UNCONDITIONAL REFLEXIVE POLYTOPES

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ABSTRACT. A convex body is unconditional if it is symmetric with respect to reflections in all coordinate hyperplanes. In this paper, we investigate unconditional lattice polytopes with respect to geometric, combinatorial, and algebraic properties. In particular, we characterize unconditional reflexive polytopes in terms of perfect graphs. As a prime example, we study the signed Birkhoff polytope. Moreover, we derive constructions for Gale-dual pairs of polytopes and we explicitly describe Gröbner bases for unconditional reflexive polytopes coming from partially ordered sets.

1. INTRODUCTION

A d-dimensional convex lattice polytope $P \subset \mathbb{R}^d$ is called **reflexive** if its polar dual P^* is again a lattice polytope. Reflexive polytopes were introduced by Batyrev [Bat94] in the context of mirror symmetry as a reflexive polytope and its dual give rise to a mirror-dual pair of Calabi–Yau manifolds (c.f. [Cox15]). As thus, the results of Batyrev, and the subsequent connection with string theory, have stimulated interest in the classification of reflexive polytopes both among mathematical and theoretical physics communities. As a consequence of a well-known result of Lagarias and Ziegler [LZ91], there are only finitely many reflexive polytopes in each dimension, up to unimodular equivalence. In two dimensions, it is a straightforward exercise to verify that there are precisely 16 reflexive polygons, as depicted in Figure 1. While still finite, there are significantly more reflexive polytopes in higher dimensions. Kreuzer and Skarke [KS98, KS00] have completely classified reflexive polytopes in dimensions 3 and 4, noting that there are exactly 4319 reflexive polytopes in dimension 3 and 473800776 reflexive polytopes in dimension 4. The number of reflexive polytopes in dimension 5 is not known.

In recent years, there have been a number of results characterizing reflexive polytopes in known classes of polytopes coming from combinatorics or optimization; see, for example, [BHS09, Tag10, Ohs14, HMT15, CFS17]. The purpose of this paper is to study a class of reflexive polytopes motivated by convex geometry and relate it to combinatorics. A convex body $K \subset \mathbb{R}^d$ is **unconditional** if $\boldsymbol{p} \in K$ if and only if $\sigma\boldsymbol{p} := (\sigma_1 p_1, \sigma_2 p_2, \dots, \sigma_d p_d) \in K$ for all $\sigma \in \{-1,+1\}^d$. Unconditional convex bodies, for example, arise as unit balls in the theory of Banach spaces with a 1-unconditional basis. They constitute a restricted yet surprisingly interesting class of convex bodies for which a number of claims have been verified; cf. [BGVV14]. For example, we mention that the *Mahler conjecture* is known to hold for unconditional convex bodies; see Section 3. In this paper, we investigate unconditional lattice polytopes and their relation to anti-blocking polytopes from combinatorial optimization. In particular, we completely characterize unconditional reflexive polytopes.

The structure of this paper is as follows. In Section 2, we briefly review notions and results from discrete geometry and Ehrhart theory.

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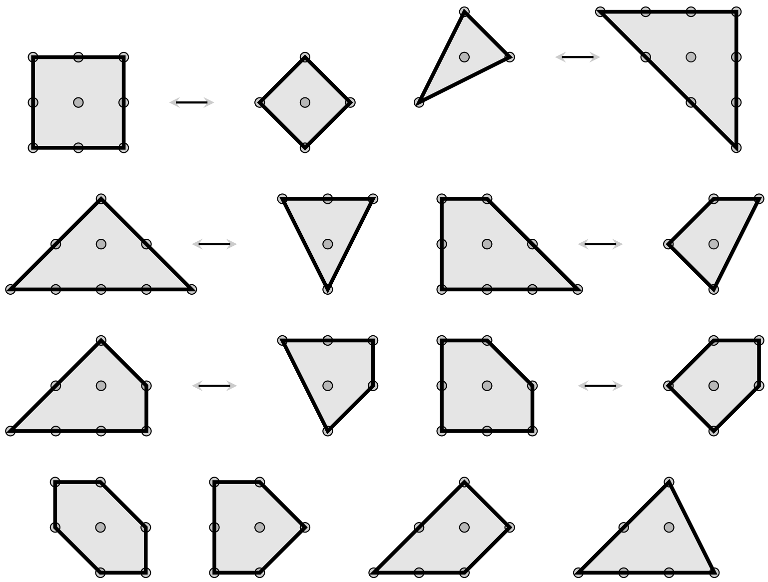

FIGURE 1. All 16 reflexive 2-dimensional polytopes. This is Figure 1.5 in [Hof18].

In Section 3, we introduce and study unconditional and, more generally, locally anti-blocking polytopes. The main result is Theorem 3.2 that relates regular, unimodular, and flag triangulations to the associated anti-blocking polytopes.

In Section 4, we associate an unconditional lattice polytope $\mathsf{U}P_G$ to every finite graph G. We show in Theorems 4.6 and 4.9 that an unconditional polytope P is reflexive if and only if $P = \mathsf{U}P_G$ for some unique perfect graph G. This also implies that unconditional reflexive polytopes have regular, unimodular triangulations.

Section 5 is devoted to a particular family of unconditional reflexive polytopes and is of independent interest: We show that the **type-B Birkhoff polytope** or **signed Birkhoff polytope** $\mathcal{BB}(n)$, that is, the convex hull of signed permutation matrices, is an unconditional reflexive polytope. We compute normalized volumes and h^*-vectors of $\mathcal{BB}(n)$ and its dual $\mathcal{C}(n) = \mathcal{BB}(n)^*$ for small values of n.

The usual Birkhoff polytope and the Gardner polytope of [FHSS] appear as faces of $\mathcal{BB}(n)$ and $\mathcal{C}(n)$, respectively. These two polytopes form a *Gale-dual pair* in the sense of [FHSS]. In Section 6, we give a general construction for compressed Gale-dual pairs coming from CIS graphs.

In Section 7, we investigate unconditional polytopes associated to comparability graphs of posets. In particular, we explicitly describe a quadratic square-free Gröbner basis for the corresponding toric ideal.

We close with open questions and future directions in Section 8.

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2. BACKGROUND

In this section, we provide a brief introduction to polytopes and Ehrhart theory. For additional background and details, we refer the reader to the excellent books [BR15, Zie95]. A **polytope** in $\mathbb{R}^d$

is the inclusion-minimal convex set $P = \operatorname{conv}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ containing a given collection of points $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{R}^d$. If $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n \in \mathbb{Z}^d$, then P is called a **lattice polytope**. The unique inclusion-minimal set $V \subseteq P$ such that $P = \operatorname{conv}(V)$ is called the **vertex set** and is denoted by $V(P)$. By the Minkowski–Weyl theorem, polytopes are precisely the bounded sets of the form

$$P \;=\; \{\boldsymbol{x} \in \mathbb{R}^d : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \le b_i \text{ for } i = 1, \dots, m\}$$

for some $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in \mathbb{R}^d$ and $b_1, \dots, b_m \in \mathbb{R}$. If $\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \le b_i$ is irredundant, then $F = P \cap \{\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle = b_i\}$ is a facet and the inequality is facet-defining.

The **dimension** of a polytope P is defined to be the dimension of its affine span. A d-dimensional polytope has at least $d+1$ vertices and a d-polytope with exactly $d+1$ many vertices is called a d**-simplex**. A d-simplex $\Delta = \operatorname{conv}\{\boldsymbol{v}_0, \boldsymbol{v}_1, \dots, \boldsymbol{v}_d\}$ is called **unimodular** if $\boldsymbol{v}_1 - \boldsymbol{v}_0$, $\boldsymbol{v}_2 - \boldsymbol{v}_0, \dots,$ $\boldsymbol{v}_d - \boldsymbol{v}_0$ form a basis for the lattice $\mathbb{Z}^d$, or equivalently if $\operatorname{vol}(\Delta) = \frac{1}{d!}$, where vol is the Euclidean volume. For lattice polytopes $P \subset \mathbb{R}^d$, we define the **normalized volume** $\operatorname{Vol}(P) := d! \operatorname{vol}(P)$. So unimodular simplices are the lattice polytopes with normalized volume 1. We say that two lattice polytopes $P, P' \subset \mathbb{R}^d$ are **unimodularly equivalent** if $P' = T(P)$ for some transformation $T(\boldsymbol{x}) = W\boldsymbol{x} + \boldsymbol{v}$ with $W \in \mathrm{SL}_d(\mathbb{Z})$ and and $\boldsymbol{v} \in \mathbb{Z}^d$. In particular, any two unimodular simplices are unimodularly equivalent.

Given a lattice d-polytope P and $t \in \mathbb{Z}_{\ge 1}$, let $tP := \{t \cdot \boldsymbol{x} \,:\, \boldsymbol{x} \in P\}$ be the t^{th} dilate of P. The lattice-point enumeration function

$$\operatorname{ehr}_P(t) \;:=\; |tP \cap \mathbb{Z}^d|.$$

is called the **Ehrhart polynomial**. By a famous result of Ehrhart [Ehr62, Thm. 1], this function agrees with a polynomial in the variable t of degree d with leading coefficient $\operatorname{vol}(P)$.

This also implies that the formal generating function

$$1 + \sum_{t \ge 1} \operatorname{ehr}_P(t) z^t \;=\; \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1-z)^{d+1}}$$

is a rational function with denominator $(1-z)^{d+1}$ and that the degree of the numerator is at most d (see, e.g., [BR15, Lem 3.9]). We call the numerator the h^***-polynomial** of P. The coefficient vector $h_P^* = (h_0^*, h_1^*, \dots, h_d^*) \in \mathbb{Z}^{d+1}$ is called the h^***-vector** of P. One should note that the Ehrhart polynomial is invariant under unimodular transformations.

Theorem 2.1 ([Sta80, Sta93])**.** *Let $P \subseteq Q$ be a lattice polytopes. Then*

$$0 \;\le\; h_i^*(P) \;\le\; h_i^*(Q)$$

for all $i = 0, \dots, d$.

The h^*-vector encodes a lot of information about the underlying polytope. This is nicely illustrated in the case of reflexive polytopes. For a d-polytope $P \subset \mathbb{R}^d$ with 0 in the interior, we define the **(polar) dual polytope**

$$P^* \;:=\; \{\mathbf{y} \in \mathbb{R}^d \;:\; \langle \mathbf{y}, \boldsymbol{x} \rangle \le 1 \text{ for all } \boldsymbol{x} \in P\}\,.$$

Definition 2.2. Let $P \subset \mathbb{R}^d$ be a d-dimensional lattice polytope that contains the origin in its interior. We say that P is **reflexive** if P^* is also a lattice polytope. Equivalently, P is reflexive if it has a description of the form

$$P \;=\; \{\boldsymbol{x} \in \mathbb{R}^d \colon A\boldsymbol{x} \le 1\},$$

where A is an integral matrix.

Quite surprisingly, reflexivity can be completely characterized by enumerative data of the h^*-vector.

Theorem 2.3 ([Hib92, Thm. 2.1])**.** *Let $P \subset \mathbb{R}^d$ be a d-dimensional lattice polytope with $h^*(P) = (h_0^*, \dots, h_d^*)$. Then P is unimodularly equivalent to a reflexive polytope if and only if $h_k^* = h_{d-k}^*$ for all $0 \le k \le \lfloor \frac{d}{2} \rfloor$.*

The reflexivity property is also deeply related to commutative algebra. A polytope P is reflexive if the canonical module of the associated graded algebra $\Bbbk[P]$ is (up to a shift in grading) isomorphic to $\Bbbk[P]$ and its minimal generator has degree 1. If one allows the unique minimal generator to have arbitrary degree, one arrives at the notion of Gorenstein rings, for details we refer to [BG09, Sec 6.C]. We say that P is **Gorenstein** if there exists a $c \in \mathbb{Z}_{\ge 1}$ such that cP is unimodularly equivalent to a reflexive polytope. This is equivalent to saying that $\Bbbk[P]$ is Gorenstein. The dilation factor c is often called the **codegree**. In particular, reflexive polytopes are Gorenstein of codegree 1. By combining results of Stanley [Sta78] and De Negri–Hibi [DNH97], we have a characterization of the Gorenstein property in terms of the h^*-vector. Namely, P is Gorenstein if and only if $h_i^* = h_{d-c+1-i}^*$ for all i.

Aside from examining algebraic properties of lattice polytopes, one can also investigate discrete geometric properties. Every lattice polytope admits a subdivision into lattice simplices. Even more, one can guarantee that every lattice point contained in a polytope corresponds to a vertex of such a subdivision. However, one cannot guarantee the existence of a subdivision where all simplices are unimodular when the dimension is greater than 2. This leads us to our next definition:

Definition 2.4. A **triangulation** $\mathcal{T}$ of a lattice d-dimensional polytope P with vertices in V is a collection of lattice d-dimensional simplices with vertices in V covering P and such that any two simplices meet in a common face. We call $\mathcal{T}$ unimodular if all simplices are unimodular.

A triangulation $\mathcal{T}$ is **regular** if there is a convex, piecewise-linear function $\overline{\omega} : P \to \mathbb{R}$ whose domains of linearity are exactly the simplices in $\mathcal{T}$. Such a function is completely described by assigning values $\omega(v)$ for $v \in V$.

A triangulation is **flag** if the inclusion-minimal sets of vertices not forming a face in $\mathcal{T}$ are all of cardinality 2.

Given a lattice polytope P, a **pulling triangulation** is a triangulation obtained by a sequence of **pulling refinements**. Given $\boldsymbol{v} \in P \cap \mathbb{Z}^d$ and a lattice subdivision $\mathcal{S}$, $\mathrm{pull}_{\boldsymbol{v}} P$ is the refined lattice subdivision induced by replacing every face $F \in \mathcal{S}$ such that $v \in F$ with the pyramids $\mathrm{conv}(\boldsymbol{v}, F')$, for each face F' of F that does not contain $\boldsymbol{v}$. Such refinements preserve regularity and thus a triangulation constructed by a sequence of pulling refinements is a regular triangulation. The reader should consult [DLRS10, HPPS14] for more details.

A special class of polytopes which possess regular, unimodular triangulations are compressed polytopes. A polytope P is **compressed** if every pulling triangulation is unimodular [Sta80]. In the interest of providing a useful characterization of compressed polytopes, we must define the notion of width of a facet. Let $P \subset \mathbb{R}^d$ be a d-dimensional lattice polytope and $F_i = P \cap \{\boldsymbol{x} : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle = b_i\}$ a facet. We assume that $\boldsymbol{a}_i$ is primitive, that is, its coordinates are coprime. The **width** of F_i is

$$\max_{\boldsymbol{p} \in P} \langle \boldsymbol{a}_i, \boldsymbol{p} \rangle - \min_{\boldsymbol{p} \in P} \langle \boldsymbol{a}_i, \boldsymbol{p} \rangle \, .$$

Theorem 2.5 ([OH01, Thm. 1.1] [Sul06, Thm. 2.4])**.** *Let $P \subset \mathbb{R}^d$ be a full-dimensional lattice polytope. The following are equivalent:*

(1) P is compressed;
(2) P has width one with respect to all its facets;
(3) P is unimodularly equivalent to the intersection of a unit cube with an affine space.

Definition 2.6. A lattice polytope P has the **integer decomposition property** (IDP) if for any positive integer t and for all $\boldsymbol{x} \in tP \cap \mathbb{Z}^d$, there exists $\boldsymbol{v}_1, \dots, \boldsymbol{v}_t \in P \cap \mathbb{Z}^d$ such that $\boldsymbol{x} = \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_d$.

One should note that if P has a unimodular triangulation, then P has the IDP. However, there are examples of polytopes which have the IDP, yet do not even admit a **unimodular cover**, that is, a covering of P by unimodular simplices, see [BG99, Sec. 3]. A more complete hierarchy of covering properties can be found in [HPPS14].

We say that h^*_P is **unimodal** if there exists a k such that $h^*_0 \le h^*_1 \le \cdots \le h^*_k \ge \cdots \ge h^*_{d-1} \ge h^*_d$. Unimodality appears frequently in combinatorial settings and it often hints at a deeper underlying algebraic structure, see [AHK18, Bre94, Sta89]. One famous instance is given by Gorenstein polytopes that admit a regular, unimodular triangulation.

Theorem 2.7 ([BR07, Thm. 1])**.** *If P is Gorenstein and has a regular, unimodular triangulation, then h^*_P is unimodal.*

The following conjecture is commonly attributed to Ohsugi and Hibi [OH06]:

Conjecture 2.8. *If P is Gorenstein and has the IDP, then h^*_P is unimodal.*

3. Unconditional and anti-blocking polytopes

For $\sigma \in \{-1,+1\}^d$ and $\boldsymbol{p} \in \mathbb{R}^d$, let us write $\sigma\boldsymbol{p} = (\sigma_1 p_1, \sigma_2 p_2, \ldots, \sigma_d p_d)$. A convex polytope $P \subseteq \mathbb{R}^d$ is called 1**-unconditional** or simply **unconditional** if $\boldsymbol{p} \in P$ implies $\sigma\boldsymbol{p} \in P$ for all $\sigma \in \{-1,+1\}^d$. So, P is a polytope that is symmetric with respect to all coordinate hyperplanes. It is apparent that P can be recovered from its restriction to the first orthant, which we denote by $P_+ = P \cap \mathbb{R}^d_+$. The polytope P_+ has the property that for any $\boldsymbol{q} \in P_+$ and $\boldsymbol{p} \in \mathbb{R}^d$ with $0 \le p_i \le q_i$ for all i, it holds that $\boldsymbol{p} \in P_+$. Polytopes in $\mathbb{R}^d_+$ with this property are called **anti-blocking polytopes**. Anti-blocking polytopes were studied and named by Fulkerson [Ful71, Ful72] in the context of combinatorial optimization, but they are also known as *convex corners* or *down-closed* polytopes; see, for example, [BB00].

Let us also write $\overline{\boldsymbol{p}} = (|p_1|, |p_2|, \ldots, |p_d|)$. Given an anti-blocking polytope $Q \subset \mathbb{R}^d_+$ it is straightforward to verify that

$$\mathsf{U}Q \ := \ \{\boldsymbol{p} \in \mathbb{R}^d : \overline{\boldsymbol{p}} \in Q\}$$

is an unconditional convex body. Every full-dimensional anti-blocking polytope has an irredundant inequality description of the form

$$Q \ = \ \{\boldsymbol{x} \in \mathbb{R}^d_+ : \langle \boldsymbol{a}_i, \boldsymbol{x} \rangle \le 1 \text{ for } i = 1, \ldots, m\} \tag{1}$$

for some $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^d_+$. Following [Sch86, Sec. 9.3], we define

$$\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r\}^{\downarrow} \ := \ \mathbb{R}^d_+ \cap (\operatorname{conv}(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r) - \mathbb{R}^d_+)$$

as the smallest anti-blocking polytope containing $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_r \in \mathbb{R}^d_+$. Conversely, if we let $V^{\downarrow}(Q) = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ be the vertices of Q that are maximal with respect to the componentwise order, then $Q = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}^{\downarrow}$. We record the consequences for the unconditional polytopes.

Proposition 3.1. *Let $P \subset \mathbb{R}^d_+$ be an anti-blocking polytope given by* (1)*. Then an irredundant inequality description of $\mathsf{U}P$ is given by the distinct*

$$\langle \sigma\boldsymbol{a}_i, \boldsymbol{x} \rangle \le 1$$

for $i = 1, \ldots, m$ and $\sigma \in \{-1,+1\}^d$. Likewise, the vertices of $\mathsf{U}P$ are $V(\mathsf{U}P) = \{\sigma\boldsymbol{v} : \boldsymbol{v} \in V^{\downarrow}(P), \sigma \in \{-1,+1\}^d\}$.

Our first result relates properties of subdivisions of anti-blocking polytopes to that of the associated unconditional polytopes. The 2^d orthants in $\mathbb{R}^d$ are denoted by $\mathbb{R}^d_\sigma := \sigma\mathbb{R}^d_+$ for $\sigma \in \{-1,+1\}^d$.

Theorem 3.2. *Let $P \subset \mathbb{R}^d_+$ be an anti-blocking polytope with triangulation $\mathcal{T}$. Then*

$$\mathsf{U}\mathcal{T} \ := \ \{\sigma S : S \in \mathcal{T}, \sigma \in \{-1,+1\}^d\}$$

is a triangulation of $\mathsf{U}P$. Furthermore

(i) *If $\mathcal{T}$ is unimodular, then so is* $\mathsf{U}\mathcal{T}$.
(ii) *If $\mathcal{T}$ is regular, then so is* $\mathsf{U}\mathcal{T}$.
(iii) *If $\mathcal{T}$ is flag, then so is* $\mathsf{U}\mathcal{T}$.

Proof. It is clear that $\mathsf{U}\mathcal{T}$ is a triangulation of $\mathsf{U}P$ and statement (i) is obvious. If U is a collection of vertices of $\mathcal{T}$ not contained in $\mathbb{R}^d_\sigma$ for any σ, then there are $u_+, u_- \in U$ that are not contained in the same orthant. Hence if $\mathcal{T}$ is flag, then $\mathsf{U}\mathcal{T}$ is flag, which proves (iii).

To show (ii), assume that $\mathcal{T}$ is regular. Let $\omega : V(P) \to \mathbb{R}$ the corresponding heights. We extend ω to $V := \bigcup_\sigma \sigma V(P)$ by setting $\omega'(\boldsymbol{v}) := \|\boldsymbol{v}\|_1 + \epsilon\,\omega(\overline{\boldsymbol{v}})$, where $\|\boldsymbol{v}\|_1 = \sum_i |v_i|$. For $\epsilon = 0$ it is easy to see that the heights induce a subdivision of P into σP for $\sigma \in \{-1,+1\}^d$. For $\epsilon > 0$ sufficiently small, the heights ω' then induce the triangulation $\sigma\mathcal{T}$ on σP. $\square$

Let us call a polytope $P \subset \mathbb{R}^d$ **locally anti-blocking** if $(\sigma P) \cap \mathbb{R}^d_+$ is an anti-blocking polytope for every $\sigma \in \{-1,+1\}^d$. Unconditional polytopes are clearly locally anti-blocking. It follows from [CFS17, Lemma 3.12] that for any two anti-blocking polytopes $P_1, P_2 \subseteq \mathbb{R}^d_+$, the polytopes

$$P_1 + (-P_2) \qquad \text{and} \qquad P_1 \vee (-P_2) \;=\; \operatorname{conv}(P_1 \cup -P_2)$$

are locally anti-blocking. Locally anti-blocking polytopes are studied in depth in [AASS19]. The following is a simple but important observation.

Lemma 3.3. *Let $P \subset \mathbb{R}^d$ be a locally anti-blocking lattice polytope. Then P is reflexive if and only if $P_\sigma = P \cap \mathbb{R}^d_\sigma$ is compressed for all $\sigma \in \{-1,+1\}^d$.*

Proof. Let $Q \subset \mathbb{R}^d_+$ be a compressed d-dimensional anti-blocking polytope given by irredundant inequalities $\boldsymbol{x} \geq 0$ and

$$\langle \boldsymbol{a}_i, \boldsymbol{x}\rangle \;\leq\; b_i$$

for $i = 1,\ldots,m$ and $\boldsymbol{a}_i \in \mathbb{Z}^d$ primitive. Since $0 \in Q$ is a lattice point, it follows that for any $\boldsymbol{z} \in Q \cap \mathbb{Z}^d$ we have $\langle \boldsymbol{a}_i, \boldsymbol{z}\rangle \in \{0, b_i\}$. Since Q is full-dimensional, we have that the standard basis vectors $\boldsymbol{e}_i = (0,\ldots,0,1,0,\ldots,0)$ are contained in Q and thus $(\boldsymbol{a}_i)_j \in \{0, b_i\}$. Since we assume $\boldsymbol{a}_i$ primitive, it follows that $b_i = 1$.

Hence, from the definition of reflexive polytopes and Proposition 3.1 we infer that P is reflexive if and only if P_σ is of the form $\{\boldsymbol{x} : A^\sigma \boldsymbol{x} \leq 1\}$ for some integer matrix A^σ for every $\sigma \in \{-1,+1\}^d$. $\square$

Theorem 3.4. *If P is a reflexive and locally anti-blocking polytope, then P has a regular and unimodular triangulation. In particular, $h^*(P)$ is unimodal.*

Proof. By Lemma 3.3, every pulling triangulation of $P_\sigma = P \cap \mathbb{R}^d_\sigma$ for $\sigma \in \{0,1\}^d$ is a unimodular triangulation. Let $U = P \cap \mathbb{Z}^d$ and choose an ordering of the points in U such that $\boldsymbol{u}$ comes before $\boldsymbol{v}$ if the support of $\boldsymbol{u}$ is contained in the support of $\boldsymbol{v}$. This gives a consistent pulling order of the vertices of each P_σ. The same argument as in the proof of Theorem 3.2, then shows that the regular subdivision of P into the polytopes P_σ can be refined to a regular and unimodular triangulation $\mathcal{T}$. The unimodality of $h^*(P)$ now follows from Theorem 2.7. $\square$

Remark 3.5. The techniques of this section can be extended to the following class of polytopes. We say that a polytope $P \subset \mathbb{R}^d$ has the **orthant-lattice property** (OLP) if the restriction $P_\sigma := P \cap \mathbb{R}^d_\sigma$ is a (possibly empty) lattice polytope. If P is reflexive, then P_σ is full-dimensional for every σ. Now, if every P_σ has a unimodular cover, then so does P and hence is IDP. Let $P_\sigma = \{\boldsymbol{x} \in \mathbb{R}^d_\sigma : A^\sigma \boldsymbol{x} \leq b^\sigma\}$. Then some conditions that imply the existence of a unimodular cover include:

(1) P_σ is compressed;
(2) A^σ is a totally unimodular matrix;
(3) A^σ consists of of rows which are B_d roots;
(4) P_σ is the product of unimodular simplices;

(5) There exists a projection $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ such that $\pi(P_\sigma)$ has a regular, unimodular triangulation $\mathcal{T}$ such that the pullback subdivision $\pi^*(\mathcal{T})$ is lattice.

We refer to [HPPS14] for background and details.

An example of such a polytope is

$$P = \operatorname{conv}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \end{bmatrix} \subset \mathbb{R}^3.$$

This is a reflexive OLP polytope. The restriction to $\mathbb{R}^3_+$ is

$$P_+ = \operatorname{conv}\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \subset \mathbb{R}^3$$

which is not an anti-blocking polytope.

The **Mahler conjecture** in convex geometry states that every centrally-symmetric convex body $K \subset \mathbb{R}^d$ satisfies

$$\operatorname{vol}(K) \cdot \operatorname{vol}(K^*) \ \geq \ \operatorname{vol}(C_d) \cdot \operatorname{vol}(C_d^*)\,,$$

where $C_d = [-1,1]^d$ is the d-cube. The Mahler conjecture has been verified only in small dimensions and for special classes of convex bodies. In particular, Saint-Raymond [SR80] proved the following beautiful inequality. The characterization of the equality case is independently due to Meyer [Mey86] and Reisner [Rei87].

Theorem 3.6 (Saint-Raymond)**.** *Let $P \subset \mathbb{R}^d_+$ be an anti-blocking polytope. Then*

$$\operatorname{vol}(P) \cdot \operatorname{vol}(A(P)) \ \geq \ \frac{1}{d!}$$

with equality if and only if P or $A(P)$ is the cube $[0,1]^d$.

This inequality directly implies the Mahler conjecture for unconditional convex polytopes, that we record for the normalized volume.

Corollary 3.7. *Let $P \subset \mathbb{R}^d$ be an unconditional reflexive polytope. Then*

$$\operatorname{Vol}(P) \cdot \operatorname{Vol}(P^*) \ \geq \ 4^d d!$$

with equality if and only if P or P^* is the cube $[-1,1]^d$.

4. Unconditional reflexive polytopes and perfect graphs

For $A \subseteq [d]$, let $\mathbf{1}_A \in \{0,1\}^d$ be its characteristic vector. If $\Gamma \subseteq 2^{[d]}$ is a simplicial complex, i.e., a nonempty set system closed under taking subsets, then

$$P \ = \ \operatorname{conv}(\mathbf{1}_\sigma : \sigma \in \Gamma)$$

is an anti-blocking $0/1$-polytope and every anti-blocking polytope with vertices in $\{0,1\}^d$ arises that way. A prominent class of anti-blocking $0/1$-polytopes arises from graphs.

Given a graph $G = ([d], E)$ with $E \subseteq \binom{[d]}{2}$, we say that $S \subseteq [d]$ is a **stable set** (or **independent set**) of G if $uv \notin E$ for any $u, v \in S$. The **stable set polytope** of G is

$$P_G \ := \ \operatorname{conv}\{\mathbf{1}_S \ : \ S \subseteq [d] \text{ stable}\}\,.$$

Stable set polytopes played an important role in the proof of the *weak perfect graph conjecture* [Lov72]. A **clique** is a set $C \subseteq [d]$ such that every two vertices in C are joined by an edge. The **clique number** $\omega(G)$ is the largest size of a clique in G. A graph is **perfect** if $\omega(H) = \chi(H)$ for all induced subgraphs $H \subseteq G$, where $\chi(H)$ is the **chromatic number** of H.

Lovász gave the following geometric characterization of perfect graphs. For a set $C \subseteq [d]$ and $\boldsymbol{x} \in \mathbb{R}^d$, we write $\boldsymbol{x}(C) = \sum_{i \in C} x_i$.

Theorem 4.1. *A graph $G = ([d], E)$ is perfect if and only if*

$$P_G \ = \ \{\boldsymbol{x} \in \mathbb{R}^d_+ : \ \boldsymbol{x}(C) \leq 1 \text{ for all cliques } C \subseteq [d]\}.$$

For an anti-blocking polytope $P \subset \mathbb{R}^d_+$ define the **anti-blocking dual**

$$A(P) \ \coloneqq \ \{\mathbf{y} \in \mathbb{R}^d_+ : \langle \mathbf{y}, \boldsymbol{x} \rangle \leq 1 \text{ for all } \boldsymbol{x} \in P\} \, .$$

The polar $(\mathsf{U}P)^*$ is again unconditional and it follows that

$$(\mathsf{U}P)^* \ = \ \mathsf{U}A(P) \, .$$

Theorem 4.2 ([Sch86, Thm. 9.4])**.** *Let $P \subset \mathbb{R}^d_+$ be a full-dimensional anti-blocking polytope with*

$$P \ = \ \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_r\}^{\downarrow} \ = \ \{\boldsymbol{x} \in \mathbb{R}^n_+ : \langle \boldsymbol{d}_i, \boldsymbol{x} \rangle \leq 1 \text{ for all } i = 1, \dots, s\}$$

for some $\boldsymbol{c}_1, \dots, \boldsymbol{c}_r, \boldsymbol{d}_1, \dots, \boldsymbol{d}_s \in \mathbb{R}^d_+$. Then

$$A(P) \ = \ \{\boldsymbol{d}_1, \dots, \boldsymbol{d}_s\}^{\downarrow} \ = \ \{\boldsymbol{x} \in \mathbb{R}^d_+ : \langle \boldsymbol{c}_i, \boldsymbol{x} \rangle \leq 1 \text{ for all } i = 1, \dots, r\}.$$

In particular, $A(A(P)) = P$.

From Theorem 4.1 one then deduces for a perfect graph G that

$$A(P_G) \ = \ P_{\overline{G}} \, , \tag{2}$$

where $\overline{G} = ([d], \binom{[d]}{2} \setminus E)$ is the **complement graph**.

Corollary 4.3 (Weak perfect graph theorem)**.** *A graph G is perfect if and only if $\overline{G}$ is perfect.*

We note that in particular if G is perfect, then P_G is compressed.

Proposition 4.4 ([CFS17, Prop. 3.10])**.** *Let $P \subset \mathbb{R}^d_+$ be an anti-blocking polytope. Then P is compressed if and only if $P = P_G$ for some perfect graph G.*

Let us remark that Theorem 4.1 also allows us to characterize the Gorenstein stable set polytopes. For comparability graphs of posets (see Section 7) this was noted by Hibi [Hib87]. A graph G is called **well-covered** if every inclusion-maximal stable set has the same size. It is called **co-well-covered** if $\overline{G}$ is well-covered.

Proposition 4.5. *Let P_G be the stable set polytope of a perfect graph $G = ([d], E)$. Then P_G is Gorenstein if and only if G is co-well-covered.*

Proof. It follows from Theorem 4.1 that the facet-defining inequalities are of the form $x_i \geq 0$ and $x(C) \leq 1$ for every maximal clique $C \subseteq [d]$. The former set of inequalities implies that if P_G is Gorenstein, then $\mathbf{1} = \mathbf{1}_{[d]}$ is the unique interior lattice point in rP_G for some $r \geq 1$. The second set of inequalities then yields that this is the case if and only if $|C| = \mathbf{1}(C) = r$ for all maximal cliques C. □

Combining Theorem 3.3 with Proposition 4.4 yields the following characterization of reflexive locally anti-blocking polytopes.

Theorem 4.6. *Let $P \subset \mathbb{R}^d$ be a locally anti-blocking lattice polytope. Then P is reflexive if and only if for every $\sigma \in \{-1, +1\}^d$ there is a perfect graph G_σ such that $P_\sigma = P_{G_\sigma}$.*

In particular, P is an unconditional reflexive polytope if and only if $P = \mathsf{U}P_G$ for some perfect graph G.

Corollary 4.7 ([CFS17, Thm. 3.4])**.** *If G_1, G_2 are perfect graphs on the vertex set $[d]$, then $P_{G_1} + (-P_{G_2})$ and $P_{G_1} \vee (-P_{G_2})$ are reflexive polytopes.*

For $G_1 = G_2 = K_d$ the complete graph on d vertices, the polytope $P_{G_1} + (-P_{G_2})$ is the Legendre polytope studied by Hetyei et al. [Het09, EHR18].

Using NORMALIZ [BIR$^+$] and the Kreuzer–Skarke database for reflexive polytopes [KS98, KS00], we were able to verify that 72 of the 3-dimensional reflexive polytopes and at least 407 of the 4-dimensional reflexive polytopes with at most 12 vertices are locally anti-blocking. Unfortunately, our computational resources were too limited to test *most* of the 4-dimensional polytopes. However, there are only 11 4-dimensional unconditional reflexive polytopes (by virtue of Theorem 4.9).

If G, G' are perfect graphs, then $G \uplus G'$ as well as its bipartite sum $G \bowtie G' = \overline{\overline{G} \uplus \overline{G'}}$ are perfect. On the level of unconditional polytopes we note that

$$\mathsf{U}P_{G \uplus G'} \; = \; \mathsf{U}P_G \times \mathsf{U}P_{G'} \quad \text{and} \quad \mathsf{U}P_{G \bowtie G'} \; = \; \mathsf{U}P_G \oplus \mathsf{U}P_{G'} \, .$$

These observations give us the class of Hanner polytopes which are important in relation to the 3^d-conjecture; see [SWZ09]. A centrally symmetric polytope $H \subset \mathbb{R}^d$ is called a **Hanner** polytope if and only if $H = [-1,1]^d$ or H is of the form $H_1 \times H_2$ or $H_1 \oplus H_2 = (H_1^* \times H_2^*)^*$ for lower dimensional Hanner polytopes H_1, H_2. Thus, every Hanner polytope is of the form $\mathsf{U}P_G$ for some perfect graph G. Hanner polytopes were obtained from split graphs in [FHSZ13] using a different geometric construction.

Let us briefly note that Theorem 4.6 also yields bounds on the entries of the h^*-vector. Recall that $h_i^*(C_d)$ for the cube $C_d = [-1,+1]^d$ is given by the **type-B Eulerian number** $B(n,i) = \sum_{j=1}^{i} (-1)^{k-i} \binom{n}{j-i} \left(2^{j-1}\right)^{n-1}$ that counts signed permutations with i descents (see also Section 5).

Corollary 4.8. *Let $P \subset \mathbb{R}^d$ be an unconditional reflexive polytope. Then*

$$\binom{d}{i} \; \le \; h_i^*(P) \; \le \; B(n,i) \, .$$

Proof. It follows from Theorem 4.6 that every reflexive and unconditional P satisfies $C_d^* \subseteq P \subseteq C_d$, where $C_d = [-1,1]^d$. By Theorem 2.1, the entries of the h^*-vector are monotone with respect to inclusion. □

We close the section by showing that distinct perfect graphs yield distinct unconditional reflexive polytopes.

Theorem 4.9. *Let G, H be perfect graphs on vertices $[d]$. Then $\mathsf{U}P_G$ is unimodular equivalent to $\mathsf{U}P_H$ if and only if $G \cong H$.*

Proof. Assume that $T(\mathsf{U}P_G) = \mathsf{U}P_H$ for some $T(\boldsymbol{x}) = W\boldsymbol{x} + \boldsymbol{t}$ with $\boldsymbol{t} \in \mathbb{Z}^d$ and $W \in \mathrm{SL}_d(\mathbb{Z})$. Since the origin is the only interior lattice point of both polytopes, we infer that $\boldsymbol{t} = 0$. Let $W = (\mathbf{w}_1, \ldots, \mathbf{w}_d)$. Thus, $\boldsymbol{z} \in \mathbb{Z}^d$ is a lattice point in $\mathsf{U}P_H$ if and only if there is a stable set S and $\sigma \in \{-1,+1\}^S$ such that

$$\boldsymbol{z} \; = \; \sum_{i \in S} \sigma_i \mathbf{w}_i \tag{3}$$

On the one hand, this implies that $\mathbf{w}_i$ and $\mathbf{w}_j$ have disjoint supports whenever $i, j \in S$ and $i \neq j$. Indeed, if the supports of $\mathbf{w}_i$ and $\mathbf{w}_j$ are not disjoint, then $\sigma_i \mathbf{w}_i + \sigma_j \mathbf{w}_j$ has a coordinate > 1 for some choice of $\sigma_i, \sigma_j \in \{-1,+1\}$, which contradicts the fact that $\mathsf{U}P_H \subseteq [-1,1]^d$.

On the other hand, for any $h \in [d]$, the point $\boldsymbol{e}_h$ is contained in $\mathsf{U}P_H$. Hence, there is a stable set S and $\sigma \in \{-1,+1\}^S$ such that (3) holds for $\boldsymbol{z} = \boldsymbol{e}_h$. Since the supports of the vectors indexed by S are disjoint, this means that $S = \{i\}$ and $\boldsymbol{e}_h = \sigma_i \mathbf{w}_i$. We conclude that W is a signed permutation matrix and $G \cong H$. □

We can conclude that number of unconditional reflexive polytopes in $\mathbb{R}^d$ up to unimodular equivalence is precisely the number of unlabelled perfect graphs on d vertices. This number has

n	3	4	5	6	7	8	9	10	11	12	13
p(n)	4	11	33	148	906	8887	136756	3269264	115811998	5855499195	410580177259

TABLE 1. Number $p(n)$ of unlabelled perfect graphs; OEIS sequence A052431.

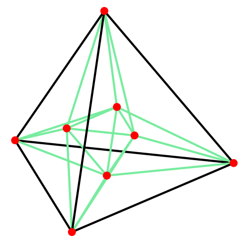

FIGURE 2. Schlegel diagram for $\mathcal{BB}(2)$.

been computed up to $d = 13$ (see [Hou06, Sec.5] and A052431 of [Slo19]). We show the sequence in Table 1.

5. THE TYPE-B BIRKHOFF POLYTOPE

Recall that the **Birkhoff polytope** $\mathcal{B}(n)$ is defined as the convex hull of all $n \times n$ permutation matrices or equivalently as the set of all **doubly stochastic matrices**, that is, nonnegative matrices M with row and column sums equal to 1, by work of Birkhoff [Bir46] and, independently, von Neumann [vN53]. This polytope has been studied quite extensively and is known to have many properties of interest (see, e.g., [Ath05, BR97, BP03, CM09, Dav15, DLLY09, Paf15]). Of particular interest to our purposes, it is known to be Gorenstein, to be compressed [Sta80], and to be h^*-unimodal [Ath05]. In this section, we will introduce a type-B analogue of this polytope corresponding to signed permutation matrices and verify many similar properties already known for $\mathcal{B}(n)$.

The **hyperoctahedral group** is defined to by $B_n := \mathbb{Z}/2\mathbb{Z} \wr \mathfrak{S}_n$, which is the Coxeter group of type-B (or type-C). Elements of this group can be thought of as permutations from $\mathfrak{S}_n$ expressed in one-line notation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, where we also associate a sign $\mathrm{sgn}(\sigma_i)$ to each σ_i. To each signed permutation $\sigma \in B_n$, we associate a matrix M_σ defined as $(M_\sigma)_{i,\sigma_i} = \mathrm{sgn}(\sigma_i)$ and $(M_\sigma)_{i,j} = 0$ otherwise. If every entry of σ is positive, then M_σ is simply a permutation matrix. This leads to the following definition:

Definition 5.1. The **type-B Birkhoff polytope** (or **signed Birkhoff polytope**) is

$$\mathcal{BB}(n) := \mathrm{conv}\,\{M_\sigma \,:\, \sigma \in B_n\} \subset \mathbb{R}^{n\times n}.$$

That is, $\mathcal{BB}(n)$ is the convex hull of all $n \times n$ signed permutation matrices.

This polytope was previously studied in [MOSZ02], though the emphasis was not on Ehrhart-theoretic questions. Since all points in the definition of $\mathcal{BB}(n)$ lie on a sphere, it follows that they are all vertices.

Proposition 5.2. *For every $\sigma \in B_n$, M_σ is a vertex of $\mathcal{BB}(n)$.*

It is clear that $\mathcal{BB}(n)$ is an unconditional lattice polytope in $\mathbb{R}^{d\times d}$ and we study it by restriction to the positive orthant.

Definition 5.3. For $n \geq 1$, we define the **positive type-B Birkhoff polytope**, $\mathcal{BB}_+(n)$, to be the polytope

$$\mathcal{BB}_+(n) := \mathcal{BB}(n) \cap \mathbb{R}^{n\times n}_+.$$

A simple way to view this as an anti-blocking polytope is via matching polytopes. Given a graph $G = ([d], E)$, a **matching** is a set $M \subseteq E$ such that $e \cap e' = \varnothing$ for any two distinct $e, e' \in M$. The corresponding **matching polytope** is

$$\mathsf{Mat}(G) \;:=\; \operatorname{conv}\{\mathbf{1}_M : M \subseteq E \text{ matching}\} \subset \mathbb{R}^E .$$

If G is a bipartite graph, then the matching polytope is easy to describe. For $v \in [d]$ let $\delta(v) \subseteq E$ denote the edges incident to v.

Theorem 5.4 ([Sch86, Sec. 8.11])**.** *For bipartite graphs G the matching polytope is given by*

$$\mathsf{Mat}(G) \;=\; \{\boldsymbol{x} \in \mathbb{R}^E_+ : x(\delta(v)) \le 1 \text{ for all } v \in [d]\} .$$

As a simple consequence, we get

Corollary 5.5. *$\mathcal{BB}_+(n)$ is the matching polytope of the complete bipartite graph $K_{n,n}$ on $2n$ vertices.*

It follows from the description given in Theorem 5.4 that matching polytopes of bipartite graphs are compressed. Hence, by Proposition 4.4 $\mathsf{Mat}(G)$ is the stable set polytope of a perfect graph. The graph in question is the **line graph** $L(G)$ on the vertex set E and edge ee' whenever $e \cap e' \neq \varnothing$. It is clear that M is a matching in G if and only if M is a stable set in $L(G)$. If $L(G)$ is perfect, then G is called a **line perfect graph**. From Lovász Theorem 4.1 one can then infer $\mathsf{Mat}(G) = P_{L(G)}$ and hence bipartite graphs are line perfect; cf. [Maf92, Thm. 2].

The polytope $\mathcal{BB}_+(n)$ is the stable set polytope of $L(K_{n,n}) = K_n \square K_n$, the Cartesian product of complete graphs, which for obvious reasons is called a **rook graph**.

Since all vertices in $K_{n,n}$ have the same degree, it follows that all maximal cliques in $K_n \square K_n$ have size n and from Proposition 4.5 we conclude the following.

Corollary 5.6. *The polytope $\mathcal{BB}_+(n)$ is Gorenstein.*

Furthermore, we can deduce that $\mathcal{BB}(n)$ is an unconditional reflexive polytope by Theorem 4.6. For two matrices $A, B \in \mathbb{R}^{d \times d}$ we denote by $\langle A, B \rangle = \operatorname{tr}(A^t B)$ the Frobenius inner product. Also, for vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$ let us write $\boldsymbol{u} \otimes \boldsymbol{v} \in \mathbb{R}^{d \times d}$ for the matrix with $(\boldsymbol{u} \otimes \boldsymbol{v})_{ij} = u_i v_j$.

Corollary 5.7. *The polytope $\mathcal{BB}(n)$ is an unconditional reflexive polytope. Its facet-defining inequalities are given by*

$$\langle A, \sigma \otimes \boldsymbol{e}_i \rangle \;\le\; 1 \quad \text{and } \langle A, \boldsymbol{e}_i \otimes \sigma \rangle \;\le\; 1$$

for all $i = 1, \dots, n$ and $\sigma \in \{-1, +1\}^n$.

The inequality description of this polytope was previously obtained in [MOSZ02] using the notion of *Birkhoff tensors*. However, we ascertain this result by applying Proposition 3.1 and Theorem 5.4.

The dual $\mathcal{C}(n) := \mathcal{BB}(n)^*$ is the unconditional reflexive polytope associated with the graph $\overline{K_n \square K_n}$. The corresponding anti-blocking polytope $\mathcal{C}_+(n) = P_{\overline{K_n \square K_n}}$ also has the nice property that all cliques have the same size n and hence Proposition 4.5 applies.

Corollary 5.8. *The polytope $\mathcal{C}_+(n)$ is Gorenstein.*

By Theorem 3.4 and Proposition 4.4, we have the following unimodality results.

Corollary 5.9. *For any $n \in \mathbb{Z}_{\ge 1}$, we have that $h^*_{\mathcal{BB}(n)}$, $h^*_{\mathcal{BB}_+(n)}$, $h^*_{\mathcal{C}(n)}$, and $h^*_{\mathcal{C}_+(n)}$ are unimodal.*

Let us conclude this section with some enumerative data. The polytope $\mathcal{BB}(n)$ has $2^n n!$ vertices and $n2^{n+1}$ facets. In contrast, the vertices of $\mathcal{BB}_+(n)$ are in bijection to partial permutations of $[n]$. Hence $\mathcal{BB}_+(n)$ has $n! \sum_{i=0}^n \frac{1}{i!}$ many vertices but only $n^2 + 2n$ facets. The polytope $\mathcal{C}_+(n)$ has $n2^{n+1} - (n+1)^2$ many vertices and $n^2 + n!$ facets. We used NORMALIZ [BIR$^+$] to compute the normalized volume and h^*-vectors of these polytopes; see Tables 2, 3, 4, and 5. Given the dimension and

n	$\mathrm{Vol}(\mathcal{BB}_+(n))$	$h^*_{\mathcal{BB}_+(n)}$
1	1	1
2	4	$(1,2,1)$
3	642	$(1,24,156,280,156,24,1)$
4	12065248	$(1,192,9534,151856,975793,2860752,4069012,$ $2860752,975793,151856,9534,192,1)$

TABLE 2. $\mathcal{BB}_+(n)$.

n	$\mathrm{Vol}(\mathcal{BB}(n))$	$h^*_{\mathcal{BB}(n)}$
1	2	$(1,1)$
2	64	$(1,12,38,12,1)$
3	328704	$(1,129,4482,40844,118950,118950,40844,4482,129,1)$
4	790708092928	?

TABLE 3. $\mathcal{BB}(n)$.

n	$\mathrm{Vol}(\mathcal{C}_+(n))$	$h^*_{\mathcal{C}_+(n)}$
1	1	1
2	6	$(1,4,1)$
3	642	$(1,24,156,280,156,24,1)$
4	2389248	$(1,88,2656,34568,201215,562112,787968$ $562112,201215,34568,2656,88,1)$
5	506289991680	?

TABLE 4. $\mathcal{C}_+(n)$.

volumes of these polytopes, our computational resources were quite quickly exhausted. Note that $\mathcal{BB}(3)$ and $\mathcal{C}(3)$ have precisely the same Ehrhart data and normalized volume and in fact it is straightforward to verify that $\mathcal{BB}(3)$ and $\mathcal{C}(3)$ are unimodularly equivalent.

Using Theorem 3.6 and Corollary 3.7, we get a lower bound on the volume of $\mathcal{BB}_+(5)$ and $\mathcal{BB}(5)$, respectively. We get that

$$\begin{array}{lcr} \mathrm{Vol}(\mathcal{BB}_+(5)) & > & 30.637.007.047.800 \\ \mathrm{Vol}(\mathcal{BB}(5)) & > & 1.028.007.369.668.940.603.880 \end{array}$$

are bounds on the number of simplices in an unimodular triangulation.

6. CIS GRAPHS AND COMPRESSED GALE-DUAL PAIRS OF POLYTOPES

The notion of Gale-dual pairs was introduced in [FHSS]. Given two polytopes $P, Q \subset \mathbb{R}^d$, we say that these polytopes are a **Gale-dual pair** if

$$\begin{aligned} P &= \{\boldsymbol{x} \in \mathbb{R}^d_+ : \langle \boldsymbol{x}, \mathbf{y}\rangle = 1 \text{ for } \mathbf{y} \in V(Q)\} \text{ and} \\ Q &= \{\boldsymbol{x} \in \mathbb{R}^d_+ : \langle \boldsymbol{x}, \mathbf{y}\rangle = 1 \text{ for } \mathbf{y} \in V(P)\}\,. \end{aligned}$$

n	$\mathrm{Vol}(\mathcal{C}(n))$	$h^*_{\mathcal{C}(n)}$
1	2	$(1, 1)$
2	96	$(1, 20, 54, 20, 1)$
3	328704	$(1, 129, 4428, 40844, 118950, 118950, 40844, 4428, 129, 1)$
4	156581756928	$(1, 592, 110136, 8093168, 222332060, 2558902352,$ $13699272072, 36553260912, 50497814342, 36553260912,$ $136992720722558902352, 222332060, 8093168, 110136, 592, 1)$
5	16988273098107125760	?

TABLE 5. $\mathcal{C}(n)$.

The prime example of a Gale-dual pair of polytopes is the Birkhoff polytope $\mathcal{B}_n$, the convex hull of permutation matrices M_τ, and the **Gardner polytope** $\mathcal{G}_n$, which is the polytope of all nonnegative matrices $A \in \mathbb{R}^{n\times n}_+$ such that $\langle M_\tau, A\rangle = 1$ for all permutation matrices M_τ. Both polytopes are compressed Gorenstein lattice polytopes of codegree n. The question raised in [FHSS] was if there other Gale-dual pairs with (a subset of) these properties. In this section we briefly outline a construction for compressed Gale-dual pairs of polytopes.

Following [ABG18], we call $G = ([d], E)$ a **CIS graph** if $C \cap S \neq \varnothing$ for every inclusion-maximal clique C and inclusion-maximal stable set S. For brevity, we refer to those as maximal cliques and stable sets, respectively. For example, if B is a bipartite graph with perfect matching, then the line graph $L(G)$ is CIS. Another class of examples is given by a theorem of Grillet [Gri69]. Let $\Pi = ([d], \preceq)$ be a partially ordered set. The **comparability graph** of Π is the simple graph $G_\prec = ([d], E)$ with $ij \in E$ if $i \prec j$ or $j \prec i$. Comparability graphs are known to be perfect. The **bull graph** is the graph vertices a, b, c, d, e and edges ab, bc, cd, de, bd.

Theorem 6.1 ([Gri69]). *Let $(\Pi, \preceq)$ be a poset with comparability graph G. Then G is CIS if every induced 4-path is contained in an induced bull graph.*

The wording in graph-theoretic terms is due to Berge; see [Zan95] for extensions.

Proposition 6.2. *Let G be a perfect CIS graph. Then*

$$\begin{aligned} P &= \mathrm{conv}(\mathbf{1}_S : S \textit{ maximal stable set of } G) \\ Q &= \mathrm{conv}(\mathbf{1}_C : C \textit{ maximal clique of } G) \end{aligned}$$

is a Gale-dual pair of compressed polytopes.

Proof. Note that every stable set meeting every maximal clique is necessarily a maximal stable set. Hence, it follows from Theorem 4.1 that

$$P = \{\boldsymbol{x} \in \mathbb{R}^n_+ : \boldsymbol{x}(C) = 1 \text{ for } C \text{ maximal clique}\} .$$

Since $\overline{G}$ is also a perfect CIS graph, the same holds for Q. □

Note that both of the examples above are perfect and CIS graphs. This shows that compressed (lattice) Gale-dual pairs are not rare. Recall that a graph G is well-covered if every maximal stable set has the same size and G is co-well-covered if $\overline{G}$ is well-covered. Theorem 6.1 and its generalization in [Zan95] allow for the construction of perfect CIS graphs which are well-covered and co-well-covered (for example, by taking ordinal sums of antichains). Moreover, the recent paper [DHMV15] gives classes of examples of well-covered and co-well-covered CIS graphs. This is a potential source of compressed Gorenstein Gale-dual pairs but we were not able to identify the perfect graphs in these families.

Theorem 4.6 implies that if (F, G) is a Gale-dual pair of Proposition 6.2, then there is a (unconditional) reflexive polytope such that $F \subset P$ and $G \subset P^*$ are dual faces.

Question 6.3. Is it true that every Gale-dual pair (F, G) appears as dual faces of some reflexive polytope P?

7. Chain Polytopes and Gröbner Bases

Given a lattice polytope $P \subset \mathbb{R}^d$, the existence of regular triangulations, particularly those which are unimodular and flag, has direct applications to the associated toric ideal of P. In this section, we will discuss how the Gröbner basis of the toric ideal of anti-blocking polytope can be extended to the associated unconditional polytope. In particular, we provide an explicit description of the Gröbner bases of the unconditional polytopes arising from the special class of anti-blocking polytopes called chain polytopes. We refer the reader to the wonderful books [CLO15] and [Stu96] for background on Gröbner bases and toric ideals.

Let $Z := P \cap \mathbb{Z}^d$. The **toric ideal associated to** P is the ideal $I_P \subset \mathbb{C}[x_p : p \in Z]$ with generators

$$x_{r_1} x_{r_2} \cdots x_{r_k} - x_{s_1} x_{s_2} \cdots x_{s_k} \,,$$

where $r_1, \ldots, r_k, s_1, \ldots, s_k \in Z$ such that $r_1 + \cdots + r_k = s_1 + \cdots + s_k$. If we denote the two *multi*sets of points by R and S, we simply write $x^R - x^S$. A celebrated result of Sturmfels [Stu96, Thm. 8.3] states that the regular triangulations $\mathcal{T}$ of P (with vertices in Z) are in correspondence with (reduced) Gröbner bases of I_P. As customary we write $\underline{x^R} - x^S$ to emphasize that x^R is the leading term. In particular, if $\mathcal{T}$ is unimodular, then the leading terms of the associated Gröbner basis are square-free [Stu96, Cor. 8.9]. That is, R is an actual *set*.

Given any lattice point $\boldsymbol{p} \in 2P$ there are unique $\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)} \in Z$ such that $2\boldsymbol{p} = \boldsymbol{p}^{(1)} + \boldsymbol{p}^{(2)}$ and $\{\boldsymbol{p}^{(1)}, \boldsymbol{p}^{(2)}\}$ is an edge in $\mathcal{T}$. Let us call two points $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^d$ **separable** if for some p_i and q_i have different signs for some $i = 1, \ldots, d$. Together with Theorem 3.2, this yields the following description of a Gröbner basis for unconditional reflexive polytope.

Theorem 7.1. *Let $P \subset \mathbb{R}^d$ be an anti-blocking polytope with a regular, unimodular, flag triangulation and let $\mathsf{U}P$ be the associated unconditional polytope. Let $\underline{x^{R_i}} - x^{S_i}$ for $i = 1, \ldots, m$ be the Gröbner basis for I_P. Then the following binomials give a Gröbner basis for $I_{\mathsf{U}P}$:*

$$\underline{x^{\sigma R_i}} - x^{\sigma S_i}$$

for $i = 1, \ldots, m$ and $\sigma \in \{-1, +1\}^d$. Moreover, for any $\boldsymbol{p}, \boldsymbol{q} \in \mathsf{U}P \cap \mathbb{Z}^d$ separable, let σ such that $\sigma(\boldsymbol{p} + \boldsymbol{q}) = \boldsymbol{e} \in 2P$ and let

$$\underline{x^{\boldsymbol{p}}\, x^{\boldsymbol{q}}} \;-\; x^{\sigma \boldsymbol{e}^{(1)}}\, x^{\sigma \boldsymbol{e}^{(2)}} \,.$$

A prominent class of perfect graphs G for which regular, unimodular triangulations P_G, as well as Gröbner bases for I_{P_G}, are well understood are comparability graphs of finite posets. Let $\Pi = ([d], \preceq)$ be a partially ordered set. with comparability graph $G_\prec$. The stable set polytopes associated to comparability graphs were studied by Stanley [Sta86] under the name **chain polytopes** and denoted by $\mathcal{C}(\Pi)$. The vertices of $P_{G_\prec}$ are precisely points $\boldsymbol{e}_A$, where A is an **antichain** which is a collections of incomparable elements in Π. Let $\mathcal{A}(\Pi)$ denote the collection of antichains. The pulling triangulation of $P_{G_\preceq}$ can be explicitly described (see Section 4.1 in [CFS17] for exposition and details). The corresponding (reverse lexicographic) Gröbner basis was described by Hibi [Hib87]. Following [CFS17], we define

$$A \sqcup A' \;:=\; \min(A \cup A') \quad \text{and} \quad A \sqcap A' \;:=\; (A \cap A') \cup (\max(A \cup A') \setminus \min(A \cup A'))\,,$$

where min and max are taken with respect to the partial order $\preceq$. We call two antichains A, A' **incomparable** if there are $a \in A$ and $a' \in A'$ such that $A \cup \{a'\}, A' \cup \{a\} \in \mathcal{A}(\Pi)$. Equivalently, if $\max(A \cup A')$ is a subset of neither A nor A'. To ease notation, we identify variables x_A in $\mathbb{C}[x_A : A \in \mathcal{A}(\Pi)]$ with $[A]$.

Theorem 7.2. *A Gröbner basis for $I_{\mathcal{C}(\Pi)}$ is given by the binomials*

$$\underline{[B]\cdot[B']} - [B \sqcup B'] \cdot [B \sqcap B'] \, ,$$

for all incomparable antichains $B, B' \in \mathcal{A}(\Pi)$.

We define the **unconditional chain polytope** $\mathsf{U}\mathcal{C}(\Pi)$ as the unconditional reflexive polytope associated to $G_{\prec}$. The lattice points in $\mathsf{U}\mathcal{C}(\Pi)$ are uniquely described by

$$\boldsymbol{e}_{B-A} \;\coloneqq\; \boldsymbol{e}_B - 2\boldsymbol{e}_A$$

where $A \subseteq B$ are antichains. We also write $B - A$ for the pair $A \subset B$. The vertices of $\mathsf{U}\mathcal{C}(\Pi)$ then correspond to the pairs $B - \varnothing$ and $B - B$ for inclusion-maximal antichains B. Note that $\boldsymbol{e}_{B-A}$ and $\boldsymbol{e}_{B'-A'}$ are separable if and only if $(B \setminus A') \cup (B' \setminus A) \neq \varnothing$.

Theorem 7.3. *Let $\Pi = ([d], \preceq)$ be a finite poset and $I_{\mathsf{U}\mathcal{C}(\Pi)}$ the toric ideal associated to the unconditional chain polytope $\mathsf{U}\mathcal{C}(\Pi)$. Then a reduced Gröbner basis is given by the binomials*

$$\underline{[B-C]\cdot[B'-C]} \;-\; [(B \sqcup B') - C] \cdot [(B \sqcap B') - C] \, ,$$

for all incomparable $B, B' \in \mathcal{A}(\Pi)$ and $C \subseteq B \cup B'$. Additionally, for every separable $B - A$ and $B' - A'$

$$[B-A]\cdot[B'-A'] \;-\; [D - (D \cap (A \cup A'))] \cdot [D' - (D' \cap (A \cup A'))] \, ,$$

where $D = (B \setminus A') \sqcap (B' \setminus A)$ and $D' = (B \setminus A') \sqcup (B' \setminus A)$.

Proof. In light of Theorems 7.1 and 7.2 , we only need to argue the second collection of binomials.

It follows from Theorem 7.2 that the edges of the unimodular (pulling) triangulation of $\mathcal{C}(\Pi)$ are of the form $\{D, D'\}$ where D, D' are comparable antichains. That is, for every $b \in D$ there is $b' \in D'$ with $b \preceq b'$. For $\boldsymbol{p} \in 2\mathcal{C}(\Pi)$, there are unique comparable $D, D' \in \mathcal{A}(\Pi)$ with $2\boldsymbol{p} = \boldsymbol{e}_D + \boldsymbol{e}_{D'}$. Set $S \coloneqq \{i : p_i \geq 1\}$ and $T \coloneqq \{i : p_i = 2\}$. Then it follows from the fact that every element in $D \cup D'$ is either a minimum or maximum that $D = \min(S)$ and $D' = \max(S) \cup T$. Hence if $\boldsymbol{p} = \boldsymbol{e}_C + \boldsymbol{e}_{C'}$ for arbitrary antichains C, C', then $D = C \sqcap C'$ and $D' = C \sqcup C'$. ☐

8. Concluding remarks

8.1. **Birkhoff polytopes of other types.** It is only natural to look at Birkhoff-type polytopes of other finite irreducible Coxeter groups. Since the type-B and the type-C Coxeter groups are equal, we get the same polytope. Recall that the type-D Coxeter group D_n is the subgroup of B_n with permutations with an even number of negatives. We can construct the **type-D Birkhoff polytope**, $\mathcal{BD}(n)$, to be the convex hull of signed permutation matrices with an even number of negative entries. As one may suspect from this construction, the omission of all lattice points in various orthants which occurs in $\mathcal{BD}(n)$ ensures that it cannot be an OLP polytope and is thus not subject to any of our general theorems. When $n = 2$ and $n = 3$, $\mathcal{BD}(n)$ is a reflexive polytope, but $\mathcal{BD}(3)$ does not have the IDP. Moreover, $\mathcal{BD}(4)$ fails to be reflexive.

Additionally, one could consider Birkhoff constructions for Coxeter groups of exceptional type, in particular E_6, E_7 and E_8 (see, e.g., [BB05]). While we did not consider these polytopes in our investigation, we do raise the following question:

Question 8.1. Do the Birkhoff polytope constructions for E_6, E_7, and E_8 have the IDP? Are these polytopes reflexive? Do they have other interesting properties?

8.2. **Future directions.** In addition to considering Birkhoff polytopes of other types and connections to Gale duality as discussed above, there are several immediate avenues for further research. Coxeter groups of great interest in the broader community of algebraic and geometric combinatorics (see, e.g., [BB05]). Subsequently, it is natural to consider how the Ehrhart-theoretic study of the type-B Birkhoff polytope informs research area. This leads to the following question:

Question 8.2. Does the convex structure of $\mathcal{BB}(n)$ encode combinatorial or group theoretic structure of interest in Coxeter combinatorics?

An additional future direction is to consider applications of the orthant-lattice property, particularly those of Theorem 3.2 and Remark 3.5. One potentially fruitful avenue is an application to reflexive *smooth* polytopes. Recall that a lattice polytope $P \subset \mathbb{R}^d$ is **simple** if every vertex of P is contained in exactly d edges (see, e.g., [Zie95]). A simple polytope P is called **smooth** if the primitive edge direction generate $\mathbb{Z}^d$ at every vertex of P. Smooth polytopes are particularly of interest due to a conjecture commonly attributed to Oda [Oda]:

Conjecture 8.3 (Oda)**.** *If P is a smooth polytope, then P has the IDP.*

This conjecture is not only of interest in the context of Ehrhart theory, but also in toric geometry. One potential strategy is to consider similar constructions to OLP polytopes for smooth reflexive polytopes to make progress towards this problem. As a first step, we pose the following question:

Question 8.4. Are all smooth reflexive polytopes OLP polytopes?

Furthermore, regarding reflexive OLP polytopes one can ask the question:

Question 8.5. Given a reflexive OLP polytope P, under what conditions can we guarantee that P^* is a reflexive OLP polytope?

By (2), this has a positive answer when P is an unconditional reflexive polytope. However, there are multiple examples of failure in general even in dimension 2 (see Figure 1).

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