# Zooming Cautiously: Linear-Memory Heuristic Search With Node Expansion Guarantees 

Laurent Orseau ${ }^{1}$, Levi H. S. Lelis ${ }^{2}$, Tor Lattimore ${ }^{1}$<br>${ }^{1}$ DeepMind, UK<br>${ }^{2}$ Universidade Federal de Viçosa, Brazil<br>\{lorseau, lattimore\}@google.com, levi.lelis@ufv.br


#### Abstract

We introduce ${ }^{1}$ and analyze two parameter-free linear-memory tree search algorithms. Under mild assumptions we prove our algorithms are guaranteed to perform only a logarithmic factor more node expansions than $\mathrm{A}^{*}$ when the search space is a tree. Previously, the best guarantee for a linear-memory algorithm under similar assumptions was achieved by IDA*, which in the worst case expands quadratically more nodes than in its last iteration. Empirical results support the theory and demonstrate the practicality and robustness of our algorithms. Furthermore, they are fast and easy to implement.


## 1 Introduction

The A* algorithm [Hart et al., 1968] is optimal in the sense that it only expands nodes for which the cost is smaller than the goal [Dechter and Pearl, 1983]. Unfortunately, in the worst case the memory use of $\mathrm{A}^{*}$ grows linearly with its running time, which can be a bottleneck for large problems. Thus, it is often preferable to have algorithms with a smaller memory footprint, possibly at the expense of an increased search time. The best one can expect from a complete algorithm is that the memory depends linearly on the solution depth [Korf, 1985].
The landmark algorithm for low-memory search is IDA* [Korf, 1985]. At the same time, in problem domains that can be represented by a tree, IDA* is guaranteed to expand at most $O\left(N_{*}^{2}\right)$ nodes, where $N_{*}$ is the number of expansions required by $\mathrm{A}^{*}$ with ties broken for the worst case, assuming the search space is a tree. Although such theoretical guarantees are welcome, in many problems IDA* really does expand $\Omega\left(N_{*}^{2}\right)$ nodes. This occurs especially in domains with a large diversity of edge costs [Sarkar et al., 1991]. Several methods have been developed to mitigate this issue [Burns and Ruml, 2013; Sharon et al., 2014; Hatem et al., 2018; Sarkar et al., 1991; Wah and Shang, 1994], but theoretical

[^0]guarantees (when provided) depend on strong assumptions such as uniformity of the costs or of the branching factor [Hatem et al., 2015, for example (cf. Assumptions 1, 2 and 3)]. Another low-memory search algorithm is RBFS [Korf, 1993], which has the same worst-case number of node expansions as IDA*.

There is a large literature on algorithms that accept a memory budget as a parameter [Sen and Bagchi, 1989; Russell, 1992; Zhou and Hansen, 2003, and many others]. Our focus, however, is on lightweight algorithms for which the memory footprint grows linearly with the search depth. Although we restrict our analysis to tree search, we expect that techniques for graph search, such as transposition tables [Reinefeld and Marsland, 1994], can be incorporated into our approach.

Like $\mathrm{A}^{*}$, our algorithms use a cost function $f$ on the nodes that we assume to be nondecreasing from parent to child. Importantly, and in contrast to previous works, we make no regularity assumptions on the search tree, such as polynomial or exponential growth, or on the distribution of $f$-values in the search tree.

We propose two novel parameter-free algorithms that enjoy theoretical guarantees on the number of node expansions relative to $\mathrm{A}^{*}$ for tree search and use memory that grows linearly with the search depth. The first algorithm, Zoomer, expands at most $O\left(N_{*} \log \left(\theta^{*} / \delta_{\text {min }}\right)\right)$ nodes, where $\theta^{*}$ is the cost of an optimal solution and $\delta_{\text {min }}$ is the minimum difference of cost thresholds between two iterations of IDA*-it is bounded below by the numerical precision, and for integer costs $\delta_{\min }=1$. The log factor corresponds to the number of bits necessary to encode $\theta^{*}$ within precision $\delta_{\text {min }}$. The central idea of this algorithm is to cap the number of node expansions at $2^{k}$ nodes at iteration $k$, and perform at each such iteration a binary search for the cost threshold.

The second algorithm, ZigZagZoomer (Z3), interleaves the iterations of Zoomer instead of performing them sequentially. This is achieved by using a scheduler that may be of independent interest. If Zoomer finds the solution in $M$ node expansions, Z3 will not take more than a factor $\log M$ compared to Zoomer, and removes the explicit dependence on $\delta_{\text {min }}$ to replace it with the difference $\delta^{*}$ of $f$ cost between the solution cost and the next node of higher cost.

Empirical results in heuristic search domains and in two domains we introduce show that although previous
algorithms can perform well in some domains, they can fail in others, depending on the underlying structure of the problem. By contrast, both Z3 and Zoomer are robust to the structure of the problems, performing well in all domains.

## 2 Notation and Background

Let $\mathbb{N}_{1}:=\{1,2,3 \ldots\}, \mathbb{N}_{0}:=\{0,1,2 \ldots\}$ and $\lceil x\rceil^{+}:=$ $\max (0,\lceil x\rceil)$. The set of all nodes in the underlying search tree describing the problem is $\mathcal{N}$, which may be infinite. $\mathcal{C}(n) \subset \mathcal{N}$ is a function that returns the set of children of a node $n$. The maximum branching factor of the search tree is $b:=\max _{n \in \mathcal{N}}|\mathcal{C}(n)|$. We are given a cost function, $f: \mathcal{N} \rightarrow(0, \infty)$ and assume that $f$ is nondecreasing so that $f(m) \geq f(n)$ for all $n \in \mathcal{N}$ and $m \in \mathcal{C}(n)$. Let $\mathcal{N}(\theta):=\{n \in \mathcal{N}: f(n) \leq \theta\}$ be the set of nodes for which the cost does not exceed $\theta$. Let $\mathcal{M}(\theta):=$ $\cup_{n \in \mathcal{N}(\theta)} \mathcal{C}(n) \backslash \mathcal{N}(\theta)$ be the nodes at the fringe. The set of all solution nodes is $\mathcal{G} \subseteq \mathcal{N}$. The cost of an optimal solution is $\theta^{*}:=\min _{n \in \mathcal{G}} f(n)$. Let $N(\theta):=|\mathcal{N}(\theta)|$ and $N_{*}:=$ $N\left(\theta^{*}\right)$. Define $\delta(\theta):=\min _{n \in \mathcal{M}(\theta)} f(n)-\theta$, which is strictly positive by the assumption that the $f$ cost is nondecreasing. Furthermore, $N\left(\theta^{\prime}\right)=N(\theta)$ for all $\theta^{\prime} \in[\theta, \theta+\delta(\theta))$. Let $\delta_{\text {min }}:=\min _{n \in \mathcal{N}, f(n) \leq \theta^{*}} \delta(f(n))$, which corresponds to the minimum difference in cost thresholds between two iterations of IDA* before the optimal solution. Finally, let $\delta^{*}:=\delta\left(\theta^{*}\right)$.
Remark 1. In many definitions of tree search the edges in the tree are associated with costs and $f=g+h$ where $g(n)$ is the cumulative cost from the root to $n$ and $h$ : $\mathcal{N} \rightarrow \mathbb{R}$ is a consistent heuristic, which guarantees that $f$ is nondecreasing. In case $h$ is admissible but not consistent, $f$ can be made monotone nondecreasing using pathmax [Mérõ, 1984; Felner et al., 2011]. In our analysis it is convenient to deal only with $f$, however.
Iterative deepening An iterative deepening tree search algorithm makes repeated depth-first searches (DFS) with increasing cost threshold. IDA* and its variants are all based on this idea, as are our algorithms, but with a few new twists.

The pseudo-code for DFS is given in Algorithm 1, which is largely the classic implementation with branch-and-bound optimization to avoid visiting provably suboptimal paths. It returns an optimal solution or no solution, the number of nodes expanded, the maximum cost $\theta^{-}$among the visited nodes whose costs do not exceed $\theta$, and the minimum cost $\theta^{+}$of the visited nodes whose costs exceed $\theta$. Thus we have $\theta^{-} \leq \theta<\theta^{+}$. A node is said to be expanded when DFS passes through Line 8.

Slightly less usual, it also takes as an argument a budget on the number of nodes to expand, and immediately terminates with "budget exceeded" when too many nodes have been expanded. If the budget is not exceeded, then $\theta^{-}=$ $\max \{f(n): n \in \mathcal{N}(\theta)\}$ and $\theta^{+}=\min \{f(n): n \in \mathcal{M}(\theta)\}$. Observe that if during one call to DFS a suboptimal solution is found and the search budget is not exceeded, then necessarily an optimal solution will be returned. This optimality property is transferred to all the algorithms presented in this paper.

IDA* always calls DFS with unlimited budged and it starts with threshold $\theta=f$ (root), subsequently using $\theta=\theta^{+}$ from the previous iteration.

Algorithm 1: Depth-First Search (DFS) starting at the given node with a cost bound $\theta$ and an expansion budget N . Returns "budget exceeded" when the budget of node expansions is exceeded, otherwise returns the descendant solution of minimum cost if one exists, or none otherwise. It also returns the maximum $\operatorname{cost} \theta^{-}$ below $\theta$ of expanded nodes, and the minimum $\operatorname{cost} \theta^{+}$above $\theta$ of non-expanded nodes among the visited nodes.

```
def DFS(node, \(\theta, N\) ):
    c := \(f\) (node)
    if c > \(\theta\) : return none, \(0,-\infty\), c
    if is_goal(node): return node, \(0, \mathrm{c},+\infty\)
    if \(\mathrm{N}==0\) : return "budget exceeded", \(0, \mathrm{c},+\infty\)
    \# Node expansion
    n_used := 1 \# number of expansions
    best_desc = none \# best descendant solution
    \(\theta^{+}\):= + \# min cost among nodes in fringe
    \(\theta^{-}\):= c \# max cost among nodes expanded
    for child in \(\mathcal{C}\) (node):
        res, \(m, \theta_{2}^{-}, \theta_{2}^{+}:=\)DFS(child, \(\theta, \mathrm{N}-\mathrm{n} \_\)used)
        n_used += m
        \(\theta^{-}:=\max \left(\theta^{-}, \theta_{2}^{-}\right)\)
        \(\theta^{+}:=\min \left(\theta^{+}, \theta_{2}^{+}\right)\)
        if res is a Node and ( best_desc is none or
                \(f(\) res \()<f(\) best_desc) ):
            best_desc := res \# better solution found
            \(\theta\) := f(res) \# branch and bound
        elif res == "budget exceeded":
            return res, n_used, \(\theta^{-}, \theta^{+}\)
    return best_desc, n_used, \(\theta^{-}, \theta^{+}\)
```

If the cost $\theta^{*}$ of the optimal solution were known in advance, then a single call to DFS with threshold $\theta^{*}$ and an unlimited budget would find the optimal solution with no overhead relative to $\mathrm{A}^{*}$ in trees. Of course the optimal cost is usually unknown, which is overcome by calling DFS repeatedly with increasing thresholds. Algorithms of this type over-expand relative to A* for two reasons. First, in early iterations they re-expand many of the same nodes. Second, in the final iteration when $\theta \geq \theta^{*}$, they tend to overshoot. Overshooting is generally more costly than undershooting because trees usually grow quite fast, which is why IDA* increases $\theta$ in the most conservative manner possible.

To illustrate a typical case, suppose that DFS is called with an unlimited budget and threshold $\theta=\theta^{*}+c$ for $c \geq \delta_{\text {min }}$. Since $f$ can be constant with depth, that the number of nodes $N(\theta)$ can be arbitrarily large compared to $N\left(\theta^{*}\right)$. Even when insisting that $f$ has a minimum edge increment $\sigma$, the number of nodes would still grow exponentially fast as $N_{*} b^{c / \sigma}$. Algorithms that update the threshold heuristically without budget constraints are not protected against serious over-expansion and do not effectively control the number of node expansions in the worst case.

Algorithm 2: The Zoomer algorithm.

```
def Zoomer(root):
    lower := f(root) # assumed > 0
    # N N: number of expansions at the root
    # up_min: lower bound on upper
    res, N0, _, up_min := DFS(root, lower, \infty)
    if res is a Node: return res
    for k := 1, 2, ...:
        upper := \infty
        while upper }\not=\mathrm{ up_min:
            if upper == \infty:
                0 := lower x 2 # sky is the limit
            else:
                0:= (upper + lower) / 2
            0 := max( }0\mathrm{ , up_min)
            res, _, }\mp@subsup{0}{}{-},\mp@subsup{0}{}{+}:= DFS(root, 0, N N 2 % 
            if res is a Node:
                return res # solution found
            elif res == "budget exceeded":
                upper := 㫜 # reduce upper bound
            else: # within budget, no solution found
                lower := 0 # increase lower bound
            up_min := 㫜
```


## 3 Zoomer

The insight behind Zoomer is that the risk of calling DFS with a large threshold can be mitigated by limiting the number of expanded nodes in each iteration. The pseudocode is provided in Algorithm 2.

Zoomer operates in iterations $k \in \mathbb{N}_{0}$. In the zeroth iteration it makes a single call to DFS with an unlimited budget and threshold $\theta=f$ (root) (Line 5). The number of nodes expanded by this search is denoted by $N_{0}$, which is usually quite small and by definition satisfies $N_{0} \leq N^{*}$. In subsequent iterations $k \in \mathbb{N}_{1}$ the algorithm makes multiple calls to DFS (Line 17), all with a budget of $N_{0} 2^{k}$, to perform an exponential search on $\theta$ (Lines 12 and 14) to identify whether there exists a feasible solution within the budget (Line 10) [Bentley and Yao, 1976].

Let $N_{k}$ be the total number of nodes expanded by Zoomer during iteration $k$, which includes multiple calls to DFS.
Theorem 2. Assuming $f$ (root) $>0$, Zoomer returns an optimal solution after expanding no more nodes than

$$
\sum_{k=0}^{\infty} N_{k} \leq \max \left\{1,4 \omega_{1}\right\} N^{*}
$$

where $\omega_{1}:=\left\lceil\log _{2}\left(\theta^{*} / \theta_{0}\right)\right\rceil+\left\lceil\log _{2}\left(\theta^{*} / \delta_{\min }\right)\right\rceil$.
Proof. Let $B_{k}:=N_{0} 2^{k}$ be the expansion budget at iteration $k$ and define the minimum cost $\theta_{k}$ at which the budget is exceeded: $\theta_{k}:=\min _{n \in \mathcal{N}}\left\{f(n): N(f(n))>B_{k}\right\}$, and let $K:=\min \left\{k \in \mathbb{N}_{0}: B_{k} \geq N^{*}\right\}=\left\lceil\log _{2}\left(N^{*} / N_{0}\right)\right\rceil$ be the first iteration with enough budget to find a solution.

Observe that DFS with budget $B_{k}$ ensures that when it returns with "budget exceeded" along with the return value $\theta^{-}$, if we call again DFS(root, $\theta^{-}, B_{k}$ ) it will also return with "budget exceeded", which means that $\theta_{k} \leq \theta^{-}$and so we always have upper $\geq \theta_{k}$ (Line 22). Similarly, the algorithm also ensures that we always have lower $<\theta_{k}$.
Iterations $k<K$ Now, suppose that at some point we have upper - lower $\leq \delta_{\text {min }}$. Then we have upper $\leq$ lower $+\delta_{\text {min }}<\theta_{k}+\delta\left(\theta_{k}\right)$ which by Line 22 implies that upper $=\theta_{k}$. We also have lower $\geq$ upper $-\delta_{\min }=\theta_{k}-\delta_{\text {min }}$ which by Line 25 implies that up_min $=\theta_{k}=$ upper, which means that no solution exists in [lower, upper] since lower $+\delta$ (lower) $=$ up_min $=$ upper. Therefore, iteration $k<K$ terminates no later than when upper - lower $\leq \delta_{\text {min }}$ (Line 10).

For $a_{k}$ times the algorithm goes through Line 12 before calling DFS, and thus lower is doubled until $\theta=2 \times$ lower $\geq$ $\theta_{k}$ when DFS returns with "budget exceeded", at which point upper is set to $\theta^{-} \leq \theta$; Together with lower $<\theta_{k}$ (by definition of $\theta_{k}$ ), this implies that upper - lower $\leq$ $\theta-$ lower $=$ lower $\leq \theta_{k}$. Thus, starting at worst from $\theta_{0}$,

$$
\begin{aligned}
a_{k} \leq \min \left\{d \in \mathbb{N}_{0}: \theta_{0} 2^{d} \geq \theta_{k}\right\} & =\left\lceil\log _{2}\left(\theta_{k} / \theta_{0}\right)\right\rceil \\
& \leq\left\lceil\log _{2}\left(\theta^{*} / \theta_{0}\right)\right\rceil
\end{aligned}
$$

Thereafter, for $b_{k}$ times the algorithm goes through Line 14 before calling DFS until upper - lower $\leq \delta_{\text {min }}$. Hence

$$
\begin{aligned}
b_{k} \leq \min \left\{d \in \mathbb{N}_{0}: \theta_{k} / 2^{d} \leq \delta_{\min }\right\} & =\left\lceil\log _{2}\left(\theta_{k} / \delta_{\min }\right)\right\rceil \\
& \leq\left\lceil\log _{2}\left(\theta^{*} / \delta_{\min }\right)\right\rceil
\end{aligned}
$$

Iteration $K$ For iteration $K$ things are slightly different. We still have lower $<\theta^{*}$ (otherwise a solution would have already been found) and upper $\geq \theta_{K}$. Assume that at some point upper - lower $\leq 2 \bar{\delta}^{*}$. Since upper is set, the algorithm goes through Line 14 before calling DFS. Thus $\theta=$ (upper + lower) $/ 2 \leq$ lower $+\delta^{*}<\theta^{*}+\delta^{*}$ and similarly $\theta \geq$ upper $-\delta^{*} \geq \theta_{K}-\delta^{*} \geq \theta^{*}$ where the last inequality is because there is enough budget for $\theta^{*}$ by definition of $K$ but not for $\theta_{K}$. This implies that $\theta \in\left[\theta^{*}, \theta^{*}+\delta^{*}\right)$ and thus the call to DFS returns with an optimal solution within budget.

Starting from the lower bound $\theta_{0}$, the number of calls to DFS after going through Line 12 before $\theta \geq \theta^{*}$ is less than $\left\lceil\log _{2}\left(\theta^{*} / \theta_{0}\right)\right\rceil$. At this point, either $\theta \in\left[\theta^{*}, \theta_{K}\right)$ and an optimal solution is returned, or $\theta \geq \theta_{K}$. In the latter case, upper $:=\theta^{-} \leq \theta=2$ lower, and since lower $<\theta^{*}$ we have upper - lower $<\theta^{*}$. Subsequently the number of calls to DFS after going through Line 14 is at most

$$
\begin{aligned}
\min \left\{d \in \mathbb{N}_{0}: \theta^{*} / 2^{d} \leq 2 \delta^{*}\right\} & \leq\left\lceil\log _{2}\left(\theta^{*} /\left(2 \delta^{*}\right)\right)\right\rceil^{+} \\
& \leq\left\lceil\log _{2}\left(\theta^{*} / \delta_{\min }\right)\right\rceil^{+}
\end{aligned}
$$

Therefore, over all iterations $k \leq K$ and including the first call to DFS, when $\omega_{1} \geq 1$ the number of node expansions is bounded by

$$
N_{0}+\sum_{k=1}^{K}\left(a_{k}+b_{k}\right) B_{k} \leq \sum_{k=0}^{K} \omega_{1} B_{k} \leq 2 \omega_{1} N_{0} 2^{K} \leq 4 \omega_{1} N^{*}
$$



Table 1: Sequence A6519: https://oeis.org/A006519.

Remark 3. Because the ordering of the nodes expanded by DFS is invariant under translation of the cost function $f$, if $f$ (root) is close to zero or negative, the algorithm can simply be run with cost function $f^{\prime}=f+1-f$ (root). If $\theta^{*}$ is known up to constant factors, then the bound can be improved by translating costs so that $\theta^{*} / \theta_{0}=O(1)$.
Remark 4. Consider a class of search trees with increasing solution depth $d$ and suppose that $\theta^{*}=O(d)$ and $\delta_{\text {min }}$ is constant, both of which are typical. When the tree has polynomial growth, then $N_{*}=O\left(d^{p}\right)$ and Zoomer expands a factor of $O\left(\log N_{*}\right)$ more nodes than $N_{*}$. In exponential domains with $N_{*}=O\left(b^{d}\right)$, Zoomer expands a factor of $O\left(\log \log N_{*}\right)$ more nodes than $N_{*}$.

## 4 Uniform Doubling Scheduling

Iterations $k<K$ in Zoomer can take a long time to terminate when $\delta_{\min }$ is small, where $K$ is the first iteration of Zoomer with enough budget to find a solution. This can be mitigated by interleaving the iterations in a careful manner. We introduce the Uniform Doubling Scheduler (UDS), which provides the correct interleaving and may be of independent interest. UDS is inspired by Luby's scheduling algorithm for speeding up randomized algorithms [Luby et al., 1993]. Suppose you have access to a countably infinite number of deterministic programs indexed by $k \in \mathbb{N}_{0}$ with unknown running times $\left(T_{k}\right)_{k=0}^{\infty}$. UDS operates in blocks $j \in \mathbb{N}_{1}$. In the $j$ th block it runs program $k_{j}$ for $A(j)$ computation steps where $k_{j}=\log _{2} A(j)$ and $A(j)$ is the $j$ th value of integer sequence A6519, which are all powers of 2 (see Table 1 and Algorithm 3). Note, the state of programs that are not running in the current block are stored in memory.

Theorem 5. The total number of computation steps before any program $k$ halts is at most $\left(4+k+\log _{2}(n)\right) n 2^{k}$, where $n=\left\lceil T_{k} / 2^{k}\right\rceil$.

Provided that program $k$ is not expected to run in time much less than $2^{k}$, then Theorem 5 shows that the overhead incurred by UDS is not especially severe. The proof of Theorem 5, provided below, depends on two lemmas.

Lemma 6 (Orseau et al., 2018, Lemma 10). For all $j \in \mathbb{N}_{1}$ and $k \in \mathbb{N}_{0}: A(j)=2^{k} \Leftrightarrow\left(\exists!n \in \mathbb{N}_{1}: j=(2 n-1) 2^{k}\right)$.
Lemma 7. $\sum_{i=1}^{j} A(i) \leq \frac{j}{2}\left(3+\left\lfloor\log _{2} j\right\rfloor\right)$ for all $j \in \mathbb{N}_{1}$.
Proof. By Lemma 6, the number of blocks program $k$ has been run for after the scheduler has run $j$ blocks is $n=$ $\max \left\{n \in \mathbb{N}_{1}:(2 n-1) 2^{k} \leq j\right\}=\left\lfloor\left(j / 2^{k}+1\right) / 2\right\rfloor$. Furthermore, the program of largest $k$ that could run up to $j$ is $\max \left\{k \in \mathbb{N}_{0}: 2^{k} \leq j\right\}=\left\lfloor\log _{2} j\right\rfloor$. That is, at block $j$, exactly $1+\left\lfloor\log _{2} j\right\rfloor$ programs have computed for at least one step. Therefore the total number of computation steps over

Algorithm 3: The Uniform Doubling Scheduling algorithm.

```
def A6519(j):
```

    return ((j XOR (j - 1)) + 1) / 2
    def uniform_doubling_scheduling(prog):
states := [] \# growable vector
for $j:=1,2, \ldots:$
$\mathrm{k}:=\log _{2}(\mathrm{~A} 6519(\mathrm{j}))$ \# exact
if $k$ >= len(states):
state[k] := make_state(states, k)
state[k] := run(prog(states, k), $2^{k}$ )
if is_goal(state[k]):
return state[k]
all started programs after block $j$ is

$$
\sum_{i=1}^{j} A(i)=\sum_{k=0}^{\left\lfloor\log _{2} j\right\rfloor}\left\lfloor\frac{j / 2^{k}+1}{2}\right\rfloor 2^{k} \leq \frac{j}{2}\left(3+\left\lfloor\log _{2} j\right\rfloor\right) .
$$

Proof of Theorem 5. UDS always runs program $k$ in blocks of length $2^{k}$. By Lemma 6, program $k$ will be run for the $n$th block once $j=(2 n-1) 2^{k}$. Therefore by Lemma 7, the total computation before program $k$ halts is at most

$$
\sum_{i=1}^{j} A(i) \leq \frac{j}{2}\left(3+\left\lfloor\log _{2}(j)\right\rfloor\right) \leq\left(4+k+\left\lfloor\log _{2}(n)\right\rfloor\right) n 2^{k} .
$$

## 5 ZigZagZoomer

Using UDS to interleave the iterations of Zoomer allows us to replace $\delta_{\min }$ with $\delta^{*}$ in the analysis. The theoretical price for the improvement is at most a logarithmic factor, with the worst case when $\delta^{*}=\delta_{\text {min }}$. See Algorithm 4 (an optimized version is provided in the supplementary material). The following theorem follows from the analysis in the previous two sections.
Theorem 8. Assuming $\theta_{0}:=f($ root $)>0$, Algorithm 4 ensures that an optimal solution is found within a number of node expansions bounded by

$$
\sum_{k=0}^{\infty} N_{k} \leq N_{0}+2\left(4+\left\lceil\log _{2}\left(N_{*} / N_{0}\right)\right\rceil+\log _{2}\left(\omega_{2}\right)\right) \omega_{2} N_{*}
$$

where $\omega_{2}:=\left\lceil\log _{2}\left(\theta^{*} / \theta_{0}\right)\right\rceil+\left\lceil\log _{2}\left(\theta^{*} /\left(2 \delta^{*}\right)\right)\right\rceil^{+}$.
Proof. Algorithm 4 terminates at the latest when program $K=\left\lceil\log _{2}\left(N_{*} / N_{0}\right)\right\rceil$ terminates, since $2^{K} N_{0} \geq N_{*}$. From the proof of Theorem 2 for the last iteration $K$ (only), we know that the number of calls to DFS after going through Line 11 before $\theta \geq \theta^{*}$ is less than $\left\lceil\log _{2}\left(\theta^{*} / \theta_{0}\right)\right\rceil$. Subsequently the number of calls to DFS after going through Line 13 is at most $\left\lceil\log _{2}\left(\theta^{*} /\left(2 \delta^{*}\right)\right)\right\rceil^{+}$. Hence the number $n_{K}$ of blocks of size $2^{K}$ macro steps (each of $N_{0}$ steps) performed by program $K$ before terminating is bound by $n_{K} \leq \omega_{2}$. Thus, applying Theorem 5 with $n=n_{K}$ and

Algorithm 4: The Z 3 algorithm (simple version). $f$ (root) is assumed strictly positive.

```
def ZigZagZoomer(root):
    lower := [f(root)] \# growable vector
    upper := [] \# growable vector
    res, \(N_{0, ~, ~}, \quad\) : \(=\) DFS(root, lower, \(+\infty\) ) \#4
    if res is a Node: return res
    for \(j:=1,2, \ldots\) :
        \(\mathrm{k}:=\log _{2}(\mathrm{~A} 6519(\mathrm{j}))\) \# exact
        if \(k>=\) len(lower):
            lower[k] := \(f\) (root)
        if \(k>=\) len(upper):
            \(\theta:=2 \times\) lower[k]
        else:
            \(\theta\) := (upper[k] + lower[k]) / 2
        res, _, _, _ := DFS(root, \(\left.\theta, N_{0} 2^{k}\right)\)
        if res is a Node:
            return res \# solution found
        elif res == "budget exceeded":
            upper[k] := \(\theta\) \# set or reduce upper bound
        else:
                lower[k] := \(\theta\) \# increase lower bound
```

$k=K$ gives that the number of macro steps performed by the scheduler is at most

$$
\begin{aligned}
& \left(4+K+\left\lfloor\log _{2}\left(n_{K}\right)\right\rfloor\right) n_{K} 2^{K} \\
& \quad \leq 2\left(4+\left\lceil\log _{2}\left(N_{*} / N_{0}\right)\right\rceil+\left\lfloor\log _{2} \omega_{2}\right\rfloor\right) \omega_{2} N_{*} / N_{0}
\end{aligned}
$$

Multiplying by the number $N_{0}$ of steps per macro step and adding the first call to DFS (Line 4) leads to the result.

Remark 9. Sometimes an iteration $k>K$ terminates earlier than iteration $K$. The analysis can be improved slightly to reflect this, as we discuss in the supplementary material.
Remark 10. Algorithm 4 is simple but a little wasteful. See the supplementary material for an optimized version that terminates iterations $k<K$ for which the budget is provably insufficient.

## 6 Experiments

Algorithms tested We test IDA* [Korf, 1985], Zoomer, Z3 (optimized), EDA* [Sharon et al., 2014], and IDA*_CR [Sarkar et al., 1991]. $\operatorname{EDA}^{*}(\gamma)$ is a variant of IDA* that repeatedly calls DFS with unlimited budget and increasing thresholds. In the $k$ th iteration it uses a threshold of $\gamma^{k}$ where $\gamma>1$. In our experiments we take $\gamma=2$. In a given iteration, IDA*_CR collects the costs of nodes in the fringe into buckets. It then selects the cost for its next iteration based on the information stored in the buckets. The cost is chosen such that IDA*_CR is likely to expand more than $b^{k}$ nodes in the $k+1$-th iteration. Similarly to EDA*, we set $b=2$. We also present the results of $\operatorname{DFS}\left(\theta^{*}\right)$, which is DFS with its initial bound set to the optimal $\operatorname{cost} \theta^{*}$. Our goal is to verify to how close each algorithm is from $\operatorname{DFS}\left(\theta^{*}\right)$. For a given heuristic function and without duplicate detection, the number of nodes
expanded by $\operatorname{DFS}\left(\theta^{*}\right)$ is a lower bound on the number of nodes expanded by the algorithms we evaluate.
Problem domains We use three traditional problem domains in our experiment: 15-Puzzle [Pekonen et al., 2007; Doran and Michie, 1966], (12, 4)-TopSpin [Lammertink, 1989], and the 15-Pancake puzzle [Dweighter, 1977]. All these domains were implemented in PSVN [Holte et al., 2014]. We use pattern database heuristics (PDBs) [Culberson and Schaeffer, 1996] for all three domains. The PDBs for the 15 -Puzzle and (12, 4)-TopSpin are generated by projecting tiles 6-15 and tokens $8-12$, respectively. The PDB for the 15 -Pancake is generated with domain abstraction by mapping pancakes $2-8$ to pancake 1 (see the PSVN manual). We show results both when operator costs are unitary, as usual, and also when they are chosen uniformly at random in [1..10000] in order to test the robustness of the algorithms. We also include the Chain problem, where the search tree is a simple chain (branching factor is 1 ) and the solution is placed at a depth chosen uniformly at random between 1 and 10000.

The branching factor of the search tree of the domains described above does not vary much. Moreover, all three domains have a relatively large density of solutions; there is a solution in any subtree of the search tree. These properties make the problem of overshooting less pronounced. In order to evaluate the robustness of the search algorithms, we introduce the Coconut problem, which is a domain with varied branching factor and small solution density. In the Coconut problem we set the heuristic $h$ to 0 and define $f=g$ as follows. The root has $f$ cost 1 . There are $b$ coconut trees, and only one has a coconut in its branches. The agent must climb the trees and find the coconut. The $b$ trunks all have the same size $D$, sampled uniformly in [1..10000]. Moving along one trunk (using the same action repeatedly) costs 1 , jumping from one trunk to another costs $2 D$. After the trunk come the branches, where each branch is a $b$-ary tree. Moving along the branches or jumping from one branch to another costs $1 / 10$. The depth of the coconut in the branches is sampled from a geometric distribution with parameter $1 / 4$.
Experimental setup We use 100 instances for each problem domain and report the number of nodes expanded and number of problems solved with a time limit of 5 hours for the PSVN domains, 1 hour for the Coconut problem and no limit for the Chain problem. We use the number of nodes expanded instead of running time because all search algorithms expand roughly the same number of nodes per second. The results are shown in Fig. 1, where the $y$-axis shows the number of nodes expanded in log scale with the dashed lines being powers of 2 . The $x$-axis depicts the instances from easiest to hardest. To ease visualization, the instances in the $x$-axis are sorted independently for each algorithm. The curves are thus not directly comparable for a particular instance, but they allow one to compare the growth in difficulty for the various solvers.
Discussion Only Zoomer and Z3 perform reliably across all domains. IDA* performs very well on domains with unit costs and a branching factor greater than 1 , sometimes terminating before $\operatorname{DFS}\left(\theta^{*}\right)$ because it stops as soon as it finds a solution. But due to its conservative selection


Figure 1: Profiles of node expansions on $\log$ scale for 100 instances on four different domains. Values are sorted for each solver individually. Lines that terminate early mean the time limit is reached for all subsequent instances. Dashed gray lines are powers of 2. Other dashed lines, when displayed, show the number of node expansions when the algorithm is terminated before a solution is found. The 3 r.h.s. plots are with random-costs operators on the first row, and with unit-costs operators on the second row.
of the threshold, it also expands too many nodes on the other domains. IDA*_CR performs best on the 15-Puzzle, (12, 4)-Topspin and 15-Pancake with random operator costs. On these same problems, EDA*, Zoomer and Z3 perform similarly. However, both IDA*_CR and EDA* perform very poorly on the Coconut problem, each of them solving only 4 instances, and are outperformed even by IDA*.

To explain the behaviour of IDA*_CR and EDA* on the Coconut problem, consider an instance where the depth of the trunk is $D=2690$ and the depth of the coconut in the branches is $q=6$. The cost set by EDA*(2) in the last iteration is 4096, resulting in a search tree with approximately $3^{(4096-2690) / 0.1} \approx 10^{6700}$ nodes. IDA*_CR is not saved by choosing a threshold in the range of costs observed in the fringe: Due to the large cost for jumping from one tree to the other, the threshold in the last iteration can be as large as $2 \times 2689$. This is not a carefully selected example. On the contrary, care must be taken to choose a Coconut problem for which these algorithms work. Even EDA* (1.01) would be only marginally better. For comparison, $A^{*}$ and IDA* expand about $10^{3}$ and $10^{6}$ nodes respectively on this instance.

By contrast, Zoomer and Z3 perform both well on the Coconut problem: even though they require a nonnegligible factor compared to $\operatorname{DFS}\left(\theta^{*}\right)$, it appears this factor is independent of the difficulty of the instance, in line with the theoretical results. Zoomer performs slightly better than Z3 because the gaps $\{\delta(f(n)): n \in \mathcal{N}\}$ are relatively large.

## 7 Conclusion

We derived two linear-memory heuristic search algorithms that require no parameter tuning and come with guarantees on the number of node expansions in the worst case relative to $\mathrm{A}^{*}$. Zoomer is guaranteed to find optimal solutions in $O\left(N_{*} \log \left(\theta^{*} / \delta_{\min }\right)\right)$ node expansions, where $\theta^{*}$ is the cost of the solution and $\delta_{\text {min }}$ is the smallest difference in cost between any two nodes that may not be on the same branch. The second algorithm, ZigZagZoomer (Z3), expands at most logarithmically more nodes than Zoomer in the worst case, but replaces $\delta_{\min }$ with a 'local version' $\delta^{*}$ for which $\delta^{*} \gg$ $\delta_{\text {min }}$ often holds. Theoretical guarantees are summarized in Table 2. Zoomer and Z3 perform very robustly in all domains tested, whereas all other tested algorithms perform poorly in at least one domain.

Table 2: All algorithms return a minimal-cost solution and require memory linear with the depth of the search; $\omega_{1}=O\left(\log \left(\theta^{*} / \delta_{\text {min }}\right)\right)$ and $\omega_{2}=O\left(\log \left(\theta^{*} / \delta^{*}\right)\right)$.

| Algorithm | Worst case |
| :--- | :--- |
| EDA $^{*}$ | $\Omega\left(b^{N_{*}}\right)$ |
| IDA $^{*}$ CR | $\Omega\left(b^{N_{*}}\right)$ |
| IDA* | $\Omega\left(N_{*}^{2}\right)$ |
| Zoomer | $O\left(\omega_{1} N_{*}\right)$ |
| Z3 | $O\left(N_{*} \omega_{2} \log \left(N_{*} \omega_{2}\right)\right)$ |

Although Zoomer and Z3 are not the fastest linear-memory
heuristic search algorithms for all problems, they do perform well consistently and their robustness makes them a safe choice. More aggressive algorithms sometimes perform better on certain problems, but can also fail catastrophically, as evidenced by the Coconut problem. When prior knowledge is available about the structure of the tree it may be preferable to run a safe algorithm like Zoomer or Z3 in parallel with a more aggressive choice.

The are many interesting directions for future research, including the use of transposition tables to prevent node reexpansion on graphs [Reinefeld and Marsland, 1994].

## Acknowledgements

Many thanks to Csaba Szepesvári, András György, János Kramár, Roshan Shariff, and the anonymous reviewers for their feedback on this work.

## References

[Bentley and Yao, 1976] Jon Louis Bentley and Andrew Chi-Chih Yao. An almost optimal algorithm for unbounded searching. Information Processing Letters, 5(3):82-87, 1976.
[Burns and Ruml, 2013] Ethan Burns and Wheeler Ruml. Iterative-deepening search with on-line tree size prediction. Annals of Mathematics and Artificial Intelligence, 69(2):183-205, Oct 2013.
[Culberson and Schaeffer, 1996] Joseph C. Culberson and Jonathan Schaeffer. Searching with pattern databases. In Canadian Conference on Artificial Intelligence, pages 402-416, 1996.
[Dechter and Pearl, 1983] Rina Dechter and Judea Pearl. The optimality of $\mathrm{A}^{*}$ revisited. In $A A A I$, pages $95-99$, 1983.
[Doran and Michie, 1966] James E Doran and Donald Michie. Experiments with the graph traverser program. Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences, 294(1437):235-259, 1966.
[Dweighter, 1977] Harry Dweighter. E2569. The American Mathematical Monthly, 84:296, 041977.
[Felner et al., 2011] Ariel Felner, Uzi Zahavi, Robert Holte, Jonathan Schaeffer, Nathan R. Sturtevant, and Zhifu Zhang. Inconsistent heuristics in theory and practice. Artificial Intelligence, 175(9-10):1570-1603, 2011.
[Hart et al., 1968] P. E. Hart, N. J. Nilsson, and B. Raphael. A formal basis for the heuristic determination of minimum cost paths. IEEE Transactions on Systems Science and Cybernetics, 4(2):100-107, 1968.
[Hatem et al., 2015] Matthew Hatem, Scott Kiesel, and Wheeler Ruml. Recursive best-first search with bounded overhead. In Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, AAAI'15, pages 1151-1157. AAAI Press, 2015.
[Hatem et al., 2018] Matthew Hatem, Ethan Burns, and Wheeler Ruml. Solving large problems with heuristic search: General-purpose parallel external-memory search. Journal of Artificial Intelligence Research, 62:233-268, jun 2018.
[Helmert et al., 2019] Malte Helmert, Tor Lattimore, Levi H. S. Lelis, Laurent Orseau, and Nathan R. Sturtevant. Iterative budgeted exponential search. In IJCAI. Forthcoming, 2019.
[Holte et al., 2014] Robert C. Holte, Broderick Arneson, and Neil Burch. PSVN Manual (June 20, 2014). Technical Report TR14-03, Computing Science Department, University of Alberta, 2014.
[Korf, 1985] Richard E. Korf. Depth-first iterativedeepening: An optimal admissible tree search. Artificial Intelligence, 27(1):97-109, 1985.
[Korf, 1993] Richard E. Korf. Linear-space best-first search. Artificial Intelligence, 62(1):41-78, 1993.
[Lammertink, 1989] Ferdinand Lammertink. Puzzle or game having token filled track and turntable, October 31989. US Patent 4,871,173.
[Luby et al., 1993] Michael Luby, Alistair Sinclair, and David Zuckerman. Optimal speedup of Las Vegas algorithms. Inf. Process. Lett., 47(4):173-180, September 1993.
[Mérõ, 1984] László Mérõ. A heuristic search algorithm with modifiable estimate. Artificial Intelligence, 23(1):13 -27, 1984.
[Orseau et al., 2018] Laurent Orseau, Levi Lelis, Tor Lattimore, and Theophane Weber. Single-agent policy tree search with guarantees. In NeurIPS, pages 3205-3215, 2018.
[Pekonen et al., 2007] Osmo Pekonen, Jerry Shcum, Die Sonneveld, and Aaron Archer. The 15 puzzle: how it drove the world crazy. The Mathematical Intelligencer, 29(2):83-85, 2007.
[Reinefeld and Marsland, 1994] Alexander Reinefeld and T. Anthony Marsland. Enhanced iterative-deepening search. IEEE Transactions on Pattern Analysis and Machine Intelligence, 16(7):701-710, 1994.
[Russell, 1992] Stuart Russell. Efficient memory-bounded search methods. ECAI-1992, Vienna, Austria, 1992.
[Sarkar et al., 1991] U.K. Sarkar, P.P. Chakrabarti, S. Ghose, and S.C. De Sarkar. Reducing reexpansions in iterativedeepening search by controlling cutoff bounds. Artificial Intelligence, 50(2):207-221, 1991.
[Sen and Bagchi, 1989] Anup K Sen and Amitava Bagchi. Fast recursive formulations for best-first search that allow controlled use of memory. In IJCAI, pages 297-302, 1989.
[Sharon et al., 2014] Guni Sharon, Ariel Felner, and Nathan R. Sturtevant. Exponential deepening A* for real-time agent-centered search. In Carla E. Brodley and Peter Stone, editors, Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence, July 27-31,

2014, Québec City, Québec, Canada., pages 871-877. AAAI Press, 2014.
[Wah and Shang, 1994] Benjamin W. Wah and Yi Shang. Comparision and evaluation of a class of IDA* algorithms. International Journal on Artificial Intelligence Tools, 03(04):493-523, dec 1994.
[Zhou and Hansen, 2003] Rong Zhou and Eric A Hansen. Sweep A*: space-efficient heuristic search in partially ordered graphs. In Proceedings. 15th IEEE International Conference on Tools with Artificial Intelligence, pages 427-434. IEEE, 2003.

## A Generalized analysis of ZigZagZoomer

The bound in Theorem 8 can be improved slightly. Let $K=\min \left\{k: N_{0} 2^{k} \geq N^{*}\right\}$. All iterations $k<K$ will never find the solution. Iterations $k \geq K$ will eventually find the solution. In the analysis of Z3 we simply bounded the number of node expansions of iteration $K$, but it can happen that an even larger $k>K$ will find the solution earlier. Precisely, the $k$ th program will halt once it calls DFS with $\theta \in\left[\theta^{*}, \theta_{k}\right]$ with $\theta_{k}=\max \left\{\theta: N(\theta) \leq N_{0} 2^{k}\right\}$. The number of nodes expanded before iteration $k$ halts is

$$
T_{k} \leq N_{0} 2^{k} \omega\left(\theta^{*}, \theta_{k}\right)
$$

where $\omega(a, b)$ is the number of 'zooms' before iteration $k$ calls DFS with $\theta \in[a, b]$, which satisfies

$$
\omega\left(\theta^{*}, \theta_{k}\right)=O\left(\log \left(\frac{\theta^{*}}{\theta_{0}}\right)+\log \left(\frac{\theta^{*}}{\theta_{k}-\theta^{*}}\right)\right)
$$

Then by Theorem 5, Z3 will find the solution after

$$
O\left(\min _{k \geq K} T_{k} \log \left(T_{k}\right)\right)
$$

node expansions. Exactly which program $k \geq K$ minimizes $T_{k} \log \left(T_{k}\right)$ depends on the blowup of $N(\theta)$ about $\theta=\theta^{*}$.

## B Optimized Z3

Algorithm 5 features a few logical optimizations compared to Algorithm 4. With a little work, they can be used to improve the bound of Theorem 8 , but we will not do this here.

The first improvement is to stop the search using up_min as is done for Zoomer, since once the interval upper - lower is smaller than the gap $\delta$ (lower), provably no solution can be found.

The second one is to replace the individual lower bounds with a global lower bound that any program can raise: indeed, when a program raises the lower bound this means that it has explored all nodes below it without success and thus no other program can find a solution below that cost either. Then, to stop testing the programs that have been proven to not having enough budget to find a solution, we simply skip the indices $j$ of these programs, using the properties of A6519, so that skipped programs take zero computation time.
More improvements A further improvement (valid for Zoomer as well) would gather the cost of a solution if one is found but the budget is exceeded. This cost could then be used as a global upper bound. This would require modifying the DFS algorithm.

For most iterative deepening based algorithms, It is also common to avoid re-evaluating whether a node is a solution. This can be done by passing the previous threshold $\theta$ to DFS and evaluating only nodes of a larger cost. The number of evaluation calls would then only be equal to the number of nodes expanded during the call to DFS with the largest threshold.

In DFS, when a solution has been found, children of cost equal to the solution cost need not be expanded. It is not clear however if the saving in node expansions would be worth the computation cost of the test.

Algorithm 5: The optimized Z3 algorithm.

```
def ZigZagZoomerV2(root):
    upper := [] \# growable vector
    lower := \(f\) (root)
    res, \(N_{0}, \quad\), up_min := DFS(root, lower, \(+\infty\) )
    if res is a Node: return res
    kmin \(:=0\)
    j := 0
    repeat forever:
        \# Skip steps with provably no solution
        \(j+=2^{\mathrm{kmin}}\)
        \(\mathrm{k}:=\log _{2}(\mathrm{~A} 6519(\mathrm{j}))\) \# exact
        if \(k<l e n(u p p e r):\)
            if upper[k] <= up_min:
                \# Will skip all progs \(<=k\)
                kmin := k+1
                \# Will move to next factor of \(2^{k m i n}\)
                \(j-=2^{\mathrm{k}}\)
                continue
        \(\theta:=\) (upper[k] + lower) / 2
    else:
        \(\theta\) := \(2 \times\) lower
    \(\theta:=\max (\theta\), up_min \()\)
    res, _, \(\theta^{-}, \theta^{+}:=\operatorname{DFS}\left(\right.\) root, \(\left.\theta, N_{0} 2^{k}\right)\)
    if res is a Node:
        return res \# solution found
    elif res == "budget exceeded":
        upper[k] := \(\theta^{-}\)\# set or reduce upper bound
    else:
        \# Search terminated within budget without
        \# a solution. No program can find a
        \# solution below this cost.
        lower := \(\theta\)
        up_min \(:=\theta^{+}\)
```

Detection of duplicate states can be performed along the current trajectory to avoid loops in the underlying graph, while keeping a memory that grows only linearly with the depth of the search, but is now a multiple of the size of a state.


[^0]:    ${ }^{1}$ This paper and another independent IJCAI 2019 submission have been merged into a single paper that subsumes both of them [Helmert et al., 2019]. This paper is placed here only for historical context. Please only cite the subsuming paper.

