THE h^*-POLYNOMIALS OF LOCALLY ANTI-BLOCKING LATTICE POLYTOPES AND THEIR γ-POSITIVITY

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ABSTRACT. A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ is called a locally anti-blocking polytope if for any closed orthant $\mathbb{R}^d_\varepsilon$ in $\mathbb{R}^d$, $\mathscr{P} \cap \mathbb{R}^d_\varepsilon$ is unimodularly equivalent to an anti-blocking polytope by reflections of coordinate hyperplanes. In the present paper, we give a formula of the h^*-polynomials of locally anti-blocking lattice polytopes. In particular, we discuss the γ-positivity of the h^*-polynomials of locally anti-blocking reflexive polytopes.

INTRODUCTION

A *lattice polytope* is a convex polytope all of whose vertices have integer coordinates. A lattice polytope $\mathscr{P} \subset \mathbb{R}^d_{\geq 0}$ of dimension d is called *anti-blocking* if for any $\mathbf{y} = (y_1, \dots, y_d) \in \mathscr{P}$ and $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$ with $0 \leq x_i \leq y_i$ for all i, it holds that $\mathbf{x} \in \mathscr{P}$. Anti-blocking polytopes were introduced and studied by Fulkerson [9, 10] in the context of combinatorial optimization. See, e.g., [32]. For $\varepsilon \in \{-1, 1\}^d$ and $\mathbf{x} \in \mathbb{R}^d$, set $\varepsilon\mathbf{x} := (\varepsilon_1 x_1, \dots, \varepsilon_d x_d) \in \mathbb{R}^d$. Given an anti-blocking lattice polytope $\mathscr{P} \subset \mathbb{R}^d_{\geq 0}$ of dimension d, we define

$$\mathscr{P}^{\pm} := \{\varepsilon\mathbf{x} \in \mathbb{R}^d : \varepsilon \in \{-1,1\}^d, \ \mathbf{x} \in \mathscr{P}\}.$$

Since $\mathscr{P}$ is an anti-blocking lattice polytope, $\mathscr{P}^{\pm}$ is convex (and a lattice polytope). Moreover, for any $\varepsilon \in \{-1, 1\}^d$ and $\mathbf{x} \in \mathscr{P}^{\pm}$, we have $\varepsilon\mathbf{x} \in \mathscr{P}^{\pm}$. The polytope $\mathscr{P}^{\pm}$ is called an *unconditional lattice polytope* ([21]). In general, $\mathscr{P}^{\pm}$ is symmetric with respect to all coordinate hyperplanes. In particular, the origin $\mathbf{0}$ of $\mathbb{R}^d$ is in the interior $\mathrm{int}(\mathscr{P}^{\pm})$. Given $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d) \in \{-1, 1\}^d$, let $\mathbb{R}^d_\varepsilon$ denote the closed orthant $\{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i\varepsilon_i \geq 0 \text{ for all } 1 \leq i \leq d\}$. A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d is called *locally anti-blocking* ([21]) if, for each $\varepsilon \in \{-1, 1\}^d$, there exists an anti-blocking lattice polytope $\mathscr{P}_\varepsilon \subset \mathbb{R}^d_{\geq 0}$ of dimension d such that $\mathscr{P} \cap \mathbb{R}^d_\varepsilon = \mathscr{P}^{\pm}_\varepsilon \cap \mathbb{R}^d_\varepsilon$. Unconditional polytopes are locally anti-blocking.

In the present paper, we investigate the h^*-polynomials of locally anti-blocking lattice polytopes. First, we give a formula of the h^*-polynomials of locally anti-blocking lattice polytopes in terms of that of unconditional lattice polytopes. In fact,

Theorem 0.1. *Let $\mathscr{P} \subset \mathbb{R}^d$ be a locally anti-blocking lattice polytope of dimension d and for each $\varepsilon \in \{-1, 1\}^d$, let $\mathscr{P}_\varepsilon$ be an anti-blocking lattice polytope of dimension d such*

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that $\mathscr{P} \cap \mathbb{R}^d_\varepsilon = \mathscr{P}^\pm_\varepsilon \cap \mathbb{R}^d_\varepsilon$*. Then the* h^**-polynomial of* $\mathscr{P}$ *satisfies*

$$h^*(\mathscr{P},x) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} h^*(\mathscr{P}^\pm_\varepsilon, x).$$

In particular, $h^*(\mathscr{P},x)$ *is* γ*-positive if* $h^*(\mathscr{P}^\pm_\varepsilon,x)$ *is* γ*-positive for all* $\varepsilon \in \{-1,1\}^d$.

Second, we discuss the γ-positivity of the h^*-polynomials of locally anti-blocking reflexive polytopes. A lattice polytope is called *reflexive* if the dual polytope is also a lattice polytope. Many authors have studied reflexive polytopes from viewpoints of combinatorics, commutative algebra and algebraic geometry. In [12], Hibi characterized reflexive polytopes in terms of their h^*-polynomials. To be more precise, a lattice polytope of dimension d is (unimodularly equivalent to) a reflexive polytope if and only if the h^*-polynomial is a palindromic polynomial of degree d. On the other hand, in [21], locally anti-blocking reflexive polytopes were characterized. In fact, a locally anti-blocking lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d is reflexive if and only if for each $\varepsilon \in \{-1,1\}^d$, there exists a perfect graph G_ε on $[d] := \{1,\dots,d\}$ such that $\mathscr{P} \cap \mathbb{R}^d_\varepsilon = \mathscr{Q}^\pm_{G_\varepsilon} \cap \mathbb{R}^d_\varepsilon$, where $\mathscr{Q}_{G_\varepsilon}$ is the stable set polytope of G_ε. Moreover, every locally anti-blocking reflexive polytope possesses a regular unimodular triangulation. This fact and the result of Bruns–Römer [4] imply that its h^*-polynomial is unimodal.

In the present paper, we discuss whether the h^*-polynomial of a locally anti-blocking reflexive polytope has a stronger property, which is called γ*-positivity*. In [28], a class of lattice polytopes $\mathscr{B}_G$ arising from finite simple graphs G on $[d]$, which are called *symmetric edge polytopes of type B*, was given. Symmetric edge polytopes of type B are unconditional, and they are reflexive if and only if the underlying graphs are bipartite. Moreover, when they are reflexive, the h^*-polynomials are always γ-positive. On the other hand, in [29], another family of lattice polytopes $\mathscr{C}_P^{(e)}$ arising from finite partially ordered sets P on $[d]$, which are called *enriched chain polytopes*, was given. Enriched chain polytopes are unconditional and reflexive, and their h^*-polynomials are always γ-positive. Combining these facts and Theorem 0.1, we know that, for a locally anti-blocking reflexive polytope $\mathscr{P}$, if every $\mathscr{P} \cap \mathbb{R}^d_\varepsilon$ is the intersection of $\mathbb{R}^d_\varepsilon$ and either an enriched chain polytope or a symmetric edge reflexive polytope of type B, then the h^*-polynomial of $\mathscr{P}$ is γ-positive (Corollary 3.2). By using this result, we show that the h^*-polynomials of several classes of reflexive polytopes are γ-positive.

In Section 4, we will discuss the γ-positivity of the h^*-polynomials of *symmetric edge polytopes of type A*, which are reflexive polytopes arising from finite simple graphs. In [19], it was shown that the h^*-polynomials of the symmetric edge polytopes of type A of complete bipartite graphs are γ-positive. We will show that for a large class of finite simple graphs, which includes complete bipartite graphs, the h^*-polynomials of the symmetric edge polytopes of type A are γ-positive (Subsection 4.1). Moreover, by giving explicit h^*-polynomials of del Pezzo polytopes and pseudo-del Pezzo polytopes, we will show that the h^*-polynomial of every pseudo-symmetric simplicial reflexive polytope is γ-positive (Theorem 4.8).

In Section 5, we will discuss the γ-positivity of h^*-polynomials of *twinned chain polytopes* $\mathscr{C}_{P,Q} \subset \mathbb{R}^d$, which are reflexive polytopes arising from two finite partially ordered sets P and Q on $[d]$. In [36], it was shown that twinned chain polytopes $\mathscr{C}_{P,Q}$ are locally anti-blocking and each $\mathscr{C}_{P,Q} \cap \mathbb{R}^d_\varepsilon$ is the intersection of $\mathbb{R}^d_\varepsilon$ and an enriched chain polytopes. Hence the h^*-polynomials of $\mathscr{C}_{P,Q}$ are γ-positive. We will give a formula of the h^*-polynomials of twinned chain polytopes in terms of the left peak polynomials of finite partially ordered sets (Theorem 5.3). Moreover, we will define *enriched* (P,Q)*-partitions* of P and Q, and show that the Ehrhart polynomial of the twined chain polytope $\mathscr{C}_{P,Q}$ of P and Q coincides with a counting polynomial of enriched (P,Q)-partitions (Theorem 5.8).

This paper is organized as follows: In Section 1, we will review the theory of Ehrhart polynomials, h^*-polynomials, and reflexive polytopes. In Section 2, we will introduce several classes of anti-blocking polytopes and unconditional polytopes. In Section 3, we will investigate the h^*-polynomials of locally anti-blocking lattice polytopes. In particular, we will prove Theorem 0.1. We will discuss symmetric edge polytope of type A in Section 4, and twinned chain polytopes in Section 5.

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1. Ehrhart theory and Reflexive polytopes

In this section, we review the theory of Ehrhart polynomials, h^*-polynomials, and reflexive polytopes. Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d. Given a positive integer m, we define

$$L_{\mathscr{P}}(m) = |m\mathscr{P} \cap \mathbb{Z}^d|.$$

Ehrhart [8] proved that $L_{\mathscr{P}}(m)$ is a polynomial in m of degree d with the constant term 1. We say that $L_{\mathscr{P}}(m)$ is the *Ehrhart polynomial* of $\mathscr{P}$. The generating function of the lattice point enumerator, i.e., the formal power series

$$\mathrm{Ehr}_{\mathscr{P}}(x) = 1 + \sum_{k=1}^{\infty} L_{\mathscr{P}}(k)x^k$$

is called the *Ehrhart series* of $\mathscr{P}$. It is well known that it can be expressed as a rational function of the form

$$\mathrm{Ehr}_{\mathscr{P}}(x) = \frac{h^*(\mathscr{P},x)}{(1-x)^{d+1}}.$$

Then $h^*(\mathscr{P},x)$ is a polynomial in x of degree at most d with nonnegative integer coefficients ([33]) and it is called the *h^*-polynomial* (or the *δ-polynomial*) of $\mathscr{P}$. Moreover, one has $\mathrm{Vol}(\mathscr{P}) = h^*(\mathscr{P},1)$, where $\mathrm{Vol}(\mathscr{P})$ is the normalized volume of $\mathscr{P}$.

A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ of dimension d is called *reflexive* if the origin of $\mathbb{R}^d$ is a unique lattice point belonging to the interior of $\mathscr{P}$ and its dual polytope

$$\mathscr{P}^\vee := \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x},\mathbf{y}\rangle \leq 1 \text{ for all } \mathbf{x} \in \mathscr{P}\}$$

is also a lattice polytope, where $\langle \mathbf{x},\mathbf{y}\rangle$ is the usual inner product of $\mathbb{R}^d$. It is known that reflexive polytopes correspond to Gorenstein toric Fano varieties, and they are related to mirror symmetry (see, e.g., [2, 6]). In each dimension there exist only finitely many reflexive polytopes up to unimodular equivalence ([23]) and all of them are known up to dimension 4 ([22]). In [12], Hibi characterized reflexive polytopes in terms of their h^*-polynomials. We recall that a polynomial $f \in \mathbb{R}[x]$ of degree d is said to be *palindromic* if

$f(x) = x^d f(x^{-1})$. Note that if a lattice polytope of dimension d has interior lattice points, then the degree of its h^*-polynomial is equal to d.

Proposition 1.1 ([12])**.** *Let $\mathscr{P} \subset \mathbb{R}^d$ be a lattice polytope of dimension d with $\mathbf{0} \in \operatorname{int}(\mathscr{P})$. Then $\mathscr{P}$ is reflexive if and only if $h^*(\mathscr{P},x)$ is a palindromic polynomial of degree d.*

Next, we review properties of polynomials. Let $f = \sum_{i=0}^{d} a_i x^i$ be a polynomial with real coefficients and $a_d \neq 0$. We now focus on the following properties.

(RR) We say that f is *real-rooted* if all its roots are real.
(LC) We say that f is *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for all i.
(UN) We say that f is *unimodal* if $a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_d$ for some k.

If all its coefficients are nonnegative, then these properties satisfy the implications

$$\text{(RR)} \Rightarrow \text{(LC)} \Rightarrow \text{(UN)}.$$

On the other hand, the polynomial f is *γ-positive* if f is palindromic and there are $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor} \geq 0$ such that $f(x) = \sum_{i \geq 0} \gamma_i \, x^i (1+x)^{d-2i}$. The polynomial $\sum_{i \geq 0} \gamma_i \, x^i$ is called *γ-polynomial of f*. We can see that a γ-positive polynomial is real-rooted if and only if its γ-polynomial is real-rooted. If f is a palindromic and real-rooted, then it is γ-positive. Moreover, if f is γ-positive, then it is unimodal.

For a given lattice polytope, a fundamental problem within the field of Ehrhart theory is to determine if its h^*-polynomial is unimodal. One famous instance is given by reflexive polytopes that possess a regular unimodular triangulation.

Proposition 1.2 ([4])**.** *Let $\mathscr{P} \subset \mathbb{R}^d$ be a reflexive polytope of dimension d. If P possesses a regular unimodular triangulation, then $h^*(\mathscr{P},x)$ is unimodal.*

It is known that if a reflexive polytope possesses a flag regular unimodular triangulation all of whose maximal simplices contain the origin, then the h^*-polynomial coincides with the h-polynomial of a flag triangulation of a sphere ([4]). For the h-polynomial of a flag triangulation of a sphere, Gal ([11]) conjectured the following:

Conjecture 1.3 (Gal Conjecture)**.** The h-polynomial of any flag triangulation of a sphere is γ-positive.

2. Classes of anti-blocking polytopes and unconditional polytopes

In this section, we introduce several classes of anti-blocking polytopes and unconditional polytopes. Throughout this section, we associate each subset $F \subset [d]$ with a $(0,1)$-vector $\mathbf{e}_F = \sum_{i \in F} \mathbf{e}_i \in \mathbb{R}^d$, where each $\mathbf{e}_i$ is ith unit coordinate vector in $\mathbb{R}^d$.

2.1. $(0,1)$-polytopes arising from simplicial complices.

Let Δ be a simplicial complex on the vertex set $[d]$. Then Δ is a collection of subsets of $[d]$ with $\{i\} \in \Delta$ for all $i \in [d]$ such that if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$. In particular $\emptyset \in \Delta$ and $\mathbf{e}_\emptyset = \mathbf{0}$. Let $\mathscr{P}_\Delta$ denote the convex hull of $\left\{\mathbf{e}_F \in \mathbb{R}^d : F \in \Delta\right\}$. The following is an important observation.

Proposition 2.1. *Let $\mathscr{P} \subset \mathbb{R}^d_{\geq 0}$ be a (0,1)-polytope of dimension d. Then $\mathscr{P}$ is anti-blocking if and only if there exists a simplicial complex Δ on $[d]$ such that $\mathscr{P} = \mathscr{P}_\Delta$.*

2.2. **Stable set polytopes.** Let G be a finite simple graph on the vertex set $[d]$ and $E(G)$ the set of edges of G. (A finite graph G is called simple if G possesses no loop and no multiple edge.) A subset $W \subset [d]$ is called *stable* if, for all i and j belonging to W with $i \neq j$, one has $\{i,j\} \notin E(G)$. We remark that a stable set is often called an *independent set*. Let $S(G)$ denote the set of stable sets of G. One has $\emptyset \in S(G)$ and $\{i\} \in S(G)$ for each $i \in [d]$. The *stable set polytope* $\mathscr{Q}_G \subset \mathbb{R}^d$ of G is the $(0,1)$-polytope defined by

$$\mathscr{Q}_G := \mathrm{conv}(\{\mathbf{e}_W \in \mathbb{R}^d : W \in S(G)\}).$$

Then one has $\dim \mathscr{Q}_G = d$. Since we can regard $S(G)$ as a simplicial complex on $[d]$, $\mathscr{Q}_G$ is an anti-blocking polytope.

Locally anti-blocking reflexive polytopes are characterized by stable set polytopes. A *clique* of G is a subset $W \subset [d]$ which is a stable set of the complementary graph $\overline{G}$ of G. The *chromatic number* of G is the smallest integer $t \geq 1$ for which there exist stable set $W_1, \ldots, W_t$ of G with $[d] = W_1 \cup \cdots \cup W_t$. A finite simple graph G is said to be *perfect* if, for any induced subgraph H of G including G itself, the chromatic number of H is equal to the maximal cardinality of cliques of H. See, e.g., [7] for details on graph theoretical terminologies.

Proposition 2.2 ([21]). *Let $\mathscr{P} \subset \mathbb{R}^d$ be a locally anti-blocking lattice polytope of dimension d. Then $\mathscr{P} \subset \mathbb{R}^d$ is reflexive if and only if, for each $\varepsilon \in \{-1,1\}^d$, there exists a perfect graph G_ε on $[d]$ such that $\mathscr{P} \cap \mathbb{R}^d_\varepsilon = \mathscr{Q}^{\pm}_{G_\varepsilon} \cap \mathbb{R}^d_\varepsilon$.*

2.3. **Chain polytopes and enriched chain polytopes.** Let $(P, <_P)$ be a partially ordered set (poset, for short) on $[d]$. A subset A of $[d]$ is called an *antichain* of P if all i and j belonging to A with $i \neq j$ are incomparable in P. In particular, the empty set $\emptyset$ and each 1-element subset $\{i\}$ are antichains of P. Let $\mathscr{A}(P)$ denote the set of antichains of P. In [34], Stanley introduced the *chain polytope* $\mathscr{C}_P$ of P defined by

$$\mathscr{C}_P := \mathrm{conv}(\{\mathbf{e}_A \in \mathbb{R}^d : A \in \mathscr{A}(P)\}).$$

It is known that chain polytopes are stable set polytopes. Indeed, let G_P be the finite simple graph on $[d]$ such that $\{i,j\} \in E(G_P)$ if and only if $i <_P j$ or $j <_P i$. We call G_P the *comparability graph* of P. It then follows that $\mathscr{A}(P) = S(G_P)$. Hence the chain polytope $\mathscr{C}_P$ is the stable set polytope of $\mathscr{Q}_{G_P}$. Therefore, chain polytopes are anti-blocking polytopes. We remark that any comparability graph is perfect.

On the other hand, the *enriched chain polytope* $\mathscr{C}^{(e)}_P$ of P is the unconditional lattice polytope defined by

$$\mathscr{C}^{(e)}_P := \mathscr{C}^{\pm}_P.$$

In [29], it was shown that the Ehrhart polynomial of $\mathscr{C}^{(e)}_P$ coincides with a counting polynomial of left enriched P-partitions. We assume that P is naturally labeled. Let $[m]^{\pm} := \{1,-1,2,-2,\ldots,m,-m\}$ and $[m]^{\pm}_0 := \{0\} \cup [m]^{\pm}$ for $0 < m \in \mathbb{Z}$. A map $f : P \to [m]^{\pm}$ is called an *enriched P-partition* ([35]) if, for all $x,y \in P$ with $x <_P y$, f satisfies

(i) $|f(x)| \leq |f(y)|$;

(ii) $|f(x)| = |f(y)| \Rightarrow f(y) > 0$.

A map $f : P \to [m]^{\pm}_0$ is called a *left enriched P-partition* ([31]) if, for all $x,y \in P$ with $x <_P y$, f satisfies

(i) $|f(x)| \leq |f(y)|$;
(ii) $|f(x)| = |f(y)| \Rightarrow f(y) \geq 0$.

We denote $\Omega_P^{(\ell)}(m)$ the number of left enriched P-partitions $f : P \to [m]_0^{\pm}$, which is called the *left enriched order polynomial* of P.

Proposition 2.3 ([29])**.** *Let P be a naturally labeled finite poset on $[d]$. Then one has*

$$L_{\mathscr{C}_P^{(e)}}(m) = \Omega_P^{(\ell)}(m).$$

Given a linear extension $\pi = (\pi_1, \ldots, \pi_d)$ of a finite poset P on $[d]$, a *left peak* of π is an index $1 \leq i \leq d-1$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$. Let $\mathrm{pk}^{(\ell)}(\pi)$ denote the number of left peaks of π. Then the *left peak polynomial* $W_P^{(\ell)}(x)$ of P is defined by

$$W_P^{(\ell)}(x) = \sum_{\pi \in \mathscr{L}(P)} x^{\,\mathrm{pk}^{(\ell)}(\pi)},$$

where $\mathscr{L}(P)$ is the set of linear extensions of P.

Proposition 2.4 ([29])**.** *Let P be a naturally labeled finite poset on $[d]$. Then the h^*-polynomial of $\mathscr{C}_P^{(e)}$ is*

$$h^*(\mathscr{C}_P^{(e)}, x) = (x+1)^d \, W_P^{(\ell)}\left(\frac{4x}{(x+1)^2}\right).$$

In particular, $h^*(\mathscr{C}_P^{(e)}, x)$ is γ-positive.

Note that if Q is a finite poset which is obtained from P by reordering the label, then $\mathscr{C}_P^{(e)}$ and $\mathscr{C}_Q^{(e)}$ are unimodularly equivalent. Hence the h^*-polynomials of enriched chain polytopes are always γ-positive.

2.4. **Symmetric edge polytopes of type B.** Let G be a finite simple graph on $[d]$. We set

$$B_G := \mathrm{conv}(\{\mathbf{0}, \mathbf{e}_1, \ldots, \mathbf{e}_d\} \cup \{\mathbf{e}_i + \mathbf{e}_j : \{i, j\} \in E(G)\}).$$

Then $B_G = \mathscr{P}_\Delta$ where Δ is a simplicial complex on $[d]$ obtained by regarding G as a 1-dimensional simplicial complex. The *symmetric edge polytope of type B* of G is the unconditional lattice polytope defined by

$$\mathscr{B}_G := B_G^{\pm}.$$

Proposition 2.5 ([28])**.** *Let G be a finite simple graph on $[d]$. Then $\mathscr{B}_G$ is reflexive if and only if G is bipartite.*

A *hypergraph* is a pair $\mathscr{H} = (V, E)$, where $E = \{e_1, \ldots, e_n\}$ is a finite multiset of non-empty subsets of $V = \{v_1, \ldots, v_m\}$. Elements of V are called vertices and the elements of E are the hyperedges. Then we can associate $\mathscr{H}$ to a bipartite graph $\mathrm{Bip}\mathscr{H}$ with a bipartition $V \cup E$ such that $\{v_i, e_j\}$ is an edge of $\mathrm{Bip}\mathscr{H}$ if $v_i \in e_j$. Assume that $\mathrm{Bip}\mathscr{H}$ is connected. A *hypertree* in $\mathscr{H}$ is a function $\mathbf{f} : E \to \{0, 1, \ldots\}$ such that there exists a spanning tree Γ of $\mathrm{Bip}\mathscr{H}$ whose vertices have degree $\mathbf{f}(e) + 1$ at each $e \in E$. Then we say that Γ induce $\mathbf{f}$. Let $B_{\mathscr{H}}$ denote the set of all hypertrees in $\mathscr{H}$. A hyperedge $e_j \in E$ is said to be *internally active* with respect to the hypertree $\mathbf{f}$ if it is not possible to decrease

$\mathbf{f}(e_j)$ by 1 and increase $\mathbf{f}(e_{j'})$ ($j' < j$) by 1 so that another hypertree results. We call a hyperedge *internally inactive* with respect to a hypertree if it is not internally active and denote the number of such hyperedges of $\mathbf{f}$ by $\bar{\iota}(\mathbf{f})$. Then the *interior polynomial* of $\mathscr{H}$ is the generating function $I_{\mathscr{H}}(x) = \sum_{\mathbf{f}\in B_{\mathscr{H}}} x^{\bar{\iota}(\mathbf{f})}$. It is known [20, Proposition 6.1] that $\deg I_{\mathscr{H}}(x) \le \min\{|V|,|E|\}-1$. If $G = \mathrm{Bip}\mathscr{H}$, then we set $I_G(x) = I_{\mathscr{H}}(x)$.

Assume that G is a bipartite graph with a bipartition $V_1 \cup V_2 = [d]$. Then let $\widetilde{G}$ be a connected bipartite graph on $[d+2]$ whose edge set is

$$E(\widetilde{G}) = E(G) \cup \{\{i,d+1\} : i \in V_1\} \cup \{\{j,d+2\} : j \in V_2 \cup \{d+1\}\}.$$

Proposition 2.6 ([28]). *Let G be a bipartite graph on $[d]$. Then h^*-polynomial of the reflexive polytope $\mathscr{B}_G$ is*

$$h^*(\mathscr{B}_G,x) = (x+1)^d I_{\widetilde{G}}\left(\frac{4x}{(x+1)^2}\right).$$

In particular, $h^*(\mathscr{B}_G,x)$ is γ-positive.

3. h^*-POLYNOMIALS OF LOCALLY ANTI-BLOCKING LATTICE POLYTOPES

In the present section, we prove Theorem 0.1, that is, a formula of the h^*-polynomials of locally anti-blocking lattice polytopes in terms of that of unconditional lattice polytopes. Given a subset $J = \{j_1,\dots,j_r\}$ of $[d]$, let $\pi_J : \mathbb{R}^d \to \mathbb{R}^r$, $\pi_J((x_1,\dots,x_d)) = (x_{j_1},\dots,x_{j_r})$ denote the projection map. (Here $\pi_\emptyset$ is the zero map.)

Proposition 3.1. *Let $\mathscr{P} \subset \mathbb{R}^d_{\ge 0}$ be an anti-blocking lattice polytope. Then we have*

$$h^*(\mathscr{P}^{\pm},x) \;=\; \sum_{j=0}^{d} 2^j (x-1)^{d-j} \sum_{J\subset[d],\ |J|=j} h^*(\pi_J(\mathscr{P}),x).$$

Proof. The proof is similar to the discussion in [28, Proof of Proposition 3.1]. The intersection of $\mathscr{P}^{\pm} \cap \mathbb{R}^d_{\varepsilon}$ and $\mathscr{P}^{\pm} \cap \mathbb{R}^d_{\varepsilon'}$ is of dimension $d-1$ if and only if $\varepsilon - \varepsilon' \in \{\pm 2\mathbf{e}_1,\dots,\pm 2\mathbf{e}_d\}$. Moreover, if $\varepsilon - \varepsilon' = 2\mathbf{e}_k$, then

$$(\mathscr{P}^{\pm} \cap \mathbb{R}^d_{\varepsilon}) \cap (\mathscr{P}^{\pm} \cap \mathbb{R}^d_{\varepsilon'}) = \mathscr{P}^{\pm} \cap \mathbb{R}^d_{\varepsilon} \cap \mathbb{R}^d_{\varepsilon'} \simeq \pi_{[d]\setminus\{k\}}(\mathscr{P}^{\pm}) \cap \mathbb{R}^{d-1}_{\pi_{[d]\setminus\{k\}}(\varepsilon)} \simeq \pi_{[d]\setminus\{k\}}(\mathscr{P}).$$

Hence the Ehrhart polynomial $L_{\mathscr{P}^{\pm}}(m)$ satisfies the following:

$$L_{\mathscr{P}^{\pm}}(m) = \sum_{j=0}^{d} 2^j (-1)^{d-j} \sum_{J\subset[d],\ |J|=j} L_{\pi_J(\mathscr{P})}(m).$$

Thus the Ehrhart series satisfies

$$\frac{h^*(\mathscr{P}^{\pm},x)}{(1-x)^{d+1}} \;=\; \sum_{j=0}^{d} 2^j (-1)^{d-j} \sum_{J\subset[d],\ |J|=j} \frac{h^*(\pi_J(\mathscr{P}),x)}{(1-x)^{j+1}},$$

as desired. □

We now prove Theorem 0.1.

Proof of Theorem 0.1. Given $J = \{j_1, \ldots, j_r\} \subset [d]$ and $\varepsilon \in \{-1,1\}^r$, let

$$\mathbb{R}^d_{J,\varepsilon} = \{\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d : \pi_J(\mathbf{x}) \in \mathbb{R}^r_\varepsilon \text{ and } x_j = 0 \text{ for all } j \notin J\}.$$

It then follows that $\mathscr{P} \cap \mathbb{R}^d_{J,\varepsilon}$ is equal to $\pi_J(\mathscr{P}_{\varepsilon'})^{\pm} \cap \mathbb{R}^r_\varepsilon$, where $\pi_J(\varepsilon') = \varepsilon$. Note that, given $J = \{j_1, \ldots, j_r\} \subset [d]$ and $\varepsilon \in \{-1,1\}^r$, we have $|\{\varepsilon' \in \{-1,1\}^d : \pi_J(\varepsilon') = \varepsilon\}| = 2^{d-r}$. Thus

$$\begin{aligned}
h^*(\mathscr{P},x) &= \sum_{j=0}^{d} (x-1)^{d-j} \sum_{J\subset[d],\ |J|=j} \ \sum_{\varepsilon\in\{-1,1\}^j} h^*(\mathscr{P}\cap\mathbb{R}^d_{J,\varepsilon},x) \\
&= \sum_{j=0}^{d} (x-1)^{d-j} \sum_{\varepsilon\in\{-1,1\}^d} \ \sum_{J\subset[d],\ |J|=j} \frac{1}{2^{d-j}} h^*(\pi_J(\mathscr{P}_\varepsilon),x) \\
&= \frac{1}{2^d} \sum_{\varepsilon\in\{-1,1\}^d} \ \sum_{j=0}^{d} 2^j (x-1)^{d-j} \sum_{J\subset[d],\ |J|=j} h^*(\pi_J(\mathscr{P}_\varepsilon),x) \\
&= \frac{1}{2^d} \sum_{\varepsilon\in\{-1,1\}^d} h^*(\mathscr{P}^{\pm}_\varepsilon,x)
\end{aligned}$$

by Proposition 0.1. □

Combining Theorem 0.1 and Propositions 2.4 and 2.6, we have the following.

Corollary 3.2. *Let $\mathscr{P} \subset \mathbb{R}^d$ be a locally anti-blocking reflexive polytope. If every $\mathscr{P} \cap \mathbb{R}^d_\varepsilon$ is the intersection of $\mathbb{R}^d_\varepsilon$ and either an enriched chain polytope or a symmetric edge reflexive polytope of type B, then the h^*-polynomial of $\mathscr{P}$ is γ-positive.*

Finally, we conjecture the following:

Conjecture 3.3. The h^*-polynomial of any locally anti-blocking reflexive polytope is γ-positive.

Thanks to Theorem 0.1 and Proposition 2.2, in order to prove Conjecture 3.3, it is enough to study unconditional lattice polytopes $\mathscr{Q}^{\pm}_G$ where $\mathscr{Q}_G$ is the stable set polytope of a perfect graph G.

4. Symmetric edge polytopes of type A

Let G be a finite simple graph on the vertex set $[d]$ and the edge set $E(G)$. The *symmetric edge polytope $\mathscr{A}_G \subset \mathbb{R}^d$* of type A is the convex hull of the set

$$A(G) = \{\pm(\mathbf{e}_i - \mathbf{e}_j) \in \mathbb{R}^d : \{i,j\} \in E(G)\}.$$

The polytope $\mathscr{A}_G$ is introduced in [24, 26] and called a "symmetric edge polytope of G."

Example 4.1. Let G be a tree on $[d]$. Then $\mathscr{A}_G$ is unimodularly equivalent to a $(d-1)$-dimensional cross polytope. Hence we have $h^*(\mathscr{A}_G,x) = (x+1)^{d-1}$.

It is known [24, Proposition 4.1] that the dimension of $\mathscr{A}_G$ is $d-1$ if and only if G is connected. Higashitani [18] proved that $\mathscr{A}_G$ is simple if and only if $\mathscr{A}_G$ is smooth if and only if G contains no even cycles. It is known [24, 26] that $\mathscr{A}_G$ is unimodularly equivalent to a reflexive polytope having a regular unimodular triangulation. In particular, h^*-polynomial of $\mathscr{A}_G$ is palindromic and unimodal. For a complete bipartite graph $K_{\ell,m}$, it is known [19] that the h^*-polynomial of $\mathscr{A}_{K_{\ell,m}}$ is real-rooted and hence γ-positive.

4.1. **Recursive formulas for h^*-polynomials.** In this section, we give several recursive formulas of h^*-polynomials of $\mathscr{A}_G$ when G belongs to certain classes of graphs. By the following fact, we may assume that G is 2-connected if needed.

Proposition 4.2. *Let G be a graph and let $G_1,\dots,G_s$ be 2-connected components of G. Then the h^*-polynomial of $\mathscr{A}_G$ satisfies*

$$h^*(\mathscr{A}_G,x) = h^*(\mathscr{A}_{G_1},x)\cdots h^*(\mathscr{A}_{G_s},x).$$

Proof. Since $\mathscr{A}_G$ is the free sum of reflexive polytopes $\mathscr{A}_{G_1},\dots,\mathscr{A}_{G_s}$, a desired conclusion follows from [3, Theorem 1]. □

The *suspension* $\widehat{G}$ of a graph G is the graph on the vertex set $[d+1]$ and the edge set

$$E(G)\cup\{\{i,d+1\} : i\in[d]\}.$$

We now study the h^*-polynomial of $\mathscr{A}_{\widehat{G}}$. Given a subset $S\subset[d]$,

$$E_S := \{e\in E(G) : |e\cap S| = 1\}$$

is called a *cut* of G. For example, we have $E_\emptyset = E_{[d]} = \emptyset$. In general, it follows that $E_S = E_{[d]\setminus S}$. We identify E_S with the subgraph of G on the vertex set $[d]$ and the edge set E_S. By definition, E_S is a bipartite graph. Let $\mathrm{Cut}(G)$ be the set of all cuts of G. Note that $|\mathrm{Cut}(G)| = 2^{d-1}$. From Theorem 0.1 and Proposition 2.6, we have the following.

Theorem 4.3. *Let G be a finite graph on $[d]$. Then $\mathscr{A}_{\widehat{G}}$ is unimodularly equivalent to a locally anti-blocking reflexive polytope whose h^*-polynomial is*

$$h^*(\mathscr{A}_{\widehat{G}},x) = \frac{1}{2^{d-1}}\sum_{H\in\mathrm{Cut}(G)} h^*(\mathscr{B}_H,x) = (x+1)^d f_G\left(\frac{4x}{(x+1)^2}\right),$$

where

$$f_G(x) = \frac{1}{2^{d-1}}\sum_{H\in\mathrm{Cut}(G)} I_{\widetilde{H}}(x).$$

In particular, $h^*(\mathscr{A}_{\widehat{G}},x)$ is γ-positive. Moreover, $h^*(\mathscr{A}_{\widehat{G}},x)$ is real-rooted if and only if $f_G(x)$ is real-rooted.

Proof. Let $\mathscr{P}\subset\mathbb{R}^d$ be the convex hull of

$$\{\pm\mathbf{e}_1,\dots,\pm\mathbf{e}_d\}\cup\{\pm(\mathbf{e}_i-\mathbf{e}_j) : \{i,j\}\in E(G)\}.$$

Then $\mathscr{A}_{\widehat{G}}$ is lattice isomorphic to $\mathscr{P}$. Given $\varepsilon = (\varepsilon_1,\dots,\varepsilon_d)\in\{-1,1\}^d$, let $S_\varepsilon = \{i\in[d] : \varepsilon_i = 1\}$. Then $\mathscr{P}\cap\mathbb{R}^d_\varepsilon$ is the convex hull of

$$\{\mathbf{0}\}\cup\{\varepsilon_i\mathbf{e}_i : i\in[d]\}\cup\{\mathbf{e}_i-\mathbf{e}_j : \{i,j\}\in E_{S_\varepsilon}, i\in S_\varepsilon\}.$$

Hence $\mathscr{P}\cap\mathbb{R}^d_\varepsilon = \mathscr{B}_{E_{S_\varepsilon}}\cap\mathbb{R}^d_\varepsilon$. Thus $\mathscr{P}$ is a locally anti-blocking polytope and

$$h^*(\mathscr{A}_{\widehat{G}},x) = \frac{1}{2^{d-1}}\sum_{H\in\mathrm{Cut}(G)} h^*(\mathscr{B}_H,x)$$

by Theorem 0.1. $\square$

Let G be a graph and let $e = \{i, j\}$ be an edge of G. Then the graph G/e obtained by the procedure

(i) Delete e and identify the vertices i and j;
(ii) Delete the multiple edges that may be created while (i)

is called the graph obtained from G by *contracting* the edge e. Next, we will show that, for any bipartite graph G and $e \in E(G)$, $h^*(\mathscr{A}_G, x)$ is γ-positive if and only if so is $h^*(\mathscr{A}_{G/e}, x)$. In order to show this fact, we need the theory of Gröbner bases of toric ideals. Given a graph G on the vertex set $[d]$ and the edge set $E(G) = \{e_1, \ldots, e_n\}$, let

$$\mathscr{R} = K[t_1, t_1^{-1}, \ldots, t_d, t_d^{-1}, s]$$

be the Laurent polynomial ring over a field K and let

$$\mathscr{S} = K[x_1, \ldots, x_n, y_1, \ldots, y_n, z]$$

be the polynomial ring over K. We define the ring homomorphism $\pi : \mathscr{S} \to \mathscr{R}$ by setting $\pi(z) = s$, $\pi(x_k) = t_i t_j^{-1} s$ and $\pi(y_k) = t_i^{-1} t_j s$ if $e_k = \{i, j\} \in E(G)$ and $i < j$. The *toric ideal* $I_{\mathscr{A}_G}$ of $\mathscr{A}_G$ is the kernel of π. (See, e.g., [13] for details on toric ideals and Gröbner bases.) We now define the notation given in [19]. For any oriented edge e_i, let p_i denote the corresponding variable, i.e. $p_i = x_i$ or $p_i = y_i$ depending on the orientation and let $\{p_i, q_i\} = \{x_i, y_i\}$. Let $\mathscr{G}(G)$ be the set of all binomials f satisfying one of the following:

$$f = \prod_{e_i \in I} p_i - \prod_{e_i \in C \setminus I} q_i, \tag{1}$$

where C is an even cycle in G of length $2k$ with a fixed orientation, and I is a k-subset of C such that $e_\ell \notin I$ for $\ell = \min\{i : e_i \in C\}$;

$$f = \prod_{e_i \in I} p_i - z \prod_{e_i \in C \setminus I} q_i, \tag{2}$$

where C is an odd cycle in G of length $2k+1$ and I is a $(k+1)$-subset of C;

$$f = x_i y_i - z^2, \tag{3}$$

where $1 \le i \le n$. Then $\mathscr{G}(G)$ is a Gröbner basis of $I_{\mathscr{A}_G}$ with respect to a reverse lexicographic order $<$ induced by the ordering $z < x_1 < y_1 < \cdots < x_n < y_n$ ([19, Proposition 3.8]). Here the initial monomial of each binomial is the first monomial. Using this Gröbner basis, we have the following.

Proposition 4.4. *Let G be a bipartite graph on $[d]$ and let $e \in E(G)$. Then we have*

$$h^*(\mathscr{A}_G, x) = (x+1) h^*(\mathscr{A}_{G/e}, x).$$

Proof. Let $E(G) = \{e_1, \ldots, e_n\}$ with $e = e_1 = \{i, j\}$. Since G is a bipartite graph, the Gröbner basis $\mathscr{G}(G)$ above consists of the binomials of the form (1) and (3).

Since G has no triangles, the procedure (ii) does not occur when we contract e of G. Hence $E(G/e) = \{e'_2, \ldots, e'_n\}$ where e'_k is obtained from e_k by identifying i with j. Let G'

be a graph obtained by adding an edge $e'_1 = \{d+1, d+2\}$ to the graph G/e. Then $\mathscr{G}(G')$ consists of all binomials f satisfying one of the following:

$$f = \prod_{e_i \in I} p_i - \prod_{e_i \in C \setminus I} q_i, \tag{4}$$

where C is an even cycle in G of length $2k$ with a fixed orientation and $e_1 \notin C$, and I is a k-subset of C such that $e_\ell \notin I$ for $\ell = \min\{i : e_i \in C\}$;

$$f = \prod_{e_i \in I} p_i - z \prod_{e_i \in C \setminus I} q_i, \tag{5}$$

where $C \cup \{e_1\}$ is an even cycle in G of length $2k+2$ and I is a $(k+1)$-subset of C;

$$f = x_i y_i - z^2, \tag{6}$$

where $1 \le i \le n$. Hence $\{\mathrm{in}_<(f) : f \in \mathscr{G}(G)\} = \{\mathrm{in}_<(f) : f \in \mathscr{G}(G')\}$. By a similar argument as in the proof of [17, Theorem 3.1], it follows that

$$h^*(\mathscr{A}_G, x) = h^*(\mathscr{A}_{G'}, x) = h^*(\mathscr{A}_{\{e'_1\}}, x) h^*(\mathscr{A}_{G/e}, x) = (x+1) h^*(\mathscr{A}_{G/e}, x),$$

as desired. □

From Theorem 4.3, Propositions 4.2 and 4.4 we have the following immediately.

Corollary 4.5. *Let G be a bipartite graph on $[d]$. Then we have the following*:

(a) *The h^*-polynomial $h^*(\mathscr{A}_{\widetilde{G}}, x) = (x+1) h^*(\mathscr{A}_{\widehat{G}}, x)$ is γ-positive.*
(b) *If G is obtained by gluing bipartite graphs G_1 and G_2 along with an edge e, then*

$$\begin{aligned} h^*(\mathscr{A}_G, x) &= (x+1) h^*(\mathscr{A}_{G/e}, x) \\ &= (x+1) h^*(\mathscr{A}_{G_1/e}, x) h^*(\mathscr{A}_{G_2/e}, x) \\ &= h^*(\mathscr{A}_{G_1}, x) h^*(\mathscr{A}_{G_2}, x)/(x+1). \end{aligned}$$

4.2. **Pseudo-symmetric simplicial reflexive polytopes.** A lattice polytope $\mathscr{P} \subset \mathbb{R}^d$ is called *pseudo-symmetric* if there exists a facet $\mathscr{F}$ of $\mathscr{P}$ such that $-\mathscr{F}$ is also a facet of $\mathscr{P}$. Nill [25] proved that any pseudo-symmetric simplicial reflexive polytope $\mathscr{P}$ is a free sum of $\mathscr{P}_1, \dots, \mathscr{P}_s$, where each $\mathscr{P}_i$ is one of the following:

- cross polytope;
- del Pezzo polytope $V_{2m} = \mathrm{conv}(\pm \mathbf{e}_1, \dots, \pm \mathbf{e}_{2m}, \pm(\mathbf{e}_1 + \cdots + \mathbf{e}_{2m}))$;
- pseudo-del Pezzo polytope $\widetilde{V}_{2m} = \mathrm{conv}(\pm \mathbf{e}_1, \dots, \pm \mathbf{e}_{2m}, -\mathbf{e}_1 - \cdots - \mathbf{e}_{2m})$.

Note that a del Pezzo polytope is unimodularly equivalent to $\mathscr{A}_{C_{2m+1}}$ where C_{2m+1} is an odd cycle of length $2m+1$ (see [18]). The h^*-polynomial of $\mathscr{A}_{C_d}$ was essentially studied in the following papers (see also the OEIS sequence A204621):

- Conway–Sloane [5, p.2379] computed $h^*(\mathscr{A}_{C_d}, x)$ for small d by using results of O'Keeffe [30] and gave a conjecture on the γ-polynomial of $h^*(\mathscr{A}_{C_d}, x)$ (coincides with the γ-polynomial in Proposition 4.7 below).
- General formulas for the coefficients of $h^*(\mathscr{A}_{C_d}, x)$ were given by Ohsugi–Shibata [27] and Wang–Yu [37].

In order to give the h^*-polynomial of $\widetilde{V}_{2m}$, we need the following lemma.

Lemma 4.6. *Let G be a connected graph. Suppose that an edge $e = \{i, j\}$ of G is not a bridge. Let $\mathscr{P}_e$ be the convex hull of $A(G) \setminus \{\mathbf{e}_i - \mathbf{e}_j\}$. Then we have*

$$h^*(\mathscr{P}_e, x) = \frac{1}{2}(h^*(\mathscr{A}_G, x) + h^*(\mathscr{A}_{G\setminus e}, x)),$$

where $G \setminus e$ is the graph obtained by deleting e from G.

Proof. Note that $\mathscr{A}_{G\setminus e} \subset \mathscr{P}_e \subset \mathscr{A}_G$. Since G is connected and e is not a bridge of G, the dimension of each of $\mathscr{A}_G$ and $\mathscr{A}_{G\setminus e}$ is $d-1$. Let $\mathscr{P}'_e$ denote the convex hull of $A(G) \setminus \{-\mathbf{e}_i + \mathbf{e}_j\}$, which is unimodularly equivalent to $\mathscr{P}_e$. Then $\mathscr{A}_G$ and $\mathscr{P}_e$ are decomposed into the following disjoint union:

$$\begin{aligned} \mathscr{A}_G &= \mathscr{A}_{G\setminus e} \cup (\mathscr{P}_e \setminus \mathscr{A}_{G\setminus e}) \cup (\mathscr{P}'_e \setminus \mathscr{A}_{G\setminus e}), \\ \mathscr{P}_e &= \mathscr{A}_{G\setminus e} \cup (\mathscr{P}_e \setminus \mathscr{A}_{G\setminus e}). \end{aligned}$$

Since $\mathscr{P}_e \setminus \mathscr{A}_{G\setminus e}$ is unimodularly equivalent to $\mathscr{P}'_e \setminus \mathscr{A}_{G\setminus e}$, we have a desired conclusion. □

The h^*-polynomials of V_{2m} and $\widetilde{V}_{2m}$ are as follows:

Proposition 4.7. *Let C_d denote a cycle of length $d \geq 3$ and let $1 \leq m \in \mathbb{Z}$. Then we have*

$$\begin{aligned} h^*(\mathscr{A}_{C_d}, x) &= \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{2i}{i} x^i (x+1)^{d-2i-1}, \\ h^*(V_{2m}, x) &= \sum_{i=0}^{m} \binom{2i}{i} x^i (x+1)^{2m-2i}, \\ h^*(\widetilde{V}_{2m}, x) &= (x+1)^{2m} + \sum_{i=1}^{m} \binom{2i-1}{i-1} x^i (x+1)^{2m-2i}. \end{aligned}$$

In particular, the h^*-polynomials of $\mathscr{A}_{C_d}$, V_{2m} and $\widetilde{V}_{2m}$ are γ-positive.

Proof. The proof for C_d is induction on d. First, we have $h^*(\mathscr{A}_{C_3}, x) = x^2 + 4x + 1 = (x+1)^2 + \binom{2}{1}x$. If $d \geq 4$ is even, then

$$h^*(\mathscr{A}_{C_d}, x) = (x+1)h^*(\mathscr{A}_{C_{d-1}}, x) = \sum_{i=0}^{\frac{d-2}{2}} \binom{2i}{i} x^i (x+1)^{d-2i-1} = \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} \binom{2i}{i} x^i (x+1)^{d-2i-1}.$$

Moreover, if $d = 2m+1$ ($2 \leq m \in \mathbb{Z}$), then the coefficient of x^m in

$$\sum_{i=0}^{\frac{d-1}{2}} \binom{2i}{i} x^i (x+1)^{d-2i-1} = (x+1)h^*(\mathscr{A}_{C_{d-1}}, x) + \binom{2m}{m} x^m$$

is $\sum_{i=0}^{m} \binom{2i}{i}\binom{2m-2i}{m-i} = 4^m = 2^{d-1}$ and other coefficient is arising from $(x+1)h^*(\mathscr{A}_{C_{d-1}}, x)$. By a recursive formula in [27, Theorem 2.3], we have

$$h^*(\mathscr{A}_{C_d}, x) = \sum_{i=0}^{\frac{d-1}{2}} \binom{2i}{i} x^i (x+1)^{d-2i-1}.$$

Since V_{2m} is unimodularly equivalent to $\mathscr{A}_{C_{2m+1}}$, we have $h^*(V_{2m},x) = h^*(\mathscr{A}_{C_{2m+1}},x)$. By Lemma 4.6, it follows that

$$\begin{aligned} h^*(\widetilde{V}_{2m},x) &= \frac{1}{2}(h^*(\mathscr{A}_{C_{2m+1}},x)+h^*(\mathscr{A}_{P_{2m+1}},x)) \\ &= \frac{1}{2}\left(\sum_{i=0}^{m}\binom{2i}{i}x^i(x+1)^{2m-2i}+(x+1)^{2m}\right) \\ &= (x+1)^{2m}+\sum_{i=1}^{m}\binom{2i-1}{i-1}x^i(x+1)^{2m-2i}. \end{aligned}$$

□

Thus it turns out that any pseudo-symmetric simplicial reflexive polytope is a free sum of reflexive polytopes whose h^*-polynomial are γ-positive. By [3, Theorem 1], we have the following.

Theorem 4.8. *The h^*-polynomial of any pseudo-symmetric simplicial reflexive polytope is γ-positive.*

4.3. **Classes of graphs such that $h^*(\mathscr{A}_G,x)$ is γ-positive.** Using results in the present section, for example, $h^*(\mathscr{A}_G,x)$ is γ-positive if one of the following holds:

- $G = \widehat{H}$ for some graph H (e.g., G is a complete graph, a wheel graph);
- $G = \widetilde{H}$ for some bipartite graph H (e.g., G is a complete bipartite graph);
- G is a cycle;
- G is an outer planar bipartite graph.

Moreover, we can compute $h^*(\mathscr{A}_G,x)$ explicitly in some cases. We give examples of such calculations for known formulas (for complete graphs [1], and for complete bipartite graphs [19]).

Example 4.9 ([1])**.** For a complete graph K_d, we have

$$\begin{aligned} h^*(\mathscr{A}_{K_d},x) &= h^*(\mathscr{A}_{\widehat{K_{d-1}}},x) \\ &= \frac{1}{2^{d-1}}\sum_{k=0}^{d-1}\binom{d-1}{k}\sum_{i=0}^{\lfloor\frac{d-1}{2}\rfloor}4^i\binom{k}{i}\binom{d-k-1}{i}x^i(x+1)^{d-1-2i} \\ &= \frac{1}{2^{d-1}}\sum_{i=0}^{\lfloor\frac{d-1}{2}\rfloor}4^ix^i(x+1)^{d-1-2i}\sum_{k=i}^{d-i-1}\binom{d-1}{k}\binom{k}{i}\binom{d-k-1}{i}. \end{aligned}$$

Since

$$\sum_{k=i}^{d-i-1}\binom{d-1}{k}\binom{k}{i}\binom{d-k-1}{i} = \sum_{k=i}^{d-i-1}\binom{d-1}{2i}\binom{2i}{i}\binom{d-1-2i}{k-i} = 2^{d-1-2i}\binom{d-1}{2i}\binom{2i}{i},$$

we have

$$h^*(\mathscr{A}_{K_d},x) = \sum_{i=0}^{\lfloor\frac{d-1}{2}\rfloor}\binom{d-1}{2i}\binom{2i}{i}x^i(x+1)^{d-1-2i}.$$

Example 4.10 ([19])**.** Let $G = K_{m,n}$. Then $\widetilde{G} = K_{m+1,n+1}$ and

$$
\begin{aligned}
& h^*(\mathscr{A}_{K_{m+1,n+1}}, x) = (x+1)h^*(\mathscr{A}_{\widehat{K_{m,n}}}, x) = \\
= \;& \frac{x+1}{2^{m+n}} \sum_{k=0}^{m} \sum_{\ell=0}^{n} \binom{m}{k} \binom{n}{\ell} \left(\sum_{i=0}^{\min(k,\ell)} 4^i \binom{k}{i} \binom{\ell}{i} x^i (x+1)^{k+\ell-2i} \right) \\
& \qquad \left(\sum_{j=0}^{\min(m-k,n-\ell)} 4^j \binom{m-k}{j} \binom{n-\ell}{j} x^j (x+1)^{m+n-k-\ell-2j} \right) \\
= \;& \frac{1}{2^{m+n}} \sum_{i,j \geq 0} 4^{i+j} x^{i+j} (x+1)^{n+m-2(i+j)+1} \sum_{k=i}^{m-j} \binom{m}{k} \binom{k}{i} \binom{m-k}{j} \sum_{\ell=i}^{n-j} \binom{n}{\ell} \binom{\ell}{i} \binom{n-\ell}{j}.
\end{aligned}
$$

Since

$$
\sum_{k=i}^{m-j} \binom{m}{k} \binom{k}{i} \binom{m-k}{j} = \sum_{k=i}^{m-j} \binom{m}{i+j} \binom{i+j}{i} \binom{m-(i+j)}{k-i} = 2^{m-(i+j)} \binom{m}{i+j} \binom{i+j}{i},
$$

we have

$$
\begin{aligned}
h^*(\mathscr{A}_{K_{m+1,n+1}}, x) &= \sum_{i \geq 0} \sum_{j \geq 0} \binom{i+j}{i}^2 \binom{m}{i+j} \binom{n}{i+j} x^{i+j} (x+1)^{m+n-2(i+j)+1} \\
&= \sum_{\alpha=0}^{\min(m,n)} \sum_{i=0}^{\alpha} \binom{\alpha}{i}^2 \binom{m}{\alpha} \binom{n}{\alpha} x^{\alpha} (x+1)^{m+n-2\alpha+1} \\
&= \sum_{\alpha=0}^{\min(m,n)} \binom{2\alpha}{\alpha} \binom{m}{\alpha} \binom{n}{\alpha} x^{\alpha} (x+1)^{m+n-2\alpha+1}.
\end{aligned}
$$

Finally, we conjecture the following:

Conjecture 4.11. The h^*-polynomial of any symmetric edge polytope of type A is γ-positive.

5. Twinned chain polytopes

In this section, we will apply Theorem 0.1 to twinned chain polytopes. For two lattice polytopes $\mathscr{P}, \mathscr{Q} \subset \mathbb{R}^d$, we set

$$\Gamma(\mathscr{P}, \mathscr{Q}) := \mathrm{conv}(\mathscr{P} \cup (-\mathscr{Q})) \subset \mathbb{R}^d.$$

Let P and Q be two finite posets on $[d]$. The *twinned chain polytope* of P and Q is the lattice polytope defined by

$$\mathscr{C}_{P,Q} := \Gamma(\mathscr{C}_P, \mathscr{C}_Q).$$

Then $\mathscr{C}_{P,Q}$ is reflexive. Moreover, $\mathscr{C}_{P,Q}$ has a flag, regular unimodular triangulation all of whose maximal simplices contain the origin ([14, Proposition 1.2]). Hence we obtain the following:

Corollary 5.1. *Let P and Q be two finite posets. Then the h^*-polynomial of $\mathscr{C}_{P,Q}$ coincides with the h-polynomial of a flag triangulation of a sphere.*

In [36, Proposition 2.2] it was shown that $\mathscr{C}_{P,Q}$ is locally anti-blocking. In general, for two finite posets $(P,<_P)$ and $(Q,<_Q)$ with $P\cap Q=\emptyset$, the *ordinal sum* of P and Q is the poset $(P\oplus Q,<_{P\oplus Q})$ on $P\oplus Q=P\cup Q$ such that $i<_{P\oplus Q} j$ if and only if (a) $i,j\in P$ and $i<_P j$, or (b) $i,j\in Q$ and $i<_Q j$, or (c) $i\in P$ and $j\in Q$. Given a subset I of $[d]$, we define the *induced subposet* of P on I to be the finite poset $(P_I,<_{P_I})$ on W such that $i<_{P_I} j$ if and only if $i<_P j$. For $I\subset[d]$, let $\overline{I}:=[d]\setminus I$.

Proposition 5.2 ([36, Proposition 2.2])**.** *Let P and Q be two finite posets on $[d]$. Then for each $\varepsilon\in\{-1,1\}^d$, it follows that*

$$\mathscr{C}_{P,Q}\cap\mathbb{R}^d_{\varepsilon}=\mathscr{C}^{\pm}_{P_{I_\varepsilon}\oplus Q_{\overline{I_\varepsilon}}}\cap\mathbb{R}^d_{\varepsilon},$$

where $I_\varepsilon=\{i\in[d]:\varepsilon_i=1\}$.

From this result, Theorem 0.1 and Proposition 2.4 we obtain the following:

Theorem 5.3. *Let P and Q be two finite posets on $[d]$. Then one has*

$$h^*(\mathscr{C}_{P,Q},x)=\frac{1}{2^d}\sum_{\varepsilon\in\{-1,1\}^d}h^*(\mathscr{C}^{(e)}_{R_\varepsilon},x)=(x+1)^d f_{P,Q}\left(\frac{4x}{(x+1)^2}\right),$$

where $I_\varepsilon=\{i\in[d]:\varepsilon_i=1\}$ and R_ε is a naturally labeled poset which is obtained from $P_{I_\varepsilon}\oplus Q_{\overline{I_\varepsilon}}$ by reordering the label and

$$f_{P,Q}(x)=\frac{1}{2^d}\sum_{\varepsilon\in\{-1,1\}^d}W^{(\ell)}_{R_\varepsilon}(x)$$

In particular, $h^*(\mathscr{C}_{P,Q},x)$ is γ-positive. Moreover, $h^*(\mathscr{C}_{P,Q},x)$ is real-rooted if and only if $f_{P,Q}(x)$ is real-rooted.

On the other hand, it is known that, from $h^*(\mathscr{C}_{P,Q},x)$, we obtain the h^*-polynomials of several non-locally anti-blocking lattice polytopes arising from the posets P and Q. The *order polytope $\mathscr{O}_P$* ([34]) of P is the $(0,1)$-polytope defined by

$$\mathscr{O}_P:=\{\mathbf{x}\in[0,1]^d : x_i\le x_j \text{ if } i<_P j\}.$$

Given two lattice polytopes $\mathscr{P},\mathscr{Q}\subset\mathbb{R}^d$, we define

$$\mathscr{P}*\mathscr{Q}:=\mathrm{conv}((\mathscr{P}\times\{0\})\cup(\mathscr{Q}\times\{1\}))\subset\mathbb{R}^{d+1},$$

which are called the *Cayley sum* of $\mathscr{P}$ and $\mathscr{Q}$, and define

$$\Omega(\mathscr{P},\mathscr{Q}):=\mathrm{conv}((\mathscr{P}\times\{1\})\cup(-\mathscr{Q}\times\{-1\}))\subset\mathbb{R}^{d+1}.$$

Proposition 5.4 ([14, Theorem 1.1])**.** *Let P and Q be two finite posets on $[d]$. Then one has*

$$h^*(\mathscr{C}_{P,Q},x)=h^*(\Gamma(\mathscr{O}_P,\mathscr{C}_Q),x).$$

Furthermore, if P and Q has a common linear extension, then we obtain

$$h^*(\mathscr{C}_{P,Q},x)=h^*(\Gamma(\mathscr{O}_P,\mathscr{O}_Q),x).$$

Proposition 5.5 ([16, Theorem 1.4]). *Let P and Q be two finite posets on $[d]$. Then one has*

$$(1+x)h^*(\mathscr{C}_{P,Q},x) = h^*(\Omega(\mathscr{O}_P,\mathscr{C}_Q),x).$$

Furthermore, if P and Q has a common linear extension, then we obtain

$$(1+x)h^*(\mathscr{C}_{P,Q},x) = h^*(\Omega(\mathscr{O}_P,\mathscr{O}_Q),x).$$

Proposition 5.6 ([15, Theorem 4.1]). *Let P and Q be two finite posets on $[d]$. Then one has*

$$h^*(\mathscr{C}_{P,Q},x) = h^*(\mathscr{O}_P * \mathscr{C}_Q,x).$$

From these propositions and Theorem 5.3, we obtain the following:

Corollary 5.7. *Let P and Q be two finite posets on $[d]$. Then the h^*-polynomials of $\Gamma(\mathscr{O}_P,\mathscr{C}_Q)$, $\Omega(\mathscr{O}_P,\mathscr{C}_Q)$, $\mathscr{O}_P * \mathscr{C}_Q$ and $\Omega(\mathscr{C}_P,\mathscr{C}_Q)$ are γ-positive. Furthermore, if P and Q has a common linear extension, then the h^*-polynomials of $\Gamma(\mathscr{O}_P,\mathscr{O}_Q)$ and $\Omega(\mathscr{O}_P,\mathscr{O}_Q)$ are also γ-positive.*

In the rest of section, we introduce enriched (P,Q)-partitions and we show that the Ehrhart polynomial of $\mathscr{C}_{P,Q}$ coincides with a counting polynomial of enriched (P,Q)-partitions. Assume that P and Q are naturally labeled. We say that a map $f : [d] \to \mathbb{Z}$ is an *enriched (P,Q)-partition* if, for all $x,y \in [d]$, f satisfies

- $x <_P y$, $f(x) \geq 0$ and $f(y) \geq 0 \Rightarrow f(x) \leq f(y)$;
- $x <_Q y$, $f(x) \leq 0$ and $f(y) \leq 0 \Rightarrow f(x) \geq f(y)$.

For each $0 < m \in \mathbb{Z}$, let $\Omega^{(e)}_{P,Q}(m)$ denote the number of enriched (P,Q)-partitions $f : [d] \to [a,b]_{\mathbb{Z}}$, where a and b are integers with $a \leq 0 \leq b$ and $b-a = m$, and $[a,b]_{\mathbb{Z}} := [a,b] \cap \mathbb{Z}$.

Theorem 5.8. *Let P and Q be two finite posets on $[d]$. Then one has*

$$L_{\mathscr{C}_{P,Q}}(m) = \Omega^{(e)}_{P,Q}(m).$$

Proof. Let a and b be integers with $a \leq 0 \leq b$ and $b-a = m$, and denote $F(m)$ the set of enriched (P,Q)-partitions $f : [d] \to [a,b]_{\mathbb{Z}}$. We show that there exists a bijection from $m\mathscr{C}_{P,Q} \cap \mathbb{Z}^d$ to $F(m)$.

Let $f : [d] \to [a,b]_{\mathbb{Z}}$ be an enriched (P,Q)-partition, where a and b are integers with $a \leq 0 \leq b$ and $b-a = m$. We set

$$I = \{i \in [d] : f(i) \geq 0\}.$$

Let

$$x_i = \begin{cases} f(i) & \text{if } i \in I \text{ is minimal in } P_I, \\ \min\{f(i)-f(j) : i \text{ covers } j \text{ in } P_I\} & \text{if } i \in I \text{ is not minimal in } P_I, \\ -|f(i)| & \text{if } i \in \overline{I} \text{ is minimal in } Q_{\overline{I}}, \\ -\min\{|f(i)|-|f(j)| : i \text{ covers } j \text{ in } Q_{\overline{I}}\} & \text{if } i \in \overline{I} \text{ is not minimal in } Q_{\overline{I}}. \end{cases}$$

Assume that $I = \{1,\dots,k\}$ and $\overline{I} = \{k+1,\dots,d\}$. Then we have $(x_1,\dots,x_k) \in b\mathscr{C}_{P_I}$ and $(x_{k+1},\dots,x_d) \in a\mathscr{C}_{Q_{\overline{I}}}$ by a result of Stanley [34, Theorem 3.2]. Hence one obtains

$(x_1,\dots,x_d) \in b\mathscr{C}_{P_I} \oplus a\mathscr{C}_{Q_{\overline{I}}} \subset m\mathscr{C}_{P,Q}$, where $b\mathscr{C}_{P_I} \oplus a\mathscr{C}_{Q_{\overline{I}}}$ is the free sum of $b\mathscr{C}_{P_I}$ and $a\mathscr{C}_{Q_{\overline{I}}}$. Similarly, in general, it follows that $(x_1,\dots,x_d) \in m\mathscr{C}_{P,Q}$. Therefore, the map $\varphi : F(m) \to m\mathscr{C}_{P,Q} \cap \mathbb{Z}^d$ defined by $\varphi(f) = (x_1,\dots,x_d)$ for each $f \in F(m)$ is well-defined.

Take $(x_1,\dots,x_d) \in m\mathscr{C}_{P,Q} \cap \mathbb{Z}^d$. We set

$$I = \{i \in [d] : x_i \geq 0\}.$$

We define a map $f : [d] \to \mathbb{Z}$ by

$$f(i) = \begin{cases} \max\{x_{j_1} + \cdots + x_{j_k} : j_1 <_{P_I} \cdots <_{P_I} j_k = i\} & \text{if } i \in I, \\ \\ -\max\{|x_{j_1}| + \cdots + |x_{j_k}| : j_1 <_{Q_{\overline{I}}} \cdots <_{Q_{\overline{I}}} j_k = i\} & \text{if } i \in \overline{I}. \end{cases}$$

Assume that $I = \{1,\dots,k\}$ and $\overline{I} = \{k+1,\dots,d\}$. Then one has $(x_1,\dots,x_d) \in m(\mathscr{C}_{P_I} \oplus (-\mathscr{C}_{Q_{\overline{I}}})) \cap \mathbb{Z}^d$. Moreover, for some integers a and b with $a \leq 0 \leq b$ and $b - a = m$, it follows that $(x_1,\dots,x_k) \in b\mathscr{C}_{P_I}$ and $(x_{k+1},\dots,x_d) \in a\mathscr{C}_{Q_{\overline{I}}}$. We define $f_1 : I \to [b]_0$ by $f_1(i) = f(i)$, and $f_2 : \overline{I} \to [-a]_0$ by $f_2(i) = -f(i)$. From [34, Proof of Theorem 3.2], it follows that $f_1(x) \leq f_1(y)$ if $x <_{P_I} y$, and $f_2(x) \leq f_2(y)$ if $x <_{Q_{\overline{I}}} y$. Therefore, $f : [d] \to [a,b]_{\mathbb{Z}}$ is an enriched (P,Q)-partition, namely, $f \in F(m)$. Similarly, in general, it follows that $f \in F(m)$. Thus, the map $\psi : m\mathscr{C}_{P,Q} \cap \mathbb{Z}^d \to F(m)$ defined by $\psi(\mathbf{x})(i) = f(i)$ for each $\mathbf{x} = (x_1,\dots,x_d) \in m\mathscr{C}_{P,Q} \cap \mathbb{Z}^d$ is well-defined.

Finally, we show that φ is a bijection. However, this immediately follows by the above and the argument in [34, Proof of Theorem 3.2]. □

Since $\mathscr{C}_{P,Q}$ is reflexive, we obtain the following:

Corollary 5.9. *Let P and Q be two finite naturally labeled posets on $[d]$. Then $\Omega^{(e)}_{P,Q}(m)$ is a polynomial in m of degree d and one has*

$$\Omega^{(e)}_{P,Q}(m) = (-1)^d \Omega^{(e)}_{P,Q}(-m-1).$$

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