HANKEL CONTINUED FRACTIONS AND HANKEL DETERMINANTS OF THE EULER NUMBERS

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Abstract. The Euler numbers occur in the Taylor expansion of $\tan(x) + \sec(x)$. Since Stieltjes, continued fractions and Hankel determinants of the even Euler numbers, on the one hand, of the odd Euler numbers, on the other hand, have been widely studied separately. However, no continued fractions and Hankel determinants of the (mixed) Euler numbers have been obtained and explicitly calculated. The reason for that is that some Hankel determinants of the Euler numbers are null. This implies that the Jacobi continued fraction of the Euler numbers does not exist. In the present paper, this obstacle is bypassed by using the Hankel continued fraction, instead of the J-fraction. Consequently, an explicit formula for the Hankel determinants of the Euler numbers is being derived, as well as a full list of Hankel continued fractions and Hankel determinants involving Euler numbers. Finally, a new q-analog of the Euler numbers $E_n(q)$ based on our continued fraction is proposed. We obtain an explicit formula for $E_n(-1)$ and prove a conjecture by R. J. Mathar on these numbers.

1. Introduction

The *Euler numbers* $E_n (n \geq 0)$ are defined by their generating function

$$\tan(x) + \sec(x) = \sum E_n \frac{x^n}{n!}. \tag{1.1}$$

The even (resp. odd) Euler numbers E_{2n} (resp. E_{2n+1}) are also called *secant* (resp. *tangent*) numbers, and their first values read:

n	$=$	0	1	2	3	4	5	6	7	8	9
E_n	$=$	1	1	1	2	5	16	61	272	1385	7936

As already proved back in 1879 by André [2], the Euler numbers count the alternating permutations and satisfy the following recurrence relation

$$E_0 = 1;\ E_1 = 1; \quad E_n = \frac{1}{2}\sum_{k=1}^{n} \binom{n-1}{k-1} E_{k-1}E_{n-k}. \tag{1.2}$$

The Euler numbers have been widely studied in Combinatorics (see [2, 8, 28, 39, 55]) and are closely connected with the Bernoulli and Genocchi numbers [55, 1].

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Stieltjes derived the continued fractions for the ordinary generating functions of the tangent and secant numbers (see [54], [9], [56, p. 206, (53.11)], [56, p. 369])

$$\sum_{n\ge 0} E_{2n}x^{2n} = \frac{1}{1} - \frac{1^2x^2}{1} - \frac{2^2x^2}{1} - \frac{3^2x^2}{1} - \cdots \tag{1.3}$$

$$\sum_{n\ge 0} E_{2n+1}x^{2n+1} = \frac{x}{1} - \frac{1\cdot 2x^2}{1} - \frac{2\cdot 3x^2}{1} - \frac{3\cdot 4x^2}{1} - \cdots \tag{1.4}$$

Thanks to the seminal paper by Flajolet [9] on combinatorial aspects of continued fractions, the above two continued fractions have become very classical, and been generalized in several directions [45, 17, 27, 50, 19, 26], including their q-analogs.

Although a lot of continued fractions involving either the tangent numbers (E_{2n+1}), or secant numbers (E_{2n}) have been studied *separately*, the continued fraction of the ordinary generating function $\sum_{n\ge 0} E_n x^n$ for the *mixed* Euler numbers has never been derived. The reason for that is that *some* Hankel determinants of the Euler number sequence are null. This implies that the corresponding Jacobi continued fraction does not exist. In the present paper, this obstacle is bypassed by using the Hankel continued fraction [21], instead of the J-fraction. Consequently, we derive an explicit formula for the Hankel determinants of the Euler numbers. See Section 2 for the basic definition and properties of Hankel continued fractions. We establish the following two theorems about the Hankel continued fraction and the Hankel determinants of the Euler numbers. As expected, we see that some of the Hankel determinants are zero.

Theorem 1.1. *We have the following Hankel continued fraction expansion of the (mixed) Euler numbers:*

$$\begin{aligned}\sum_{n\ge 0} E_n x^n &= \frac{1}{1-x} - \frac{x^3}{1-2x-4x^2} - \frac{9x^3}{1-5x} - \frac{4x^2}{1-7x} \\ &\quad - \frac{75x^3}{1-6x-36x^2} - \frac{147x^3}{1-11x} - \frac{16x^2}{1-13x} - \cdots \\ &= b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots\end{aligned} \tag{1.5}$$

the general patterns for the coefficients a_j and b_j being:

$$\begin{aligned} a_1 &= 1,\\ a_{3k} &= -(4k-1)^2(2k-1)x^3,\\ a_{3k+1} &= -4k^2x^2,\\ a_{3k+2} &= -(4k+1)^2(2k+1)x^3;\\ b_0 &= 0,\\ b_{3k} &= -(6k-1)x+1,\\ b_{3k+1} &= -(6k+1)x+1,\\ b_{3k+2} &= -4(2k+1)^2x^2-2(2k+1)x+1.\end{aligned}$$

Throughout this paper we use the following convention, called "*index convention*", saying that each expression given by cases is only valid for integers which have not been considered as previous special values. In the above example, the expression b_{3k+1} is valid for $k \geq 0$; However, the expression a_{3k+1} is valid for $k \geq 1$, but not for $k = 0$, since a_1 is already listed above.

The *Hankel determinant* of order n of the formal power series $f(x) = c_0 + c_1 x + c_2 x^2 + \cdots$ (or of the sequence $(c_0, c_1, c_2, \cdots)$ is defined by

$$H_n(c_0, c_1, c_2, \ldots) := \det(c_{i+j})_{0 \leq i,j \leq n-1}$$

for $n \geq 1$, and $H_0(c_0, c_1, c_2, \ldots) = 1$ if $n = 0$.

Theorem 1.2. *The Hankel determinants H_n of the (mixed) Euler numbers*

$$(E_0, E_1, E_2, E_3, \ldots)$$

are given by the following formulas:

$$\begin{aligned} H_0 &= 1, \\ H_{4k+1} &= (-1)^k \frac{(2k)!^2}{2^{4k(2k-1)}} \prod_{j=1}^{2k-1} (2j+1)!^4, \\ H_{4k+2} &= 0, \\ H_{4k+3} &= (-1)^{k+1} \frac{(2k+1)!^2}{2^{4k(2k+1)}} \prod_{j=1}^{2k} (2j+1)!^4, \\ H_{4k+4} &= (-1)^{k+1} \frac{(2k+1)!^2 (4k+3)!^2}{2^{2(2k+1)^2}} \prod_{j=1}^{2k} (2j+1)!^4. \end{aligned}$$

The proofs of Theorems 1.1 and 1.2 are given in Section 4 by making use of the Flajolet continued fraction for permutation statistics [9] and some combinatorial models for the Euler numbers described in Section 3. We provide a large list of Hankel continued fractions and Hankel determinants involving Euler numbers. The formulas involving the ordinary generating functions of Euler numbers, most of them stated and proved in Section 5, are resumed in Table 1. In Section 6 we first obtain a continued fraction of the exponential generating function of a quadruple permutation statistic, see Theorem 6.1. Then, we derive the Hankel continued fractions and Hankel determinants for the exponential generating functions involving the Euler numbers. We resume these formulas in Table 2. There, we write $e_n = E_n/n!$ for short. In the last section, a new q-analog of the Euler numbers $E_n(q)$ based on the continued fraction is proposed. We obtain an explicit formula for $E_n(-1)$ and prove a conjecture by R. J. Mathar on these numbers.

Some further remarks are in order for a better understanding of our motivation, as well as various methods and notation used in this paper.

Remark 1. Some continued fractions obtained in the paper are of *Jacobi* type, as well as others need to be expressed as Hankel continued fractions. These two situations are indicated by the letters "J" or "H" in the second column in Tables 1 and 2.

Remark 2. Some of these formulas are known or easy to prove. We list them here for a quick view and comparison. In fact, we can find (H1) in [1], (H2) and (H3) for the case $r = 1$ in [1, 46], [29, (3.52-53)] and [37, (4.58-59)]; (H22) and (H23) in [38], (F9) in [37, (3.120)].

TABLE 1. Formulas for the ordinary generating functions

Sequence	H-Fraction	Hankel det.	Exp. g. f.
$(E_0, E_1, E_2, E_3, E_4, \ldots)$	Thm1.1, H	Thm1.2	$\tan(x) + \sec(x)$
$(E_1, E_2, E_3, E_4, \ldots)$	(F7), J	(H7)	$(\tan(x) + \sec(x))'$
$(E_2, E_3, E_4, \ldots)$	(F10), H	(H10)	$(\tan(x) + \sec(x))''$
$(E_0, 0, E_2, 0, E_4, 0, \ldots)$	(F1), J	(H1)	$\sec(x)$
$(0, E_2, 0, E_4, 0, \ldots)$	(F8), H	(H8)	$\sec(x)'$
$(E_0, E_2, E_4, E_6, \ldots)$	(F2), J	(H2)	–
$(E_2, E_4, E_6, \ldots)$	(F3), J	(H3)	–
$(0, E_1, 0, E_3, 0, E_5, \ldots)$	(F9), H	(H9)	$\tan(x)$
$(E_1, 0, E_3, 0, E_5, \ldots)$	(F4), J	(H4)	$\tan(x)'$
$(0, E_3, 0, E_5, \ldots)$	(F11), H	(H11)	$\tan(x)''$
$(E_1, E_3, E_5, E_7, \ldots)$	(F5), J	(H5)	–
$(E_3, E_5, E_7, \ldots)$	(F6), J	(H6)	–

TABLE 2. Formulas for the exponential generating functions

Sequence	H-Fraction	Hankel det.	Function
$(e_0, e_1, e_2, e_3, e_4, e_5, \ldots)$	(F17), J	(H17)	$\tan(x) + \sec(x)$
$(e_1, e_2, e_3, e_4, e_5, \ldots)$	(F16), J	(H16)	$(\tan(x) + \sec(x) - 1)/x$
$(e_2, e_3, e_4, e_5, \ldots)$	(F18), J	(H18)	$\cdots$
$(e_3, e_4, e_5, \ldots)$	(F19), J	(H19)	$\cdots$
$(e_4, e_5, \ldots)$	(F20), J	(H20)	$\cdots$
$(0, e_1, 0, e_3, 0, e_5, \ldots)$	(F15), H	(H15)	$\tan(x)$
$(e_1, 0, e_3, 0, e_5, \ldots)$	(F12), J	(H12)	$\tan(x)/x$
$(0, e_3, 0, e_5, \ldots)$	(F14), H	(H14)	$(\tan(x) - x)/x^2$
$(e_3, 0, e_5, \ldots)$	(F21), J	(H21)	$\cdots$
$(e_1, e_3, e_5, e_7, e_9, \ldots)$	(F13), J	(H13)	$\tan(\sqrt{x})/\sqrt{x}$
$(e_3, e_5, e_7, e_9, \ldots)$	(F22), J	(H22)	$(\tan(\sqrt{x})/\sqrt{x} - 1)/x$
$(e_5, e_7, e_9, \ldots)$	(F23), J	(H23)	$\cdots$
$(e_7, e_9, \ldots)$	(F24), J	(H24)	$\cdots$

Remark 3. The structure of the continued fractions makes no simple formula for the addition of two simple continued fractions. In fact, our main result says that the sum of the two continued fractions (1.3) and (1.4) is equal to (1.5). This is not easy to prove since no addition formula is available. In the same way, we know the continued fractions for $\tan(x) + \sec(x)$ and also $\tan(x)$, but we do not know the continued fraction for their difference $\sec(x)$.

Remark 4. There are simple continued fractions for $(E_0, 0, E_2, 0, E_4, \ldots)$ and $(E_0, E_2, E_4, \ldots)$. However, no simple continued fractions for $(e_0, 0, e_2, 0, e_4, \ldots)$ and $(e_0, e_2, e_4, \ldots)$ are known. We are convinced that there is no unified method to derive continued fractions for ordinary generating functions and exponential generating functions.

Remark 5. For the exponential generating functions, the three families in Table 2 seem to be naturally extended. However, for the ordinary generating functions, the five families cannot grow, and all continued fractions of simple form are listed in Table 1.

Remark 6. In view of Remark 5, we may think that the exponential generating functions are more adequate for the Hankel continued fractions. However, Remark 4 says the converse.

Remark 7. Theorem 1.1 and (F10) are very similar. However there is an important difference. The super 1-fraction of $\sum E_n x^n$ exists. This fact leads us to find the proof of Theorem 1.1. But there is no simple super 1-fraction of $\sum E_{n+2} x^n$. Fortunately, now we have the proof of Theorem 1.1 which gives us some indication for the proof of (F10).

2. Definitions and properties of the Continued fractions

In this section we recall some basic definitions and properties of the general continued fractions and the super continued fractions, including the Hankel continued fractions.

Let $\mathbb{K}$ be a field. In most cases, $\mathbb{K}$ will be the field $\mathbb{Q}$ of rational numbers. Consider the field of fractional fractions $\mathbb{K}(x)$. Let $\mathbf{a} = (a_1, a_2, \ldots)$ and $\mathbf{b} = (b_0, b_1, b_2, \ldots)$ be two sequences of $\mathbb{K}(x)$. The *generalized continued fraction* associated with the two sequences $\mathbf{a}$ and $\mathbf{b}$ can be written by using the natural notation

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}}$$

or Pringsheim's notation:

$$b_0 + \frac{a_1 |}{| b_1} + \frac{a_2 |}{| b_2} + \frac{a_3 |}{| b_3} + \cdots \tag{2.1}$$

For the notion about *value, partial numerator and denominator, fundamental recurrence formulas, equivalence transformations* of continued fractions, see [56, p. 13-19]. The *value* of the above continued fraction is a formal power series in x with coefficients in $\mathbb{K}$. Throughout the paper we will extensively use the following contraction formulas. The contractions with for the case $b_j = 1$ are well-known [56, 42, 54].

Theorem 2.1. *The generalized continued fraction defined in* (2.1) *has the following contraction formulas.*

(1) Even contraction:

$$\begin{aligned} (2.2)\quad & b_0 + \frac{a_1 b_2 |}{| b_1 b_2 + a_2} - \frac{a_2 a_3 b_4 |}{| b_2 b_3 b_4 + a_4 b_2 + a_3 b_4} - \frac{a_4 a_5 b_2 b_6 |}{| b_4 b_5 b_6 + a_6 b_4 + a_5 b_6} - \cdots \\ & = b'_0 + \frac{a'_1 |}{| b'_1} + \frac{a'_2 |}{| b'_2} + \frac{a'_3 |}{| b'_3} + \cdots \end{aligned}$$

The general patterns for the new coefficients a'_j and b'_j are:

$$\begin{aligned} a'_1 &= a_1 b_2, \\ a'_2 &= -a_2 a_3 b_4, \\ a'_j &= -a_{2j-2} a_{2j-1} b_{2j-4} b_{2j}; \end{aligned}$$

$$b'_0 = b_0,$$
$$b'_1 = b_1 b_2 + a_2,$$
$$b'_j = b_{2j-2} b_{2j-1} b_{2j} + a_{2j} b_{2j-2} + a_{2j-1} b_{2j}.$$

When $b_0 = 0$ and $b_1 = b_2 = \cdots = 1$, the even contraction formula becomes

$$\frac{a_1}{\lceil 1 + a_2} - \frac{a_2 a_3}{\lceil 1 + a_3 + a_4} - \frac{a_4 a_5}{\lceil 1 + a_5 + a_6} - \cdots \tag{2.3}$$

(2) Odd contraction:

$$\frac{b_0 b_1 + a_1}{b_1} - \frac{a_1 a_2 b_3 / b_1}{\lceil b_1 b_2 b_3 + a_3 b_1 + a_2 b_3} - \frac{a_3 a_4 b_1 b_5}{\lceil b_3 b_4 b_5 + a_5 b_3 + a_4 b_5} - \cdots \tag{2.4}$$

The general patterns for the new coefficients a'_j and b'_j are:

$$a'_1 = -a_1 a_2 \frac{b_3}{b_1},$$
$$a'_2 = -a_3 a_4 b_1 b_5,$$
$$a'_j = -a_{2j-1} a_{2j} b_{2j-3} b_{2j+1};$$
$$b'_0 = \frac{b_0 b_1 + a_1}{b_1},$$
$$b'_1 = b_1 b_2 b_3 + a_3 b_1 + a_2 b_3,$$
$$b'_j = b_{2j-1} b_{2j} b_{2j+1} + a_{2j+1} b_{2j-1} + a_{2j} b_{2j+1}.$$

When $b_0 = 0$ and $b_1 = b_2 = \cdots = 1$, the odd contraction formula becomes

$$a_1 - \frac{a_1 a_2}{\lceil 1 + a_2 + a_3} - \frac{a_3 a_4}{\lceil 1 + a_4 + a_5} - \cdots \tag{2.5}$$

(3) Chop contraction:

$$\frac{b_0 b_1 + a_1}{b_1} - \frac{a_1 a_2 / b_1}{\lceil b_1 b_2 + a_2} + \frac{a_3 b_1}{\lceil b_3} + \frac{a_4}{\lceil b_4} + \frac{a_5}{\lceil b_5} \cdots \tag{2.6}$$

(4) Haircut contraction: for each $\alpha \neq a_1/b_1$,

$$\alpha + \frac{a_1 - \alpha b_1}{\lceil b_1} + \frac{a_1 a_2}{\lceil b_2 a_1 - b_1 b_2 \alpha - a_2 \alpha} + \frac{a_3(a_1 - b_1 \alpha)}{\lceil b_3} + \frac{a_4}{\lceil b_4} \cdots \tag{2.7}$$

The even and odd contraction formulas are very classical (see [56, p. 21], [42, p. 12-13], [54, p. J3]). The chop and haircut formulas can be verified directly. For the most general contraction and extension, see [42, p. 10-16]. [1]

Let $\mathbf{u} = (u_1, u_2, \ldots)$ and $\mathbf{v} = (v_0, v_1, v_2, \ldots)$ be two sequences. Recall that the *Jacobi continued fraction* attached to $(\mathbf{u}, \mathbf{v})$, or *J-fraction*, for short, is a continued fraction of the form

$$f(x) = \frac{v_0}{\lceil 1 + u_1 x} - \frac{v_1 x^2}{\lceil 1 + u_2 x} - \frac{v_2 x^2}{\lceil 1 + u_3 x} - \cdots \tag{2.8}$$

[1] There were some errors in the first edition of the book by Perron [41, p. 199], when the author derives the formula for the general contraction. These errors had been fixed in the second edition [42, p. 11]

The basic properties on J-fractions can be found in [29, 30, 9, 56, 54, 22, 18]. We emphasize the fact that the Hankel determinants can be calculated from the J-fraction by means of the following fundamental relation, first stated by Heilermann in 1846 [22]:

$$H_n(f) = v_0^n v_1^{n-1} v_2^{n-2} \cdots v_{n-2}^2 v_{n-1}. \tag{2.9}$$

The Hankel determinants of a power series f can be calculated by the above fundamental relation if the J-fraction exists, which is equivalent to the fact that all Hankel determinants of f are nonzero. If some of the Hankel determinants are zero, we must use the *Hankel continued fraction* (H*-fraction*, for short) whose existence and uniqueness are guaranteed without any condition for the power series. The Hankel determinants can also be evaluated by using the Hankel continued fraction. Let us recall the basic definition and properties of the Hankel continued fractions [21].

Definition 2.1. For each positive integer δ, a *super continued fraction* associated with δ, called *super δ-fraction* for short, is defined to be a continued fraction of the following form

$$F(x) = \cfrac{v_0 x^{k_0}}{1+u_1(x)x} - \cfrac{v_1 x^{k_0+k_1+\delta}}{1+u_2(x)x} - \cfrac{v_2 x^{k_1+k_2+\delta}}{1+u_3(x)x} - \cdots \tag{2.10}$$

where $v_j \neq 0$ are constants, k_j are nonnegative integers and $u_j(x)$ are polynomials of degree less than or equal to $k_{j-1} + \delta - 2$. By convention, 0 is of degree -1.

When $\delta = 1$ (resp. $\delta = 2$) and all $k_j = 0$, the super δ-fraction (2.10) is the traditional S-fraction (resp. J-fraction). A super 2-fraction is called *Hankel continued fraction.*

Theorem 2.2. *(i) Let δ be a positive integer. Each super δ-fraction defines a power series, and conversely, for each power series $F(x)$, the super δ-fraction expansion of $F(x)$ exists and is unique. (ii) Let $F(x)$ be a power series such that its H-fraction is given by* (2.10) *with $\delta = 2$. Then, all non-vanishing Hankel determinants of $F(x)$ are given by*

$$H_{s_j}(F(x)) = (-1)^{\epsilon_j} v_0^{s_j} v_1^{s_j - s_1} v_2^{s_j - s_2} \cdots v_{j-1}^{s_j - s_{j-1}}, \tag{2.11}$$

where $\epsilon_j = \sum_{i=0}^{j-1} k_i(k_i+1)/2$ and $s_j = k_0 + k_1 + \cdots + k_{j-1} + j$ for every $j \geq 0$.

See [21, 47, 5, 23] for the proof of Theorem 2.2.

A short historical remark. The main idea of the determinant formula (2.11) may already be known by Magnus in 1970 [35]. The present form appeared for the first time in the Ph.D. thesis by Emmanuel Roblet in 1994 [47, p. 44-56] [2]. It was independently rediscovered by Buslaev in 2010 [5], by Boltz-Tyaglov in 2012 [23], and by the author in 2016 [21]. The name of the continued fraction called by Roblet and Buslaev is P-fraction. The P-fraction (*"principal part plus" fraction*) was introduced by Magnus in 1962 [33, 34, 35, 25], also independently by Mills-Robbins in 1986 [36] and by Boltz-Tyaglov in 2012 [23]. Notice that there are slight differences among the P-fraction used by Roblet [47], the P-fraction used by Buslaev [5], and the Hankel continued fraction used in [21]. Roblet's P-fraction

[2] The author is grateful to Xavier Viennot, who gave me the reference to the Roblet's thesis.

is faithful to the Magnus's up to some equivalent transformations, and has the following form [47, (2.9)]:

$$F(x) = f_0 + f_1x + f_2x^2 + \cdots = f_0 + \frac{\lambda_1 x^{N_1}}{|P_1(x)} - \frac{\lambda_2 x^{N_1+N_2}}{|P_2(x)} - \frac{\lambda_3 x^{N_2+N_3}}{|P_3(x)} - \cdots$$

Because of the presence of the constant term f_0 in the above fraction, it is necessary to consider the Hankel determinants of $(f_1, f_2, f_3, \ldots)$ under the Magnus P-fraction notation rather than of $(f_0, f_1, f_2, \ldots)$ [47, (2.14), (2.18)]. On the other hand, Buslaev used the P-fraction notion without quoting Magnus's papers. Moreover, the condition $N_j \geq 1$, needed in the original P-fraction notation, had been removed [5].

A special case of (2.11) for a restricted family of C-fractions was obtained by Scott-Wall in 1940 [48] and independently by Cigler in 2013 [7]. It is interesting to notice that almost all these studies were of theoretical nature with no explicit examples, except some artificial ones in Cigler's paper. Finally, note that several real-life expansions and applications are given in [21], together with an analog of Euler-Lagrange theorem about periodic continued fraction for power series over a finite field.

3. Combinatorial interpretations of the Euler numbers

It is well-known that the Euler numbers count the alternating permutations [2]. The proof of our main Theorem 1.1 leads us to derive further combinatorial interpretations of the Euler numbers.

Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ with the convention $\sigma_0 = \sigma_{n+1} = +\infty$. For each $j \in \{1, 2, \ldots, n\}$, the letter σ_j is called *descent* if $\sigma_j > \sigma_{j+1}$; *ascent* if $\sigma_j < \sigma_{j+1}$; *peak* if $\sigma_{j-1} < \sigma_j > \sigma_{j+1}$; *valley* if $\sigma_{j-1} > \sigma_j < \sigma_{j+1}$; *double ascent* if $\sigma_{j-1} < \sigma_j < \sigma_{j+1}$; *double descent* if $\sigma_{j-1} > \sigma_j > \sigma_{j+1}$ (See [9, 40, 16, 57, 58] or [52, Exercise 1.61]). Let $\mathrm{des}(\sigma), \mathrm{val}(\sigma), \mathrm{pk}(\sigma), \mathrm{da}(\sigma), \mathrm{dd}(\sigma)$ be the numbers of descents, valleys, peaks, double ascents and double descents of σ. In 1974, Carlitz and Scoville obtained the exponential generating function of the quadruple statistic $(\mathrm{val}, \mathrm{pk}, \mathrm{da}, \mathrm{dd})$ for the permutations [6, 16, 40].

$$\sum_{n\geq 1} \frac{x^n}{n!} \sum_{\sigma\in\mathfrak{S}_n} u_1^{\mathrm{val}(\sigma)} u_2^{\mathrm{pk}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} = u_1 \frac{e^{\alpha_2 x} - e^{\alpha_1 x}}{\alpha_2 e^{\alpha_1 x} - \alpha_1 e^{\alpha_2 x}}, \tag{3.1}$$

where $u_3 + u_4 = \alpha_1 + \alpha_2$ and $u_1u_2 = \alpha_1\alpha_2$.

Definition 3.1. We define four weight functions $W_1(\sigma), W_2(\sigma), W_3(\sigma), W_4(\sigma)$ for the permutations $\sigma = \sigma_1\sigma_2\ldots\sigma_n$.

$$W_1(\sigma) = \prod_{j=2}^{n} (-1)^{\chi(j \text{ is a double ascent})}, \tag{3.2}$$

$$W_2(\sigma) = \prod_{j=2}^{n} 0^{\chi(j \text{ is a double ascent})} \left(\frac{1}{2}\right)^{\chi(j \text{ is a peak})}, \tag{3.3}$$

$$W_3(\sigma) = \prod_{j=2}^{n} \left(\frac{1}{2}\right)^{\chi(j \text{ is not a peak})}, \tag{3.4}$$

$$W_4(\sigma) = \prod_{j=1}^{n-1} \left(\frac{1+i}{2}\right)^{\chi(j \text{ is a ascent})} \left(\frac{1-i}{2}\right)^{\chi(j \text{ is a descent})}, \tag{3.5}$$

where i is the imaginary unit.

Theorem 3.1. *For each positive integer n we have*

$$\sum_{\sigma\in\mathfrak{S}_n} W_2(\sigma) = \sum_{\sigma\in\mathfrak{S}_n} W_3(\sigma) = \sum_{\sigma\in\mathfrak{S}_n} W_4(\sigma) = E_n, \tag{3.6}$$

and

$$\sum_{\sigma\in\mathfrak{S}_n} W_1(\sigma) = \begin{cases} E_n; & \textit{if } n \textit{ is odd} \\ 0. & \textit{if } n \textit{ is even} \end{cases} \tag{3.7}$$

For example, with $n = 3$ and $E_n = 2$, the 6 permutations with their four weights are listed below. We check that Theorem 3.1 is true for $n = 3$.

σ	W_1	W_2	W_3	W_4
123	$(-1)\cdot(-1)$	$0\cdot 0$	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1+i}{2}\cdot\frac{1+i}{2}$
132	$1\cdot 1$	$\frac{1}{2}\cdot 1$	$1\cdot\frac{1}{2}$	$\frac{1+i}{2}\cdot\frac{1-i}{2}$
213	$1\cdot(-1)$	$1\cdot 0$	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1-i}{2}\cdot\frac{1+i}{2}$
231	$1\cdot 1$	$\frac{1}{2}\cdot 1$	$1\cdot\frac{1}{2}$	$\frac{1+i}{2}\cdot\frac{1-i}{2}$
312	$1\cdot(-1)$	$1\cdot 0$	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1-i}{2}\cdot\frac{1+i}{2}$
321	$1\cdot 1$	$1\cdot 1$	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1-i}{2}\cdot\frac{1-i}{2}$
sum	2	2	2	2

Notice that although the four results are the same, the four summations themselves are quite different:

$$\begin{aligned}
\sum W_1(\sigma) &= 1+1-1+1-1+1 = 2,\\
\sum W_2(\sigma) &= 0+\frac{1}{2}+0+\frac{1}{2}+0+1 = 2,\\
\sum W_3(\sigma) &= \frac{1}{4}+\frac{1}{2}+\frac{1}{4}+\frac{1}{2}+\frac{1}{4}+\frac{1}{4} = 2,\\
\sum W_4(\sigma) &= \frac{i}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-\frac{i}{2} = 2.
\end{aligned}$$

It is well-known [11, 52, 40] that the exponential generating function of the Eulerian polynomials

$$A_n(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{1+\mathrm{des}(\sigma)} \tag{3.8}$$

is

$$1 + \sum_{n\ge 1} A_n(t)\frac{x^n}{n!} = \frac{1-t}{1-te^{(1-t)x}}. \tag{3.9}$$

Theorem 3.1 can be proved by using (3.9) and the Carlitz-Scoville formula without difficulty. Let $P_n(t,s)$ be the ordinary generating function of peaks and double ascents for the permutations:

$$P_n(t,s) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{pk}(\sigma)} s^{\mathrm{da}(\sigma)}. \tag{3.10}$$

From the Carlitz-Scoville ([6], see also [16]), we have

$$P(x;t,s)=\sum_{n\geq 1}P_n(t,s)\frac{x^n}{n!}=\frac{-2+2e^{u(t,s)x}}{(1+s+u(t,s))-(1+s-u(t,s))e^{u(t,s)x}}, \tag{3.11}$$

where $u(t,s)=\sqrt{(1+s)^2-4t}$.

Proof of Theorem 3.1. We have the following specializations.

(1) When $t=1$ and $s=-1$, $u(1,-1)=2i$. Identity (3.11) becomes

$$P(x;1,-1)=\frac{-2+2e^{2xi}}{2i+2ie^{2xi}}=\tan(x).$$

Thus, relation (3.7) is true.

(2) When $t=1/2$ and $s=0$, $u(1/2,0)=i$. Identity (3.11) becomes

$$P(x;1/2,0)=\frac{-2+2e^{xi}}{(1+i)-(1-i)e^{xi}}=\tan(x)+\sec(x)-1.$$

Thus

$$\tan(x)+\sec(x)=P(x;1/2,0)+1$$

or

$$E_n=P_n(1/2,0). \qquad (n\geq 1)$$

(3) When $t=2$ and $s=1$, $u(2,1)=2i$. Identity (3.11) becomes

$$P(x;2,1)=\frac{-2+2e^{2xi}}{(2+2i)-(2-2i)e^{2xi}}=\frac{\tan(2x)+\sec(2x)-1}{2}.$$

Thus

$$\tan(x)+\sec(x)=2P(x/2;2,1)+1$$

or

$$E_n=2^{1-n}P_n(2,1). \qquad (n\geq 1)$$

(4) When $t=i$. Identity (3.9) becomes

$$1+\sum_{n\geq 1}A_n(i)\frac{x^n}{n!}=\frac{1-i}{1-ie^{(1-i)x}}=\frac{\tan((1+i)x)+\sec((1+i)x)-i}{1-i}.$$

Thus

$$\tan(x)+\sec(x)=1+\sum_{n\geq 1}\frac{A_n(i)}{(1+i)^{n-1}}\frac{x^n}{n!}$$

or

$$E_n=-i(1+i)^{1-n}A_n(i). \qquad (n\geq 1)$$

So that

$$E_n=\sum_{\sigma\in\mathfrak{S}_n}\left(\frac{1+i}{2}\right)^{\operatorname{asc}(\sigma)}\left(\frac{1-i}{2}\right)^{\operatorname{des}(\sigma)}. \qquad (n\geq 1)$$ □

We also provide a combinatorial proof of Theorem 3.1 by using the (modified) Foata-Strehl action. First, let x be a letter of a permutation $\sigma=\sigma_1\sigma_2\cdots\sigma_n$. The *x-factorization* of σ is defined to be the sequence (w_1,w_2,x,w_4,w_5), where (1) the juxtaposition product $w_1w_2xw_4w_5$ is equal to σ; (2) w_2 is the longest right factor of $x_1x_2\cdots x_{i-1}$, all letters of which are greater than x; (3) w_4 is the longest left factor of $x_{i+1}x_{i+2}\cdots x_n$, all letters of which are greater than x.

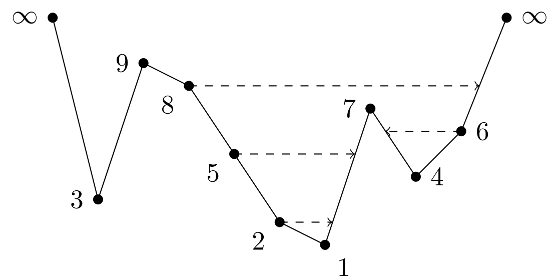

FIGURE 1. The modified Foata-Strehl bijection φ'_x

Foata and Strehl [13] introduced the involution φ_x defined by

$$\varphi_x(\sigma) = w_1 w_3 x w_2 w_4. \tag{3.12}$$

Brändén [4, 32, 49, 58] modified φ_x and defined

$$\varphi'_x(\sigma) = \begin{cases} \varphi_x(\sigma), & \text{if } x \text{ is a double ascent or double descent of } \sigma; \\ \sigma, & \text{if } x \text{ is a valley or a peak of } \sigma. \end{cases}$$

It is clear that the φ'_x's are involutions and commute. Visually, the bijection $\varphi'_x : \sigma \mapsto \sigma'$ consists of moving the letter x horizontally such that (1) if x is a double ascent (resp. double descent) of σ, it so becomes a double descent (resp. ascent) of σ'; (2) The letter x is greater than all letters it already exceeds when moving (see Figure 1).

For each subset $S \subseteq \{1, 2, \ldots, n\}$ define the function $\varphi'_S : \mathfrak{S}_n \to \mathfrak{S}_n$ by

$$\varphi'_S(\sigma) = \prod_{x \in S} \varphi'_x(\sigma).$$

Hence, the group $\mathbb{Z}_2^n$ acts on $\mathfrak{S}_n$ via the function φ'_S. This action will be called the *Modified Foata-Strehl action*.

Combinatorial proof of Theorem 3.1. (1) Consider the weight function W_1. For each permutation σ let $X(\sigma)$ be the set of double ascents and double descents. If $X(\sigma) \neq \emptyset$, let $x = \min(X(\sigma))$. Then, the map $\sigma \mapsto \sigma' = \varphi_x(\sigma)$ is an involution having the property that $\mathrm{da}(\sigma) = \mathrm{da}(\sigma') \pm 1$. Hence, $W_1(\sigma) + W_1(\sigma') = 0$. Therefore,

$$\sum_{\sigma \in \mathfrak{S}_n} W_1(\sigma) = \#\{\sigma \in \mathfrak{S}_n, X(\sigma) = \emptyset\} := \rho.$$

If n is even, there is at least one double ascent or double descent, so that $\rho = 0$. If n is odd, the permutations without any double ascent or double descent are just the alternating permutations beginning with an ascent. In this case, we have $\rho = E_n$ by André's result [2].

(2) Consider the weight function W_2. The weighted sum of the *modified* Foata-Strehl orbits $\mathrm{Orb}(\sigma)$ with the weight $(1/2)^{\mathrm{pk}(\sigma)}$ is exactly the number of the *original* Foata-Strehl orbits, which is equal to the Euler numbers [13, 14].

(3) Consider the weight function W_3. For $\sigma \in \mathfrak{S}_n$ let $\mathrm{Orb}(\sigma) = \{g(\sigma) : g \in \mathbb{Z}_2^n\}$ be the orbit of σ under the modified Foata-Strehl action. It is clear that (i) there is a unique permutation in $\mathrm{Orb}(\sigma)$ which has no double ascent; (ii) all permutations in $\mathrm{Orb}(\sigma)$ have the same numbers of peaks. By definition of W_2 we know that W_2

is evaluated for all orbits with only one representative per orbit (for example, the permutation without double ascent). Consequently, for each orbit $\mathrm{Orb}(\sigma)$ we can evaluate all the elements in the orbit, then, divide by the cardinality of the orbit, which is equal to $2^{n-1-2\,\mathrm{pk}(\sigma)}$. By (2) we have

$$E_n = \sum_{\sigma\in\mathfrak{S}_n} W_2(\sigma) = \sum_{\sigma\in\mathfrak{S}_n} \frac{(1/2)^{\mathrm{pk}(\sigma)}}{2^{n-1-2\,\mathrm{pk}(\sigma)}} = \sum_{\sigma\in\mathfrak{S}_n} \frac{1}{2^{n-1-\mathrm{pk}(\sigma)}} = \sum_{\sigma\in\mathfrak{S}_n} W_3(\sigma).$$

(4) Recall the following formula [49, 13, 4], which can be proved by using the modified Foata-Strehl action:

$$\sum_{\tau\in\mathrm{Orb}(\sigma)} t^{\mathrm{des}(\sigma)} = t^{\mathrm{pk}(\sigma)}(1+t)^{n-1-2\,\mathrm{pk}(\sigma)}.$$

For the same reason as (3) we have

$$\begin{aligned}\sum_{\sigma\in\mathfrak{S}_n} W_4(\sigma) &= \sum_{\sigma\in\mathfrak{S}_n} \left(\frac{1+i}{2}\right)^{\mathrm{asc}(\sigma)} \left(\frac{1-i}{2}\right)^{\mathrm{des}(\sigma)} \\ &= (1+i)^{1-n} A_n(i) \\ &= (1+i)^{1-n} \sum_{\sigma\in\mathfrak{S}_n} \frac{i^{\mathrm{pk}(\sigma)}(1+i)^{n-1-2\,\mathrm{pk}(\sigma)}}{2^{n-1-2\,\mathrm{pk}(\sigma)}} \\ &= \sum_{\sigma\in\mathfrak{S}_n} \frac{1}{n-1-\mathrm{pk}(\sigma)} = E_n. \qquad \square\end{aligned}$$

Remark. The two weight functions W_3 and W_4 are also connected by Stembridge's formula [53, 58, 4]

$$A_n(t) = \left(\frac{1+t}{2}\right)^{n+1} P_n\left(\frac{4t}{(1+t)^2}, 1\right).$$

4. Proofs of Theorems 1.1 and 1.2

In his work on combinatorial aspects of continued fractions Flajolet [9, Theorem 3A] obtained the continued fraction for the ordinary generating function of the quadruple statistic $(\mathrm{val}, \mathrm{pk}, \mathrm{da}, \mathrm{dd})$. [3] By adding a superfluous variable u_1, because $\mathrm{val}(\sigma) = \mathrm{pk}(\sigma) + 1$ for each permutation σ, we restate his theorem as follows.

Theorem 4.1. *We have*

$$\sum_{n\ge 1} x^n \sum_{\sigma\in\mathfrak{S}_n} u_1^{\mathrm{val}(\sigma)} u_2^{\mathrm{pk}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} \tag{4.1}$$
$$= \frac{u_1 x}{1-1(u_3+u_4)x} \, \Big| - \frac{1\cdot 2u_1u_2x^2}{1-2(u_3+u_4)x} \, \Big| - \frac{2\cdot 3u_1u_2x^2}{1-3(u_3+u_4)x} \, \Big| - \cdots$$

At this stage it is interesting to compare the previous continued fraction expression with the continued fraction derived for the *exponential* generating function, as stated in Theorem 6.1

[3] There is a typo in [9, Theorem 3A], the first numerator in the continued fraction of $P(u, v, w, z)$ should be z, instead of 1. This typo had not been fixed in the reprint [10] of the paper.

From the definitions of $W_2(\sigma)$ and $P_n(t,s)$ given in (3.3) and (3.10), respectively, we have $E_n = P_n(1/2, 0)$ by Theorem 3.1. Hence, the specialization of identity (4.1) with $u_1 = 1, u_2 = 1/2, u_3 = 0, u_4 = 1$ leads the following Theorem.

Theorem 4.2. *We have the following continued fraction the Euler numbers*

$$\sum_{n\geq 0} E_n x^n = 1 + \frac{x}{1-x} - \frac{x^2}{1-2x} - \frac{3x^2}{1-3x} - \frac{6x^2}{1-4x} - \cdots \tag{4.2}$$

The general pattern for the coefficients a_k and b_k are:

$$a_1 = x, \quad a_k = -\binom{k}{2} x^2; \qquad b_0 = 1, \quad b_k = 1 - kx.$$

Notice that the above continued fraction is neither a super 1-fraction nor a super 2-fraction. Fortunately, we can use it to derive a super 1-fraction by a series of chop contractions at appropriate positions.

Theorem 4.3. *We have the following super 1-fraction expansion:*

$$\sum E_n x^n = \frac{1}{1} - \frac{x}{1} - \frac{x^2}{1-2x} - \frac{3x^2}{1} - \frac{3x}{1} - \frac{2x}{1} - \frac{2x}{1} - \frac{5x}{1} - \frac{15x^2}{1-6x} - \frac{21x^2}{1} - \frac{7x}{1} - \frac{4x}{1} - \frac{4x}{1} - \cdots \tag{4.3}$$

The general patterns for the coefficients a_j and b_j are:

$$\begin{aligned} a_1 &= 1, & b_0 &= 0,\\ a_{6k} &= -2kx, & b_{6k} &= 1,\\ a_{6k+1} &= -2kx, & b_{6k+1} &= 1,\\ a_{6k+2} &= -(4k+1)x, & b_{6k+2} &= 1,\\ a_{6k+3} &= -(4k+1)(2k+1)x^2, & b_{6k+3} &= 1-2(2k+1)x,\\ a_{6k+4} &= -(4k+3)(2k+1)x^2, & b_{6k+4} &= 1,\\ a_{6k+5} &= -(4k+3)x; & b_{6k+5} &= 1. \end{aligned}$$

Proof. The theorem will be proved by using the chop contraction defined in (2.6) at specific positions. Chop contraction at the first position of the continued fraction on the right-hand side of (4.3), we get

$$1 + \frac{x}{1-x} - \frac{x^2}{1-2x} - \frac{3x^2}{1} - \frac{3x}{1} - \frac{2x}{1} - \frac{2x}{1} - \frac{5x}{1} - \frac{15x^2}{1-6x} - \cdots$$

Then, chop at the 4th position:

$$1 + \frac{x}{1-x} - \frac{x^2}{1-2x} - \frac{3x^2}{1-3x} - \frac{6x^2}{1-2x} - \frac{2x}{1} - \frac{5x}{1} - \frac{15x^2}{1-6x} - \cdots$$

and chop at the 5th position:

$$1 + \frac{x}{1-x} - \frac{x^2}{1-2x} - \frac{3x^2}{1-3x} - \frac{6x^2}{1-4x} - \frac{10x^2}{1-5x} - \frac{15x^2}{1-6x} - \cdots$$

The next chop contraction is to be applied at position 8. In general, we contract the second numerator that is a monomial in x of degree 1, and repeat. This will work since

$$\begin{aligned}
&-\frac{(4k+3)x}{1}\Big| - \frac{(2k+2)x}{1}\Big| - \frac{(2k+2)x}{1}\Big| + \cdots \\
&= -(4k+3)x - \frac{(4k+3)(2k+2)x^2}{1-(2k+2)x}\Big| - \frac{(2k+2)x}{1}\Big| + \cdots
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{(2k+2)x}{1}\Big| - \frac{(4k+5)x}{1}\Big| - \frac{(4k+5)(2k+3)x^2}{1-2(2k+3)x}\Big| + \cdots \\
&= -(2k+2)x - \frac{(4k+5)(2k+2)x^2}{1-(4k+5)}\Big| - \frac{(4k+5)(2k+3)x^2}{1-2(2k+3)x}\Big| + \cdots
\end{aligned}$$

Finally, we will get the continued fraction on the right-hand side of (4.2). Hence, (4.3) is true. We verify that the continued fraction is a super 1-fraction, under the general super continued fraction form (2.10) with $\delta = 1$ and $(k_0, k_1, k_2, \ldots) = (0, 0, 1, 0, 0, 0)^*$, where the star sign means that the sequence is periodic and obtained by repeating the underlying segment. □

Proof of Theorem 1.1. We prove the result by applying the even contraction on the super 1-fraction (4.3) given in Theorem 4.3. Let us detail only the calculations for the coefficients a'_j. From the general formula for the even contraction (2.2) we have

$$\begin{aligned}
a'_1 &= a_1 b_2 = 1, \\
a'_2 &= -a_2 a_3 b_4 = -x^3, \\
a'_{3k} &= -a_{6k-2} a_{6k-1} b_{6k-4} b_{6k} = -(4k-1)^2(2k-1)x^3, \\
a'_{3k+1} &= -a_{6k} a_{6k+1} b_{6k-2} b_{6k+2} = -4k^2 x^2, \\
a'_{3k+2} &= -a_{6k+2} a_{6k+3} b_{6k} b_{6k+4} = -(4k+1)^2(2k+1)x^3.
\end{aligned}$$

The calculations for the coefficients b'_j are similar. We verify that the continued fraction (1.5) is a super 2-fraction, under the general super continued fraction form (2.10) with $\delta = 2$ and $(k_0, k_1, k_2, \ldots) = (0, 1, 0)^*$. □

Proof of Theorem 1.2. The Hankel determinants are evaluated by using the fundamental theorem 2.2. For the Hankel continued fraction given in (1.5), we have $(k_0, k_1, k_2, \ldots) = (0, 1, 0)^*$. So that

$$(s_0, s_1, s_2, \ldots) = (0, 1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 15 \ldots)$$

and

$$(\epsilon_0, \epsilon_1, \epsilon_2, \ldots) = (0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, \ldots).$$

Comparing (1.5) and (2.10), we have

$$\begin{aligned}
v_0 &= 1, \\
v_{3k} &= 4k^2, \\
v_{3k+1} &= (4k+1)^2(2k+1), \\
v_{3k+2} &= (4k+3)^2(2k+1).
\end{aligned}$$

Put all these $(v_j), (s_j), (\epsilon_j)$ into (2.11), we obtain the explicit Hankel determinant formulas given in Theorem 1.2 after simplification. □

5. The ordinary generating functions of the Euler numbers

In this section we consider the ordinary generating functions of the Euler numbers, and derive some Hankel continued fractions and some Hankel determinants involving these numbers. Some formulas are known or easy to prove. We list them here for a quick view and comparison. Let $(\sec(x))^r = \sum_{n\geq 0} E_{2n}^{(r)} \frac{x^{2n}}{(2n)!}$.

Theorem 5.1. *We have the following Hankel continued fraction expansions.*

(F1) $$\sum_{n\geq 0} E_{2n}^{(r)} x^{2n} = \frac{1}{1} - \frac{1rx^2}{1} - \frac{2(r+1)x^2}{1} - \frac{3(r+2)x^2}{1} - \cdots$$

(F2) $$\sum_{n\geq 0} E_{2n}^{(r)} x^{n} = \frac{1}{1-rx} - \frac{2r(r+1)x^2}{1-(5r+8)x} - \frac{-12(r+2)(r+3)x^2}{1-(9r+32)x} - \cdots$$

The general patterns for the coefficients a_j and b_j are:

$$a_1 = 1, \quad a_k = -(2k+r-3)(2k+r-4)(2k-3)(2k-2)x^2;$$
$$b_0 = 0, \quad b_k = 1-(8k^2+4kr-16k-3r+8)x.$$

(F3) $$\sum_{n\geq 1} E_{2n}^{(r)} x^{n-1} = \frac{r}{1-(2+3r)x} - \frac{6(r+2)(r+1)x^2}{1-(18+7r)x} + \cdots$$

The general patterns for the coefficients a_j and b_j are:

$$a_1 = rx, \quad a_k = -2(2k+r-2)(2k+r-3)(2k-1)(k-1)x^2;$$
$$b_0 = 1, \quad b_k = 1-(8k^2+4kr-8k-r+2)x.$$

(F4) $$\sum_{n\geq 0} E_{2n+1} x^{2n} = \frac{1}{1} - \frac{1\cdot 2x^2}{1} - \frac{2\cdot 3x^2}{1} - \frac{3\cdot 4x^2}{1} - \cdots$$

(F5) $$\sum_{n\geq 0} E_{2n+1} x^{n} = \frac{1}{-2x+1} + \frac{-12x^2}{-18x+1} + \frac{-240x^2}{-50x+1} + \cdots$$

The general patterns for the coefficients a_j and b_j are:

$$a_1 = 1, \quad a_k = -4(2k-1)(2k-3)(k-1)^2x^2;$$
$$b_0 = 0, \quad b_1 = -2(2k-1)^2x+1.$$

(F6) $$\sum_{n\geq 1} E_{2n+1} x^{n-1} = \frac{2}{1-8x} - \frac{72x^2}{1-32x} - \frac{600x^2}{1-72x} + \cdots$$

The general patterns for the coefficients a_j and b_j are:

$$a_1 = 2, \quad a_k = -4(2k-1)^2(k-1)kx^2; \qquad b_0 = 0, \quad b_k = 1-8k^2x.$$

(F7) $$\sum_{n\geq 0} E_{n+1} x^{n} = \frac{1}{1-x} - \frac{x^2}{1-2x} - \frac{3x^2}{1-3x} - \frac{6x^2}{1-4x} - \cdots$$

The general patterns for the coefficients a_k and b_k are given by

$$a_1 = 1, \quad a_k = -\binom{k}{2}x^2; \qquad b_0 = 0, \quad b_k = 1-kx.$$

(F8)
$$\sum_{n\geq 1}\frac{E_{2n}^{(r)}}{r}x^{2n-1}=\frac{x}{\lfloor 1-(3r+2)x^2}-\frac{2\cdot 3(r+2)(r+1)x^4}{\lfloor 1-(7r+18)x^2}-\frac{4\cdot 5(r+4)(r+3)x^4}{\lfloor 1-(11r+50)x^2}-\cdots$$

The general patterns for the coefficients a_j and b_j are:

$$a_1=x,\quad a_k=-(2k-1)(2k-2)(2k-3+r)(2k-2+r)x^4;$$
$$b_0=0,\quad b_k=1-(8k^2-8k+4rk+2-r)x^2.$$

(F9)
$$\sum_{n\geq 0}E_{2n+1}x^{2n+1}=\frac{x}{\lfloor 1-2\cdot 1^2x^2}-\frac{1\cdot 2^2\cdot 3x^4}{\lfloor 1-2\cdot 3^2x^2}-\frac{3\cdot 4^2\cdot 5x^4}{\lfloor 1-2\cdot 5^2x^2}-\cdots$$

The general patterns for the coefficients a_j and b_j are:

$$a_1=x,\quad a_k=-(2k-1)(2k-2)^2(2k-3)x^4;$$
$$b_0=0,\quad b_k=1-2(2k-1)^2x^2.$$

(F10)
$$\sum_{n\geq 0}E_{n+2}x^n=\frac{1}{\lfloor 1-2x}-\frac{x^2}{\lfloor 1-4x}-\frac{18x^3}{\lfloor 1-4x-16x^2}-\frac{50x^3}{\lfloor 1-8x}-\frac{9x^2}{\lfloor 1-10x}-\frac{196x^3}{\lfloor 1-8x-64x^2}-\frac{324x^3}{\lfloor 1-14x}-\cdots$$

The general patterns for the coefficients a_j and b_j are:

$$\begin{aligned}
a_1&=1, & b_0&=0,\\
a_{3k+0}&=-2(4k-1)^2kx^3, & b_{3k+0}&=-16k^2x^2-4kx+1,\\
a_{3k+1}&=-2(4k+1)^2kx^3, & b_{3k+1}&=-2(3k+1)x+1,\\
a_{3k+2}&=-(2k+1)^2x^2; & b_{3k+2}&=-2(3k+2)x+1.
\end{aligned}$$

(F11)
$$\sum_{n\geq 1}E_{2n+1}x^{2n-1}=\frac{2x}{\lfloor 1-8x^2}-\frac{72x^4}{\lfloor 1-32x^2}-\frac{600x^4}{\lfloor 1-72x^2}+\cdots$$

The general patterns for the coefficients a_j and b_j are:

$$a_1=2x,\quad a_k=-4(2k-1)^2(k-1)kx^4;$$
$$b_0=0,\quad b_k=1-8k^2x^2.$$

Proof. (F1) This is a well-known formula (see [56, p. 206], [9]).

(F2) First, replace x^2 by x in (F1) we get

(5.1)
$$\sum_{n\geq 0}E_{2n}^{(r)}z^n=\frac{1}{\lfloor 1}-\frac{1rz}{\lfloor 1}-\frac{2(r+1)z}{\lfloor 1}-\frac{3(r+2)z}{\lfloor 1}-\cdots$$

Even contraction on (5.1) yields (F2).

(F3) Odd contraction on (5.1). Then, subtract by 1 and divide by x.

(F4) Divide x in (1.4).

(F5) First, replace x^2 by x in (F4), we get

(5.2)
$$\sum_{n\geq 0}E_{2n+1}z^n=\frac{1}{\lfloor 1}-\frac{1\cdot 2z}{\lfloor 1}-\frac{2\cdot 3z}{\lfloor 1}-\frac{3\cdot 4z}{\lfloor 1}-\cdots$$

Even contraction on (5.2) yields (F5).

(F6) Odd contraction on (5.2). Then, subtract by 1 and divide by x.

(F7) Subtract by 1 and divide by x in (4.2).

(F8) Chop contraction on (F1) yields

$$\sum_{n\geq 0} E_{2n}^{(r)} x^{2n} = 1 + \frac{rx^2}{1-rx^2} - \frac{2(r+1)x^2}{1} - \frac{3(r+2)x^2}{1} - \frac{4(r+3)x^2}{1} - \cdots$$

Then, subtract by 1 and divide by rx in the above identity. We get

$$\sum_{n\geq 1} \frac{E_{2n}^{(r)}}{r} x^{2n-1} = \frac{x}{1-rx^2} - \frac{2(r+1)x^2}{1} - \frac{3(r+2)x^2}{1} - \frac{4(r+3)x^2}{1} - \cdots$$

with the general patterns

$$a_1 = x, \quad a_k = -(k+r-1)kx^2;$$
$$b_0 = 0, \quad b1 = 1 - rx^2, \quad b_k = 1.$$

Notice that the previous continued fraction is *not* a super 1-fraction. Even contraction on the above fraction yields (F8), which is a H-fraction.

(F9) Take $w = 0$ in (F8). We get (F9), by using the fact that

$$\sum_{n\geq 1} \frac{E_{2n}^{(r)}}{r} \frac{x^{2n-1}}{(2n-1)!} = \frac{1}{r}(\sec(x)^r)' = \tan(x)\sec(x)^r.$$

(F10) This continued fraction is very similar to that given in Theorem 1.1. However, unlike (1.5), which has a super 1-fraction (see Theorem 4.3), (F10) does not have a super 1-fraction. Thanks to the similarity of (F10) and (1.5), this proof is suggested by the proof of Theorem 1.1.

Apply the haircut contraction as defined in (2.7) to (F7) with $\alpha = 1$. We get

$$\sum_{n\geq 0} E_{n+1}x^n = 1 + \frac{x}{1-x} - \frac{x}{1-x} - \frac{3x^2}{1-3x} - \frac{6x^2}{1-4x} - \cdots$$

Hence

$$\sum_{n\geq 0} E_{n+2}x^n = \frac{1}{1-x} - \frac{x}{1-x} - \frac{3x^2}{1-3x} - \frac{6x^2}{1-4x} - \cdots \tag{5.3}$$

This is neither a super 1-fraction, nor a super 2-fraction. Now, we claim that

$$\sum_{n\geq 0} E_{n+2}x^n = \frac{1}{1-x} - \frac{x}{1} - \frac{x}{1} - \frac{3x}{1} - \frac{6x^2}{1-4x} - \frac{10x^2}{1} - \frac{5x}{1} - \frac{3x}{1} - \frac{3x}{1} - \cdots \tag{5.4}$$

with the general patterns

$$\begin{aligned} a_1 &= 1, & b_0 &= 0, \quad b_1 = 1-x, \\ a_{6k+0} &= -2(4k+1)kx^2, & b_{6k+0} &= 1, \\ a_{6k+1} &= -(4k+1)x, & b_{6k+1} &= 1, \\ a_{6k+2} &= -(2k+1)x, & b_{6k+2} &= 1, \\ a_{6k+3} &= -(2k+1)x, & b_{6k+3} &= 1, \end{aligned}$$

$$a_{6k+4} = -(4k+3)x, \qquad b_{6k+4} = 1,$$
$$a_{6k+5} = -2(4k+3)(k+1)x^2; \qquad b_{6k+5} = 1-4(k+1)x.$$

Notice that (5.4) is neither a super 1-fraction nor a super 2-fraction.

By using the chop contractions at appropriate positions in the right-hand side of (5.4), we successively get

$$\frac{1}{1-x} - \frac{x}{1} - \frac{x}{1} - \frac{3x}{1} - \frac{6x^2}{1-4x} - \frac{10x^2}{1} - \frac{5x}{1} - \frac{3x}{1} - \cdots$$
$$= \frac{1}{1-x} - \frac{x}{1-x} - \frac{3x^2}{1-3x} - \frac{6x^2}{1-4x} - \frac{10x^2}{1} - \frac{5x}{1} - \frac{3x}{1} - \cdots$$
$$= \frac{1}{1-x} - \frac{x}{1-x} - \frac{3x^2}{1-3x} - \frac{6x^2}{1-4x} - \frac{10x^2}{1-5x} - \frac{15x^2}{1-3x} - \cdots$$

By using the general patterns we can obtain the right-hand side of (5.3), so that (5.4) is true. Finally, an even contraction on (5.4) yields (F10). We verify that (F10) is a H-fraction.

(F11) Replace x by x^2 and multiply by x in (F6). □

By using Theorem 2.2, the Hankel continued fractions (F1–F11) listed in Theorem 5.1 implies the Hankel determinants formulas (H1–H11) in the next theorem respectively.

Theorem 5.2. *We have the following formulas for the Hankel determinants.*

(H1) *The Hankel determinants of* $(E_0^{(r)}, 0, E_2^{(r)}, 0, E_4^{(r)}, \ldots)$ *are*

$$H_n = \prod_{k=1}^{n-1} k! r(r+1)(r+2)\cdots(r+k-1).$$

In particular, when $r=1$, *the Hankel determinants of* $(E_0, 0, E_2, 0, E_4, \ldots)$ *are*

$$H_n = \prod_{k=1}^{n-1} k!^2.$$

(H2) *The Hankel determinants of* $(E_0^{(r)}, E_2^{(r)}, E_4^{(r)}, \ldots)$ *are*

$$H_n = \prod_{k=1}^{n-1} (2k)! r(r+1)\cdots(r+2k-1).$$

In particular, when $r=1$, *the Hankel determinants of* $(E_0, E_2, E_4, \ldots)$ *are*

$$H_n = \prod_{k=1}^{n-1} (2k)!^2.$$

(H3) *The Hankel determinants of* $(E_2^{(r)}, E_4^{(r)}, \ldots)$ *are*

$$H_n = \prod_{k=0}^{n-1} (2k+1)! r(r+1)\cdots(r+2k).$$

In particular, when $r=1$, *the Hankel determinants of* $(E_2, E_4, \ldots)$ *are*

$$H_n = \prod_{k=1}^{n-1} (2k+1)!^2$$

(H4) *The Hankel determinants of* $(E_1, 0, E_3, 0, E_5, \ldots)$ *are*

$$H_n = n! \prod_{k=1}^{n-1} k!^2.$$

(H5) *The Hankel determinant of* $(E_1, E_3, E_5, \ldots)$ *are*

$$H_n = \prod_{k=1}^{2n-1} k!.$$

(H6) *The Hankel determinant of* $(E_3, E_5, \ldots)$ *are*

$$H_n = \prod_{k=1}^{2n} k!.$$

(H7) *The Hankel determinants of* $(E_1, E_2, E_3, E_4, \ldots)$ *are*

$$H_n = \frac{n!}{2^{n(n-1)/2}} \prod_{k=2}^{n-1} k!^2$$

(H8) *The Hankel determinants of* $(0, E_2^{(r)}/r, 0, E_4^{(r)}/r, \ldots)$ *are*

$$H_{2n+1} = 0, \quad H_{2n} = (-1)^n \prod_{k=1}^{n-1} \left((2k+1)!(r+1)(r+2)\cdots(r+2k)\right)^2.$$

Or equivalently, the Hankel determinants of $(0, E_2^{(r)}, 0, E_4^{(r)}, \ldots)$ *are*

$$H_{2n+1} = 0, \quad H_{2n} = (-1)^n r^2 \prod_{k=1}^{n-1} \left((2k+1)! r(r+1)(r+2)\cdots(r+2k)\right)^2.$$

In particular, when $r = 1$*, the Hankel determinants of* $(0, E_2, 0, E_4, \ldots)$ *are*

$$H_{2n+1} = 0, \quad H_{2n} = (-1)^n \prod_{k=1}^{n-1} (2k+1)!^4.$$

(H9) *The Hankel determinants of* $(0, E_1, 0, E_3, 0, E_5, \ldots)$ *are*

$$H_{2n+1} = 0, \quad H_{2n} = (-1)^n \prod_{k=1}^{2n-1} k!^2.$$

(H10) *The Hankel determinants of* $(E_2, E_3, E_4, E_5, \ldots)$ *are*

$$H_0 = 1,$$

$$H_{4k} = \frac{(-1)^k k^2 (2k-1)!^2}{2^{8k^2-4k-2}} \prod_{j=1}^{2k-1} (2j+1)!^4$$

$$H_{4k+1} = \frac{(-1)^k (2k)!^2}{2^{8k^2}(4k+1)!^2} \prod_{j=1}^{2k} (2j+1)!^4$$

$$H_{4k+2} = \frac{(-1)^k (2k+1)!^2}{2^{8k^2+4k}} \prod_{j=1}^{2k} (2j+1)!^4$$

$$H_{4k+3} = 0$$

(H11) *The Hankel determinant of* $(0, E_3, 0, E_5, \ldots)$ *are*

$$H_n = (-1)^n \prod_{k=1}^{2n} k!^2.$$

6. The exponential generating functions of the Euler numbers

In this section we consider the exponential generating functions of the Euler numbers. They are $\tan(x), \sec(x), \tan(x) + \sec(x)$ and their variants. Although most continued fractions and Hankel determinants involving the Euler numbers are for the *ordinary* generating functions (see Section 5), a few of them are about the *exponential* generating functions. In 1761, Lambert [31] proved that π is irrational by first deriving the following continued fraction expansion of $\tan(x)$ (see [56, p. 349 (91.7)])

$$\tan(x) = \frac{x}{\lceil 1} - \frac{x^2}{\lceil 3} - \frac{x^2}{\lceil 5} - \frac{x^2}{\lceil 7} - \cdots \tag{6.1}$$

or [56, p. 349 (91.6)] [4]

$$\tanh(x) = \frac{x}{\lceil 1} + \frac{x^2}{\lceil 3} + \frac{x^2}{\lceil 5} + \frac{x^2}{\lceil 7} + \cdots \tag{6.2}$$

Also, Hankel determinants of the Euler numbers divided by the factorial numbers $E_n/n!$ are studied in [38].

We have seen the Flajolet continued fraction (4.1) for the ordinary generating function of the quadruple statistic $(\mathrm{val}, \mathrm{pk}, \mathrm{da}, \mathrm{dd})$ [9], and the Carlitz-Scoville exponential generating function (3.1) for the same statistic [6]. The next continued fraction for their exponential generating function seems to be new.

Theorem 6.1. *We have the following continued fraction of the exponential generating function for the quadruple statistic* $(\mathrm{val}, \mathrm{pk}, \mathrm{da}, \mathrm{dd})$*:*

$$\sum_{n\geq 1} \frac{x^n}{n!} \sum_{\sigma \in \mathfrak{S}_n} u_1^{\mathrm{val}(\sigma)} u_2^{\mathrm{pk}(\sigma)} u_3^{\mathrm{da}(\sigma)} u_4^{\mathrm{dd}(\sigma)} \tag{6.3}$$

$$= \frac{u_1 x}{\lceil 1 - cx} + \frac{(c^2 - u_1u_2)x^2}{\lceil 3} + \frac{(c^2 - u_1u_2)x^2}{\lceil 5} + \frac{(c^2 - u_1u_2)x^2}{\lceil 7} + \cdots$$

where $c = (u_3 + u_4)/2$*.*

Proof. Let $F(x)$ be the left-hand side of (6.3). By using Carlitz-Scoville formula (3.1), with $u_3 + u_4 = \alpha_1 + \alpha_2$, $u_1u_2 = \alpha_1\alpha_2$, and $\tau = (\alpha_1 - \alpha_2)/2$, we have

$$\begin{aligned} F(x) &= \frac{u_1}{x} \cdot \frac{e^{\alpha_2 x} - e^{\alpha_1 x}}{\alpha_2 e^{\alpha_1 x} - \alpha_1 e^{\alpha_2 x}}, \\ &= \frac{u_1}{x} \cdot \frac{e^{\tau x} - e^{-\tau x}}{\tau(e^{\tau x} + e^{-\tau x}) - c(e^{\tau x} - e^{-\tau x})}, \\ &= \frac{u_1}{\frac{\tau x}{\tanh(\tau x)} - cx}. \end{aligned}$$

[4] There are two typos in the equalities (91.6). The middle side $\frac{z\Psi(\frac{3}{2};\frac{z}{4})^2}{\Psi(\frac{3}{2};\frac{z}{4})^2}$ should be $\frac{z\Psi(\frac{3}{2};\frac{z^2}{4})}{\Psi(\frac{3}{2};\frac{z^2}{4})}$.

By using Lambert's continued fraction (6.1), we obtain

$$F(x) = \frac{u_1}{1-cx} + \frac{\tau^2 x^2}{3} + \frac{\tau^2 x^2}{5} + \frac{\tau^2 x^2}{7} + \cdots \tag{6.4}$$

Replace τ^2 by $(\alpha_1-\alpha_2)^2/2^2 = c^2 - u_1u_2$, we obtain (6.3). □

Comparing (6.3) and (4.1), we can roughly say that the *formal Laplace transformation* converts the continued fraction on the right-hand side of (6.3) to the continued fraction on the right-hand side of (4.1).

Theorem 6.2. *We have the following Hankel fraction expansions.*

$$\sum_{n\ge 0} E_{2n+1}\frac{x^{2n}}{(2n+1)!} = \frac{1}{1} - \frac{\frac{1}{1\cdot 3}x^2}{1} - \frac{\frac{1}{3\cdot 5}x^2}{1} - \frac{\frac{1}{5\cdot 7}x^2}{1} - \cdots \tag{F12}$$

$$\sum_{n\ge 0} E_{2n+1}\frac{x^{n}}{(2n+1)!} = \frac{1}{1-\frac{1}{3}x} - \frac{\frac{1}{1\cdot 3^2\cdot 5}x^2}{1-\frac{2}{3\cdot 7}x} - \frac{\frac{1}{5\cdot 7^2\cdot 9}x^2}{1-\frac{2}{7\cdot 11}x} + \cdots \tag{F13}$$

$$\sum_{n\ge 0} E_{2n+3}\frac{x^{2n+1}}{(2n+3)!} = \frac{\frac{1}{3}x}{1-\frac{2}{1\cdot 5}x^2} - \frac{\frac{1}{3\cdot 5^2\cdot 7}x^4}{1-\frac{2}{5\cdot 9}x^2} - \frac{\frac{1}{7\cdot 9^2\cdot 11}x^4}{1-\frac{2}{9\cdot 11}x^2} - \cdots \tag{F14}$$

$$\tan(x) = \frac{x}{1-\frac{1}{3}x^2} - \frac{\frac{1}{1\cdot 3^2\cdot 5}x^4}{1-\frac{2}{3\cdot 7}x^2} - \frac{\frac{1}{5\cdot 7^2\cdot 9}x^4}{1-\frac{2}{7\cdot 11}x^2} - \frac{\frac{1}{9\cdot 11^2\cdot 13}x^4}{1-\frac{2}{11\cdot 15}x^2} - \cdots \tag{F15}$$

$$\sum_{n\ge 0} E_{n+1}\frac{x^{n}}{(n+1)!} = \frac{1}{1-\frac{1}{2}x} - \frac{\frac{1}{2\cdot 6}x^2}{1} - \frac{\frac{1}{6\cdot 10}x^2}{1} - \frac{\frac{1}{10\cdot 14}x^2}{1} - \cdots \tag{F16}$$

$$\tan(x)+\sec(x) = \frac{1}{1-x} + \frac{\frac{1}{2}x^2}{1+\frac{2}{3}x} + \frac{\frac{1}{6^2}x^2}{1-\frac{4}{3\cdot 5}x} + \frac{\frac{1}{10^2}x^2}{1+\frac{6}{5\cdot 7}x} + \cdots \tag{F17}$$

$$\sum_{n\ge 0} E_{n+2}\frac{x^{n}}{(n+2)!} = \frac{\frac{1}{2}}{1-\frac{2}{3}x} + \frac{\frac{1}{6^2}x^2}{1+\frac{4}{3\cdot 5}x} + \frac{\frac{1}{10^2}x^2}{1-\frac{6}{5\cdot 7}x} + \frac{\frac{1}{14^2}x^2}{1+\frac{8}{7\cdot 9}x} + \cdots \tag{F18}$$

$$\sum_{n\ge 0} E_{n+3}\frac{x^{n}}{(n+3)!} = \frac{\frac{1}{3}}{1-\frac{5}{8}x} - \frac{\frac{3}{320}x^2}{1+\frac{13}{72}x} - \frac{\frac{16}{2835}x}{1-\frac{25}{288}x} - \cdots \tag{F19}$$

The general patterns of the coefficients a_j and b_j are:

$$a_1 = \frac{1}{3},\ a_j = -\frac{(j-1)^2(j+1)^2x^2}{4j^4(2j-1)(2j+1)};\ b_0 = 0,\ b_j = 1 + (-1)^j\frac{(2j^2+2j+1)x}{2j^2(j+1)^2}.$$

$$\sum_{n\ge 0} E_{n+4}\frac{x^{n}}{(n+4)!} = \frac{\frac{5}{24}}{1-\frac{16}{25}x} + \frac{\frac{11}{3750}x^2}{1+\frac{432}{1925}x} + \frac{\frac{475}{142296}x^2}{1-\frac{640}{4389}x} + \cdots \tag{F20}$$

The general patterns for the coefficients a_j and b_j are:

$$a_1 = \frac{5}{24}, \quad a_j = \frac{\big(j^2+3\,j+1\big)\big(j^2-j-1\big)(j+2)(j-1)\cdot x^2}{4\,(j^2+j-1)^2(2\,j+1)^2(j+1)j};$$

$$b_0 = \frac{1}{3}, \quad b_j = 1 + \frac{2\,(-1)^j\,(j+2)(j+1)^3 j\cdot x}{(j^2+3\,j+1)(j^2+j-1)(2\,j+3)(2\,j+1)}.$$

(F21) $$\sum_{n\ge 0} E_{2n+3}\frac{x^{2n}}{(2n+3)!} = \frac{\frac{1}{3}}{1} - \frac{\frac{2}{5}x^2}{1} - \frac{\frac{1}{210}x^2}{1} - \frac{\frac{5}{126}x^2}{1} - \cdots$$

The general patterns for the coefficients a_j are:

$$a_{2k} = \frac{-(2k+1)(k+1)}{(4k-1)(4k+1)(2k-1)k},$$

$$a1 = \frac{1}{3}; \quad a_{2k+1} = \frac{-k(2k-1)}{(4k+1)(4k+3)(2k+1)(k+1)}.$$

(F22) $$\sum_{n\ge 0} E_{2n+3}\frac{x^{n}}{(2n+3)!} = \frac{\frac{1}{3}}{1-\frac{2}{1\cdot 5}x} - \frac{\frac{1}{3\cdot 5^2\cdot 7}x^2}{1-\frac{2}{5\cdot 9}x} - \frac{\frac{1}{7\cdot 9^2\cdot 11}x^2}{1-\frac{2}{9\cdot 11}x} - \cdots$$

(F23) $$\sum_{n\ge 0} E_{2n+5}\frac{x^{n}}{(2n+5)!} = \frac{\frac{2}{15}}{1-\frac{17}{42}x} - \frac{\frac{1}{5292}x^2}{1-\frac{101}{2310}x} - \frac{\frac{56}{1061775}x^2}{1-\frac{73}{4620}x} - \cdots$$

The general patterns for the coefficients a_j and b_j are:

$$a_1 = \frac{2}{15}, \quad a_j = -\frac{(2\,j+1)(2\,j-3)(j+1)(j-1)x^2}{(4\,j+1)(4\,j-1)^2(4\,j-3)(2\,j-1)^2 j^2};$$

$$b_0 = 0, \quad b_j = 1 - \frac{(8\,j^4+8\,j^3+22\,j^2+10\,j+3)x}{(4\,j+3)(4\,j-1)(2\,j+1)(2\,j-1)(j+1)j}.$$

(F24) $$\sum_{n\ge 0} E_{2n+7}\frac{x^{n}}{(2n+7)!} = \frac{\frac{17}{315}}{1-\frac{62}{153}x} - \frac{\frac{26}{1287495}x^2}{1-\frac{1150}{25857}x} - \cdots$$

The general patterns for the coefficients a_j and b_j are:

$$a_j = \frac{-\big(4\,j^2+10\,j+3\big)\big(4\,j^2-6\,j-1\big)(2\,j+3)(2\,j-3)(j+2)(j-1)x^2}{(4\,j^2+2\,j-3)^2(4\,j+3)(4\,j+1)^2(4\,j-1)(2\,j+1)(2\,j-1)(j+1)j};$$

$$a_1 = \frac{17}{315}; \quad b_0 = 0,\ b_j = 1 - \frac{2\,\big(16\,j^4+48\,j^3+164\,j^2+192\,j+45\big)x}{(4\,j^2+10\,j+3)(4\,j^2+2\,j-3)(4\,j+5)(4\,j+1)}.$$

Proof. (F12) Apply the equivalence transformations [56, p. 19] on the Lambert continued fraction (6.1) and divide by x.

(F13) Replace x^2 by x in (F12) we get

(f13) $$\sum E_{2n+1}\frac{x^{n}}{(2n+1)!} = \frac{1}{1} - \frac{\frac{1}{1\cdot 3}x}{1} - \frac{\frac{1}{3\cdot 5}x}{1} - \frac{\frac{1}{5\cdot 7}x}{1} - \cdots$$

Then, even contraction on (f13) yields (F13).

(F14) Odd contraction on (F12); subtract by 1; divide by x.

(F15) Replace x by x^2 in (F13), and multiply by x. We get the H-fraction (F15).

(F16) By Theorem 3.1 with the weight W_2, we have $E_n = P_n(1/2, 0)$. Hence, with the specialization $u_2 = 1/2, u_3 = 0, u_1 = u_4 = 1$ in Theorem 6.1, we have $c = 1/2, c^2 - u_1u_2 = 1/4 - 1/2 = -1/4$. Identity (6.3) becomes.

$$\sum_{n\geq 1} E_n \frac{x^n}{n!} = \frac{x}{1-\frac{1}{2}x} - \frac{\frac{1}{4}x^2}{3} - \frac{\frac{1}{4}x^2}{5} - \frac{\frac{1}{4}x^2}{7} - \cdots$$

Divide the above continued fraction by x, and normalize to J-fraction by equivalence transformations [56, p. 19], we get (F16).

(F17) We claim that

$$\tan(x) + \sec(x) = \frac{1}{1} - \frac{x}{1} + \frac{\frac{1}{2}x}{1} + \frac{\frac{1}{6}x}{1} - \frac{\frac{1}{6}x}{1} - \frac{\frac{1}{10}x}{1} + \cdots \tag{f17}$$

The general patterns for the coefficients a_j are:

$$a_1 = 1, \quad a_2 = -x,$$
$$a_{4k} = \frac{x}{8k-2}, \quad a_{4k+1} = -\frac{x}{8k-2}, \quad a_{4k+2} = -\frac{x}{8k+2}, \quad a_{4k+3} = \frac{x}{8k+2}.$$

Odd contraction on (f17), subtract by 1, divide by x, we get (F16). Hence, (f17) is true. Even contraction on (f17) implies (F17).

(F18) Let $F(x) = \tan(x) + \sec(x)$ and $G(x) = ((F(x) - 1)/x - 1)/x$. By (F17), it suffices to verify that

$$F(-x) = \frac{1}{1 + x + x^2G(x)}.$$

This is true, since $F(x)F(-x) = 1$.

(F19) We have

$$\sum_{n\geq 0} E_{n+2} \frac{x^n}{(n+2)!} = \frac{\frac{1}{2}}{1} - \frac{\frac{2}{3}x}{1} + \frac{\frac{1}{24}x}{1} + \frac{\frac{9}{40}x}{1} - \cdots \tag{f18}$$

The general patterns for the coefficients a_j are:

$$a_1 = 1/2, \quad a_{2j} = \frac{(-1)^j(j+1)^2x}{2j^2(2j+1)}, \quad a_{2j+1} = \frac{(-1)^{j+1}j^2x}{2(j+1)^2(2j+1)},$$

since, even contraction on (f18) implies (F18). Now, odd contraction on (f18), subtract by 1/2, divide by x, we get (F19).

(F20) We have

$$\sum_{n\geq 0} E_{n+3} \frac{x^n}{(n+3)!} = \frac{\frac{1}{3}}{1} - \frac{\frac{5}{8}x}{1} - \frac{\frac{3}{200}x}{1} + \frac{\frac{44}{225}x}{1} + \cdots \tag{f19}$$

The general patterns for the coefficients a_j are:

$$a_{2j} = \frac{(-1)^j j(j+2)(j^2+3j+1)x}{2(j^2+j-1)(2j+1)(j+1)^2},$$
$$a_1 = 1/3, \qquad a_{2j+1} = \frac{(-1)^j j(j+2)(j^2+j-1)x}{2(j^2+3j+1)(2j+3)(j+1)^2},$$

since, even contraction on (f19) implies (F19). Now odd contraction on (f19), subtract by 1/3, divide by x, we get (F20).

(F21) Even contraction on (F21) is the same as (F14) divided by x.

(F22) From (F14), divide by x, replace x^2 by x.

(F23) Replace x^2 by x in (F21), we get the S-fraction:

$$\sum_{n\geq 0} E_{2n+3}\frac{x^n}{(2n+3)!} = \frac{\frac{1}{3}}{1} - \frac{\frac{2}{5}x}{1} - \frac{\frac{1}{210}x}{1} - \frac{\frac{5}{126}x}{1} - \cdots \tag{f22}$$

For the general patterns see (F21). Odd contraction on (f22), subtract by 1/3, divide by x. We get (F23).

(F24) We claim the S-fraction:

$$\sum_{n\geq 0} E_{2n+5}\frac{x^n}{(2n+5)!} = \frac{\frac{2}{15}}{1} - \frac{\frac{17}{42}x}{1} - \frac{\frac{1}{2142}x}{1} - \frac{\frac{364}{8415}x}{1} - \cdots \tag{f23}$$

The general patterns for the coefficients a_j are:

$$a_{2j} = \frac{-(4j+6)(4j+8)(4j^2+10j+3)x}{(4j+1)(4j+2)(4j+3)(4j+4)(4j^2+2j-3)},$$

$$a_1 = \frac{2}{15}, \qquad a_{2j+1} = \frac{-(4j-2)(4j)(4j^2+2j-3)x}{(4j+2)(4j+3)(4j+4)(4j+5)(4j^2+10j+3)}.$$

Even contraction on (f23), we get (F23). So that (f23) is true. Now, odd contraction on (f23), subtract by 2/15, divide by x. We get (F24). □

By using Theorem 2.2, the Hankel continued fractions (F12–F24) listed in Theorem 6.2 imply the Hankel determinants formulas (H12–H24) in the next theorem respectively.

Theorem 6.3. *We have the following formulas for the Hankel determinants.*

(H12) *The Hankel determinants of* $(E_1/1!, 0, E_3/3!, 0, E_5/5!, 0, \ldots)$ *are*

$$H_0 = 1, \quad H_n = 2^{(n-1)^2}\frac{(n-1)!}{(2n-1)!}\prod_{k=1}^{n-1}\frac{(k-1)!^2}{(2k-1)!^2}.$$

(H13) *The Hankel determinants of* $(E_1/1!, E_3/3!, E_5/5!, \ldots)$ *are*

$$H_0 = 1, \quad H_n = 2^{(n-1)(2n-1)}\prod_{k=1}^{2n-2}\frac{k!}{(2k+1)!}.$$

(H14) *The Hankel determinants of* $(0, E_3/3!, 0, E_5/5!, \ldots)$ *are*

$$H_{2n+1} = 0; \quad H_{2n} = (-1)^n 2^{2n(2n-1)}\prod_{k=1}^{2n-1}\frac{k!^2}{(2k+1)!^2}.$$

(H15) *The Hankel determinants of* $(0, E_1/1!, 0, E_3/3!, 0, E_5/5!, \ldots)$ *are*

$$H_{2n+1} = 0; \quad H_0 = 1, \quad H_{2n} = (-1)^n 2^{2(n-1)(2n-1)}\prod_{k=1}^{2n-2}\frac{k!^2}{(2k+1)!^2}.$$

(H16) *The Hankel determinant of* $(E_1/1!, E_2/2!, E_3/3!, E_4/4!, \ldots)$ *are*

$$H_0 = 1, \quad H_n = \frac{(n-1)!}{2^{n-1}(2n-1)!}\prod_{k=1}^{n-2}\frac{k!^2}{(2k+1)!^2}.$$

(H17) *The Hankel determinants of* $(E_0/0!, E_1/1!, E_2/2!, E_3/3!, \ldots)$ *are*

$$H_0 = 1, \quad H_n = (-1)^{n(n-1)/2}\frac{1}{2^{n-1}}\prod_{k=2}^{n-1}\frac{(k-1)!^2}{(2k-1)!^2}.$$

(H18) *The Hankel determinant of* $(E_2/2!, E_3/3!, E_4/4!, \ldots)$ *are*

$$H_n = (-1)^{n(n-1)/2} \frac{1}{2^n} \prod_{k=2}^{n} \frac{(k-1)!^2}{(2k-1)!^2}.$$

(H19) *The Hankel determinants of* $(E_3/3!, E_4/4!, E_5/5!, \ldots)$ *are*

$$H_n = \frac{(n+1)(n+1)!}{2^n(2n+1)!} \prod_{k=1}^{n-1} \frac{k!^2}{(2k+1)!^2}$$

(H20) *The Hankel determinants of* $(E_4/4!, E_5/5!, E_6/6!, \ldots)$ *are*

$$H_n = (-1)^{n(n-1)/2} \frac{(n+1)(n+2)(n^2+3n+1)}{2^{n+1}} \prod_{k=1}^{n} \frac{k!^2}{(2k+1)!^2}.$$

(H21) *The Hankel determinants of* $(E_3/3!, 0, E_5/5!, \ldots)$ *are*

$$H_{2n} = 2^{4n^2-1} \frac{(2n+2)!}{(4n+1)!} \prod_{k=1}^{2n-1} \frac{k!^2}{(2k+1)!^2},$$

$$H_{2n+1} = 2^{4n(n+1)} \frac{(2n+1)(2n+2)!}{(4n+3)!} \prod_{k=1}^{2n} \frac{k!^2}{(2k+1)!^2}.$$

(H22) *The Hankel determinants of* $(E_3/3!, E_5/5!, E_7/7!, \ldots)$ *are*

$$H_n = 2^{n(2n-1)} \prod_{k=1}^{2n-1} \frac{k!}{(2k+1)!}.$$

(H23) *The Hankel determinants of* $(E_5/5!, E_7/7!, \ldots)$ *are*

$$H_n = 2^{n(2n+1)}(n+1)(2n+1) \prod_{k=1}^{2n} \frac{k!}{(2k+1)!}.$$

(H24) *The Hankel determinants of* $(E_7/7!, E_9/9!, \ldots)$ *are*

$$H_n = 2^{n(2n+3)}(2n+1)(4n^2+10n+3) \frac{(n+1)(n+2)(2n+3)}{3} \prod_{k=1}^{2n+1} \frac{k!}{(2k+1)!}.$$

7. New q-analog of the Euler numbers

There exist three kinds of q-analogs of the tangent and secant numbers, defined via (i) the q-sine and q-cosine functions introduced by Jackson [24, 12, 3, 45]; (ii) Lambert's continued fraction (6.1) of $\tan(x)$ (see [17, 44]); (iii) The continued fractions (1.3) and (1.4) of the ordinary generating functions of these numbers [27, 50, 19, 26].

Let $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$, $[n]!_q = [1]_q[2]_q \cdots [n]_q$ and

$$\binom{n}{k}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}.$$

Version (iii) of the q-secant and q-tangent numbers are defined by

$$\sum_{n\geq 0} \hat{E}_{2n}(q)x^{2n} = \frac{1}{1} - \frac{[1]_q^2x^2}{1} - \frac{[2]_q^2x^2}{1} - \frac{[3]_q^2x^2}{1} - \cdots \tag{7.1}$$

$$\sum_{n\geq 0}\hat{E}_{2n+1}(q)x^{2n+1} = \frac{x}{1} - \frac{[1]_q\cdot[2]_q x^2}{1} - \frac{[2]_q\cdot[3]_q x^2}{1} - \cdots \tag{7.2}$$

In the same manner, we now use our theorem 4.2 to define a *new q-analog* of the Euler numbers as follows

$$\sum_{n\geq 0}E_n(q)x^n = 1+\frac{x}{1-x} - \frac{\binom{2}{2}_q x^2}{1-[2]_q x} - \frac{\binom{3}{2}_q x^2}{1-[3]_q x} - \frac{\binom{4}{2}_q x^2}{1-[4]_q x} - \cdots \tag{7.3}$$

The general pattern for the coefficients a_k and b_k are:

$$a_1 = x,\quad a_k = -\binom{k}{2}_q x^2;\qquad b_0=1,\quad b_k = 1-[k]_q x.$$

The first values of $E_n(q)$ are listed below:

$$\begin{aligned} &E_0(q)=E_1(q)=E_2(q)=1,\quad E_3(q)=2,\\ &E_4(q)=q+4,\\ &E_5(q)=2q^2+5q+9,\\ &E_6(q)=q^4+5q^3+14q^2+20q+21.\end{aligned}$$

The specializations for $q=1,0,-1$ are

n	=	0	1	2	3	4	5	6	7	8	9
$E_n(1)$	=	1	1	1	2	5	16	61	272	1385	7936
$E_n(0)$	=	1	1	1	2	4	9	21	51	127	323
$E_n(-1)$	=	1	1	1	2	3	6	11	24	51	122

Of course $(E_n(1))_{n\geq 0}$ are just the Euler numbers. Also, it is easy to see that $(E_n(0))_{n\geq 1}$ are the Motzkin numbers (see [9, Proposition 5], [51, p. 238]) . The most interesting case is $q=-1$. We have a non-trivial explicit formula for $E_n(-1)$, as stated next.

Theorem 7.1. *We have $E_0(-1)=1$ and*

$$E_n(-1) = \sum_{k=0}^{n-1}\binom{n-k-1}{k}k! \tag{7.4}$$

or equivalently,

$$\sum_{n\geq 0}\sum_{k=0}^{n}\binom{n-k}{k}k!x^n = \frac{1}{1-x} - \frac{x^2}{1} - \frac{x^2}{1-x} - \frac{2x^2}{1} - \frac{2x^2}{1-x} - \cdots \tag{7.5}$$

The general patterns for the new coefficients a_j and b_j are:

$$a_1=1,\ a_{2k}=-kx^2,\ a_{2k+1}=-kx^2;\quad b_0=0,\ b_{2k}=1,\ b_{2k+1}=1-x.$$

Proof. To prove identity (7.5), we need to guess a unified property for the following continued fraction with one more parameter u:

$$F_u(x) = \frac{1}{1-x} - \frac{(u+1)x^2}{1} - \frac{(u+1)x^2}{1-x} - \frac{(u+2)x^2}{1} - \frac{(u+2)x^2}{1-x} - \cdots$$

It is clear that $F_0(x)$ is the continued fraction in (7.5). We claim that $F_u(x)$ satisfies the following differential equation

$$(x-1)x^3F_u'(x) - (x-1)(x-2)x^2uF_u(x)^2 + (x-1)(2x^2+x-2)F_u(x) + (x-2) = 0.$$

Since we do not have efficient tools for guessing the above equation with the parameter u, our method consists of two steps: first guessing formulas for specific values $u = 0, 1, 2, 3, \ldots$, then finding a unified pattern. This method was fully detailed with another example in [20].

To prove the above differential equation we let $S_u(x)$ be its left-hand side. By using the following relation between $F_u(x)$ and $F_{u+1}(x)$

$$F_u(x) = \cfrac{1}{1 - x - \cfrac{(u+1)x^2}{1-(u+1)x^2F_{u+1}(x)}},$$

we can express $S_u(x)$ in function of $F_{u+1}(x)$ and $F'_{u+1}(x)$. After simplification we obtain

$$S_u(x) = \frac{(u+1)^2x^4S_{u+1}(x)}{\left((u+1)(x-1)x^2F_{u+1}(x) + (1-x-(u+1)x^2)\right)^2}. \tag{7.6}$$

Since the constant term in x of the denominator in the above fraction is equal to 1, applying relation (7.6) iteratively yields that $S_u(x) = 0$. Notice that we cannot prove that $S_u(x) = 0$ by induction, since the base case for induction would be $S_\infty(x) = 0$. This is impossible to prove because $F_\infty(x)$ does not exist. Now take $u = 0$. We have $S_0(x) = 0$, or

$$(x-1)x^3F'_0(x) + (x-1)(2x^2+x-2)F_0(x) + (x-2) = 0.$$

Write $\alpha_n := E_{n+1}(-1)$ for short. Comparing the coefficient of x^n in the above equation, we know that the α_n's satisfy the recurrence relation

$$2\alpha_n = 3\alpha_{n-1} + (n-1)\alpha_{n-2} - (n-1)\alpha_{n-3}, \tag{7.7}$$

with initial values $\alpha_0 = \alpha_1 = 1$ and $\alpha_2 = 2$. Finally we can prove (7.1) by Zeilberger's algorithm [43]. □

Remark. Under the sequence A122852 in the OEIS [15], formula (7.4) is given by Paul Barry without proof and reference; as well as recurrence (7.7) is stated by R. J. Mathar as a conjecture. The above proofs of (7.4) and (7.7) are unexpectedly non-trivial.

Applying Heilermann's formula (2.9) to the J-fraction (7.5) we obtain

$$\det\left(\sum_{k=0}^{i+j}\binom{i+j-k}{k}k!\right)_{i,j=0}^{2n-1} = \prod_{k=1}^{n}(k-1)!^3k!$$

and

$$\det\left(\sum_{k=0}^{i+j}\binom{i+j-k}{k}k!\right)_{i,j=0}^{2n} = \prod_{k=1}^{n}(k-1)!k!^3.$$

The continued fractions (7.1) and (7.2) lead to several combinatorial interpretations of the polynomials $\hat{E}_{2n}(q)$ and $\hat{E}_{2n+1}(q)$, see [27, 50, 19, 26]. It would be interesting to find a combinatorial model for the new q-Euler numbers $E_q(n)$.

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