

The enumeration of coverings of closed orientable Euclidean manifolds \mathcal{G}_3 and \mathcal{G}_5 .

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Abstract

There are only 10 Euclidean forms, that is flat closed three dimensional manifolds: six are orientable and four are non-orientable. The aim of this paper is to describe all types of n -fold coverings over orientable Euclidean manifolds \mathcal{G}_3 and \mathcal{G}_5 , and calculate the numbers of non-equivalent coverings of each type. We classify subgroups in the fundamental groups $\pi_1(\mathcal{G}_3)$ and $\pi_1(\mathcal{G}_5)$ up to isomorphism and calculate the numbers of conjugated classes of each type of subgroups for index n . The manifolds \mathcal{G}_3 and \mathcal{G}_5 are uniquely determined among the others orientable forms by their homology groups $H_1(\mathcal{G}_3) = \mathbb{Z}_3 \times \mathbb{Z}$ and $H_1(\mathcal{G}_5) = \mathbb{Z}$.

Key words: Euclidean form, platycosm, flat 3-manifold, non-equivalent coverings, crystallographic group.

2010 Mathematics Subject Classification: 20H15, 57M10, 55R10.

Introduction

Let \mathcal{M} be a manifold with fundamental group $\Gamma = \pi_1(\mathcal{M})$. Two coverings

$$p_1 : \mathcal{M}_1 \rightarrow \mathcal{M} \text{ and } p_2 : \mathcal{M}_2 \rightarrow \mathcal{M}$$

*This work was supported by the Russian Foundation for Basic Research (grant 18–01–00036/18).

†This work was supported by the Russian Foundation for Basic Research (grant 16-31-00138).

are said to be equivalent if there exists a homeomorphism $h : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ such that $p_1 = p_2 \circ h$. According to the general theory of covering spaces, any n -fold covering is uniquely determined by a subgroup of index n in the group Γ . The equivalence classes of n -fold covering of \mathcal{M} are in one-to-one correspondence with the conjugacy classes of subgroups of index n in the fundamental group $\pi_1(\mathcal{M})$. See, for example, ([7], p. 67). In such a way the following two natural problems arise. The first one is to calculate the number of subgroups of given finite index n in $\pi_1(\mathcal{M})$. The second problem is to find the number of conjugacy classes of subgroups of index n in $\pi_1(\mathcal{M})$.

The problem of enumeration for nonequivalent coverings over a Riemann surface with given branch type goes back to the paper [8] by Hurwitz, in which the number of coverings over the Riemann sphere with given number of simple (of order two) branching points was determined. Later, in [9], it has been proved that this number can be expressed in the terms of irreducible characters of symmetric groups. The Hurwitz problem was studied by many authors. A detailed survey of the related results is contained in ([13], [10]). For closed Riemann surfaces, this problem was completely solved in [16]. However, of most interest is the case of unramified coverings. Let $s_\Gamma(n)$ denote the number of subgroups of index n in the group Γ , and let $c_\Gamma(n)$ be the number of conjugacy classes of such subgroups. According to what was said above, $c_\Gamma(n)$ coincides with the number of nonequivalent n -fold coverings over a manifold \mathcal{M} with fundamental group Γ . The numbers $s_\Gamma(n)$ and $c_\Gamma(n)$ for the fundamental group of a closed surface (orientable or not) were found in ([14], [15], [17]). In the paper [18], a general method for calculating the number $c_\Gamma(n)$ of conjugacy classes of subgroups in an arbitrary finitely generated group Γ was given. Asymptotic formulas for $s_\Gamma(n)$ in many important cases were obtained by T. W. Müller and his collaborators ([19], [20], [21]).

In the three-dimensional case, for a large class of Seifert fibrations, the value of $s_\Gamma(n)$ was calculated in [11] and [12]. In our previous papers [1] and [2] the numbers $s_\Gamma(n)$ and $c_\Gamma(n)$ were determined for the fundamental group of non-orientable Euclidian manifolds \mathcal{B}_1 and \mathcal{B}_2 whose homologies are $H_1(\mathcal{B}_1) = \mathbb{Z}_2 \oplus \mathbb{Z}^2$ and $H_1(\mathcal{B}_2) = \mathbb{Z}^2$ and for the orientable Euclidean manifolds \mathcal{G}_2 and \mathcal{G}_4 with $H_1(\mathcal{G}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$ and $H_1(\mathcal{G}_4) = \mathbb{Z}_2 \oplus \mathbb{Z}$ respectively.

The aim of the present paper is to investigate n -fold coverings over orientable Euclidean three dimensional manifolds \mathcal{G}_3 and \mathcal{G}_5 , with the first homologies $H_1(\mathcal{G}_3) = \mathbb{Z}_3 \oplus \mathbb{Z}$ and $H_1(\mathcal{G}_5) = \mathbb{Z}$. We classify subgroups of finite index in the fundamental groups of $\pi_1(\mathcal{G}_3)$ and $\pi_1(\mathcal{G}_5)$ up to isomorphism. Then we calculate the number of subgroups and the number of conjugacy classes of subgroups of each isomorphism type for a given index n .

We note that numerical methods to solve these and similar problems for the three-dimensional crystallogical groups were developed by the Bilbao group [4]. The first homologies of all the three-dimensional crystallogical groups are determined in [22].

Notations

We use the following notations: $s_{H,G}(n)$ is the number of subgroups of index n in the group G , isomorphic to the group H ; $c_{H,G}(n)$ is the number conjugacy classes of sub-

groups of index n in the group G , isomorphic to the group H .

Also we will need the following combinatorial functions:

$$\sigma_0(n) = \sum_{k|n} 1, \quad \sigma_1(n) = \sum_{k|n} k, \quad \sigma_2(n) = \sum_{k|n} \sigma_1(k), \quad \omega(n) = \sum_{k|n} k\sigma_1(k),$$

$$\theta(n) = |\{(p, q) | p > 0, q \geq 0, p^2 - pq + q^2 = n\}|.$$

In all cases we consider the function vanished if $n \notin \mathbb{N}$.

Remark. It can be shown that $\theta(n) = \sum_{k|n} \left(\frac{k}{3}\right)$, where $\left(\frac{k}{3}\right)$ is the Legendre symbol, see [6] p.112. This representation clarifies the analogy between the functions $\sigma_1(n)$ and $\theta(n)$, and makes the appearance of the latter one less amazing. However, this representation will not be used further.

1 Overview

The main goal of this paper is to prove the following results.

The first theorem provides the complete solution of the problem of enumeration of subgroups of a given finite index in $\pi_1(\mathcal{G}_3)$.

Theorem 1. *Every subgroup Δ of finite index n in $\pi_1(\mathcal{G}_3)$ is isomorphic to either $\pi_1(\mathcal{G}_3)$ or $\pi_1(\mathcal{G}_1) \cong \mathbb{Z}^3$. The respective numbers of subgroups are*

$$(i) \quad s_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_3)}(n) = \sum_{k|n} k\theta(k) - \sum_{k|\frac{n}{3}} k\theta(k),$$

$$(ii) \quad s_{\mathbb{Z}^3, \pi_1(\mathcal{G}_3)}(n) = \omega\left(\frac{n}{3}\right).$$

The next theorem provides the number of conjugacy classes of subgroups of index n in $\pi_1(\mathcal{G}_3)$ for each isomorphism type. That is the number of non-equivalent n -fold covering \mathcal{G}_3 , which have a prescribe fundamental group.

Theorem 2. *Let $\mathcal{N} \rightarrow \mathcal{G}_3$ be an n -fold covering over \mathcal{G}_3 . If n is not divisible by 3 then \mathcal{N} is homeomorphic to \mathcal{G}_3 . If n is divisible by 3 then \mathcal{N} is homeomorphic to either \mathcal{G}_3 or \mathcal{G}_1 . The corresponding numbers of nonequivalent coverings are given by the following formulas:*

$$(i) \quad c_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_3)}(n) = \sum_{k|n} \theta(k) + \sum_{k|\frac{n}{3}} \theta(k) - 2 \sum_{k|\frac{n}{9}} \theta(k)$$

$$(ii) \quad c_{\mathbb{Z}^3, \pi_1(\mathcal{G}_3)}(n) = \frac{1}{3} \left(\omega\left(\frac{n}{3}\right) + 2 \sum_{k|\frac{n}{3}} \theta(k) + 4 \sum_{k|\frac{n}{9}} \theta(k) \right).$$

The next two theorems are analogues of Theorem 1 and Theorem 2 respectively for the manifold \mathcal{G}_5 .

Theorem 3. *Every subgroup Δ of finite index n in $\pi_1(\mathcal{G}_5)$ is isomorphic to either $\pi_1(\mathcal{G}_5)$ or $\pi_1(\mathcal{G}_3)$ or $\pi_1(\mathcal{G}_2)$ or $\pi_1(\mathcal{G}_1) \cong \mathbb{Z}^3$. The respective numbers of subgroups are*

$$(i) \quad s_{\pi_1(\mathcal{G}_5), \pi_1(\mathcal{G}_5)}(n) = \sum_{k|n, (\frac{n}{k}, 6)=1} k\theta(k)$$

$$(ii) \quad s_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_5)}(n) = \sum_{k|\frac{n}{2}} k\theta(k) - \sum_{k|\frac{n}{6}} k\theta(k)$$

$$(iii) \quad s_{\pi_1(\mathcal{G}_2), \pi_1(\mathcal{G}_5)}(n) = \omega\left(\frac{n}{3}\right) - \omega\left(\frac{n}{6}\right)$$

$$(iv) \quad s_{\mathbb{Z}^3, \pi_1(\mathcal{G}_5)}(n) = \omega\left(\frac{n}{6}\right).$$

Theorem 4. *The numbers of n -fold covering over \mathcal{G}_5 is given by the following formulas:*

$$(i) \quad c_{\pi_1(\mathcal{G}_5), \pi_1(\mathcal{G}_5)}(n) = \sum_{k|n, (\frac{n}{k}, 6)=1} \theta(k)$$

$$(ii) \quad c_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_5)}(n) = \sum_{k|\frac{n}{2}} \theta(k) - \sum_{k|\frac{n}{18}} \theta(k)$$

$$(iii) \quad c_{\pi_1(\mathcal{G}_2), \pi_1(\mathcal{G}_5)}(n) = \frac{1}{3} \left(\sigma_2\left(\frac{n}{3}\right) + 2\sigma_2\left(\frac{n}{6}\right) - 3\sigma_2\left(\frac{n}{12}\right) + 2 \sum_{k|\frac{n}{3}} \theta(k) - 2 \sum_{k|\frac{n}{6}} \theta(k) \right)$$

$$(iv) \quad c_{\mathbb{Z}^3, \pi_1(\mathcal{G}_5)}(n) = \frac{1}{6} \left(\omega\left(\frac{n}{6}\right) + \sigma_2\left(\frac{n}{6}\right) + 3\sigma_2\left(\frac{n}{12}\right) + 4 \sum_{k|\frac{n}{6}} \theta(k) + 4 \sum_{k|\frac{n}{18}} \theta(k) \right)$$

In the Appendix we present the Dirichlet generating functions for the above sequences.

2 Preliminaries

Further we use the representations for the fundamental groups $\pi(\mathcal{G}_3)$ and $\pi(\mathcal{G}_5)$ given in [25] and [3].

$$\pi_1(\mathcal{G}_3) = \langle x, y, z : xyx^{-1}y^{-1} = 1, zxz^{-1} = y, zyz^{-1} = (xy)^{-1} \rangle. \quad (2.1)$$

$$\pi_1(\mathcal{G}_5) = \langle \tilde{x}, \tilde{y}, \tilde{z} : \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = 1, \tilde{z}\tilde{x}\tilde{z}^{-1} = \tilde{x}\tilde{y}, \tilde{z}\tilde{y}\tilde{z}^{-1} = \tilde{x}^{-1} \rangle. \quad (2.2)$$

We will widely use the following statements.

Proposition 1. *Let Δ be a subgroup of finite index n in \mathbb{Z}^2 . Then Δ have a pair of generators of the form $(a, 0)$ and (μ, b) where a and b are positive integers with $ab = n$ and μ is a nonnegative integer with $0 \leq \mu < a$. Furthermore, the set of subgroups Δ with $|\mathbb{Z}^2 : \Delta| = n$ bijectively corresponds to the set of pairs of generators of described form. The number of such subgroups Δ is $\sigma_1(n)$.*

Let Δ be a subgroup of finite index n in \mathbb{Z}^3 . Then Δ have a set of three generators $(a, 0, 0)$, $(\mu, b, 0)$ and (ν, λ, c) where a, b, c are positive integers with $abc = n$, μ, ν are integers with $0 \leq \mu, \nu < a$ and λ is an integer with $0 \leq \lambda < b$. Furthermore, the set of subgroups Δ with $|\mathbb{Z}^3 : \Delta| = n$ bijectively corresponds to the set of triplets of generators of described form. The number of such subgroups Δ is $\omega(n)$.

Corollary 1. *Given an integer n , by $S(n)$ denote the number of pairs (H, ν) , where H is a subgroup of index n in \mathbb{Z}^2 and ν is a coset of \mathbb{Z}^2/H with $2\nu = 0$ (we use the additive notation). Then*

$$S(n) = \sigma_1(n) + 3\sigma_1\left(\frac{n}{2}\right).$$

Lemma 1. *Let $H \leq G$ be an abelian group and its subgroup of finite index. Let $\phi : G \rightarrow G$ be an endomorphism of G , such that $\phi(H) \leq H$ and the index $|G : \phi(G)|$ is also finite. Then the cardinality of kernel of $\phi : G/H \rightarrow G/H$ equals to the index $|G : (H + \phi(G))|$.*

For the proofs, see Lemma 1 in [2]

Remark 1. *Combining Lemma 1 and Corollary 1 we get the following observation. Given a subgroup $H \leq \mathbb{Z}^2$, the number of $\nu \in \mathbb{Z}^2/H$, such that $2\nu = 0$, is equal to $|\mathbb{Z}^2 / \langle (2, 0), (0, 2), H \rangle|$. Indeed, taking \mathbb{Z}^2 as G , H as H and $\phi : g \rightarrow 2g, g \in \mathbb{Z}^2$ as ϕ one gets the desired equality. Since for each H the numbers $|\{\nu | \nu \in \mathbb{Z}^2/H, 2\nu = 0\}|$ and $|\mathbb{Z}^2 / \langle (2, 0), (0, 2), H \rangle|$ coincide, their sums taken over all subgroups H also coincide, that is*

$$S(n) = \sum_{H \leq \mathbb{Z}^2, |\mathbb{Z}^2/H|=n} |\{\nu | \nu \in \mathbb{Z}^2/H, 2\nu = 0\}| = \sum_{H \leq \mathbb{Z}^2, |\mathbb{Z}^2/H|=n} |\mathbb{Z}^2 / \langle (2, 0), (0, 2), H \rangle|.$$

Definition 1. *Consider the group \mathbb{Z}^2 . By ℓ denote the automorphism $\ell : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$ given by $(x, y) \rightarrow (-y, x - y)$.*

Lemma 2. *A subgroup $H \leq \mathbb{Z}^2$ is preserved by ℓ if and only if H is generated by a pair of elements of the form $(p, q), (-q, p - q)$. In this case $|\mathbb{Z}^2/H| = p^2 - pq + q^2$. For a given integer n the number of invariant under ℓ subgroups H of index n in \mathbb{Z}^2 is $\theta(n)$.*

Proof. Suppose H is generated by elements (p, q) and $(-q, p - p) = \ell((p, q))$. Then obviously $\ell(H) = H$. Also $|\mathbb{Z}^2/H| = p^2 - pq + q^2$, since $p^2 - pq + q^2$ is the number of integer points in a fundamental domain of H on the plane.

Vice versa, suppose $\ell(H) = H$. Denote $d(x, y) = x^2 - xy + y^2$. Let $u = (p, q) \in H \setminus \{0\}$ be an element with the minimal value of $d(u)$. Consider the subgroup $H_1 = \langle u, \ell(u) \rangle \leq H$. Assume $H_1 \neq H$ and $v \in H \setminus H_1$. Since $H_1 = \langle u, \ell(u) \rangle$, the fundamental domain of H_1 is a parallelogram with vertices $0, u, \ell(u), u + \ell(u)$. That means that the plane splits

into the parallelograms of the form $w, w + u, w + \ell(u), w + u + \ell(u), w \in H_1$, each of them splits into two right triangles. One of this triangles contains v . Note that the distance from a point inside a right triangle to one of its vertices is not greater than the side of this triangle. This contradicts the minimality of $d(u)$, thus $H_1 = H$.

To find the number of subgroups H note that the number of pairs (p, q) with $p^2 - pq + q^2 = n, p > 0, q \geq 0$ is $\theta(n)$. As it was proven above, for each pair (p, q) of the above type two pairs (p, q) and $\ell((p, q))$ generate a subgroup H of the required type. Moreover $d(p, q)$ takes the minimal value among $d(v), v \in H \setminus \{0\}$. Suppose two different pairs (p, q) and (p', q') correspond to the same subgroup H . Then $(p - p', q - q') \in H$ and $0 < d(p - p', q - q') < d(p, q)$, which contradiction proves that there is a one-to-one correspondence between pairs (p, q) and subgroups H . \square

Before formulating the next corollary note that $\ell(\nu), \nu \in \mathbb{Z}^2/H$ is well-defined if $\ell(H) \leq H$.

Corollary 2. *Let n be an integer. Consider the set of all subgroups H of \mathbb{Z}^2 of index n . Given an integer n , by $T(n)$ denote the number of pairs (H, ν) , where H is a subgroup of index n in \mathbb{Z}^2 with $\ell(H) = H$ and ν is a coset of \mathbb{Z}^2/H with $\ell(\nu) = \nu$. Then*

$$T(n) = \theta(n) + 2\theta\left(\frac{n}{3}\right).$$

Proof. Consider subgroup $H \leq \mathbb{Z}^2$ with $|\mathbb{Z}^2 : H| = n$ and $\ell(H) = H$. Lemma 2 claims that H has a pair of generators $(p, q), (-q, p - q)$, where $p^2 - pq + q^2 = n$. Suppose $\ell(\nu) = \nu$ holds for some coset $\nu \in \mathbb{Z}^2/H$. Let $(a, b) \in \mathbb{Z}^2$ be a representative of coset ν . Then $\nu - \ell(\nu) = (a + b, -a + 2b) \in i(p, q) + j(-q, p - q)$. That is $(a, b) = i\left(\frac{2p-q}{3}, \frac{p+q}{3}\right) + j\left(\frac{-p-q}{3}, \frac{p-2q}{3}\right)$ for some integer i, j . Then modulo $\langle (p, q), (-q, p) \rangle$ there are only three different choices for pairs (a, b) corresponding to $i = j = 0, i = j = 1$ and $i = j = 2$. The first pair is always integer, the latter two are integer if and only if $p + q \equiv 0 \pmod{3}$. Also, $p + q \equiv 0 \pmod{3}$ if and only if $3 \mid p^2 - pq + q^2 = n$. That is, for a fixed H there is one choice of ν if $3 \nmid n$ and three choices if $3 \mid n$. By Lemma 2, the number of possible subgroups H is $\theta(n)$. So $R(n) = \theta(n)$ if $3 \nmid n$ and $R(n) = 3\theta(n)$ if $3 \mid n$. Finally note $\theta\left(\frac{n}{3}\right) = \theta(n)$ if $3 \mid n$ and $\theta\left(\frac{n}{3}\right) = 0$ otherwise. Then we have the required $R(n) = \theta(n) + 2\theta\left(\frac{n}{3}\right)$. \square

Remark 2. *Similar to Remark 1 get*

$$T(n) = \sum_{\ell(H)=H < \mathbb{Z}^2, |\mathbb{Z}^2/H|=n} |\{\nu | \nu \in \mathbb{Z}^2/H, \ell(\nu) = \nu\}| = \sum_{\ell(H)=H < \mathbb{Z}^2, |\mathbb{Z}^2/H|=n} |\mathbb{Z}^2 / \langle (1, -1), (1, 2), H \rangle|.$$

3 On the covering of \mathcal{G}_3

3.1 The structure of the group $\pi_1(\mathcal{G}_3)$

The following proposition provides the canonical form of an element in $\pi_1(\mathcal{G}_3) = \langle x, y, z : xyx^{-1}y^{-1} = 1, zxz^{-1} = y, zyz^{-1} = (xy)^{-1} \rangle$. The proof is similar to the proof of Proposition 2 in [2].

Proposition 2. (i) Each element of $\pi_1(\mathcal{G}_3)$ can be represented in the canonical form $x^a y^b z^c$ for some integer a, b, c .

(ii) The product of two canonical forms is given by the formula

$$x^a y^b z^c \cdot x^d y^e z^f = \begin{cases} x^{a+d} y^{b+e} z^{c+f} & \text{if } c \equiv 0 \pmod{3} \\ x^{a-e} y^{b+d-e} z^{c+f} & \text{if } c \equiv 1 \pmod{3} \\ x^{a-d+e} y^{b-d} z^{c+f} & \text{if } c \equiv 2 \pmod{3} \end{cases} \quad (3.3)$$

(iii) The canonical epimorphism $\phi_{\mathcal{G}_3} : \pi_1(\mathcal{G}_3) \rightarrow \pi_1(\mathcal{G}_3)/\langle x, y \rangle \cong \mathbb{Z}$, given by the formula $x^a y^b z^c \rightarrow c$ is well-defined.

(iv) The representation in the canonical form $g = x^a y^b z^c$ for each element $g \in \pi_1(\mathcal{G}_3)$ is unique.

Routinely follows from the definition of the group.

Notation. By Γ denote the subgroup of $\pi_1(\mathcal{G}_3)$ generated by x, y .

Our goal is to introduce some easy invariants, similar to those of Proposition 1.

Definition 2. Suppose all elements of $\pi_1(\mathcal{G}_3)$ are represented in the canonical form. Let Δ be a subgroup of finite index n in $\pi_1(\mathcal{G}_3)$. Put $H(\Delta) = \Delta \cap \Gamma$. By $a(\Delta)$ denote the minimal positive exponent of z among all the elements of Δ . Choose an element Z_Δ with such exponent of z , represented in the form $Z_\Delta = h z^{a(\Delta)}$, where $h \in \Gamma$. By $\nu(\Delta) = hH(\Delta)$ denote the coset in the coset decomposition $\Gamma/H(\Delta)$.

Note that the invariants $a(\Delta)$, $H(\Delta)$ and $\nu(\Delta)$ are well-defined. In particular the latter one does not depends on a choice of Z_Δ . Also $a(\Delta)[\Gamma : H(\Delta)] = [\pi_1(\mathcal{G}_2) : \Delta]$.

Definition 3. A 3-plet (a, H, ν) is called n -essential if the following conditions holds:

- (i) a is a positive divisor of n ,
- (ii) H is a subgroup of index n/a in Γ with $H \triangleleft \pi_1(\mathcal{G}_3)$ if $3 \nmid a$,
- (iii) ν is an element of Γ/H .

The next proposition show that the introduced invariants are sufficient to enumerate the subgroups of finite index.

Proposition 3. There is a bijection between the set of n -essential 3-plets (a, H, ν) and the set of subgroups Δ of index n in $\pi_1(\mathcal{G}_5)$, such that $(a, H, \nu) = (a(\Delta), H(\Delta), \nu(\Delta))$, given by the correspondence $\Delta \leftrightarrow (a(\Delta), H(\Delta), \nu(\Delta))$. Moreover, $\Delta \cong \mathbb{Z}^3$ if $3 \mid a(\Delta)$ and $\Delta \cong \pi_1(\mathcal{G}_3)$ otherwise.

The next few lemmas are auxiliary statements needed for the proof of Proposition 3.

Lemma 3. If $3 \nmid a(\Delta)$ then $H(\Delta) \triangleleft \pi_1(\mathcal{G}_3)$.

Proof. Recall $Z_\Delta = hz^{a(\Delta)} \in \Delta$. Then $H(\Delta)^{Z_\Delta} = H(\Delta)$. Since $H(\Delta)^x = H(\Delta)^y = H(\Delta)^{z^3} = H(\Delta)$, the former fact yields $H(\Delta)^g = H(\Delta)$, $g \in \pi_1(\mathcal{G}_3)$. \square

Lemma 4. *For arbitrary n -essential 3-plet (a, H, ν) there exists a subgroup Δ in the group $\pi_1(\mathcal{G}_3)$ such that $(a, H, \nu) = (a(\Delta), H(\Delta), \nu(\Delta))$.*

Proof. Take an n -essential 3-plet (a, H, ν) . In case $3 \nmid a$ consider the following construction

$$\Delta = \{hz^{(3l+1)a(\Delta)}H | l \in \mathbb{Z}\} \cup \{hz^{a(\Delta)}hz^{-a(\Delta)}z^{(3l+2)a(\Delta)}H | l \in \mathbb{Z}\} \cup \{z^{3la(\Delta)}H | l \in \mathbb{Z}\}.$$

One can check that Δ is a subgroup of index n in $\pi_1(\mathcal{G}_3)$ and $(a(\Delta), H(\Delta), \nu(\Delta)) = (a, H, \nu)$.

Similarly, in case $3 \mid a$ the set

$$\Delta = \{h^l z^{la(\Delta)}H | l \in \mathbb{Z}\}$$

is the required subgroup. \square

Proof of Proposition 3. Consider the family of subgroups Δ of index n in $\pi_1(\mathcal{G}_3)$, and the family of n -essential 3-plets. The definition of notions $a(\Delta)$, $H(\Delta)$ and $\nu(\Delta)$ together with Lemma 3 provide the correspondence from subgroups to n -essential 3-plets. Since the above invariants are well-defined, each subgroup Δ corresponds to only one 3-plet. By virtue of Lemma 4 each 3-plet corresponds to some subgroup, also different 3-plets correspond to different subgroups. The bijection part is proven.

If $3 \mid a(\Delta)$ Lemma 4 implies that Δ is a subgroup of $\langle x, y, z^3 \rangle$. Thus Δ is a subgroup of finite index in \mathbb{Z}^3 , hence Δ is isomorphic to \mathbb{Z}^3 itself.

Consider case $a(\Delta) \equiv 1 \pmod{3}$. Since $H(\Delta)$ is a subgroup of finite index in $\langle x, y \rangle \cong \mathbb{Z}^2$, we have $H(\Delta) \cong \mathbb{Z}^2$. By Lemmas 2 and 3 there is a pair of elements, that generates $H(\Delta)$, that have the form $(x^p y^q, x^{-q} y^{p-q})$. Let h be an arbitrary element in the coset $\nu(\Delta)$. Put $X = x^p y^q$, $Y = x^{-q} y^{p-q}$ and $Z = hz^{a(\Delta)}$. Direct verification shows that the relations $XYX^{-1}Y^{-1} = 1$, $ZXZ^{-1} = Y$, and $ZYZ^{-1} = (XY)^{-1}$ holds. Further we call this relations *the proper relations of the subgroup Δ* . Thus the map $x \rightarrow X$, $y \rightarrow Y$, $z \rightarrow Z$ can be extended to an epimorphism $\pi_1(\mathcal{G}_3) \rightarrow \Delta$. To prove that this epimorphism is really an isomorphism we need to show that each relation in Δ is a corollary of proper relations. We call a relation, that is not a corollary of proper relations an *improper relation*.

Assume the contrary, i.e. there are some improper relations in Δ . Since in Δ the proper relations holds, each element can be represented in the canonical form, given by Proposition 2 in terms of X, Y, Z , by using just the proper relations. That is each element g can be represented as

$$g = X^a Y^b Z^c.$$

If there is an improper relation then there is an equality

$$X^a Y^b Z^c = X^{a'} Y^{b'} Z^{c'}, \tag{3.4}$$

where at least one of the inequalities $a \neq a'$, $b \neq b'$, $c \neq c'$ holds. Applying $\phi_{\mathcal{G}_3}$ to both parts we get $ca(\Delta) = c'a(\Delta)$, thus $c = c'$. Then $U^aV^b = U^{a'}V^{b'}$, that means

$$\begin{cases} ap - bq = a'p - b'q \\ aq + b(p - q) = a'q + b'(p - q) \end{cases} \quad (3.5)$$

Since $\det \begin{pmatrix} p & -q \\ q & p - q \end{pmatrix} = p^2 - pq + q^2 = \frac{n}{a(\Delta)} \neq 0$, the system 3.5 implies $a = a'$ and $b = b'$, this contradiction proves that $\pi_1(\mathcal{G}_3) \cong \Delta$.

The proof in case $a(\Delta) \equiv 2 \pmod{3}$ is similar with the only difference we take $Y = x^p y^q$ and $X = x^{-q} y^{p-q}$. \square

3.2 The proof of Theorem 1

Proceed to the proof of Theorem 1. Proposition 3 claims that each subgroup Δ of finite index n is isomorphic to $\pi_1(\mathcal{G}_3)$ or \mathbb{Z}^3 , depending upon whether $a(\Delta)$ is a multiple of 3. Consider these two cases separately.

Case (i). Let Δ be a subgroup of $\pi_1(\mathcal{G}_3)$ isomorphic to $\pi_1(\mathcal{G}_3)$. To find the number of such subgroups, by Proposition 3 we need to calculate the cardinality of the set of n -essential 3-plets with $3 \nmid a$.

For each $3 \nmid a$ there are $\theta(\frac{n}{a})$ subgroups H in Γ such that $|\Gamma : H| = \frac{n}{a}$ and $H \triangleleft \pi_1(\mathcal{G}_3)$. Also there are $\frac{n}{a}$ different choices of a coset ν . Thus, for each $3 \nmid a$ the number of n -essential 3-plets is $\frac{n}{a}\theta(\frac{n}{a})$. So, the total number of subgroups is given by

$$s_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_3)}(n) = \sum_{a|n, 3 \nmid a} \frac{n}{a}\theta(\frac{n}{a}) = \sum_{a|n} \frac{n}{a}\theta(\frac{n}{a}) - \sum_{3a|n} \frac{n}{3a}\theta(\frac{n}{3a}) = \sum_{k|n} k\theta(k) - \sum_{k|\frac{n}{3}} k\theta(k).$$

Case (ii). Similarly to the previous case, we get the formula

$$s_{\mathbb{Z}^3, \pi_1(\mathcal{G}_3)}(n) = \sum_{3a|n} \frac{n}{3a}\sigma_1(\frac{n}{3a}) = \omega(\frac{n}{3}).$$

3.3 The proof of Theorem 2

3.3.1 Overall scheme of the proof

The proof of both cases follows the general scheme that we describe first. Recall that a subgroup G of finite index in $\pi_1(\mathcal{G}_3)$ has one of the following isomorphism types: \mathbb{Z}^3 or $\pi_1(\mathcal{G}_3)$. We use the standard notation $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$, $g_1, g_2 \in \pi_1(\mathcal{G}_3)$. Also, given subgroups $G_1, G_2 \leq \pi_1(\mathcal{G}_3)$ by $[G_1, G_2]$ we denote the subgroup, generated by the elements $[g_1, g_2]$, $g_1 \in G_1, g_2 \in G_2$. Fix an isomorphism type of a subgroup. Further Δ will always denote a subgroup of this isomorphism type of index n in $\pi_1(\mathcal{G}_3)$. In each case we point a normal subgroup of finite index $\Lambda \trianglelefteq \pi_1(\mathcal{G}_3)$ such that two conditions are met:

1° for any $\lambda \in \Lambda$ and any Δ holds $Ad_\lambda(H(\Delta)) = H(\Delta)$,

2° $[\Lambda, Z(\Delta)] = [H(\Lambda), Z(\Delta)]$, where $Z(\Delta)$ is given by Definition 2.

Call *an intermediate conjugacy class* Δ^Λ of Δ the set of subgroups Δ^λ , $\lambda \in \Lambda$. Denote the number of such classes by $c_{G, \pi_1(\mathcal{G}_3)}^\Lambda$, where G isomorphic to one of \mathbb{Z}^3 or $\pi_1(\mathcal{G}_3)$.

We propose an algorithm to uniformly calculate $c_{G, \pi_1(\mathcal{G}_3)}^\Lambda$. Given Δ , a subgroup $\Delta' \in \Delta^\Lambda$ have the following invariants: $a(\Delta') = a(\Delta)$, $H(\Delta') = H(\Delta)$ and $\nu(\Delta') \in \nu(\Delta)[\Lambda, Z(\Delta)]$. Keep in mind that $[\Lambda, Z(\Delta)] \leq \Gamma$, since Γ is normal in $\pi_1(\mathcal{G}_3)$ and $\pi_1(\mathcal{G}_3)/\Gamma$ is abelian. Thus for a fixed pair (a, H) there are $|\Gamma : \langle [\Lambda, Z(\Delta)], H \rangle|$ partial conjugacy classes Δ^Λ , each corresponding to the pair (a, H) . This let us to enumerate partial conjugacy classes.

The factor-group $\pi_1(\mathcal{G}_3)/\Lambda$ acts by conjugation on partial conjugacy classes. An orbit of partial conjugacy classes form a conjugacy class, thus we use the Burnside's lemma to calculate the number of conjugacy classes. To do this, we introduce one more definition.

Definition 4. Given $u \in \pi_1(\mathcal{G}_3)/\Lambda$, by $B(u)$ denote the number of partial conjugacy classes, preserved by the conjugation with u : $B(u) = |\{\Delta^\Lambda | (\Delta^\Lambda)^u = \Delta^\Lambda\}|$. In particular, $B(1)$ is the number of partial conjugacy classes.

Now we are done with the general scheme and proceed to its realization in specific cases.

3.3.2 Case (i)

Put $\Lambda = \pi_1(\mathcal{G}_3)$. Proposition 3 claims $H(\Delta) \triangleleft \pi_1(\mathcal{G}_3)$ in case $\Delta \cong \pi_1(\mathcal{G}_3)$, thus (1°) holds. Recall that $Z(\Delta) = hz^{a(\Delta)}$. Direct calculation through (3.3) shows that $[\Lambda, Z(\Delta)] = \langle xy^{-1}, xy^2 \rangle = [H(\Lambda), Z(\Delta)]$.

This means, firstly, that (2°) holds, and secondly that $|\Gamma : \langle [\Lambda, Z(\Delta)], H \rangle|$ equals 1 if $3 \nmid n$ and equals 3 if $3 \mid n$. For a fixed $a(\Delta)$ the number of subgroups $H(\Delta)$ is $\theta(\frac{n}{a(\Delta)})$.

Keep in mind that $\theta(\frac{k}{3}) = \theta(k)$ if $\frac{k}{3}$ is integer, and $\theta(\frac{k}{3}) = 0$ otherwise. So the function $k \mapsto \begin{cases} \theta(k) & \text{if } 3 \nmid k \\ 3\theta(k) & \text{if } 3 \mid k \end{cases}$ is given by $\theta(k) + 2\theta(\frac{k}{3})$. Applying this and summing achieved number of pairs (H, ν) over all values of a one gets:

$$c_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_3)}^\Lambda(n) = \sum_{a|n, 3 \nmid a} \theta(\frac{n}{a}) + 2\theta(\frac{n}{3a}) = \sum_{a|n} \theta(\frac{n}{a}) + \sum_{a|\frac{n}{3}} \theta(\frac{n}{3a}) - 2 \sum_{a|\frac{n}{9}} \theta(\frac{n}{9a}) =$$

$$\sum_{k|n} \theta(k) + \sum_{k|\frac{n}{3}} \theta(k) - 2 \sum_{k|\frac{n}{9}} \theta(k).$$

Since $\Lambda = \pi_1(\mathcal{G}_3)$, the Burnside's lemma is not needed to conclude $c_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_3)}^\Lambda(n) = c_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_3)}^\Lambda(n)$.

3.3.3 Case (ii)

Put $\Lambda = \langle x, y, z^3 \rangle$. Since $\Lambda \cong \mathbb{Z}^3$ and $\Delta \leq \Lambda$ for any Δ , conditions (1°) and (2°) hold. Also each partial conjugacy class consists of one subgroup, i.e. $c_{\mathbb{Z}^3, \pi_1(\mathcal{G}_3)}^\Lambda(n) = s_{\mathbb{Z}^3, \pi_1(\mathcal{G}_3)}(n) = \omega\left(\frac{n}{3}\right)$.

The factor $\pi_1(\mathcal{G}_3)/\Lambda$ consists of three elements, represented by 1, z , and z^2 respectively.

The condition $\Delta^z = \Delta$ is equivalent to $(H(\Delta))^z = H(\Delta)$ and $(\nu(\Delta))^z = \nu(\Delta)$ met simultaneously. Corollary 2 provides the number of such pairs (H, ν) for a given value of a . Summing over all the possible values (recall that $3 \mid a$) one gets $B(z) = B(z^2) = \sum_{k|\frac{n}{3}} \theta(k) + 2 \sum_{k|\frac{n}{9}} \theta(k)$.

By making use of Burnside's lemma obtain

$$c_{\mathbb{Z}^3, \pi_1(\mathcal{G}_3)}(n) = \frac{1}{3} \left(\omega\left(\frac{n}{3}\right) + 2 \sum_{k|\frac{n}{3}} \theta(k) + 4 \sum_{k|\frac{n}{9}} \theta(k) \right).$$

4 On the coverings of \mathcal{G}_5

4.1 The structure of the group $\pi_1(\mathcal{G}_5)$

Recall that the fundamental group of \mathcal{G}_5 is given by: $\pi_1(\mathcal{G}_5) = \langle \tilde{x}, \tilde{y}, \tilde{z} : \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = 1, \tilde{z}\tilde{x}\tilde{z}^{-1} = \tilde{x}\tilde{y}, \tilde{z}\tilde{y}\tilde{z}^{-1} = \tilde{x}^{-1} \rangle$. The following proposition provides the canonical form of an element in $\pi_1(\mathcal{G}_5)$.

Proposition 4. (i) *Each element of $\pi_1(\mathcal{G}_5)$ can be represented in the canonical form $\tilde{x}^a \tilde{y}^b \tilde{z}^c$ for some integer a, b, c .*

(ii) *The product of two canonical forms is given by the formula*

$$\tilde{x}^a \tilde{y}^b \tilde{z}^c \cdot \tilde{x}^d \tilde{y}^e \tilde{z}^f = \begin{cases} \tilde{x}^{a+d} \tilde{y}^{b+e} \tilde{z}^{c+f} & \text{if } c \equiv 0 \pmod{6} \\ \tilde{x}^{a+d-e} \tilde{y}^{b+d} \tilde{z}^{c+f} & \text{if } c \equiv 1 \pmod{6} \\ \tilde{x}^{a-e} \tilde{y}^{b+d-e} \tilde{z}^{c+f} & \text{if } c \equiv 2 \pmod{6} \\ \tilde{x}^{a-d} \tilde{y}^{b-e} \tilde{z}^{c+f} & \text{if } c \equiv 3 \pmod{6} \\ \tilde{x}^{a-d+e} \tilde{y}^{b-d} \tilde{z}^{c+f} & \text{if } c \equiv 4 \pmod{6} \\ \tilde{x}^{a+e} \tilde{y}^{b-d+e} \tilde{z}^{c+f} & \text{if } c \equiv 5 \pmod{6} \end{cases} \quad (4.6)$$

(iii) *The canonical epimorphism $\phi_{\mathcal{G}_5} : \pi_1(\mathcal{G}_5) \rightarrow \pi_1(\mathcal{G}_5)/\langle \tilde{x}, \tilde{y} \rangle \cong \mathbb{Z}$, given by the formula $\tilde{x}^a \tilde{y}^b \tilde{z}^c \rightarrow c$ is well-defined.*

(iv) *The representation in the canonical form $g = \tilde{x}^a \tilde{y}^b \tilde{z}^c$ for each element $g \in \pi_1(\mathcal{G}_5)$ is unique.*

Routinely follows from the definition of the group.

Notation. Let $\Gamma = \langle \tilde{x}, \tilde{y} \rangle$ be the subgroup of $\pi_1(\mathcal{G}_5)$ generated by \tilde{x}, \tilde{y} .

Our goal is to introduce some easy invariants, similar to those in Proposition 1. That will let us to enumerate the subgroups.

Definition 5. Suppose all elements of $\pi_1(\mathcal{G}_5)$ are represented in the canonical form. Let Δ be a subgroup of finite index n in $\pi_1(\mathcal{G}_5)$. Put $H(\Delta) = \Delta \cap \Gamma$. By $a(\Delta)$ denote the minimal positive exponent of \tilde{z} among all the elements $\tilde{x}^a \tilde{y}^b \tilde{z}^c \in \Delta$. Choose an element $Z(\Delta)$ with such exponent of \tilde{z} , represented in the form $Z(\Delta) = h \tilde{z}^{a(\Delta)}$, where $h \in \Gamma$. By $\nu(\Delta) = hH(\Delta)$ denote the coset in $\Gamma/H(\Delta)$ containing h .

Definition 6. A 3-plet (a, H, ν) is called n -essential if the following conditions holds:

- (i) a is a positive divisor of n ,
- (ii) H is a subgroup of index n/a in Γ with $H \triangleleft \pi_1(\mathcal{G}_5)$ if $3 \nmid a$,
- (iii) ν is an element of Γ/H .

Proposition 5. There is a bijection between the set of n -essential 3-plets (a, H, ν) and the set of subgroups Δ of index n in $\pi_1(\mathcal{G}_5)$, given by the correspondence $\Delta \leftrightarrow (a, H, \nu) = (a(\Delta), H(\Delta), \nu(\Delta))$. Moreover, $\Delta \cong \mathbb{Z}^3$ if $(a, 6) = 6$, $\Delta \cong \pi_1(\mathcal{G}_2)$ if $(a, 6) = 3$, $\Delta \cong \pi_1(\mathcal{G}_3)$ if $(a, 6) = 2$ and $\Delta \cong \pi_1(\mathcal{G}_5)$ if $(a, 6) = 1$.

Proof. The proof of Proposition 5 is similar to the proof of Proposition 3. \square

4.2 The proof of Theorem 3

In case (i) the argument similar to the proof of Theorem 1 leads to the formula:

$$s_{\pi_1(\mathcal{G}_5), \pi_1(\mathcal{G}_5)}(n) = \sum_{a|n, (a,6)=1} \frac{n}{a} \theta\left(\frac{n}{a}\right) = \sum_{k|n, (\frac{n}{k},6)=1} k \theta(k).$$

The last equality is obtained by applying the inclusion–exclusion principle. The cases (ii), (iii) and (iv) can be treated in the similar way.

4.3 The proof of Theorem 4

The proof uses the overall scheme from section 3.3.1, so we just proceed to its realization in specific cases.

4.3.1 Case (i)

Put $\Lambda = \pi_1(\mathcal{G}_5)$. Proposition 5 claims $H(\Delta) \triangleleft \pi_1(\mathcal{G}_5)$ in case $\Delta \cong \pi_1(\mathcal{G}_5)$, thus (1°) holds. Direct calculation using (4.6) in case $a(\Delta) \equiv 1 \pmod{6}$ gives $[\tilde{x}, Z(\Delta)] = \tilde{y}^{-1}$ and $[\tilde{y}, Z(\Delta)] = \tilde{x}\tilde{y}$. In case $c(\Delta) \equiv 5 \pmod{6}$ we respectively get $[\tilde{x}, Z(\Delta)] = \tilde{x}\tilde{y}$ and $[\tilde{y}, Z(\Delta)] = \tilde{x}^{-1}$. So in both cases $[\Gamma, Z(\Delta)] = \Gamma$. Firstly this means that (2°) holds. Secondly, conjugacy classes of subgroups Δ are in one-to-one correspondence with pairs (a, H) . Summing the number of choices of H over all the possible values of a get

$$c_{\pi_1(\mathcal{G}_5), \pi_1(\mathcal{G}_5)}(n) = \sum_{a|n, (6,a)=1} \theta\left(\frac{n}{a}\right) = \sum_{k|n} \theta(k) - \sum_{k|\frac{n}{2}} \theta(k) - \sum_{k|\frac{n}{3}} \theta(k) + \sum_{k|\frac{n}{6}} \theta(k).$$

4.3.2 Case (ii)

Put $\Lambda = \langle \tilde{x}, \tilde{y}, \tilde{z}^2 \rangle$. Proposition 5 claims $H(\Delta) \triangleleft \pi_1(\mathcal{G}_5)$ in case $\Delta \cong \pi_1(\mathcal{G}_5)$, thus (1°) holds. Recall that $Z(\Delta) = h\tilde{z}^{a(\Delta)}$. Direct calculation through (4.6) shows that $[\Lambda, Z(\Delta)] = \langle \tilde{x}\tilde{y}^{-1}, \tilde{x}\tilde{y}^2 \rangle = [H(\Lambda), Z(\Delta)]$.

This means, firstly, that (2°) holds, and secondly that $|\Gamma : \langle [\Lambda, Z(\Delta)], H \rangle|$ equals 1 if $3 \nmid n$ and equals 3 if $3 \mid n$.

The factor $\pi_1(\mathcal{G}_5)/\Lambda$ consists of two elements, represented by 1 and \tilde{x}^3 respectively. The conjugation with these elements preserves (a, H) , thus in case $3 \nmid n$ the partial conjugacy classes coincide with conjugacy classes, and there is only one conjugacy class for a fixed pair (a, H) . In case $3 \mid n$ for a fixed pair (a, H) there are 3 partial conjugacy classes: namely Δ_0^Λ , Δ_1^Λ and Δ_2^Λ , where $\Delta_0 \leftrightarrow (a, H, 1)$, $\Delta_1 \leftrightarrow (a, H, \tilde{y})$ and $\Delta_2 \leftrightarrow (a, H, \tilde{y}^2)$. Note that the conjugation with \tilde{x}^3 swaps the partial conjugacy classes Δ_1^Λ and Δ_2^Λ . Thus for a fixed pair (a, H) there are two conjugacy classes: $\Delta_0^{\pi_1(\mathcal{G}_5)} = \Delta_0^\Lambda$ and $\Delta_1^{\pi_1(\mathcal{G}_5)} = \Delta_1^\Lambda \cup \Delta_2^\Lambda$.

Keep in mind that $\theta(\frac{n}{3}) = \theta(n)$ if $\frac{n}{3}$ is integer, and $\theta(\frac{n}{3}) = 0$ otherwise. Applying this and summing achieved number of conjugacy classes over all values of a one gets:

$$\begin{aligned} c_{\pi_1(\mathcal{G}_3), \pi_1(\mathcal{G}_5)}(n) &= \sum_{a|\frac{n}{2}, a|\frac{n}{6}, 3|a} \theta\left(\frac{n}{2a}\right) + 2 \sum_{a|\frac{n}{6}, 3|a} \theta\left(\frac{n}{6a}\right) = \sum_{a|\frac{n}{2}, 3|a} \theta\left(\frac{n}{2a}\right) + \sum_{a|\frac{n}{6}, 3|a} \theta\left(\frac{n}{6a}\right) = \\ &= \sum_{a|\frac{n}{2}} \theta\left(\frac{n}{2}\right) - \sum_{a|\frac{n}{18}} \theta\left(\frac{n}{18}\right) = \sum_{k|\frac{n}{2}} \theta(k) - \sum_{k|\frac{n}{18}} \theta(k). \end{aligned}$$

4.3.3 Case (iii)

Put $\Lambda = \langle \tilde{x}, \tilde{y}, \tilde{z}^3 \rangle$. $Ad_{\tilde{x}}$ and $Ad_{\tilde{y}}$ are the identity transformation on Γ . $Ad_{\tilde{z}^3}$ is given by $g \rightarrow g^{-1}$, $g \in \Gamma$. That is $Ad_{\tilde{x}}$, $Ad_{\tilde{y}}$ and $Ad_{\tilde{z}^3}$ preserves $H(\Delta)$, i.e. (1°) holds.

Further $[H(\Lambda), Z(\Delta)] = \langle \tilde{x}^2, \tilde{y}^2 \rangle$. Recall the notation $Z(\Delta) = h\tilde{z}^{a(\Delta)}$. Then $[\tilde{z}^3, Z(\Delta)] = h^{-2} \in \langle \tilde{x}^2, \tilde{y}^2 \rangle$, so (2°) holds.

Applying Remark 1 and Corollary 1 and summing over all possible values of a find

$$c_{\pi_1(\mathcal{G}_2), \pi_1(\mathcal{G}_5)}^\Lambda(n) = \sum_{\substack{a|n, \\ 3|a, 6|a}} \left(\sigma_1\left(\frac{n}{a}\right) + 3\sigma_1\left(\frac{n}{2a}\right) \right) = \sigma_2\left(\frac{n}{3}\right) + 2\sigma_2\left(\frac{n}{6}\right) - 3\sigma_2\left(\frac{n}{12}\right).$$

The factor $\pi_1(\mathcal{G}_5)/\Lambda$ consists of three elements, represented by 1, \tilde{x}^2 and \tilde{x}^4 respectively. Obviously the numbers of partial conjugacy classes, preserved by $Ad_{\tilde{x}^2}$ and $Ad_{\tilde{x}^4}$ coincide: $B(\tilde{x}^2) = B(\tilde{x}^4)$. To find them note the following. A partial conjugacy class Δ^Λ is preserved by $Ad_{\tilde{x}^2}$ iff the following conditions are met simultaneously:

$$(*) \quad (H(\Delta))^{\tilde{z}^2} = H(\Delta)$$

$$(**) \quad (\nu(\Delta))^{\tilde{z}^2} \text{ must belong to the same conjugacy class in } \Gamma/\langle \tilde{x}^2, \tilde{y}^2, H \rangle \text{ as } \nu(\Delta).$$

By Lemma 2 the condition (*) implies that $H(\Delta)$ have a pair of generators $(\tilde{x}^p \tilde{y}^q, \tilde{x}^{-q} \tilde{y}^{p-q})$. The matrix $\begin{pmatrix} p & q \\ -q & p-q \end{pmatrix}$ modulo 2 have the rank 0 or 2, never 1. So $\Gamma / \langle \tilde{x}^2, \tilde{y}^2, \tilde{x}^p \tilde{y}^q, \tilde{x}^{-q} \tilde{y}^{p-q} \rangle$ is either trivial or isomorphic to \mathbb{Z}_2^2 . In the first case the condition (**) holds for the sole element, in the second case the condition (**) holds only for the coset of 0, and does not holds for three other cosets.

So the conjugacy classes Δ^Λ with $\Delta^{\tilde{z}^2} \in \Delta^\Lambda$ are in one-to-one correspondence with the normal subgroups $H(\Delta) \triangleleft \pi_1(\mathcal{G}_5)$. Applying Lemma 2 and summing over all the possible values of a yields $B(\tilde{x}^2) = B(\tilde{x}^4) = \sum_{k|\frac{n}{3}} \theta(k) - \sum_{k|\frac{n}{6}} \theta(k)$. Substituting to Burnside's lemma obtain

$$c_{\pi_1(\mathcal{G}_2), \pi_1(\mathcal{G}_5)}(n) = \frac{1}{3} \left(\sigma_2\left(\frac{n}{3}\right) + 2\sigma_2\left(\frac{n}{6}\right) - 3\sigma_2\left(\frac{n}{12}\right) + 2 \sum_{k|\frac{n}{3}} \theta(k) - 2 \sum_{k|\frac{n}{6}} \theta(k) \right).$$

4.3.4 Case (iv)

Put $\Lambda = \langle \tilde{x}, \tilde{y}, \tilde{z}^6 \rangle$. Since $\Lambda \cong \mathbb{Z}^3$ and $\Delta \leq \Lambda$ for any Δ , conditions (1°) and (2°) hold. Also each partial conjugacy class consists of one subgroup, i.e. $c_{\mathbb{Z}^3, \pi_1(\mathcal{G}_5)}(n)^\lambda = s_{\mathbb{Z}^3, \pi_1(\mathcal{G}_5)}(n) = \omega\left(\frac{n}{6}\right)$.

The factor $\pi_1(\mathcal{G}_5)/\Lambda$ consists of six elements, represented by 1, \tilde{x} , \tilde{x}^2 , \tilde{x}^3 , \tilde{x}^4 and \tilde{x}^5 respectively.

The condition $\Delta^{\tilde{z}^3} = \Delta$ is equivalent to $2\nu(\Delta) = 0$. Corollary 1 provides the number of such pairs (H, ν) for a given value of a , summing over all possible values (recall that $6 \mid a$) one gets $B(\tilde{z}^3) = \sigma_2\left(\frac{n}{6}\right) + 3\sigma_2\left(\frac{n}{12}\right)$.

The condition $\Delta^{\tilde{z}^2} = \Delta$ is equivalent to $(H(\Delta))^{\tilde{z}^2} = H(\Delta)$ and $(\nu(\Delta))^{\tilde{z}^2} = \nu(\Delta)$ met simultaneously. Corollary 1 provides the number of such pairs (H, ν) for a given value of a . Summing over all the possible values (recall that $6 \mid a$) one gets $B(\tilde{z}^2) = B(\tilde{z}^4) = \sum_{k|\frac{n}{6}} \theta(k) + 2 \sum_{k|\frac{n}{18}} \theta(k)$.

Finally, $\Delta^{\tilde{z}} = \Delta$ implies $(H(\Delta))^{\tilde{z}} = H(\Delta)$ and $(\nu(\Delta))^{\tilde{z}^3} = (\nu(\Delta))^{\tilde{z}^2} = \nu(\Delta)$. The latter two equalities imply $\nu(\Delta) = 0$, so the unique subgroup Δ correspond to a $H(\Delta)$. Summing over all the possible values of a yields $B(\tilde{z}) = B(\tilde{z}^5) = \sum_{k|\frac{n}{6}} \theta(k)$.

Substituting to Burnside's lemma obtain

$$c_{\mathbb{Z}^3, \pi_1(\mathcal{G}_5)}(n) = \frac{1}{6} \left(\omega\left(\frac{n}{6}\right) + \sigma_2\left(\frac{n}{6}\right) + 3\sigma_2\left(\frac{n}{12}\right) + 4 \sum_{k|\frac{n}{6}} \theta(k) + 4 \sum_{k|\frac{n}{18}} \theta(k) \right).$$

5 Appendix

Given a sequence $\{f(n)\}_{n=1}^\infty$, a formal power series

$$\widehat{f}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is called Dirichlet generating function for $\{f(n)\}_{n=1}^{\infty}$, see for example, [23]. For the way to reconstruct the sequence $f(n)$ by $\widehat{f}(s)$ see Perron's formula (for example [24]).

Here we present the Dirichlet generating functions for the calculated sequences $s_{H,G}(n)$ and $c_{H,G}(n)$. Since theorems 1–4 provides the explicit formulas, the remaining can be done by direct calculations which we omit here.

Notations. By $\zeta(s)$ we denote the Riemann zeta function. Define sequence $\{\chi(n)\}_{n=1}^{\infty}$ by $\chi(n) = \frac{1}{\sqrt{3}} \left(\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^n - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^n \right)$ or equivalently $\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{3} \\ -1 & \text{if } n \equiv 2 \pmod{3} \end{cases}$. For the sake of brevity denote $\vartheta(s) = \widehat{\chi}(s)$. Note that $\vartheta(s)$ is the Dirichlet L-series for the multiplicative character $\chi(n)$.

H \ G		\mathcal{G}_3	\mathcal{G}_5
		\mathbb{Z}^3	$\widehat{s}_{H,G}$
	$\widehat{c}_{H,G}$	$3^{-s-1}\zeta(s)\left(\zeta(s-1)\zeta(s-2) + 2(1+2\cdot 3^{-s})\zeta(s)\vartheta(s)\right)$	$6^{-s-1}\zeta(s)\left(\zeta(s-1)\zeta(s-2) + (1+3\cdot 2^{-s})\zeta(s)\zeta(s-1) + 4(1+3^{-s})\zeta(s)\vartheta(s)\right)$
\mathcal{G}_2	$\widehat{s}_{H,G}$		$3^{-s}(1-2^{-s})\zeta(s)\zeta(s-1)\zeta(s-2)$
	$\widehat{c}_{H,G}$		$3^{-s}(1-2^{-s})\zeta(s)^2\left((1+3\cdot 2^{-s})\zeta(s-1) + 2\vartheta(s)\right)$
\mathcal{G}_3	$\widehat{s}_{H,G}$	$(1-3^{-s})\zeta(s-1)^2\vartheta(s-1)$	$2^{-s}(1-3^{-s})\zeta(s-1)^2\vartheta(s-1)$
	$\widehat{c}_{H,G}$	$(1-3^{-s})(1+2\cdot 3^{-s})\zeta(s)^2\vartheta(s)$	$2^{-s}(1-3^{-s})(1+3^{-s})\zeta(s)^2\vartheta(s)$
\mathcal{G}_5	$\widehat{s}_{H,G}$		$(1-2^{-s})(1-3^{-s})\zeta(s-1)^2\vartheta(s-1)$
	$\widehat{c}_{H,G}$		$(1-2^{-s})(1-3^{-s})\zeta(s)^2\vartheta(s)$

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