TERNARY QUADRATIC FORMS REPRESENTING ARITHMETIC PROGRESSIONS

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Abstract. A positive quadratic form is (k,ℓ)-universal if it represents all natural numbers $\equiv \ell \pmod{k}$, and almost (k,ℓ)-universal if it represents all but finitely many of them. We prove that for any k,ℓ such that $k \nmid \ell$ there exists an almost (k,ℓ)-universal diagonal ternary form. We also conjecture that there are only finitely many primes p for which a (p,ℓ)-universal diagonal ternary form exists (for any $\ell < p$) and we show the results of computer experiments that speak in favor of the conjecture.

1. Introduction

The sum of three squares $x^2 + y^2 + z^2$ does not represent any integer of the form $8n + 7$ and similarly every other positive ternary quadratic form fails to represent some arithmetic sequence. Conversely, around 1797 Legendre showed that $x^2+y^2+z^2$ represents all positive integers that are not of the form $4^k(8n+7)$ (with $k,n \geq 0$) and there have been numerous results concerning the integers represented by ternary quadratic forms. Before discussing some of them, let us introduce some basic notions.

A *positive ternary quadratic form* is a form $Q(x,y,z) = ax^2+by^2+cz^2+dyz+exz+fxy$, where a,b,c,d,e,f are integers and $Q(x,y,z) > 0$ for all real numbers x,y,z, not all of them 0.

For positive integers k,ℓ consider the arithmetic sequence

$$S_{k,\ell} := \{\, kx+\ell : x \in \mathbb{Z},\, x \geq 0 \,\}.$$

We say that a positive quadratic form with $\mathbb{Z}$-coefficients is *(k,ℓ)-universal* if it represents all elements of $S_{k,\ell}$ over the ring of integers $\mathbb{Z}$. A quadratic form is *almost (k,ℓ)-universal* if it represents almost all elements of $S_{k,\ell}$, i.e., if there are at most finitely many elements of $S_{k,\ell}$ that are not represented.

Kaplansky [Kap95] showed that there are at most 23 ternary forms that represent all odd positive integers (i.e., that are $(2,1)$-universal) and proved the $(2,1)$-universality of 19 of them. Jagy [Jag96] dealt with one of the remaining candidates and, assuming the $(2,1)$-universality of the 3 other forms, Rouse [Rou14] proved the 451-theorem: A positive quadratic form (of any rank) is $(2,1)$-universal if and only if it represents all the integers $1,3,5,\dots,451$.

2010 *Mathematics Subject Classification.* 11E12, 11E20, 11E25, 11-04, 11Y50, 11Y55.
Key words and phrases. Quadratic form, ternary, universal, diagonal, arithmetic progression.
Both authors acknowledge support by Czech Science Foundation (GAČR) grant 17-04703Y.

Oh [Oh11b] then proved that for any (k, ℓ), there are only finitely many equivalence classes of (k, ℓ)-universal ternaries (that are moreover classical, i.e., d, e, f are all even).

This was followed by investigations of all the (k, ℓ)-universal diagonal ternary forms for small values of k and $1 \leq \ell < k$. Independently, Pehlivan and Williams [PW18] computed all such possible candidates for $k \leq 11$, and Sun [Sun17] found all such candidates with $k \leq 30$. Pehlivan and Williams established the (k, ℓ)-universality of a number of their candidates and then Wu and Sun [WS18] proved this for more of these forms.

Notably, when $k = p \geq 11$ is a prime, then no (k, ℓ)-universal diagonal ternary forms appears in these lists!

(k, ℓ)-universal forms are intimately connected to regular forms, i.e., quadratic forms that represent over $\mathbb{Z}$ all the integers that they represent over $\mathbb{R}$ and the ring $\mathbb{Z}_p$ of p-adic integers for all primes p. Jagy, Kaplansky, and Schiemann [JKS97] proved that there are at most 913 regular ternary forms and established the regularity of all but 22 of them. Oh [Oh11a] then proved the regularity of 8 of these, and then Lemke Oliver [LO14] dealt with the 14 remaining cases under the assumption of Generalized Riemann Hypothesis, using the method of Ono and Soundrajaran [OS97].

The problem of precisely determining the set of integers represented by a given ternary quadratic form is still open, although it has been thoroughly studied. Let us mention only the results by Kneser [Kne61], Duke and Schulze-Pillot [DSP90], and by Earnest, Hsia, and Hung [EHH94], and point the interested reader to the very nice survey by Hanke [Han04].

In this short paper, we study (k, ℓ)-universality of diagonal ternary quadratic forms. Considering *almost* (k, ℓ)-universal diagonal ternaries, we show that they always exist.

Theorem 1. *Let k, ℓ be positive integers such that $k \nmid \ell$. Then there is a diagonal ternary positive quadratic form that is almost (k, ℓ)-universal.*

We prove the theorem in § 2 by first dealing with almost (p, ℓ)-universal forms; in fact, we show that for each prime p, there is a prime q (or $q = 1$) such that the form $x^2 + qy^2 + pz^2$ is anisotropic precisely at p (and ∞) and that this form is then almost (p, ℓ)-universal. This then quickly implies the theorem for general k.

In § 3 we expand on the observation (based on the results of Pehlivan and Williams [PW18] and Sun [Sun17]) that when p is a prime satisfying $11 \leq p \leq 29$, then there is no (p, ℓ)-universal diagonal ternary form for $1 \leq \ell < p$.

We first search for (p, ℓ)-universal diagonal ternaries and obtain that, for $11 \leq p \leq 1237$, the only case when they can exist is $(101, 98)$, when $x^2 + 2y^2 + 101z^2$ appears to be $(101, 98)$-universal.

To obtain more refined understanding of the situation, we then consider the number of "gaps" of a given form (that satisfies the necessary anisotropy conditions), i.e., of (small) integers that are not represented, see § 3.3. Our computations suggest that the number of gaps is always larger than $p \log p$, which provides heuristic argument in favor of the following conjecture (details are discussed in § 3).

Conjecture 2. *There are only finitely many primes p and $1 \le \ell < p$ possessing a diagonal ternary positive (p,ℓ)-universal quadratic form.*

In fact, the data suggest even the stronger conjecture that the form $x^2+2y^2+101z^2$ was the last missing one and that now the knowledge is exhaustive:

Conjecture 3. *The following table gives the complete list of diagonal ternary positive (p,ℓ)-universal quadratic forms for a prime p and $1 \le \ell < p$ (here $\langle a,b,c\rangle$ stands for the form $ax^2+by^2+cz^2$):*

p	ℓ	(p,ℓ)-*universal forms*
2	1	$\langle 1,1,2\rangle$, $\langle 1,2,3\rangle$, $\langle 1,2,4\rangle$
3	1	$\langle 1,1,3\rangle$, $\langle 1,1,6\rangle$, $\langle 1,3,3\rangle$, $\langle 1,3,9\rangle$, $\langle 1,6,9\rangle$
	2	$\langle 1,1,3\rangle$, $\langle 1,1,6\rangle$, $\langle 2,3,3\rangle$
5	1	$\langle 1,2,5\rangle$, $\langle 1,5,10\rangle$
	2	$\langle 1,2,5\rangle$
	3	$\langle 1,2,5\rangle$
	4	$\langle 1,2,5\rangle$, $\langle 1,5,10\rangle$
7	1	$\langle 1,2,7\rangle$, $\langle 1,7,14\rangle$
	2	$\langle 1,2,7\rangle$
	3	$\langle 1,2,7\rangle$
101	98	$\langle 1,2,101\rangle$

The universality of the forms for $p = 2,3,5$ has been established (see, e.g., [PW18]), whereas for $p = 7,101$ it is only conjectural.

Note that the forms $\langle 1,2,p\rangle$ appear frequently in the preceding table. This is not an accident, as it seems that these forms are the most likely candidates for (p,ℓ)-universality (see § 3.3). In fact, it turns out that the heuristic argument that we use for dealing with the other forms fails in this case! Hence we have to consider the forms $\langle 1,2,p\rangle$ (together with $\langle 1,1,p\rangle$ and $\langle 1,3,p\rangle$) separately in detail in § 3.4.

It is of course very interesting to consider the existence of (p,ℓ)-universal ternaries without the restriction that $\ell < p$ and without the diagonality assumption. As we have shown that almost (p,ℓ_0)-universal ternaries always exists, it trivially follows that also (p,ℓ)-universal ternaries exist once $\ell \equiv \ell_0 \pmod p$ is sufficiently large. Nevertheless, if we set a bound $\ell < cp$ for a fixed positive integer c, our heuristics suggest that there again should be only finitely many (p,ℓ)-universal diagonal ternaries with $\ell < cp$ (see Figure 3.4). We have not done almost any computations with non-diagonal forms. Nevertheless, our (mostly unfounded) guess might be that there are only finitely many (p,ℓ)-universal *non-diagonal* ternaries as well.

2. Existence of almost (k,ℓ)-universal forms

In the rest of the article we will consider only diagonal ternary positive forms, i.e., quadratic forms $Q(x,y,z) = ax^2+by^2+cz^2 =: \langle a,b,c\rangle$, where a,b,c are positive integers. We denote $\Delta_Q = abc$ the discriminant of Q.

Let v be a place of $\mathbb{Q}$. A quadratic form $Q(x,y,z)$ is *isotropic at* v if it non-trivially represents 0 over the completion $\mathbb{Q}_v$, i.e., if $Q(x,y,z) = 0$ for some $x,y,z \in$

$\mathbb{Q}_v$, not all of them 0. Otherwise Q is *anisotropic at* v. Note that if $v = p$ is a finite place corresponding to a prime p, then $Q = \langle a, b, c\rangle$ can be anisotropic at p only if $p \mid 2abc$.

A positive ternary form is always anisotropic at ∞ and, by Hilbert reciprocity law, it is anisotropic at an odd number of finite places v.

Proposition 4. *Let p be a prime. Then there is a positive diagonal ternary form $Q = \langle 1, q, p\rangle$ that is anisotropic precisely at p and ∞, where $q = 1$ or a prime different from p.*

Proof. For $p = 2$ we can take $Q = \langle 1, 1, 2\rangle$, so let us assume that p is an odd prime. We will distinguish several cases according to the value $p \pmod 8$.

Case $p \equiv 3 \pmod 4$. Then we claim that $Q = \langle 1, 1, p\rangle$ works. The only candidates for primes at which Q can be anisotropic are 2 and p. By Hilbert reciprocity, it suffices to show that Q is anisotropic at p. Assume that $x^2 + y^2 + pz^2 = 0$ with $x, y, z \in \mathbb{Z}_p$. Then $x^2 + y^2 \equiv 0 \pmod p$, and so $x \equiv y \equiv 0 \pmod p$, because -1 is a quadratic non-residue mod p in this case. But then also $z \equiv 0 \pmod p$ and we have $(x/p)^2 + (y/p)^2 + p(z/p)^2 = 0$. Continuing in this way, we get that $x = y = z = 0$, i.e., that Q is anisotropic at p.

Case $p \equiv 5, 7 \pmod 8$. In this case the form $Q = \langle 1, 2, p\rangle$ works, which can be proved by the same argument as in the previous paragraph.

Case $p \equiv 1 \pmod 8$. We will show that the form $Q = \langle 1, q, p\rangle$ works if q is a prime such that $q \equiv 3 \pmod 4$ and the Legendre symbol $\left(\frac{q}{p}\right) = -1$ (such primes q clearly exist). For this form Q, the anisotropic candidates are 2, p, and q. Distinguishing the two possibilities for $q \pmod 8$, it is easy to verify that the form Q is always isotropic at 2. Hence it suffices to show that Q is anisotropic at p. As $p \equiv 1 \pmod 8$, -1 is a quadratic residue mod p, and so $\left(\frac{-q}{p}\right) = -1$. Hence $x^2 + qy^2 \equiv 0 \pmod p$ implies that $x \equiv y \equiv 0 \pmod p$ and we see as before that Q is indeed anisotropic at p. $\square$

Proposition 5. *Let p be an odd prime and $Q = \langle 1, q, p\rangle$ a positive diagonal ternary form that is anisotropic precisely at p and ∞, where $q = 1$ or a prime different from p. Then Q is almost (p, ℓ)-universal for every positive integer ℓ such that $p \nmid \ell$.*

Proof. We will use a theorem of Duke and Schulze-Pillot [DSP90], cf. [Han04, Theorem on p. 11]. For any undefined notions in the proof, see, e.g., [Han04].

We are interested in almost (p, ℓ)-universality and we have $p \nmid \ell$, so we need to show that Q locally represents all elements of the corresponding arithmetic progression and that there are no spinor exceptions.

The local representation is no problem at $\mathbb{R}$ and at the isotropic places, so we need to check it only at the anisotropic place p. It suffices to show that the binary form $x^2 + qy^2$ with $p \nmid q$ represents all non-zero classes modulo p. If q is a quadratic non-residue mod p, then qy^2 represents all the non-residues and x^2 represents all the residues. If q is a quadratic residue, then $x^2 + qy^2$ represents the same elements mod p as $x^2 + y^2$. But the latter form represents (over $\mathbb{Z}$) all the primes $\equiv 1 \pmod 4$, which cover all the classes mod p.

Let us now consider the spinor exceptions, i.e., integers a such that Q does not represent the values of the quadratic sequence ax^2 for integers x; there are always only finitely many spinor exceptions (for an overview of their properties, see [SP00, pp. 351–352]). In particular, a spinor exception can exist only if the genus of Q breaks into an even number of spinor genera. However, a necessary condition for this to happen is that the determinant of Q is not squarefree [Cas78, Ch. 11, Theorem 1.3], whereas in our case, the determinant pq is squarefree. Alternatively, one can deduce the non-existence of spinor exceptions from the explicit results of Earnest, Hsia, and Hung [EHH94]. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. When $k = p = 2$, then $\langle 1, 1, 2\rangle$ is $(2, 1)$-universal. When $k = p$ is an odd prime, the theorem was proved in Proposition 5.

If k and ℓ are coprime, then there exists a prime p such that $p \mid k$ and $p \nmid \ell$. Then $S_{k,\ell} \subset S_{p,\ell}$, and so every almost (p, ℓ)-universal form is also almost (k, ℓ)-universal.

Finally, if $d = \gcd(k, \ell)$, then let Q be an almost $(k/d, \ell/d)$-universal ternary form (which exists by the previous paragraph). Then dQ is almost (k, ℓ)-universal. □

3. Non-existence of (p, ℓ)-universal forms

The reasoning behind Conjectures 2 and 3 is based on several observations from numerical experiments. Prior to stating the observations, let us denote $X_{Q,p}$ the set of non-represented numbers (we call them *gaps*) for a ternary form Q that is anisotropic precisely at p (and ∞):

$$X_{Q,p} := \{\, n \in \mathbb{N} : n \not\equiv 0 \pmod{p},\ n \text{ not represented by } Q \,\}.$$

Note that when the discriminant Δ_Q is squarefree, $X_{Q,p}$ is finite ([Cas78], we discuss this in the proof of Proposition 5 above).

We carried out the following computations.

3.1. Full search for (p, ℓ)-universal forms.

For a specific odd prime p, there is an easy algorithm that searches for (p, ℓ)-universal diagonal ternary forms. Suppose that such a form exists. Then there certainly exists a form $Q = \langle a, b, c\rangle$ such that:

- all three coefficients are squarefree (for instance, if $a' = ad^2$, then $\langle a, b, c\rangle$ is also (p, ℓ)-universal);
- $a \le b \le c$;
- $a \le \ell$ (one of the coefficients must be less than ℓ as the form represents ℓ);
- $\langle a, b\rangle$ represents ℓ (suppose it does not; then $c \le \ell < p$ and this is a contradiction with $p \mid abc$);
- if ℓ is not squarefree, then $a \le \ell/2$ (either $\ell = ax^2$ and as ℓ is not squarefree and a is, $a < \ell$, $x \ge 2$, whence $a \le \ell/4$; or $\ell = by^2$ and $a \le b \le \ell/4$; or $\ell = ax^2 + by^2$, whence $2a \le a + b \le \ell$);
- $p \mid b$ or $p \mid c$ (the prime divides the discriminant abc and as $a \le \ell < p$, $p \nmid a$).

Moreover, we know that no unary or binary (p, ℓ)-universal forms exist. So if we denote e_Q the smallest number $\equiv \ell \pmod{p}$ not represented by Q, we know that $b \le e_{\langle a\rangle}$ and $c \le e_{\langle a,b\rangle}$.

Based on these conditions, we searched through all the possible triples (a, b, c) and obtained:

Proposition 6. *For all primes* $11 \leq p \leq 1237$ *and* $1 \leq \ell \leq p-1$ *such that* $(p, \ell) \neq (101, 98)$*, there are no diagonal ternary positive* (p, ℓ)*-universal quadratic forms.*

We carried out this computation in Python 2.7.12 on Intel Xeon machines with Ubuntu 16.04, kernel version 4.13.0. This computation took 670 CPU days to complete. We precomputed a list of primes and a list of squarefree numbers in SageMath 6.4 [Sage] as Python natively does not support these.

3.2. $(101, 98)$**-universality of** $\langle 1, 2, 101\rangle$**.** We did not manage to prove that the form $Q = \langle 1, 2, 101\rangle$ represents all elements of $S_{101,98}$, as the form is not regular (the size of the genus is 9 as computed by Magma [Magma]). We assert the following:

Proposition 7. *The form* $\langle 1, 2, 101\rangle$ *represents* $S_{101,98} \cap [0, 10^{12}]$*.*

We carried out this computation in C++ with GCC 4.8.3 on an Intel i5 PC. To verify the result, we iterated through triples x, y, z such that $x^2 + 2y^2 \leq 10^{12}$, $x^2+2y^2 \equiv 98 \pmod{101}$ and $x^2+2y^2+101z^2 \leq 10^{12}$. However, there are simply too many such triples, so we first restricted to small x, and then interactively increased the x's considered until we got representations of the whole $S_{101,98} \cap [0, 10^{12}]$.

3.3. **Number of gaps.** When a form has a squarefree discriminant, we know that there are only finitely many numbers not represented by the form (as always excluding the zero class modulo p), i.e., the set of gaps $X_{Q,p}$ is finite. We investigated the cardinality $\#X_{Q,p}$ for forms of small discriminant and we observe that it behaves very roughly as $p \log p$. The ratio $\alpha := \#X_{Q,p}/p \log p$ for forms with $p < 300$ and $\Delta_Q < 30p$ is shown in Figure 1. (Note that $X_{Q,p}$ may be finite also for non-squarefree discriminants, but then always $\alpha > 100$ in these cases. Anyway, without loss of generality as in 3.1 we consider only forms with squarefree $a \leq b \leq c$.)

This computation was performed in C++ with GCC 5.4.0 on a cluster with Intel Xeon CPU cores @ 2.00–2.40 GHz. It took less than 1 CPU day to complete. We precomputed the list of Q anisotropic precisely at $p < 300$ and with $\Delta_Q < 30p$ using SageMath [Sage].

If we assume that the elements of $X_{Q,p}$ are equidistributed modulo p, we can use the comparison of $\#X_{Q,p}$ with $p \log p$ in the following heuristic argument: For a specific form Q, denote $\alpha > 0$ such constant that $\#X_{Q,p} = \alpha p \log p$ (see Figure 1). For Q to be (p, ℓ)-universal for a specific ℓ, we need that *none* of the gaps lies in the set $S_{p,\ell}$. The probability of this is

$$\Big(1 - \frac{1}{p-1}\Big)^{\#X_{Q,p}} = \Big(1 - \frac{1}{p-1}\Big)^{\alpha p \log p} \approx e^{-\alpha \frac{p}{p-1} \log p} \approx p^{-\alpha}.$$

Then the expected number of ℓ's such that Q is (p, ℓ)-universal is $(p-1)p^{\alpha} \approx p^{1-\alpha}$. This shows that the larger the value of α, the smaller the chance that a form is (p, ℓ)-universal for some ℓ.

We can even use this to estimate the total expected number of (p, ℓ)-universal forms: Oh [Oh11b, Theorem 2.3] proved an upper bound for the discriminant of

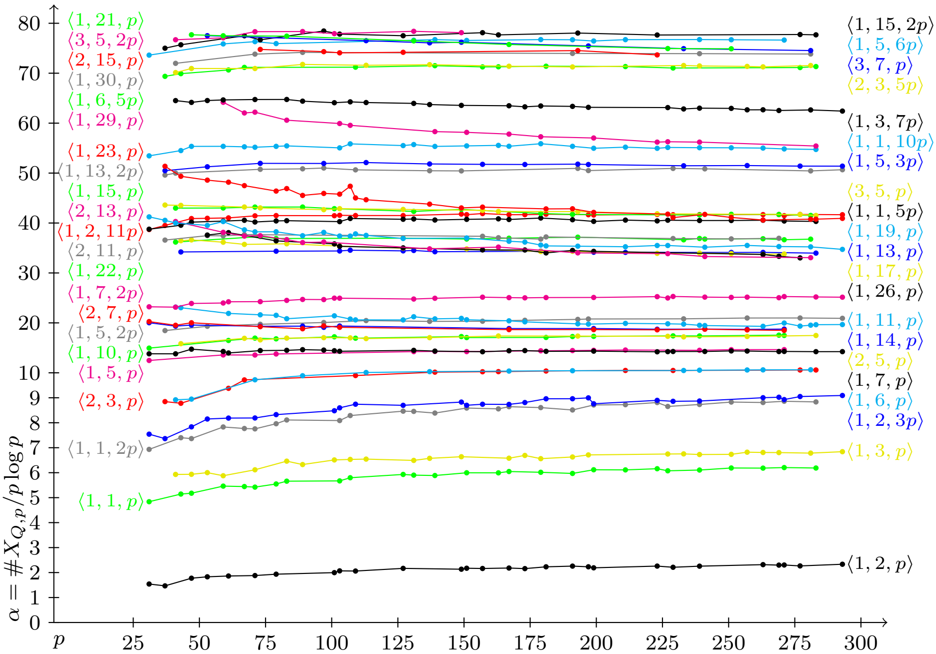

FIGURE 1. Comparison of a lower estimate of $\#X_{Q,p}$ to $p\log p$ for $30 < p < 300$ and $\Delta_Q < 30p$. Note that only forms with $\#X_{Q,p} < 100p\log p$ are shown.

a (p,ℓ)-universal ternary, which implies $\Delta_Q < Cp^6$ for a constant C. As $p \mid \Delta_Q$, there are at most Cp^5 possible discriminants. Thus asymptotically, for each p there are at most $p^{5+\varepsilon}$ candidates for (p,ℓ)-universal forms, as the number of ways of factoring $\Delta_Q = abc$ is below Δ_Q^{ε} for any ε. Therefore, for a fixed p the expected number of ℓ's and Q's such that Q is (p,ℓ)-universal is asymptotically smaller than $p^{6+\varepsilon-\min\alpha}$.

Let us exclude the forms $\langle 1,1,p\rangle$, $\langle 1,2,p\rangle$, and $\langle 1,3,p\rangle$ for now. The data behind Figure 1 suggest that we eventually have $\alpha > 7$ for each of the remaining forms. Thus for given sufficiently large p, the expected number of (non-excluded) (p,ℓ)-universal forms is less than $C'p^{1-\varepsilon'}$ for some $\varepsilon' > 0$ and a constant C'. The total expected number over all large primes p is then $C'\sum_p p^{1-\varepsilon'}$, which *converges*! Thus besides from the excluded forms $\langle 1,1,p\rangle$, $\langle 1,2,p\rangle$, and $\langle 1,3,p\rangle$, we expect to have only finitely many (p,ℓ)-universal ones.

Unfortunately, it turns out that $X_{Q,p}$ is not equidistributed modulo p. The distribution appears to be not too far from normal (and in fact, seems to be skewed in favor of even fewer universal forms), and so the preceding heuristic computation still provides non-trivial information, especially since most of the values α are much larger than $\alpha > 7$ that we needed.

Further, the above consideration shows an important aspect: the form $\langle 1,2,p\rangle$ is by orders more likely to be (p,ℓ)-universal than any other form. Thus we performed yet another experiment.

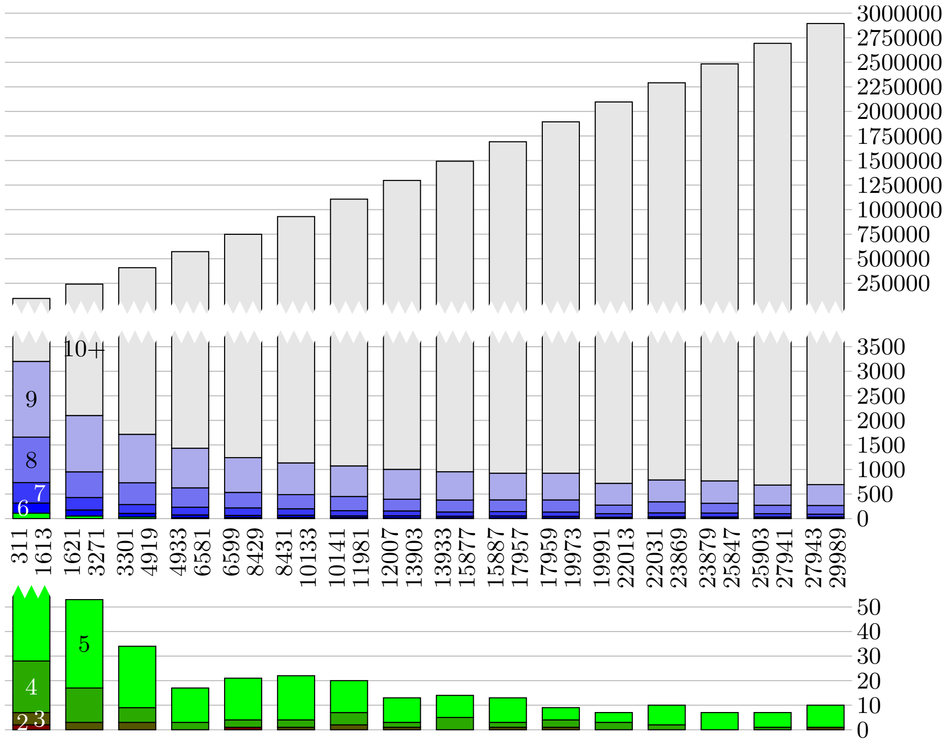

FIGURE 2. Number of (p,ℓ)'s such that $X_{Q,p} \cap [0, 120000p] \cap S_{p,\ell} = m$ for $m = 0, 1, \dots, 9$. For details, see § 3.4.

3.4. **(p,ℓ)-universality of $\langle 1,2,p\rangle$.** In order for $\langle 1,2,p\rangle$ to be (p,ℓ)-universal, it has to be anisotropic precisely at p (and ∞), which happens if and only if $p \equiv 5,7 \pmod 8$. For these primes, we calculated $X_{Q,p}$ (or rather a subset of it, namely $X_{Q,p} \cap [0, 120000p]$) using the same software as in § 3.3 and checked whether it contains elements from all classes $\not\equiv 0 \pmod p$. The computation took 290 CPU days to complete. It turns out that is $\langle 1,2,p\rangle$ is not (p,ℓ)-universal for any $103 \le p < 30000$. And not only that; we even observe that classes containing small number of elements are extremely rare, as can be seen in Figure 2.

In Figure 2, primes $300 < p < 30000$ with $p \equiv 5,7 \pmod 8$ are shown, with each bar corresponding to a group of 100 primes. For each group and $m < 10$, we show (in shades of blue) the number of (p,ℓ) such that our (lower) estimate of $X_{Q,p} \cap S_{p,\ell}$ equals m. For comparison, we show (in gray) the total number of ℓ's. Because there are very few (p,ℓ) for which $m \le 5$, we highlight these (in red and green) in the bottom chart. Note that $m \le 1$ never appears for $p \ge 103$ and that $m = 2$ appears only for $p \le 1181$ and $p = 6607$.

Of course, similar computations can be done for other forms with small α, such as $\langle 1,1,p\rangle$ or $\langle 1,3,p\rangle$. We tested $\langle 1,1,p\rangle$ and $\langle 1,3,p\rangle$ for primes $p < 15000$ getting no candidates for (p,ℓ)-universal forms (other than $p = 2,3$). This computation took 210 CPU days to complete.

3.5. **Conclusion of the experiments.** We can summarize our observations from the computations as follows:

(1) there is strong evidence that only finitely many diagonal ternary forms $Q \neq \langle 1,2,p\rangle, \langle 1,1,p\rangle, \langle 1,3,p\rangle$ are (p,ℓ)-universal (see § 3.3);

(2) $Q = \langle 1,2,p\rangle$ is not (p,ℓ)-universal for any $103 < p < 30000$, and also the cases such that the set of gaps $X_{Q,p} \cap S_{p,\ell}$ is small rapidly cease to exist as p increases (see § 3.4). Likewise for $\langle 1,1,p\rangle, \langle 1,3,p\rangle$ (that are not (p,ℓ)-universal for $3 < p < 15000$).

This convinces us that only finitely many (p,ℓ)-universal diagonal ternary forms exist; and this is the claim of Conjecture 2. Conjecture 3 is then motivated by the absence of (p,ℓ)-universal forms for $103 \leq p < 1257$ (see § 3.1) and the non-universality of $\langle 1,2,p\rangle$ up to 30000.

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