1}}.$$

Proof. We note from Equation (1) that $\mathcal{T}\left(\frac{x}{t}, \frac{-yt}{x}, yw, yz\right) = Ax/(Bx+Ct)$, where A, B, and C are functions of y, w, and z. The contour integration replaces $x \to -Ct/B$ in the factor $(1+2x)^{-1/2}$. The integration over t is interpreted as acting on the series expansion of the resulting expression. □

This is not a convergent series[4], and hence we cannot benefit from an analytic generating function with which questions about asymptotic behavior could easily be answered. In order to convert this formal generating function into a convergent series, we must take an inverse Laplace transform in y to form an exponential generating function

$$E(y, w, z) = \mathcal{L}^{-1}\left\{y^{-1}D(y^{-1}, w, z)\right\}.$$

Performing this transform is not straightforward, however in the simple case of counting only vertical dominoes, it is feasible and yields a well-known result,

$$D(y, w, 1) = \sum_{j=0}^{\infty} \frac{(2j-1)!!\, y^j}{(1+(1-w)y)^{j+1}} \to E(y, w, 1) = \frac{e^{y(w-1)}}{\sqrt{1-2y}}, \tag{4}$$

which counts the number of matchings of $2k$ people with partners (of either sex) such that exactly v couples are left together, see A055140. Unfortunately, we have been unable to perform the transform for the case of counting only horizontal dominoes (A325754, to appear)

$$D(y, 1, z) = \frac{1}{(1-(1-z)y)} \sum_{j=0}^{\infty} \frac{(2j-1)!!\, y^j}{(1+(1-z)y)^{2j+1}},$$

but in the next section we derive the exponential generating function by appealing to known results for the $1 \times 2k$ problem.

It is also interesting to consider the case where vertical and horizontal dominoes are not distinguished, i.e., $D(y, z, z)$. The present author [1] considered this sequence previously.

[4] It may be interpreted as the real part of the expansion of the following expression, asymptotic in y^{-1}: $\sqrt{\frac{2(1+(1-z)y)}{y(1-(1-z)y)(1+(1-w)y-(1-z)^2y^2)}} F\left(\sqrt{\frac{(1+(1-z)y)(1+(1-w)y-(1-z)^2y^2)}{2y(1-(1-z)y)}}\right)$, where F is Dawson's integral, see http://en.wikipedia.org/wiki/Dawson_function.

We can now provide a generating function for these numbers (A325753, to appear)

$$D(y,z,z)=\sum_{j=0}^{\infty}\frac{(2j-1)!!\,y^j\,(1-(1-z)y)^j}{(1+(1-z)y)^j\,(1+(1-z)y-(1-z)^2y^2)^{j+1}}.$$

Several conjectures were also made for generating functions for the so-called $(k-l)$-domino configurations, when the number of dominoes is l less than the maximum value k. These can now be readily calculated as follows:

$$\mathcal{F}_l(y)=\sum_k\left(\sum_{\substack{h,v\\h+v=k-l}}D_{k,v,h}\right)y^k=[z^{-l}]\lim_{z\to\infty}D\left(\frac{y}{z},z,z\right).$$

The first few such generating functions are

$$\mathcal{F}_0=\frac{1}{1-y-y^2},\qquad \mathcal{F}_1=\frac{2y^3}{(1-y)(1-y-y^2)^2},\qquad \mathcal{F}_2=\frac{y^2(1+3y+6y^2+y^3+3y^4)}{(1-y)^2(1-y-y^2)^3}.$$

The function $\mathcal{F}_0$ is the generating function for the Fibonacci numbers, giving the number of domino tilings. The function $\mathcal{F}_1$ is that for the path length of the Fibonacci tree of order k, A178523. The sequences corresponding to the cases $l=2,\ldots,5$ appear as A318267 to A318270 respectively.

5 Connections to linear chord diagrams

Kreweras and Poupard [9] solved the problem of counting the h-domino configurations on the $1\times 2k$ grid graph (i.e., a path of length $2k$). Cameron and Killpatrick [5] recently revisited this case in the context of linear chord diagrams, and provided a derivation of the corresponding exponential generating function. Let $L_{k,h}$ be the number of h-domino configurations on the $1\times 2k$ grid graph. We seek to establish a correspondence with the number $D_{k,h}$ of h horizontal domino configurations on the $2\times k$ grid graph, where we allow any number of vertical dominoes.

Theorem 4. *The numbers $L_{k,h}$ and $D_{k,h}$ are related by the following recursion relation:*

$$D_{k,h}=D_{k-1,h}+L_{k,h}-D_{k-1,h-1}.$$

Proof. We begin by unfolding the the vertices of the $2\times k$ grid graph, as shown in Figure 7, to give the vertices of the $1\times 2k$ grid graph. We then note that the central pair of vertices does not correspond to a horizontal domino in the $2\times k$ graph, but rather to a vertical one. Thus the configurations counted by $L_{k,h}$ may be divided into two sets: those with a domino on the central pair and those without. Those configurations with a domino on the central pair are counted by $D_{k-1,h-1}$, as the central pair is effectively deleted, leaving the $2\times(k-1)$ grid graph with $h-1$ horizontal dominoes, see Figure 8. The configurations counted by $D_{k,h}$ can be similarly divided, but since the central pair this time represents a vertical domino, those configurations with this vertical domino are equal in number to $D_{k-1,h}$. □

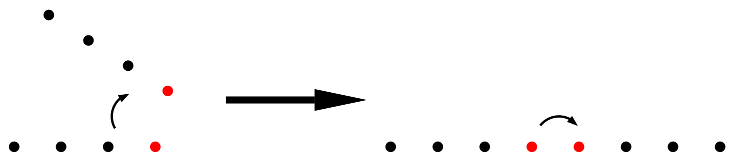

Figure 7: The $2 \times k$ grid graph is unfolded to produce the $1 \times 2k$ grid graph. The vertices marked in red comprise a vertical domino which becomes horizontal upon unfolding.

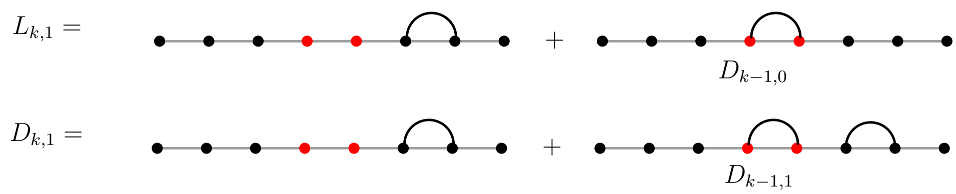

Figure 8: The relationship between the numbers, $L_{k,h}$ and $D_{k,h}$, of h-horizontal-domino configurations on the $1 \times 2k$ and $2 \times k$ grid graphs respectively, is shown for the case $h = 1$.

Jovovic [10], and Cameron and Killpatrick [5] provide the exponential generating function

$$L(y,z) = \sum_{k,h} L_{k,h} \frac{y^k}{k!} z^h = \frac{e^{(\sqrt{1-2y}-1)(1-z)}}{\sqrt{1-2y}}.$$

Translating the recursion relation from Theorem 4 into a differential equation for the exponential generating function, we obtain

$$\frac{\partial E(y,1,z)}{\partial y} - (1-z)E(y,1,z) = \frac{\partial L(y,z)}{\partial y}. \tag{5}$$

This elementary non-homogeneous first-order ODE may be solved using an integrating factor.

Corollary 5. *The exponential generating function for $D_{k,h}$ is as follows:*

$$E(y,1,z) = \frac{e^{(\sqrt{1-2y}-1)(1-z)}}{\sqrt{1-2y}} - e^{(y-2)(1-z)}\sqrt{\frac{\pi}{2}}\sqrt{1-z}\left(\mathrm{Erfi}\left(\frac{(\sqrt{1-2y}+1)\sqrt{1-z}}{\sqrt{2}}\right) - \mathrm{Erfi}(\sqrt{2}\sqrt{1-z})\right), \tag{6}$$

where we have expressed the result in terms of the imaginary error function Erfi.

Proof. The method of an integrating factor may be used to solve Equation (5). □

6 Asymptotic growth and distributions

In the $k \to \infty$ limit, one expects the asymptotic distribution of dominoes to be Poisson. Let

$$P_{k,p} = \frac{1}{(2k-1)!!} \sum_{\substack{v,h \\ v+h=p}} D_{k,v,h}.$$

This quantity is the probability distribution for dominoes (not distinguishing between horizontal and vertical). The present author [1] proved that the mean of this distribution, exact in k, is given by

$$\sum_p p\, P_{k,p} = \frac{3k-2}{2k-1}.$$

We therefore expect that

Conjecture 6.

$$\lim_{k\to\infty} P_{k,p} \simeq \frac{e^{-3/2}}{p!} \left(\frac{3}{2}\right)^p.$$

Without the benefit of a convergent generating function for the $D_{k,v,h}$, we cannot explore the asymptotics using the usual machinery of analytic combinatorics. We can, however, explore the asymptotics of the exponential generating functions $E(y,w,1)$ and $E(y,1,z)$ given in Equations (4) and (6) respectively. This allows us to analyze the following distributions:

$$V_{k,v} = \frac{1}{(2k-1)!!} \sum_h D_{k,v,h}, \qquad H_{k,h} = \frac{1}{(2k-1)!!} \sum_v D_{k,v,h}.$$

Taking derivatives of $E(y,w,1)$ by w, we can compute the factorial moments of $V_{k,v}$. We note that

$$\left[y^k\right] \left.\frac{\partial^m E(y,w,1)}{\partial w^m}\right|_{w=1} = \left[y^k\right] \frac{y^m}{\sqrt{1-2y}} \sim \left(\frac{1}{2}\right)^m \frac{(2k-1)!!}{k!},$$

where $\sim$ indicates asymptotic growth in k, and so the m^{th} factorial moment of $V_{k,v}$ is asymptotically $(1/2)^m$, consistent with a Poisson distribution of mean $1/2$. A similar argument can be made for $H_{k,h}$, where the corresponding mean is found to be 1. Indeed, Kreweras and Poupard [9] (see also Cameron and Killpatrick [5]) proved that the asymptotic factorial moments for the distribution of dominoes on the $1 \times 2k$ grid graph are all equal to one; it is clear that the case of horizontal dominoes on the $2 \times k$ grid graph must have the same asymptotic behavior. We therefore see that if, in the $k \to \infty$ limit, the occurrences of vertical and horizontal dominoes are independent, the combined distribution $P_{k,p}$ should indeed be Poisson with mean $1/2 + 1 = 3/2$.

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(Concerned with sequences A000045, A046741, A055140, A079267, A178523, A265167, A318243, A318244, A318267, A318268, A318269, A318270, A325753, and A325754)
