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ENUMERATING LINEAR SYSTEMS ON GRAPHS

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Abstract. The divisor theory of graphs views a finite connected graph G as a discrete version of a Riemann surface. Divisors on G are formal integral combinations of the vertices of G, and linear equivalence of divisors is determined by the discrete Laplacian operator for G. As in the case of Riemann surfaces, we are interested in the complete linear system $|D|$ of a divisor D—the collection of nonnegative divisors linearly equivalent to D. Unlike the case of Riemann surfaces, the complete linear system of a divisor on a graph is always finite. We compute generating functions encoding the sizes of all complete linear systems on G and interpret our results in terms of polyhedra associated with divisors and in terms of the invariant theory of the (dual of the) Jacobian group of G. If G is a cycle graph, our results lead to a bijection between complete linear systems and binary necklaces. The final section generalizes our results to a model based on integral M-matrices.

1. Introduction.

Let G be a finite, connected, undirected graph with vertex set V. The divisor theory of graphs uses the graph Laplacian to view G as a discrete analogue of a Riemann surface. As a reference, the reader should consult the seminal paper by Baker and Norine ([3]), a main result of which is the Riemann-Roch theorem for graphs. That work is related to a broader circle of ideas that includes chip-firing on graphs ([7]), the arithmetical groups of Lorenzini ([16]), the abelian sandpile model ([2], [9]), and parking functions in combinatorics ([20]). For general textbooks, including many references, see ([8] and [15]). The papers [14] and [1] are also recommended.

Precise definitions follow in Section 2, but for the purposes of this introduction, it is useful to think of divisor theory on graphs in terms of the *dollar game* introduced in [3]. By definition, a *divisor* D is an element of $\mathbb{Z}V$, the free abelian group on the vertices of G. Think of D as an assignment of $D(v)$ dollars to each vertex v. If the integer $D(v)$ is negative, then v is in debt. The net amount of money on the graph is the *degree*, $\deg(D) := \sum_{v\in V} D(v)$, of D. A *lending* move (or *firing*) by a vertex v consists of v giving one dollar to each of its neighbors and losing the corresponding amount itself. A *borrowing* move is the opposite, in which v takes a dollar from each of its neighbors. Two divisors are *linearly equivalent* if one may be transformed into the other by a sequence of lending and borrowing moves. The *Picard group*, $\mathrm{Pic}(G)$, is the group of divisors modulo linear equivalence. Since lending and borrowing conserve total wealth, $\mathrm{Pic}(G)$ is graded by degree. Its degree 0 part is a finite group called the *Jacobian group*, $\mathrm{Jac}(G)$, and there is an isomorphism $\mathrm{Pic}(G) \simeq \mathbb{Z} \oplus \mathrm{Jac}(G)$, depending on the choice of a vertex (cf. (1)).

The point of the dollar game is for the vertices to cooperate and, through a sequence of lending and borrowing moves, reach a state in which no vertex is in debt. If this is possible, the effect is to transform the divisor D into a new, linearly equivalent divisor E that is debt-free, i.e., such that $E(v) \geq 0$ for all vertices v. Such a debt-free divisor is said to be *effective*. The *complete linear system* of a divisor D, denoted $|D|$, is the set of all effective divisors linearly equivalent to D. In other words, $|D|$ is the set of all winning states for the dollar game starting with the initial distribution of wealth prescribed by D.

The purpose of this paper is to answer the question: What is the cardinality of the complete linear system $|D|$ for each divisor D? In other words, how many winning states are there for each dollar game on G? We know of no previous systematic study of this question.[1] To answer it, we first use the isomorphism $\mathrm{Pic}(G) \simeq \mathbb{Z} \oplus \mathrm{Jac}(G)$ to partition the collection of all effective divisors on G into sets $\mathbb{E}_{[D]}$, one for each $[D] \in \mathrm{Jac}(G)$. Let $\lambda_{[D]}(k)$ be the number of divisors in $\mathbb{E}_{[D]}$ with degree k, and let $\Lambda_{[D]}(z) := \sum_{k\geq 0} \lambda_{[D]}(k) z^k$ be its generating function. Our aim, then, is to understand the structure of

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[1] For the combinatorial structure of $|D|$ in the case of a metric graph (tropical curve), see [13].

the $\mathbb{E}_{[D]}$ and use it to find closed expressions for $\Lambda_{[D]}(z)$ for each $[D] \in \mathrm{Jac}(G)$. The following is an outline of our results:

• In Section 3, we show that each effective divisor has a decomposition into a sum of *primary* and *secondary* divisors for G and then use this idea to compute a rational expression for $\Lambda_{[D]}(z)$ for each $[D] \in \mathrm{Jac}(G)$ (Theorem 3.1 and Corollary 3.2). Proposition 3.4 provides an effective method for computing primary and secondary divisors, and hence for computing $\Lambda_{[D]}(z)$. The section ends with several examples.

• Section 4 reinterprets the results of Section 3 in terms of lattice points in a rational polyhedra cone. Generators for the cone correspond to primary divisors and lattice points in a fundamental parallelepiped correspond to secondary divisors; the rational expression for $\Lambda_{[D]}(z)$ from Section 3 is re-derived using standard lattice-point counting techniques (Theorem 4.4 and Proposition 4.5).

• Section 5 approaches our question using invariant theory. By Theorem 5.1, the elements of $\mathbb{E}_{[D]}$ may be regarded as a basis for the (relative) polynomial invariants of a certain complex representation of the dual of $\mathrm{Jac}(G)$. Molien's theorem then expresses $\Lambda_{[D]}(z)$ in a form that is substantially different from that given earlier (Corollary 5.2). Examples are given at the end of the section.

• Section 6 applies our theory to the specific case of the cycle graph C_n with n vertices, yielding a remarkable connection to binary necklaces. Let $\mathcal{N}(n,k)$ denote the set of binary necklaces with n black beads and k white beads. Theorem 6.3 sets up the relevant invariant theory, and Corollary 6.4 shows that $\lambda_{[D]}(k)$ counts the number of elements of $\mathcal{N}(n,k)$ exhibiting certain symmetry (depending on $[D]$). In particular, $\lambda_{[0]}(k)$ is the total number of binary necklaces with n black beads and k white beads. Theorem 6.7 gives a combinatorial bijection between the divisors of degree k in $\mathbb{E}_{[D]}$ and $\mathcal{N}(n,k)$ for each $[D]$ whenever n and k are relatively prime. For further work, motivated by these results, see [17].

• Section 7 provides a possible direction for further research. A Laplacian matrix for a graph is closely related to a more general type of matrix, called an M-matrix, defined by certain positivity conditions. These matrices allow one to extend much of the divisor theory of graphs to a broader context ([10], [12]). In Section 7 we show how each M-matrix gives rise to a family of matrices serving the role of the Laplacian matrix, which allows our results to be extended to this broader context. As examples, we discuss two particular cases from [6]: Cartan matrices for crystallographic root systems and McKay-Cartan matrices for faithful complex representations of arbitrary finite groups.

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2. Divisor theory preliminaries

Let $G = (V, \mathcal{E})$ be a connected, undirected multigraph with finite vertex set V and finite edge multiset $\mathcal{E}$. Many of our constructions will depend on fixing a vertex $q \in V$, which we do now, once and for all. Loops are allowed but our results are not affected if they are removed. We let $\mathbb{N} := \mathbb{Z}_{\geq 0}$ denote the natural numbers.

We recall some of the theory of divisors on graphs, referring readers unfamiliar with this theory to [3] or to the textbooks [8] and [15]. A *divisor* on G is an element of the free abelian group on the vertices of G,

$$\mathrm{Div}(G) := \mathbb{Z}V = \left\{\textstyle\sum_{v\in V} D(v)v : D(v) \in \mathbb{Z}\right\}.$$

The degree of a divisor D is the sum of its coefficients: $\deg(D) := \sum_{v\in V} D(v)$. For instance, if we consider $v \in V$ as a divisor, then $\deg(v) = 1$. We use the notation $\deg_G(v)$ to refer to the ordinary degree of a vertex—the number of edges incident on v. The set of divisors of degree k is denoted by $\mathrm{Div}^k(G)$.

The *(discrete) Laplacian operator* of G is the function $L\colon \mathbb{Z}^V \to \mathbb{Z}^V$ given by

$$L(f)(v) = \sum_{vw\in\mathcal{E}} (f(v) - f(w))$$

for each $f \in \mathbb{Z}^V$ and $v \in V$. The *divisor of a function* $f\colon V \to \mathbb{Z}$, arising by analogy from the theory of divisors on Riemann surfaces, is then

$$\operatorname{div}(f) := \sum_{v \in V} (L(f)(v))\, v \in \operatorname{Div}(G).$$

The mapping $v \mapsto \chi_v$ which sends each vertex to its corresponding characteristic function determines an isomorphism $\chi\colon \operatorname{Div}(G) \simeq \mathbb{Z}^V$, and we have $\chi \circ \operatorname{div} = L$, which we use to identify div with L.

Divisors of functions are called *principal divisors*, and they form an additive subgroup of $\operatorname{Div}(G)$ denoted $\operatorname{Prin}(G)$. Two divisors D and D' are *linearly equivalent* if $D - D' \in \operatorname{Prin}(G)$, in which case, we write $D \sim D'$. The *Picard group* of G is then the group of divisors modulo linear equivalence:

$$\operatorname{Pic}(G) := \operatorname{Div}(G)/\operatorname{Prin}(G).$$

Since principal divisors have degree zero, $\operatorname{Pic}(G)$ is graded by degree. Its degree k part is denoted $\operatorname{Pic}^k(G)$. The degree-zero part of the Picard group is a subgroup called the *Jacobian group* of G:

$$\operatorname{Jac}(G) := \operatorname{Pic}^0(G) = \operatorname{Div}^0(G)/\operatorname{Prin}(G) \subseteq \operatorname{Pic}(G).$$

We write $[D]$ for the class of a divisor D modulo $\operatorname{Prin}(G)$. With respect to our fixed vertex q, there is an isomorphism

$$\begin{aligned} \operatorname{Pic}(G) &\to \mathbb{Z} \oplus \operatorname{Jac}(G) \\ [D] &\mapsto (\deg(D), [D - \deg(D)q])\,. \end{aligned} \tag{1}$$

Fixing an ordering $v_1, \dots, v_n$ of V determines a basis for $\operatorname{Div}(G)$ and a corresponding dual basis for $\mathbb{Z}^V$, allowing us to identify both spaces with $\mathbb{Z}^n$. Thus, $D \in \operatorname{Div}(G)$ is identified with $(D(v_1), \dots, D(v_n))$, and for any $v \in V$, we may refer to the v-th coordinate of a vector in $\mathbb{Z}^n$. With respect to the chosen bases, div and L are represented by the $n \times n$ *Laplacian matrix*, which we also denote by L. This matrix is given by

$$L = \operatorname{Deg}(G) - A$$

where $\operatorname{Deg}(G) = \operatorname{diag}(\deg_G(v_1), \dots, \deg_G(v_n))$ and A is the adjacency matrix for G with i, j-th entry equal to the number of edges connecting v_i to v_j. The matrix L is symmetric since G is undirected. We then have the isomorphism

$$\begin{aligned} \operatorname{Pic}(G) &\simeq \operatorname{cok}(L) = \mathbb{Z}^n/\operatorname{im}_{\mathbb{Z}}(L) \\ [\textstyle\sum_{i=1}^n a_i v_i] &\mapsto (a_1, \dots, a_n) + \operatorname{im}_{\mathbb{Z}}(L). \end{aligned}$$

The *reduced Laplacian* matrix for G with respect to q is the $(n-1) \times (n-1)$ matrix $\tilde{L}$ formed by removing the row and column corresponding to q from L. There is an isomorphism

$$\begin{aligned} \operatorname{Jac}(G) &\simeq \mathbb{Z}^{n-1}/\operatorname{im}_{\mathbb{Z}}(\tilde{L}) \\ [D] &\to D|_{q=0} \end{aligned} \tag{2}$$

where $D|_{q=0} := \sum_{v \in V \setminus \{q\}} D(v)v$. The inverse sends the class of the v-th standard basis vector in $\mathbb{Z}^{n-1}/\operatorname{im}_{\mathbb{Z}}(\tilde{L})$ to $[v - q]$ for each $v \neq q$. Isomorphisms 1 and 2 combine to say that for $D, D' \in \operatorname{Div}(G)$,

$$D \sim D' \quad \Longleftrightarrow \quad \Big(\deg(D) = \deg(D') \quad \text{and} \quad D|_{q=0} = D'|_{q=0} \bmod \operatorname{im}_{\mathbb{Z}}(\tilde{L})\Big)\,.$$

The kernel of the Laplacian matrix is the set of constant vectors, and the reduced Laplacian has full rank $n - 1$. By the matrix-tree theorem, the number of spanning trees of G is $\det(\tilde{L})$, and thus by (2), it is also the order of $\operatorname{Jac}(G)$. We adopt the following notation: for each $v \in V$, let

$$\operatorname{ord}_q(v) := \text{order of } [v - q] \in \operatorname{Jac}(G).$$

In particular, $\operatorname{ord}_q(q) = 1$.

We now describe a standard set of representatives for the elements of $\operatorname{Jac}(G)$. A *set firing* by a subset $W \subseteq V$ on a divisor D produces a new divisor $D' = D - \operatorname{div}(\chi_W)$ where χ_W is the characteristic function of W. Having fixed an ordering of the vertices, we identify W with a 0-1 vector in $\mathbb{Z}^n$, and we have $D' = D - LW$ where L is the Laplacian matrix. A *reverse firing* would instead produce the divisor $D + LW$. Thus, two divisors are linearly equivalent if and only if they differ by a sequence of set firings and reverse firings.

Firing a set W is *legal* if $D'(w) \geq 0$ for all $w \in W$. The divisor D is *q-reduced* if

(i) $D(v) \geq 0$ for all $v \in V \setminus \{q\}$, and
(ii) D has no legal set firing by a nonempty set $W \subseteq V \setminus \{q\}$.

It turns out that each divisor is linearly equivalent to a unique q-reduced divisor. Thus, the q-reduced divisors of degree 0 form a set of representatives for the elements of $\mathrm{Jac}(G)$. There is an efficient algorithm (Dhar's algorithm) for finding the q-reduced representative of any divisor class. If D is q-reduced, then letting $\deg(D|_{q=0}) := \sum_{v \in V \setminus \{q\}} D(v)$ we have

$$0 \leq \deg(D|_{q=0}) \leq |\mathcal{E}| - |V| + 1.$$

Therefore, searching through all divisors D of degree 0 satisfying the above bound provides a fairly efficient means of calculating $\mathrm{Jac}(G)$. (For an improvement, see [4].)

2.1. **Partitioning effective divisors.** A divisor E is *effective* if $E(v) \geq 0$ for all $v \in V$, in which case we write $E \geq 0$. The *complete linear system* of a divisor D is its set of linearly equivalent effective divisors:

$$|D| := \{E \in \mathrm{Div}(G) : E \geq 0 \text{ and } E \sim D\}.$$

Note that $|D|$ depends only on the divisor class of D. Also, since linearly equivalent divisors have the same degree, $|D|$ is finite.

For each $[D] \in \mathrm{Jac}(G)$, define

$$\mathbb{E}_{[D]} := \cup_{k \geq 0} |D + kq| = \{E \in \mathrm{Div}(G) : E \geq 0 \text{ and } E - \deg(E)q \sim D\}.$$

The $\mathbb{E}_{[D]}$ partition the set of effective divisors as D runs over a set of representatives for $\mathrm{Jac}(G)$. The collection $\mathbb{E}_{[0]}$ is a commutative monoid, and it acts on each $\mathbb{E}_{[D]}$ via addition: $\mathbb{E}_{[0]} + \mathbb{E}_{[D]} = \mathbb{E}_{[D]}$. Note that $\mathbb{E}_{[D]}$ depends on q.[2]

Definition 2.1. The λ-sequence for $[D] \in \mathrm{Jac}(G)$ is the sequence with k-th term

$$\lambda_{[D]}(k) := \#|D + kq|.$$

(It does not depend on the choice of representative of the class $[D]$.) The generating function for the λ-sequence is

$$\Lambda_{[D]}(z) := \sum_{k \geq 0} \lambda_{[D]}(k) z^k.$$

Our main goal is to find closed expressions for $\Lambda_{[D]}$ for each $[D] \in \mathrm{Jac}(G)$ and thus determine the cardinality of $|F|$ for all $F \in \mathrm{Div}(G)$.

3. Primary and secondary divisors

In this section, we compute $\Lambda_{[D]}$ using *primary* and *secondary* divisors, defined as part of the following theorem.

Theorem 3.1.
(1) (Existence) *There exists a finite subset $\mathcal{P} \subset \mathbb{E}_{[0]}$ and for each $[D] \in \mathrm{Jac}(G)$, a finite subset $\mathcal{S}_{[D]} \subset \mathbb{E}_{[D]}$ such that each $E \in \mathbb{E}_{[D]}$ can be written uniquely as*

$$E = F + \sum_{P \in \mathcal{P}} a_P P$$

with $F \in \mathcal{S}_{[D]}$ and $a_P \in \mathbb{N}$ for all $P \in \mathcal{P}$. The set $\mathcal{P}$ is called a set of primary divisors *for G, and $\mathcal{S}_{[D]}$ is called the set of* $[D]$-secondary *divisors with respect to $\mathcal{P}$.*
(2) (Uniqueness) *Sets $\mathcal{P}$ and $\{\mathcal{S}_{[D]}\}_{[D] \in \mathrm{Jac}(G)}$ satisfy part (1) if and only if*

$$\mathcal{P} = \{\ell_v v : v \in V\} \quad \textit{and} \quad \mathcal{S}_{[D]} = \{E \in \mathbb{E}_{[D]} : E(v) < \ell_v \textit{ for all } v \in V\},$$

where ℓ_v is a positive multiple of $\mathrm{ord}_q(v)$ for all $v \in V$. In particular, taking $\ell_v = \mathrm{ord}_q(v)$ for all $v \in V$ produces the set of primary divisors of smallest degree and corresponding sets of secondary divisors with minimal cardinality.

[2] For $q' \in V$, writing $D + kq = D + kq' + k(q - q')$ shows the dependence is "periodic" with period equal to the order of $[q - q'] \in \mathrm{Jac}(G)$.

Proof. To prove part (1), for each $v \in V$, let ℓ_v be a positive multiple of $\operatorname{ord}_q(v)$, and define $\mathcal{P}$ and each $\mathcal{S}_{[D]}$ as in part (2) of the theorem. Given $E \in \mathbb{E}_{[D]}$, for each $v \in V$, let k_v be the largest integer such that $E(v) - k_v\ell_v \geq 0$, and define $F := E - \sum_{v\in V} k_v\ell_v v \in \mathcal{S}_{[D]}$. Then $E = F + \sum_{v\in V} k_v\ell_v v$ is a decomposition as required in part (1). For uniqueness of this decomposition, suppose $E = F' + \sum_{v\in V} k'_v v$ for some $F' \in \mathcal{S}_{[D]}$ and $k'_v \in \mathbb{N}$. Then for each $v \in V$, we have $0 \leq F(v) = E(v) - k_v\ell_v < \ell_v$ and $0 \leq F'(v) = E(v) - k'_v\ell_v < \ell_v$. Subtracting these inequalities yields $-\ell_v < (k'_v - k_v)\ell_v < \ell_v$. It follows that $k_v = k'_v$ for all v and $F = F'$.

We have shown that if $\mathcal{P}$ and $\mathcal{S}_{[D]}$ have the form displayed in (2), then they serve as sets of primary and secondary divisors, i.e., they satisfy the conditions in part (1). To show that necessity of this form and thus finish the proof of part (2), let $\mathcal{P}$ and $\{\mathcal{S}_{[D]}\}_{[D]\in\operatorname{Jac}(G)}$ be any sets of primary and $[D]$-secondary divisors. Since $\mathcal{S}_{[0]}$ is finite, for each $v \in V$, there is a smallest natural number m_v such that $m_v \operatorname{ord}(v)v \notin \mathcal{S}_{[0]}$. Consider the primary-secondary decomposition $m_v \operatorname{ord}(v)v = F + \sum_{P\in\mathcal{P}} a_P P$. Since the divisors on the right-hand side are effective and all a_P are nonnegative, considering coefficients on both sides, it follows that decomposition takes the form $m_v \operatorname{ord}(v)v = av + bv$ for some $a, b \in \mathbb{N}$ such that $av \in \mathcal{S}_{[0]}$ and $bv \in \mathcal{P}$. By definition of m_v, we must have $b > 0$. Since $\mathcal{S}_{[0]}$ and $\mathcal{P}$ are subsets of $\mathbb{E}_{[0]}$, we have $av \sim aq$ and $bv \sim bq$. Therefore, $a = a' \operatorname{ord}(v)$ and $b = b' \operatorname{ord}(v)$ for some $a', b' \in \mathbb{N}$. If $a' \neq 0$, then $b' < m_v$, which implies $b' \operatorname{ord}(v)v \in \mathcal{S}_{[0]}$ by definition of m_v. However, that is impossible since the uniqueness of decompositions described in part (1) implies $\mathcal{S}_{[0]}$ and $\mathcal{P}$ are disjoint. Defining $\ell_v := m_v \operatorname{ord}(v)$, it follows that $\ell_v v \in \mathcal{P}$ for all v. However, again by uniqueness of decompositions, the elements of $\mathcal{P}$ must be linearly independent over $\mathbb{Z}$, which implies there are no other primary divisors. So $\mathcal{P} = \{\ell_v : v \in V\}$, as claimed, and it is then straightforward to show that $\mathcal{S}_{[D]}$ must have the form stated in (2) for each $[D] \in \operatorname{Jac}(G)$. □

Corollary 3.2. *Fix primary and secondary divisors as in Theorem 3.1. For each* $[D] \in \operatorname{Jac}(G)$*,*

$$\Lambda_{[D]}(z) = \frac{S(z)}{\prod_{v\in V}(1 - z^{\ell_v})}$$

where

$$S(z) = \sum_{F\in\mathcal{S}_{[D]}} z^{\deg(F)}.$$

Proof. Introduce indeterminates $\{x_v\}_{v\in V}$, and identify each effective divisor E with a monomial $x^E := \prod_{v\in V} x_v^{E(v)}$. Define

$$\sigma_D(x) = \sum_{E\in\mathbb{E}_{[D]}} x^E.$$

By Theorem 3.1, we may uniquely write

$$x^E = x^F \cdot \prod_{P\in\mathcal{P}} x^{a_P P} = x^F \cdot \prod_{v\in V} x_v^{a_v\ell_v}$$

for some $F \in \mathcal{S}_{[D]}$ and $a_v \geq 0$. Then

$$\sigma_D(x) = \sum_{F\in\mathcal{S}_{[D]}} x^F \sum_{a\in\mathbb{N}^V} \prod_{v\in V} x_v^{a_v\ell_v} = \left(\sum_{F\in\mathcal{S}_{[D]}} x^F\right) \prod_{v\in V} \left(1 + x_v^{\ell_v} + x_v^{2\ell_v} + \dots\right)$$

$$= \left(\sum_{F\in\mathcal{S}_{[D]}} x^F\right) \prod_{v\in V} \frac{1}{1 - x_v^{\ell_v}}.$$

Now note that $\Lambda_{[D]}(z) = \sigma_D(z, z, \dots, z)$ to conclude the proof. □

Remark 3.3. Denote the numerator S in Corollary 3.2 by $S_{[D]}(z)$ to indicate its dependence on $[D] \in \operatorname{Jac}(G)$. Since the $\mathbb{E}_{[D]}$ partition the set of effective divisors, it follows that $\sum_{[D]\in\operatorname{Jac}(G)} \Lambda_{[D]}(z) = 1/(1-z)^n$, and hence,

$$\sum_{[D]\in\operatorname{Jac}(G)} S_{[D]}(z) = \frac{\prod_{v\in V}(1 - z^{\ell_v})}{(1 - z)^n} = \prod_{v\in V}(1 + z + z^2 + \dots + z^{\ell_v - 1}).$$

We now describe how to easily compute primary and secondary divisors. Recall that we have fixed an ordering of the vertices of G to identify $\mathrm{Div}(G)$ with $\mathbb{Z}^n$. Fix primary and secondary divisors as in Theorem 3.1 (2), and consider the natural projection

$$\pi\colon\ \mathrm{Div}(G)=\mathbb{Z}^n\to\mathbb{Z}^n/\prod_{v\in V}\ell_v\mathbb{Z}.$$

A *standard representative* of an element $\overline{D}\in\mathbb{Z}^n/\prod_{v\in V}\ell_v\mathbb{Z}$ is a divisor $E\in\mathrm{Div}(G)$ such that $\pi(E)=\overline{D}$ and $0\le E(v)<\ell_v$ for all v.

Assume vertex q appears last in the ordering so that

$$\mathrm{im}_{\mathbb{Z}}\,\tilde{L}\times\mathbb{Z}\subseteq\mathbb{Z}^{n-1}\times\mathbb{Z}=\mathbb{Z}^n=\mathrm{Div}(G).$$

For each $D\in\mathrm{Div}(G)$, define

$$H_{[D]}:=\pi(D+(\mathrm{im}_{\mathbb{Z}}\,\tilde{L}\times\mathbb{Z}))\subseteq\mathbb{Z}^n/\prod_{v\in V}\ell_v\mathbb{Z}.$$

Proposition 3.4. *Let $\mathcal{P}=\{\ell_v v : v\in V\}$ and $\mathcal{S}_{[D]}$ for each $[D]\in\mathrm{Jac}(G)$ be as in Theorem 3.1 (2).*

(1) *Let $\tilde{L}^{-1}$ be the inverse of the reduced Laplacian over $\mathbb{Q}$. Then, for each $v\ne q$, the integer $\mathrm{ord}_q(v)$ is the least common multiple of the denominators of the (reduced) fractions in the v-th column of $\tilde{L}^{-1}$.*

(2) *For each $[D]\in\mathrm{Jac}(G)$, there is a bijection of sets*

$$\begin{aligned}\mathcal{S}_{[D]}&\to H_{[D]}\\ E&\mapsto\pi(E|_{q=0},\deg(E)),\end{aligned}$$

and thus $\mathcal{S}_{[D]}$ is exactly a set of standard representatives for $H_{[D]}$.

(3) *For each $[D]\in\mathrm{Jac}(G)$,*

$$|\mathcal{S}_{[D]}||\,\mathrm{Jac}(G)|=\prod_{v\in V}\ell_v.$$

Proof. First note that $\tilde{L}$ has rank $n-1$, and thus has an inverse $\tilde{L}^{-1}$ over $\mathbb{Q}$. By (2) of Section 2, the order of $[v-q]\in\mathrm{Jac}(G)$ is the least positive integer k such that $kv\in\mathrm{im}_{\mathbb{Z}}\,\tilde{L}$. Therefore, $\mathrm{ord}_q(v)$ is the least positive integer k such that $\tilde{L}^{-1}(kv)\in\mathbb{Z}^{n-1}$. Part (1) follows.

Part (2) is immediate: a divisor E is a standard representative for an element in $H_{[D]}$ if and only if $E|_{q=0}=D|_{q=0}\bmod\mathrm{im}_{\mathbb{Z}}\,\tilde{L}$ and $0\le E(v)<\ell_v$ for all $v\in V$, which is exactly the requirement for being an element of $\mathcal{S}_{[D]}$.

Now consider part (3). Since $\ell_v[v-q]=[0]\in\mathrm{Jac}(G)$ for all v, there is a surjection

$$\mathbb{Z}^n/\textstyle\prod_{v\in V}\ell_v\mathbb{Z}\to\mathrm{Jac}(G)\simeq\mathbb{Z}^{n-1}/\,\mathrm{im}_{\mathbb{Z}}\,\tilde{L}$$

which sends the class of the v-th standard basis vector to $[v-q]$. Its kernel is $H_{[0]}$. So by part (2), we have $|\mathcal{S}_{[0]}||\,\mathrm{Jac}(G)|=\prod_{v\in V}\ell_v$. However, for each $[D]\in\mathrm{Jac}(G)$, there is a well-defined bijection

$$\begin{aligned}H_{[0]}&\to H_{[D]}\\ \pi(E)&\mapsto\pi(D+E).\end{aligned}$$

So $|\mathcal{S}_{[D]}|=|\mathcal{S}_{[0]}|$, and part (3) follows. ∎

Remark 3.5. (Computation of primary and secondary divisors) To summarize the above: in order to compute a set of primary divisors, use Proposition 3.4 to compute each $\mathrm{ord}_q(v)$ for $v\ne q$ from the columns of $\tilde{L}^{-1}$. Then take $\mathcal{P}=\{\ell_v v\}_{v\in V}$ where the ℓ_v are arbitrary positive multiples of the corresponding $\mathrm{ord}_q(v)$. In order to minimize the number of secondary divisors, one would take $\ell_v=\mathrm{ord}_q(v)$ for each v. In particular, this would mean $\ell_q=\mathrm{ord}_q(q)=1$.

Next, use part (2) of Proposition 3.4 to compute $\mathcal{S}_{[D]}$ for each $[D]\in\mathrm{Jac}(G)$. To ease the computation of $\mathrm{im}_{\mathbb{Z}}\,\tilde{L}$, perform invertible integer column operations on $\tilde{L}$ to compute its Hermite normal form A. (We will always take "Hermite normal form" to mean "column Hermite normal form".) Then find the set S of standard representatives for the coset $D|_{q=0}+\mathrm{im}_{\mathbb{Z}}\,A$ modulo $\prod_{v\in V\setminus\{q\}}\ell_v\mathbb{Z}$. Finally $\mathcal{S}_{[D]}=\{c+kq : c\in S\text{ and }0\le k<\ell_q\}$.

3.1. **Examples.** We now use the method outlined in Remark 3.5 to compute λ-sequence generating functions for several examples.

3.1.1. *Trees.* If G is a tree, then $\mathrm{Jac}(G)$ is trivial, and the mapping $[D] \mapsto \deg(D)$ is an isomorphism of $\mathrm{Pic}(G)$ with $\mathbb{Z}$. It follows that for any $q \in V$,

$$\mathbb{E}_{[0]} = \cup_{k\geq 0}|kq| = \{E \in \mathrm{Div}(G) : E \geq 0 \text{ and } \deg(E) \geq 0\}\,.$$

So, letting $n = |V|$, the cardinality of $|kq|$ is the number of elements of $\mathbb{N}^n$ with coordinate sum equal to k. Thus,

$$\Lambda_{[0]}(z) = \sum_{k\geq 0} \binom{n-1+k}{k} z^k = \frac{1}{(1-z)^n},$$

in agreement with Corollary 3.2 where we take $\ell_v = 1$ for all $v \in V$. In that case $\mathcal{P} = V$ and $\mathcal{S}_{[0]} = \{0\}$.

3.1.2. *Diamond graph.* Let G be the diamond graph pictured in Figure 1. The reduced Laplacian for G and

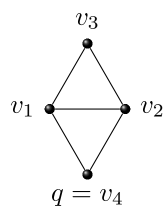


FIGURE 1. The diamond graph.

its inverse are:

$$\tilde{L} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \tilde{L}^{-1} = \begin{pmatrix} \frac{5}{8} & \frac{3}{8} & \frac{1}{2} \\ \frac{3}{8} & \frac{5}{8} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}.$$

Taking the least common multiples of denominators in the columns of $\tilde{L}^{-1}$ gives

$$(\mathrm{ord}_q(v_1), \mathrm{ord}_q(v_2), \mathrm{ord}_q(v_3), \mathrm{ord}_q(q)) = (8, 8, 2, 1).$$

To minimize the number of secondary divisors, we take $\ell_v = \mathrm{ord}_q(v)$ for all v. Thus,

$$\mathcal{P} = \{8v_1, 8v_2, 2v_3, q\}\,.$$

By Proposition 3.4 (3), we have $|\mathcal{S}_{[D]}| = 16$ for each $[D] \in \mathrm{Jac}(G)$ since $|\,\mathrm{Jac}(G)| = \det(\tilde{L}) = 8$.

To compute $\mathcal{S}_{[0]}$, perform invertible integer column operations on $\tilde{L}$ to reduce it to Hermite normal form:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Using this matrix, it is easy to find standard representatives for $\mathrm{im}_{\mathbb{Z}} A = \mathrm{im}_{\mathbb{Z}} \tilde{L}$ modulo $8\mathbb{Z} \times 8\mathbb{Z} \times 2\mathbb{Z}$. According to Remark 3.3, since $\ell_q = 1$, we then append 0 to each of these representatives to get

$$\mathcal{S}_{[0]} = \{(0,0,0,0), (1,1,0,0), (2,2,0,0), (3,3,0,0), (4,4,0,0), (5,5,0,0), (6,6,0,0), (7,7,0,0)$$
$$(0,4,1,0), (1,5,1,0), (2,6,1,0), (3,7,1,0), (4,0,1,0), (5,1,1,0), (6,2,1,0), (7,3,1,0)\}.$$

From Corollary 3.2,

$$\begin{aligned}\Lambda_{[0]}(z) &= \frac{1 + z^2 + z^4 + 2z^5 + z^6 + 2z^7 + z^8 + 2z^9 + z^{10} + 2z^{11} + z^{12} + z^{14}}{(1-z)(1-z^2)(1-z^8)^2} \\ &= \frac{1 - z + z^2 - z^3 + z^4 + z^5 - z^6 + z^7}{(1+z)^2(1+z^2)(1+z^4)(1-z)^4} \\ &= 1 + z + 3z^2 + 3z^3 + 6z^4 + 8z^5 + 12z^6 + 16z^7 + 23z^8 + 29z^9 + 39z^{10} + \cdots\,.\end{aligned}$$

For instance, the six effective divisors of degree 4 in $\mathbb{E}_{[0]}$ predicted by the generating function are

$$(0,0,4,0),(0,0,2,2),(0,0,0,4),(1,1,2,0),(1,1,0,2),(2,2,0,0),$$

which we get from $\mathcal{S}_{[0]}$ by adding appropriate multiples of elements of $\mathcal{P}=\{8v_1,8v_2,2v_3,q\}$.

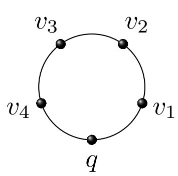
As another example, let $D=v_1-q=(1,0,0,-1)$. To find $\mathcal{S}_{[D]}$, add D to each of the divisors in $\mathcal{S}_{[0]}$, then take their standard representatives as elements of $\mathbb{Z}_8\times\mathbb{Z}_8\times\mathbb{Z}_2\times\mathbb{Z}_1$:

$$\mathcal{S}_{[D]}=\{(1,0,0,0),(2,1,0,0),(3,2,0,0),(4,3,0,0),(5,4,0,0),(6,5,0,0),(7,6,0,0),(0,7,0,0)$$
$$(1,4,1,0),(2,5,1,0),(3,6,1,0),(4,7,1,0),(5,0,1,0),(6,1,1,0),(7,2,1,0),(0,3,1,0)\}.$$

Therefore,

$$\Lambda_{[D]}(z)=\frac{z+z^3+z^4+z^5+2z^6+2z^7+2z^8+z^9+2z^{10}+z^{11}+z^{12}+z^{13}}{(1-z)(1-z^2)(1-z^8)^2}$$
$$=\frac{x(1-x+x^3)}{(1+x)^2(1+x^4)(1-x)^4}$$
$$=z+z^2+3z^3+4z^4+7z^5+10z^6+15z^7+20z^8+28z^9+35z^{10}+\cdots.$$

3.1.3. *Cycle graphs.* Let C_n be the cycle graph on n vertices, with vertices $v_1,\dots,v_n$ around the cycle. Take $q=v_n$. It is well-known that $\operatorname{Jac}(C_n)\simeq\mathbb{Z}/n\mathbb{Z}$, with generator $D_1:=[v_1-q]$ and such that $D_j:=[v_j-q]=j[v_1-q]$ for all j, where the indices are determined modulo n. Therefore, $\operatorname{ord}_q(v_i)=n/\gcd(i,n)$ for all i. For convenience, take $\ell_{v_i}=n$ for $i=1,\dots,n-1$ and $\ell_q=1$. The reduced Laplacian $\tilde{L}$ is the $(n-1)\times(n-1)$ tridiagonal matrix with 2s on the diagonal and -1s on the super and subdiagonals. It is straightforward to reduce $\tilde{L}$ to its Hermite form, which is $I_{n-1}+B_{n-1}$ where I_{n-1} is the identity matrix and B_{n-1} is a matrix whose rows are all 0-vectors except for the last row, which is the vector $(1,2,\dots,n-1)$. See Figure 2 for an example. The primary divisors are $\mathcal{P}=\{nv_1,nv_2,\dots,nv_{n-1},q\}$ and for each $j=0,1,\dots,n-1$, the



C_5

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$\tilde{L}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 5 \end{pmatrix}$$

Hermite form for $\tilde{L}$

FIGURE 2. The cycle graph C_5 (cf. Example 3.1.3).

secondary divisors for D_j are

$$\mathcal{S}_{[D_j]}=\left\{\left((a_1+j)\bmod n,a_2,\dots,a_{n-2},\textstyle\sum_{i=1}^{n-2}ia_i\bmod n,0\right):0\le a_i<n\text{ for }i=1,\dots,n-2\right\}.$$

For example, in the case $n=4$ and $j=2$, we have 16 secondary divisors for $[D_2]\in\operatorname{Jac}(C_4)$:

$$\mathcal{S}_{[D_2]}=\{(2,0,0,0),(2,1,2,0),(2,2,0,0),(2,3,2,0),(3,0,1,0),(3,1,3,0),(3,2,1,0),(3,3,3,0),$$
$$(0,0,2,0),(0,1,0,0),(0,2,2,0),(0,3,0,0),(1,0,3,0),(1,1,1,0),(1,2,3,0),(1,3,1,0)\}.$$

By Corollary 3.2,

$$\Lambda_{[D_2]}(z)=\frac{z+2z^2+2z^3+4z^4+2z^5+2z^6+2z^7+z^9}{(1-z^3)^4(1-z)}$$
$$=\frac{z+z^2-z^3+z^4}{(1+z^2)(1+z)^2(1-z)^4}$$
$$=z+3z^2+5z^3+9z^4+14z^5+22z^6+30z^7+42z^8+55z^9+73z^{10}+\cdots.$$

For instance, the term $5z^3$ in the above expression corresponds to the 5 elements of the complete linear system for $D_2 + 3q = (2, 0, 0, 1)$ pictured in Figure 3.

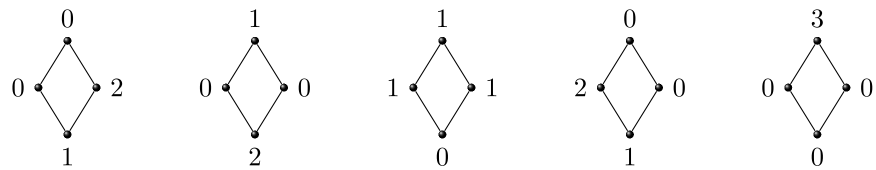
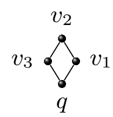
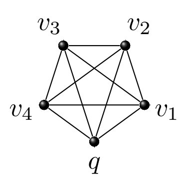


FIGURE 3. The complete linear system $|D_2 + 3q|$ on $C_4 =$ 

(cf. Example 3.1.3).

See Section 6 for the relation between complete linear systems on cycle graphs and binary necklaces.

3.1.4. *Complete graphs.* Let K_n be the complete graph with vertex set $V = \{v_1, \dots, v_n\}$, and let $q := v_n$. Its reduced Laplacian $\tilde{L}$ is the matrix $nI_{n-1} - J_{n-1}$ where J_{n-1} is the $(n-1)\times(n-1)$ matrix whose entries are all 1. The inverse is $\tilde{L}^{-1} = \frac{1}{n}(I_{n-1} + J_{n-1})$, and therefore, $\mathrm{ord}_q(v) = n$ for all vertices $v \neq q$. To reduce $\tilde{L}$ to Hermite normal form, add columns 2 through $n-1$ to the first column of $\tilde{L}$, then add the first column to each of the others. The result is the matrix formed by replacing the first column of nI_{n-1} by a column of all 1s. So the image of $\tilde{L}$ in $(\mathbb{Z}/n\mathbb{Z})^{n-1}$ is spanned by the vector of all 1s. See Figure 4 for an example.



K_5

$$\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{pmatrix}$$

$\tilde{L}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 \\ 1 & 0 & 5 & 0 \\ 1 & 0 & 0 & 5 \end{pmatrix}$$

Hermite form for $\tilde{L}$

FIGURE 4. The complete graph K_5 (cf. Example 3.1.4).

Taking $\ell_v = n$ for $v \neq q$ and $\ell_q = 1$, the primary divisors are $\mathcal{P} = \{nv\}_{v \in V \setminus \{q\}} \cup \{q\}$, and the secondary divisors for $[0] \in \mathrm{Jac}(K_n)$ are

$$\mathcal{S}_{[0]} = \{k(1, 1, \dots, 1, 0) : 0 \leq k < n\}.$$

Hence,

$$\begin{aligned} \Lambda_{[0]}(z) &= \frac{\sum_{k=0}^{n-1} z^{k(n-1)}}{(1 - z^n)^{n-1}(1 - z)} \\ &= \frac{1}{1 - z}\left(\sum_{k=0}^{n-1} z^{k(n-1)}\right) \sum_{k \geq 0} \binom{n - 2 + k}{n - 2} z^{kn}. \end{aligned}$$

The sequence of first differences of the λ-sequence for 0 has k-th term $\Delta\lambda_{[0]}(k) := \lambda_{[0]}(k+1) - \lambda_{[0]}(k)$, which is the coefficient of z^{k+1} in $(1 - z)\Lambda_{[0]}(z)$ for $k \geq 0$. A bit of calculation then shows that writing $k = an + b$ in terms of quotient and remainder by n gives

$$\Delta\lambda_{[0]}(k) = \binom{a + b}{n - 2}.$$

This means $\Delta\lambda_{[0]}$ is formed by concatenating sequences of length n:

$$\binom{m}{n - 2}, \binom{m + 1}{n - 2}, \dots, \binom{m + (n - 1)}{n - 2}$$

for $m \in \mathbb{N}$. For example, in the case $n = 5$,

$$\begin{aligned}\Lambda_{[0]}(z) &= \frac{1 + z^4 + z^8 + z^{12} + z^{16}}{(1 - z^5)^4(1 - z)} \\ &= 1 + z + z^2 + z^3 + 2z^4 + 6z^5 + 6z^6 + 6z^7 + 7z^8 + 11z^9 + \cdots.\end{aligned}$$

We have

$$\lambda_{[0]} = 1, 1, 1, 1, 2, 6, 6, 6, 7, 11, 21, 21, 22, 26, 36, 56, 57, 61, 71, 91, 126, 130, 140, 160, 195, 251, \ldots,$$

which gives

$$\Delta\lambda_{[0]} = \underline{0, 0, 0, 1, 4},\ \underline{0, 0, 1, 4, 10},\ \underline{0, 1, 4, 10, 20},\ \underline{1, 4, 10, 20, 35},\ \underline{4, 10, 20, 35, 56},\ \ldots$$

Parking functions. Parking functions are basic objects in combinatorics closely related to q-reduced divisors on K_n . We briefly recall these notions here. For details, see e.g. [8, Chapter 11]. A vector $p = (p_1, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1}$ with $1 \leq p_i \leq n$ for each i is a *parking function* of length $n-1$ if for each $j = 1, \ldots, n-1$,

$$|\{i : p_i \leq j\}| \geq j.$$

We partially order parking functions by $p' \leq p$ if $p'_i \leq p_i$ for all i. To form all parking functions of length $n - 1$, start with a set P_{n-1} containing the maximal parking function $p := (1, \ldots, n - 1)$, then add all vectors $p' \in \mathbb{Z}^{n-1}$ such that $\vec{1} \leq p' \leq p$ to P_{n-1}. Finally, for each $p' \in P_{n-1}$, add all vectors that arise from permuting the coordinates of p'. The total number of parking functions of length $n - 1$ is $n^{n-2} = |\operatorname{Jac}(K_n)|$.

As with any graph, the elements of $\operatorname{Jac}(K_n)$ are represented by q-reduced divisors of degree 0. However, on K_n it turns out that a divisor D is q-reduced if and only if $D|_{q=0} = p - \vec{1} = (p_1 - 1, \ldots, p_{n-1} - 1)$ for some parking function p. Thus, on K_n there is a bijective correspondence between parking functions and elements of $\operatorname{Jac}(K_n)$.

We have discussed the λ-sequence for the unique divisor class corresponding to the smallest parking function, $p = \vec{1}$. We will now show that the first differences of the λ-sequence for any of the $(n - 1)!$ divisor classes corresponding to a maximal parking function has a particularly nice form. By symmetry, we may assume that $D = (0, 1, \ldots, n - 2, \alpha)$ where $\alpha := -\sum_{k=0}^{n-2} k$ so that $\deg(D) = 0$ and $D|_{q=0} = (0, 1, \ldots, n - 2)$. We saw earlier that the standard representatives of the Hermite normal form for $\tilde{L}$ are $k(1, \ldots, 1) \in \mathbb{Z}^{n-1}$ for $k = 0, \ldots, n - 1$. Therefore, by Proposition 3.4, we get $\mathcal{S}_{[D]}$ by taking standard representatives for elements of the following set, working modulo n in the first $n - 1$ coordinates:

$$\{(0, 1, \ldots, n - 2, 0) + (k(1, 1, \ldots, 1, 0) : k = 0, \ldots, n - 1\}.$$

Computing the degrees of these divisors, Corollary 3.2 gives

$$\Lambda_{[D]}(z) = \frac{\sum_{i=0}^{n-1} z^{\binom{n}{2} - i}}{(1 - z^n)^{n-1}(1 - z)}.$$

An analysis like that given above for $\lambda_{[0]}$ shows that the sequence of first differences of the λ-sequence for $[D]$ starts out with $\binom{n-1}{2}$ zeroes and then is followed by the sequence $\binom{k+n-2}{n-2}_{k\geq 0}$ but with each term repeated n times. For instance, on K_5 we have $D = (0, 1, 2, 3, -6)$, and

$$\begin{aligned}\lambda_{[D]} &= 0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 9, 13, 17, 21, 25, 35, 45, 55, 65, 75, 95, 115, 135, 155, 175, 210, \ldots \\ \Delta\lambda_{[D]} &= 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 4, 4, 4, 4, 4, 10, 10, 10, 10, 10, 20, 20, 20, 20, 20, 35, \ldots\end{aligned}$$

4. Polyhedra

We now interpret the results of Section 3 in terms of lattice points in polyhedra naturally associated with divisors.

4.1. **Background.** We first recall some theory, using [5] as our reference. An *affine n-cone in $\mathbb{R}^n$*, or simply, an *n-cone*, is a set of the form

$$\mathcal{K} = \{\, p + \lambda_1\omega_1 + \cdots + \lambda_m\omega_m : \lambda_1, \ldots, \lambda_m \geq 0\}\,,$$

where $\omega_1, \ldots, \omega_m, p \in \mathbb{R}^n$ and the span of the ω_i has dimension n. The ω_i are called *generators* of the cone. Any generator that is not a nonnegative combination of the remaining generators is called an *extreme ray*. The cone is *pointed* if it contains no line, and in that case p is called its *apex*. We say $\mathcal{K}$ is *rational* if $p, \omega_1, \ldots, \omega_m \in \mathbb{Q}^n$, and then, by rescaling, we may assume the ω_i have integer coordinates. An n-cone is *simplicial* if it may be written using n generators. Simplicial cones are necessarily pointed.

Equivalently, we may define a rational pointed n-cone in $\mathbb{R}^n$ to be an n-dimensional intersection of finitely many half-planes of the form

$$\{x \in \mathbb{R}^n : a_1x_1 + \cdots + a_nx_n \geq \beta\}\,,$$

where $a_1, \ldots, a_n, \beta \in \mathbb{Z}$ and such that the hyperplanes

$$\{x \in \mathbb{R}^n : a_1x_1 + \cdots + a_nx_n = \beta\}$$

meet in a single point. In that case, we may express the cone as $\{x \in \mathbb{R}^n : Ax \geq b\}$ where A is an integral $m\times n$ matrix of rank n and $b \in \mathbb{Z}^m$.

If $\mathcal{K}$ is a simplicial n-cone in $\mathbb{R}^n$ with an integral generating set $\Omega = \{\omega_1, \ldots, \omega_n\}$ and apex p, define the *fundamental parallelepiped* for $\mathcal{K}$ with respect to Ω to be

$$\Pi := \{\, p + \textstyle\sum_{i=1}^n \lambda_i\omega_i : 0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n < 1\}\,.$$

We will need the following:

Property 4.1. *Every point $\alpha \in \mathcal{K} \cap \mathbb{Z}^n$ has a unique expression as*

$$\alpha = \pi + m_1\omega_1 + \cdots + m_d\omega_d$$

with $\pi \in \Pi$ and $m_1, \ldots, m_d \in \mathbb{N}$.

Define the *integer-point transform* of a set $S \subset \mathbb{R}^n$ by

$$\sigma_S(\vec{z}\,) = \sigma_S(z_1, \ldots, z_n) := \sum_{\alpha \in S\cap\mathbb{Z}^n} \vec{z}^{\;\alpha},$$

where $\vec{z}^{\;\alpha} := \prod_{i=1}^n z_i^{\alpha_i}$.

Theorem 4.2. ([5, Theorem 3.5]) *Let*

$$\mathcal{K} = \{\, p + \lambda_1\omega_1 + \cdots + \lambda_m\omega_n : \lambda_1, \ldots, \lambda_n \geq 0\}$$

be a simplicial n-cone in $\mathbb{R}^n$ with $\omega_1, \ldots, \omega_n \in \mathbb{Z}^n$ and $p \in \mathbb{R}^n$. Then

$$\sigma_{\mathcal{K}}(\vec{z}\,) = \frac{\sigma_\Pi(\vec{z}\,)}{\prod_{i=1}^n (1 - \vec{z}^{\;\omega_i})},$$

where Π is the fundamental parallelepiped of $\mathcal{K}$ with respect to the ω_i.

4.2. **Linear systems and polyhedra.** As usual, fix an ordering $v_1, \ldots, v_n$ of the vertices of G with $q = v_n$, and then identify both $\mathrm{Div}(G)$ and $\mathbb{Z}^V$ with $\mathbb{Z}^n$.

Note: Throughout this section, we fix the embedding

$$\mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n.$$

In this way, if $D \in \mathrm{Div}(G) = \mathbb{Z}^n$, then we may regard $D|_{q=0}$ as an element of either $\mathbb{Z}^{n-1}$ or $\mathbb{Z}^n$. Similarly, given $f \in \mathbb{R}^{n-1}$, we write Lf in place of $L\binom{f}{0}$.

Divisors D and D' on G are linearly equivalent exactly when there is a function $f \in \mathbb{Z}^V$ such that $D' = D + \mathrm{div}(f)$. In this context f is referred to as a *firing script*, and we express the complete linear system for D as

$$|D| = \{E \in \mathrm{Div}(G) : E = D + Lf \geq 0 \text{ for some firing script } f\}\,.$$

The set of firing scripts appearing above for the complete linear system for D form the polyhedron

$$Q_D := \{f \in \mathbb{R}^n : Lf \geq -D\} \subset \mathbb{R}^n.$$

However, the integer points of Q_D are not in bijection with elements of $|D|$ since L has a non-trivial kernel. The kernel is generated by the all-ones vector $\vec{1}$; so modulo $\ker(L)$, each firing script $f = (f_1, \dots, f_n)$ has the unique representative $f - f_n \cdot \vec{1}$ with last coordinate 0, leading us to define

$$P_D := Q_D \cap \{f \in \mathbb{R}^n : f_n = 0\} \subset \mathbb{R}^{n-1}$$

so that $Q_D = P_D + \mathbb{R}\vec{1} \subset \mathbb{R}^n$. It is straightforward to see that the integer points $P_D \cap \mathbb{Z}^{n-1}$ are in bijection with $|D|$:

$$f \in P_D \cap \mathbb{Z}^{n-1} \quad \longleftrightarrow \quad D + Lf \in |D|. \tag{3}$$

Since $|D|$ is finite, it follows that the polyhedron P_D is bounded, and hence is a *polytope*. (For a direct proof of boundedness, see [8, Proposition 2.20].)

If $D \sim D'$ with $D' = D + Lf$, then the polyhedra associated with these divisors differ by a translation: $Q_D = f + Q_{D'}$, and as discussed above, we may assume $f_n = 0$ to write $P_D = f + P_{D'}$.

The ideas presented above may be applied in order to characterize $\mathbb{E}_{[D]} = \cup_{k \geq 0} |D + kq|$ in terms of firing vectors.

Definition 4.3. The *q-cone* for a divisor $D \in \mathrm{Div}^0(G)$ is the set

$$\mathcal{K}_D := \{(f, t) \in \mathbb{R}^n \times \mathbb{R} : Lf + tq \geq -D \text{ and } f_n = 0\} \subset \mathbb{R}^{n-1} \times \mathbb{R}.$$

Theorem 4.4. *Let $D \in \mathrm{Div}^0(G)$. Then $\mathcal{K}_D$ is a rational simplicial n-cone with apex $p := \tilde{L}^{-1}(-D|_{q=0}) \in \mathbb{R}^{n-1} \times \{0\} \subset \mathbb{R}^n$ and has the following properties:*

(1) *The set of integer points of $\mathcal{K}_D$ is in bijection with $\mathbb{E}_{[D]}$ via the mapping*

$$\begin{aligned} \psi_D \colon \mathcal{K}_D \cap \mathbb{Z}^n &\xrightarrow{\sim} \mathbb{E}_{[D]} \\ (f, k) &\mapsto D + kq + Lf. \end{aligned}$$

(2) *The mapping ψ_0 restricts to a bijection between generating sets of integral extreme rays for $\mathcal{K}_D$ and sets of primary divisors for G. Let Ω be a generating set of integral extreme rays for $\mathcal{K}_D$ with corresponding set of primary divisors $\mathcal{P} = \psi_0(\Omega)$. Let Π be the corresponding fundamental parallelepiped. Then ψ_D restricts to a bijection between the integer points of Π and the secondary divisors of $[D]$ with respect to $\mathcal{P}$.*

(3) *Let Ω and Π be as in part (2). Then the λ-sequence generating function for $\mathbb{E}_{[D]}$ is*

$$\Lambda_{[D]}(z) = \sigma_{\mathcal{K}}(1, \dots, 1, z) = \frac{\sigma_\Pi(1, \dots, 1, z)}{\prod_{\omega \in \Omega}(1 - z^{\deg(\omega)})},$$

where $\deg(\omega)$ is the sum of the coordinates of ω. The numerator and denominator of the expression on the right are the same as those appearing in Corollary 3.2.

Proof. Let $\tilde{r}_n \in \mathbb{R}^{n-1}$ denote the last row of L with its last entry removed. Then

$$\mathcal{K}_D = \left\{(f, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \tilde{L}f \geq -D|_{q=0} \text{ and } \tilde{r}_n \cdot f + t \geq -D(q)\right\}. \tag{4}$$

Since $\tilde{L}$ is invertible, these defining conditions are independent, and it follows that $\mathcal{K}_D$ is a rational n-cone. To find the apex, first solve $\tilde{L}f = -D|_{q=0}$ to find $f = \tilde{L}^{-1}(-D|_{q=0})$. Next, since the sum of the rows of L is 0, if follows that $\tilde{r}_n \cdot f = -\tilde{L}f$, and it is now easy to verify that the last coordinate of the apex is $t = 0$ using the fact that $\deg(D) = 0$.

The rest follows immediately from the discussion preceding the theorem. Part (1) uses the fact that every firing script has a unique representative modulo $\ker L$ having final coordinate 0. Part (2) relies on Property 4.1. Part (3) follows since $\deg \psi_D(f, k) = k$. □

The following proposition shows that the essential information encoded in $\mathcal{K}_D$ is contained in its bottom (with respect to the last coordinate) face:

Proposition 4.5. *Given $D \in \mathrm{Div}^0(G)$, take the union of the nested sequence of polytopes $P_D \subset P_{D+q} \subset P_{D+2q} \subset \dots$ to define*

$$\widetilde{\mathcal{K}}_D := \bigcup_{k \in \mathbb{N}} P_{D+kq} \subset \mathbb{R}^{n-1}.$$

(1) $\widetilde{\mathcal{K}}_D$ *is a rational simplicial* $(n-1)$*-cone, and*

$$\widetilde{\mathcal{K}}_D = \left\{ f \in \mathbb{R}^{n-1} : \tilde{L}f \geq -D|_{q=0} \right\}.$$

The apex of $\widetilde{\mathcal{K}}_D$ *is* $\tilde{p} := \tilde{L}^{-1}(-D|_{q=0}) \in \mathbb{R}^{n-1}$.

(2) *Let* $\tilde{r}_n \in \mathbb{R}^{n-1}$ *be the last row of the Laplacian matrix with its final entry removed. Then there is a injection*

$$\begin{aligned} \imath_D \colon \widetilde{\mathcal{K}}_D &\to \mathcal{K}_D \subset \mathbb{R}^{n-1} \times \mathbb{R} \\ f &\mapsto (f, -D(q) - (\tilde{r}_n \cdot f)). \end{aligned}$$

The image of $\imath_D$ *is the facet of* $\mathcal{K}_D$ *which is the intersection of* K_D *with the hyperplane*

$$H := \left\{ (f,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : (\tilde{r}_n \cdot f) + t = -D(q) \right\},$$

and

$$\mathcal{K}_D = \imath_D(\widetilde{\mathcal{K}}_D) + \mathbb{R}_{\geq 0} q.$$

(3) *Write*

$$\widetilde{\mathcal{K}}_D = \{ \tilde{p} + \lambda_1 \tilde{\omega}_1 + \cdots + \lambda_{n-1} \tilde{\omega}_{n-1} : \lambda_1, \ldots, \lambda_{n-1} \geq 0 \}$$

with integral generating set $\tilde{\Omega} := \{\tilde{\omega}_1, \ldots, \tilde{\omega}_{n-1}\} \subset \mathbb{Z}^{n-1}$. *Let* $\tilde{v}_i$ *denote the* i*-th standard basis vector for* $\mathbb{R}^{n-1}$. *Then up to re-indexing,* $\tilde{\omega}_i = \tilde{L}^{-1}(\ell_i \tilde{v}_i)$ *where* ℓ_i *is a positive integer multiple of* $\mathrm{ord}_q(v_i)$. *The set* $\Omega := \imath_0(\tilde{\Omega}) \cup \{\ell_q q\}$ *is an integral generating set of extreme rays for* $\mathcal{K}_D$ *for any choice of positive integer* ℓ_q. *Every integral generating set of extreme rays for* $\mathcal{K}_D$ *arises in this manner. With this notation, let* $\widetilde{\Pi}$ *and* Π *be the fundamental parallelepipeds for* $\tilde{\Omega}$ *and* Ω*, respectively. Then*

$$\Pi \cap \mathbb{Z}^n = \left\{ \imath_D(\tilde{\pi}) + \ell q : \tilde{\pi} \in \widetilde{\Pi} \cap \mathbb{Z}^{n-1} \text{ and } 0 \leq \ell < \ell_q \right\}.$$

Proof. We have $f \in \widetilde{\mathcal{K}}_D$ if and only if

$$\tilde{L}f \geq -D|_{q=0} \quad \text{and} \quad \tilde{r}_n \cdot f \geq -D(q) - k$$

for some $k \in \mathbb{N}$. The second condition is superfluous since k can be arbitrarily large. The fact that $\widetilde{\mathcal{K}}$ is a rational simplicial $(n-1)$-cone with apex $\tilde{L}^{-1}(-D|_{q=0})$ follows since $\tilde{L}$ is invertible. This establishes part (1). Part (2) then follows from part (1) and the description of $\mathcal{K}_D$ given in (4) in the proof of Theorem 4.4. Since $K_D = \imath_D \widetilde{\mathcal{K}}_D + \mathbb{R}_{\geq 0} q$, part (3) follows from Theorem 4.4 (2) and Theorem 3.1 (2). □

Example 4.6. Let graph $G = C_3 = K_3$ with vertex set v_1, v_2, and $v_3 = q$, and consider the divisor $D = (1, 0, -1)$ of degree 0. The Laplacian matrix is

$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

The cone $\widetilde{\mathcal{K}}_D$ is defined by the system of inequalities

$$\begin{aligned} 2x_1 - x_2 &\geq -1 \\ -x_1 + 2x_2 &\geq 0. \end{aligned}$$

To find generators for $\widetilde{\mathcal{K}}_D$ and $\mathcal{K}_D$, we use Proposition 4.5 (3). We have

$$\tilde{L}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Multiply the first column of $\tilde{L}^{-1}$ by $\mathrm{ord}_q(v_1) = 3$ to get $\tilde{w}_1 = (2, 1)$. Multiply the second column by $\mathrm{ord}_q(v_2) = 3$ to get $\tilde{w}_2 = (1, 2)$. Every set of integral extreme rays for $\widetilde{\mathcal{K}}_D$ will be positive integer multiples of these. Since $D|_{q=0} = (1, 0)$, the apex of the cone is

$$\tilde{L}^{-1}(-D|_{q=0}) = (-2/3, -1/3).$$

Thus,

$$\widetilde{\mathcal{K}}_D = \{ (-2/3, -1/3) + \lambda_1 (2, 1) + \lambda_2 (1, 2) \}.$$

Using the notation of Proposition 4.5 (2), we have $\tilde{r}_n = (-1,-1)$. Therefore, taking $\ell_q = 1$, we get the set of extreme rays for $\mathcal{K}_D$:

$$\Omega = \{\imath_0(\tilde{\omega}_1), \imath_0(\tilde{\omega}_2), q\} = \{(2,1,3), (1,2,3), (0,0,1)\},$$

and

$$\mathcal{K}_D = \{(-2/3,-1/3,0) + \lambda_1(2,1,3) + \lambda_2(1,2,3) + \lambda_3(0,0,1)\}.$$

The cone $\widetilde{\mathcal{K}}_D$ is pictured in Figure 5 along with its fundamental parallelogram $\widetilde{\Pi}$ with respect to $\{\tilde{\omega}_1, \tilde{\omega}_2\}$.

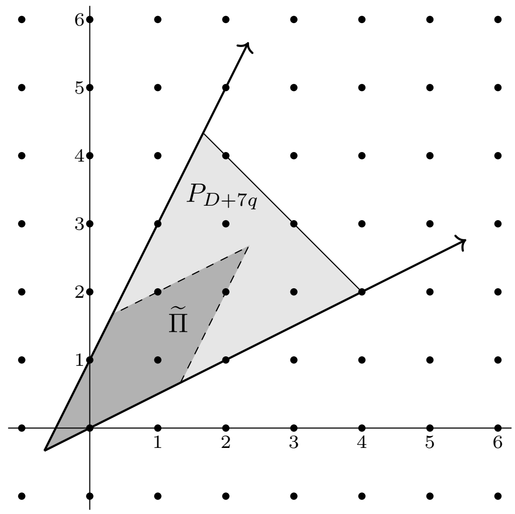


FIGURE 5. The cone $\widetilde{\mathcal{K}}_D$, a fundamental parallelogram, and the polytope P_{D+7q} for the divisor $D = (1,0,-1)$ on C_3 (cf. Example 4.6).

There are three integer points in $\widetilde{\Pi}$:

$$\widetilde{\Pi} \cap \mathbb{Z}^2 = \{(0,0), (0,1), (1,1)\},$$

and the integer points of Π are just the "lifts" of these via $\imath_D$:

$$\Pi \cap \mathbb{Z}^3 = \{(0,0,1), (0,1,2), (1,1,3)\}.$$

Thus, the integer-point transform is

$$\sigma_{\mathcal{K}}(z_1,z_2,z_3) = \frac{z_3 + z_2 z_3^2 + z_1 z_2 z_3^3}{(1 - z_1^2 z_2 z_3^3)(1 - z_1 z_2^2 z_3^3)(1 - z_3)}.$$

The λ-sequence generating function is therefore

$$\Lambda_{[D]}(z) = \sigma_{\mathcal{K}}(1,1,z) = \frac{z + z^2 + z^3}{(1-z^3)^2(1-z)}$$

$$= z + 2z^2 + 3z^3 + 5z^4 + 7z^5 + 9z^6 + 12z^7 + 15z^8 + 18z^9 + 22z^{10} + 26z^{11} + \dots$$

This is exactly what we get from Corollary 3.2 using the primary and $[D]$-secondary divisors

$$\mathcal{P} = \psi_0(\Omega) = \{(3,0,0), (0,3,0), (0,0,1\}$$
$$\mathcal{S}_{[D]} = \psi_D(\Pi \cap \mathbb{Z}^n) = \{(1,0,0), (0,2,0), (2,1,0)\}.$$

(In Example 3.1.4 we calculated $\mathcal{S}_{[0]} = \{(0,0,0), (1,1,1), (2,2,2)\}$ for $K_3 = C_3$. Proposition 3.4 (2) then says $\mathcal{S}_{[D]}$ consists of standard representatives for $D + \mathcal{S}_{[0]}$ in $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/1\mathbb{Z}$, which agrees with the above computation.)

The generating function $\Lambda_{[D]}(z)$ predicts, for example, that there are 12 elements in $|D+7q| = |(1,0,6)|$. The polytope P_{D+7q} is defined by the system of inequalities

$$\begin{aligned} 2x - y &\geq -1 \\ x - 2y &\geq 0 \\ -x - y &\geq -6. \end{aligned}$$

In Figure 5, we can see the 12 lattice points in P_{D+7q} corresponding to the elements of $|D+7q|$.

5. Invariant theory

The results in Section 3 may also be interpreted in terms of the invariant theory for a representation of the dual group $\mathrm{Jac}^*(G)$. Through this lens, primary and secondary divisors become primary and secondary invariants, and $\Lambda_D(z)$ is given a substantially different expression as a Molien series.

5.1. **Background.** We first recall basic invariant theory for finite groups with [22] and[23] as references. Given a matrix $A \in \mathrm{GL}(\mathbb{C}^n)$ and a polynomial $f \in \mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, \ldots, x_n]$, define $f \circ A$ by

$$(f \circ A)(x_1, \ldots, x_n) = f(A\vec{x})$$

where $\vec{x}$ is the column vector $[x_1, \ldots, x_n]^t$. Given a finite subgroup Γ of $\mathrm{GL}(\mathbb{C}^n)$ and a character $\chi\colon \Gamma \to \mathbb{C}^\times := \mathbb{C} \setminus \{0\}$, define the *$\chi$-relative invariants of* Γ to be elements of

$$\mathbb{C}[\mathbf{x}]^\Gamma_\chi := \{f \in \mathbb{C}[\mathbf{x}] : f \circ \gamma = \chi(\gamma) f \text{ for all } \gamma \in \Gamma\}.$$

The *χ-relative Reynolds operator* is defined for each polynomial $f \in \mathbb{C}[\mathbf{x}]$ by

$$\mathcal{R}_\chi(f) = \frac{1}{|G|} \sum_{\gamma \in \Gamma} \overline{\chi}(\gamma) f \circ \gamma.$$

It is easy to check that $\mathcal{R}_\chi$ is linear in f and that f is χ-invariant if and only if $\mathcal{R}_\chi(f) = f$. In the case $\chi = \varepsilon$, the trivial character, $\mathbb{C}[\mathbf{x}]^\Gamma := \mathbb{C}[\mathbf{x}]^\Gamma_\varepsilon$ is a subring of $\mathbb{C}[\mathbf{x}]$, graded by degree, called the *invariant subring of* Γ. It is generated by $\mathcal{R}_\varepsilon(f)$ as f ranges over all monomials of degree at most $|\Gamma|$. The elements of $\mathbb{C}[\mathbf{x}]^\Gamma$ are simply called *invariants of* Γ and $\mathcal{R} := \mathcal{R}_\varepsilon$ is the *Reynolds operator for* Γ. For arbitrary χ, the relative invariants $\mathbb{C}[\mathbf{x}]^\Gamma_\chi$ form a graded $\mathbb{C}[\mathbf{x}]^\Gamma$-module, generated by the homogeneous polynomials $\mathcal{R}_\chi(f)$ as f ranges over all monomials of degree at most $|\Gamma|$.

There exist algebraically independent homogeneous invariants $p_1, \ldots, p_n$ such that $\mathbb{C}[\mathbf{x}]^\Gamma$ is a finitely-generated free module over $\mathbb{C}[p_1, \ldots, p_n]$. For any character χ, if $q_1, \ldots, q_t$ are homogeneous polynomials forming a $\mathbb{C}$-basis for $\mathbb{C}[\mathbf{x}]^\Gamma_\chi$ modulo the submodule $\sum_{i=1}^n p_i \mathbb{C}[\mathbf{x}]^\Gamma_\chi$, then

$$\mathbb{C}[\mathbf{x}]^\Gamma_\chi = \bigoplus_{i=1}^t q_i \mathbb{C}[p_1, \ldots, p_n].$$

The p_i are called *primary invariants* and are independent of χ. The q_i are called *secondary (relative) invariants* and depend on χ. The number of secondary invariants, t, also depends on χ in general. However, letting t_ε be the number of secondary invariants for the trivial character, we have

$$t_\varepsilon |\Gamma| = \prod_{i=1}^n \deg(p_i).$$

The *Hilbert series* for $\mathbb{C}[\mathbf{x}]^\Gamma_\chi$ is

$$\Phi_{\Gamma,\chi}(z) := \sum_{d \geq 0} \dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}]^\Gamma_{\chi,d}) z^d,$$

where $\mathbb{C}[\mathbf{x}]^\Gamma_{\chi,d}$ denotes the d-th graded piece of $\mathbb{C}[\mathbf{x}]^\Gamma_\chi$. The Hilbert series is also known as the (relative) *Molien series* for Γ due to a theorem of Molien which states that

$$\Phi_{\Gamma,\chi}(z) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \frac{\overline{\chi(\gamma)}}{\det(I_n - z\gamma)}. \tag{5}$$

5.2. **Linear systems.** Order the vertices $v_1, \dots, v_n$ of G, and fix $q = v_n$. To see the relevance of invariant theory to our problem, start with the sequence of projections

$$\begin{array}{ccccc} \mathbb{Z}^n = \mathrm{Div}(G) & \twoheadrightarrow & \mathrm{Pic}(G) & \twoheadrightarrow & \mathrm{Jac}(G) \\ D & \longmapsto & [D] & \mapsto & [D - \deg(D)q]\,. \end{array}$$

Apply the functor $\mathrm{Hom}(\,\cdot\,, \mathbb{C}^\times)$ to get a sequence of dual groups

$$\mathrm{Jac}(G)^* \hookrightarrow \mathrm{Pic}(G)^* \hookrightarrow (\mathbb{C}^\times)^n \subset GL(\mathbb{C}^n),$$

identifying $(\mathbb{C}^\times)^n$ with diagonal matrices having nonzero diagonal entries. Define ρ to be the composition of these mappings:

$$\begin{array}{rccc} \rho\colon & \mathrm{Jac}(G)^* & \longrightarrow & \mathrm{GL}(\mathbb{C}^n) \\ & \chi & \mapsto & \mathrm{diag}(\chi([v_1 - q]), \chi([v_2 - q]), \dots, \chi([v_{n-1} - q]), 1). \end{array} \tag{6}$$

Theorem 5.1. *Consider each* $[D] \in \mathrm{Jac}(G)$ *as a character of* $\Gamma := \mathrm{im}(\rho) \subset \mathrm{GL}(\mathbb{C}^n)$ *via* $[D](\rho(\chi)) := \chi([D])$ *for each* $\chi \in \mathrm{Jac}(G)^*$. *Then*

(1)
$$\left\{ x^E := \textstyle\prod_{i=1}^n x_i^{E(v_i)} : E \in \mathbb{E}_{[D]} \right\}$$
is a $\mathbb{C}$*-basis for the relative invariants* $\mathbb{C}[\mathbf{x}]^\Gamma_{[D]}$,

(2) $\mathbb{C}[\mathbf{x}] = \bigoplus_{[D] \in \mathrm{Jac}(G)} \mathbb{C}[\mathbf{x}]^\Gamma_{[D]}$, *and*

(3) *the correspondence* $E \mapsto x^E$ *for effective divisors* E *gives a bijection between systems of primary and* $[D]$*-secondary divisors and systems of monomial primary and* $[D]$*-relative invariants.*

Proof. An arbitrary element of $\mathbb{C}[\mathbf{x}]$ may be written as $f = \sum_E a_E x^E$ where the sum is over all effective divisors of G and all but finitely many a_E are zero. Let $\chi \in \mathrm{Jac}(G)^*$. Then

$$\begin{aligned} f \circ \rho(\chi) &= \sum_E a_E (x^E \circ \rho(\chi)) \\ &= \sum_E a_E \left(\textstyle\prod_{v \in V} \chi([v - q])^{E(v)} \right) x^E \\ &= \sum_E a_E\, \chi\big(\textstyle\sum_{v \in V} E(v)([v - q])\big)\, x^E \\ &= \sum_E a_E\, \chi([E - \deg(E)q])\, x^E. \end{aligned}$$

Therefore, $f \in \mathbb{C}[\mathbf{x}]^\Gamma_{[D]}$ if and only if for each E such that $a_E \neq 0$, we have $\chi([E - \deg(E)q]) = \chi([D])$ for all χ, or equivalently, $[E - \deg(E)q] = [D]$, i.e., $E \in \mathbb{E}_{[D]}$. Parts (1), (2), and (3) follow. Part (2) reflects the fact that the $\mathbb{E}_{[D]}$ partition the set of effective divisors. □

As an immediate corollary, we may express λ-sequence generating functions as a Molien series. These expressions differ from those given in Corollary 3.1 and Theorem 4.2 (which are identical to each other).

Corollary 5.2. *Let* $[D] \in \mathrm{Jac}(G)$. *The generating function for the* λ*-sequence for* $\mathbb{E}_{[D]}$ *is given by the Molien series*

$$\Lambda_{[D]}(z) = \Phi_{\Gamma,[D]}(z) = \frac{1}{|\,\mathrm{Jac}(G)|} \sum_{\chi \in \mathrm{Jac}(G)^*} \frac{\overline{\chi([D])}}{\det(I_n - z\rho(\chi))}.$$

To compute with Corollary 5.2 concretely, use integer row and column operations to reduce $\tilde{L}$ to diagonal form (e.g., Smith normal form), denoting the result by $M := \mathrm{diag}(m_1, \dots, m_{n-1})$. Record the row and column operations in matrices U and W so that $U\tilde{L}W = M$. Then U descends to an isomorphism

$$\mathrm{Jac}(G) \simeq \mathbb{Z}^{n-1}/\,\mathrm{im}_{\mathbb{Z}}\, \tilde{L} \xrightarrow{U} \prod_{i=1}^{n-1} \mathbb{Z}/m_i\mathbb{Z}.$$

Having identified $\mathrm{Jac}(G)$ with $R := \prod_{i=1}^{n-1} \mathbb{Z}/m_i\mathbb{Z}$, we now describe the characters. For each $r \in R$, choose a representative lifting $(r_1, \ldots, r_{n-1}) \in \mathbb{Z}^{n-1}$, and let $\tilde{r} := \left(\frac{r_1}{m_1}, \ldots, \frac{r_{n-1}}{m_{n-1}}\right)$. Define the character $\tilde{\chi}_r \in R^*$ by

$$\tilde{\chi}_r(a) := \exp(2\pi i\, (\tilde{r} \cdot a))$$

for each $a \in R$. Then define $\chi_r \in \mathrm{Jac}^*(G)$ by

$$\chi_r([D]) := \tilde{\chi}_r(U(D|_{q=0}))$$

for each $[D] \in \mathrm{Jac}(G)$. It follows that

$$\rho(\chi_r) = \mathrm{diag}(\tilde{\chi}_r(u_1), \tilde{\chi}_r(u_2), \ldots, \tilde{\chi}_r(u_{n-1}), 1)$$

where u_j is the j-th column of U. In all of the above, if $m_k = 1$ for some k, then the k-th factor of R and the k-th *row* of U may be dropped.

5.3. **Examples.** The following examples use Corollary 5.2 to compute λ-generating functions. For a direct application of the relation between polynomial invariants and linear systems exhibited in Theorem 5.1, see Section 6.

5.3.1. *Trees.* If G is a tree, then $\mathrm{Jac}(G)$ is trivial, and $\mathrm{Jac}(G)^*$ contains only the trivial character. So Corollary 5.2 says $\Lambda_{[0](z)} = 1/(1-z)^n$.

5.3.2. *Diamond graph.* Now let G be the diamond graph of Figure 1. Letting

$$U = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -4 \end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} 3 & 1 & 5 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix},$$

we have $U\tilde{L}W = \mathrm{diag}(1, 1, 8)$ and the corresponding isomorphism

$$\begin{aligned} \mathrm{Jac}(G) \simeq \mathbb{Z}^3 / \operatorname{im}_{\mathbb{Z}} \tilde{L} &\to \mathbb{Z}/8\mathbb{Z} \\ (a, b, c) &\mapsto a - b - 4c. \end{aligned}$$

The divisor class $[v_1 - q]$ generates $\mathrm{Jac}(G)$. Let $D_j = j[v_1 - q]$ for $j = 0, \ldots, 7$, and let $\omega = \exp(2\pi i/8)$. Then, by Corollary 5.2,

$$\Lambda_{[D_j]} = \frac{1}{8(1-z)} \sum_{k=0}^{7} \frac{\omega^{-jk}}{(1-\omega^k z)(1-\omega^{-k} z)(1-\omega^{-4k} z)}$$

for $j = 0, \ldots, 7$.

5.3.3. *Cycle graphs.* Now let $G = C_n$ be a cycle graph using the notation of Example 3.1.3. Let I_n be the $n \times n$ identity matrix, and let U be the matrix formed by replacing the last row of the $-I_n$ with the row $(1, 2, \ldots, (n-2), -1)$. Let W be the $n \times n$ matrix with $W_{ij} = -\min\{i, j\}$ (so W starts with a row of -1s and ends with the row $(-1, -2, \ldots, -(n-1))$). Then $U\tilde{L}W = \mathrm{diag}(1, 1, \ldots, 1, n)$, and multiplication by U gives the isomorphism

$$\begin{aligned} \mathrm{Jac}(C_n) \simeq \mathbb{Z}^{n-1} / \operatorname{im}_{\mathbb{Z}} \tilde{L} &\to \mathbb{Z}/n\mathbb{Z} \\ (a_1, \ldots, a_{n-1}) &\mapsto a_1 + 2a_2 + \cdots + (n-2)a_{n-2} + (n-1)a_{n-1}. \end{aligned}$$

The divisor class $[v_1 - q]$ generates $\mathrm{Jac}(C_n)$. Let $D_j = j[v_1 - q]$ for $j = 0, \ldots, n-1$, and let $\omega = \exp(2\pi i/n)$. Then by Corollary 5.2,

$$\Lambda_{[D_j]}(z) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{\omega^{-jk}}{\prod_{t=0}^{n-1} (1 - \omega^{tk} z)}$$

for $j = 0, \ldots, n-1$. In particular,

$$\Lambda_{[0]}(z) = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{(1 - z^{n/\gcd(n,k)})^{\gcd(n,k)}} = \frac{1}{n} \sum_{d|n} \frac{\phi(d)}{(1 - z^d)^{n/d}}$$

where ϕ is the Euler totient function. We shall explore this example further in Section 6.

5.3.4. *Complete graphs.* Let $G = K_n$. Perform integer column operations to bring $\tilde{L}$ into Hermite normal form H as described in Example 3.1.4. Next, let U be the matrix formed by replacing the first column of the identity matrix I_{n-1} by the column $(1, -1, -1, \ldots, -1)$. Then $UH = M := \mathrm{diag}(1, n, n, \ldots, n)$. Since $M_{1,1} = 1$, alter U by removing its first row, and we get an isomorphism

$$\mathrm{Jac}(K_n) \simeq \mathbb{Z}^{n-1}/\operatorname{im}_{\mathbb{Z}} \tilde{L} \xrightarrow{U} R := (\mathbb{Z}/n\mathbb{Z})^{n-2}.$$

Let $\omega := \exp(2\pi i/n)$. Then for each $r \in R$, we have the character χ_r such that $\chi_r(a) = \omega^{r\cdot a}$ for each $a \in \mathbb{Z}^{n-1}/\operatorname{im}_{\mathbb{Z}} \tilde{L}$. By Corollary 5.2, for each $[D] \in \mathrm{Jac}(G)$, writing $D = (d_1, \ldots, d_n)$,

$$\Lambda_{[D]}(z) = \frac{1}{n^{n-2}(1-z)} \sum_{r\in(\mathbb{Z}/n\mathbb{Z})^{n-2}} \frac{\omega^{-r\cdot(d_2-d_1,\ldots,d_{n-1}-d_1)}}{(1-\omega^{-r_1-\cdots-r_{n-2}}z)(1-\omega^{r_1}z)(1-\omega^{r_2}z)\ldots(1-\omega^{r_{n-2}}z)}.$$

Remark 5.3. Corollaries 3.2 and 5.2 give two ways of expressing the generating function $\Lambda_{[D]}(z)$. One sums over elements of $\mathcal{S}_{[D]}$, and the other sums over elements of $\mathrm{Jac}^*(G)$. In practice, one of these two expressions may be much simpler than the other. For instance, the complete graph K_n has a large Jacobian group, $|\mathrm{Jac}(K_n)| = n^{n-2}$, however we can find a set of secondary divisors with only n elements. So the expression for $\Lambda_{[D]}(z)$ coming from Corollary 5.2 will have n^{n-2} summands while the numerator S appearing in Corollary 3.2 with have only n terms. In the case of the cyclic graph C_n, we have the opposite situation: $|\mathrm{Jac}(C_n)| = n$ and there are n^{n-2} secondary divisors (taking $\ell_v = n$ for all vertices $v \neq 1$).

6. Cycle graphs and necklaces

6.1. **Necklaces.** Let C be a finite set of *colors* and let C^n denote the set of all words (strings) of length n with letters in C. Let σ be the cyclic shift operator on C^n:

$$\sigma(c_1 \ldots c_n) = c_n c_1 \ldots c_{n-1}.$$

Define an equivalence relation on C^n by letting $w \sim w'$ if $w = \sigma^i(w')$ for some integer i. A *necklace* of length n on the color set C is an equivalence class $[c_1 \ldots c_n] \in C^n/\!\sim$. We think of each c_i as being a bead of color c_i. The *period* of a necklace $N = [w]$ is the smallest positive integer i such that $\sigma^i(w) = w$.

Definition 6.1. Let m be a positive integer. A necklace is *m-divisible* if its period is divisible by m. (See Figure 6 for an example.)

A *binary necklace* is a necklace for which C consists of two colors (which we take to be black and white). Let $\mathcal{N}(n, k)$ denote the set of binary necklaces with n black beads and k white beads.

Definition 6.2. Let $N = [bw^{a_1}bw^{a_2}\cdots bw^{a_n}] \in \mathcal{N}(n, k)$. The *code* for N is the necklace with n beads and with colors $\{a_1, \ldots, a_n\} \subset \mathbb{N}$,

$$\mathrm{code}(N) := [a_1 \ldots a_n].$$

6.2. **Linear systems on cycle graphs.** We use the notation of Section 3.1.3. Let C_n be the cycle graph with vertices $v_1, \ldots, v_n$ around the cycle, and take $q := v_n$. Working with subscripts modulo n, let $D_i = v_i - q \in \mathrm{Div}(C_n)$ for all $i \in \mathbb{Z}$. Then $\mathrm{Jac}(C_n)$ is the cyclic group of order n generated by $[D_1]$ with $j[D_1] = [D_j]$ for all j. The dual group $\mathrm{Jac}(C_n)^*$ is generated by the character χ_1 determined by $\chi_1([D_1]) = \omega$ where ω is a primitive n-th root of unity. The representation $\rho\colon \mathrm{Jac}(G)^* \to \mathrm{GL}(\mathbb{C}^n)$ described in Section 5.2 is determined by

$$\rho(\chi_1) = \mathrm{diag}(\omega, \omega^2, \ldots, \omega^n).[3]$$

Changing coordinates on $\mathbb{C}^n$, we conjugate this diagonal representation into a permutation representation. In detail, for any $x \in \mathbb{C}$, let $v(x) := (x, x^2, \ldots, x^n)$ and let A be the matrix with rows $v(\omega^n), v(\omega), v(\omega^2), \ldots, v(\omega^{n-1})$. We have $A^{-1} = \frac{1}{n}\overline{A}^t$. Conjugate by A to get $\rho' : \mathrm{Jac}(G)^* \to \mathrm{GL}(\mathbb{C}^n)$ where $\rho'(\chi) = A\rho(\chi)A^{-1}$ for all $\chi \in \mathrm{Jac}^*(C_n)$. Then $\rho'(\chi_1)$ is the permutation matrix P such that for each standard basis vector $e_1, \ldots, e_n$, we have $Pe_i = e_{i-1}$ (with subscripts modulo n). Let $\Gamma = \operatorname{im}(\rho)$ and $\Gamma' := \operatorname{im}(\rho')$, and define indeterminates $y = Ax$. For $f(y) \in \mathbb{C}[\mathbf{y}]$,

$$(f \circ P)(y_1, \ldots, y_n) = f(y_2, y_3, \ldots, y_n, y_1).$$

[3] We often write ω^n instead of 1 for consistency of notation.

It follows that for each $[D_j] \in \mathrm{Jac}(G)$ (considered as a character on $\mathrm{Jac}(G)^*$), there is an induced, degree-preserving, linear isomorphism of relative invariant rings

$$\begin{aligned} \mathbb{C}[\mathbf{y}]^{\Gamma'}_{[D_j]} &\xrightarrow{\sim} \mathbb{C}[\mathbf{x}]^{\Gamma}_{[D_j]} \\ f \quad &\mapsto f \circ A. \end{aligned} \tag{7}$$

For each $N \in \mathcal{N}(n,k)$, fix a representative word a_N for the necklace $\mathrm{code}(N)$, and use the Reynolds operator to define

$$f_N(y) := \mathcal{R}_{[D_j]}(y^{a_N}) = \frac{1}{n}\sum_{i=1}^{n} \omega^{-ij} y^{\sigma^i(a_N)},$$

where σ is the cyclic shift operator defined earlier and $y^{a_N} := \prod_{i=1}^n y_i^{a_{N,i}}$.

Theorem 6.3. *Let $j \in [n]$ and define*

$$m_j := \frac{n+k}{\gcd(n,k,j)} \quad \textit{and} \quad n_j := \frac{n}{\gcd(n,j)} = \mathrm{order}(\omega^j).$$

(1) *$N \in \mathcal{N}(n,k)$ is m_j-divisible if and only if $\mathrm{code}(N)$ is n_j-divisible.*
(2) *The set $\{f_N\}$ as N ranges over all m_j-divisible $N \in \mathcal{N}(n,k)$ is a basis for the $[D_j]$-relative invariants of degree k for the permutation representation ρ'.*

Proof. Let $N \in \mathcal{N}(n,k)$, and let $a = a_N = a_1 \dots a_n$ be a representative word for $\mathrm{code}(N)$. We first show that N is m_j-divisible if and only if $\mathrm{code}(N)$ is n_j-divisible. Let ℓ be the period of $\mathrm{code}(N)$. Then since the length of the necklace $\mathrm{code}(N)$ is n, there is an integer p such that $n = p\ell$. The period of N is $\ell + \alpha$ where $\alpha := \sum_{i=1}^{\ell} a_i$. Since $\sum_{i=1}^{n} a_i = k$, and the period of $\mathrm{code}(N)$ is ℓ, it follows that $p\alpha = k$. Therefore,

$$\begin{aligned} N \text{ is } m_j\text{-divisible} \quad &\iff \quad \left(\frac{n+k}{\gcd(n,k,j)}\right) \Bigg| (\ell+\alpha) \\ &\iff \quad \frac{p(\ell+\alpha)}{\gcd(p\ell,p\alpha,j)} \Bigg| (\ell+\alpha) \\ &\iff \quad \frac{\gcd(p\ell,p\alpha,j)}{p(\ell+\alpha)} \cdot (\ell+\alpha) \in \mathbb{Z} \\ &\iff \quad p|j. \end{aligned}$$

Similarly,

$$\mathrm{code}(N) \text{ is } n_j\text{-divisible} \quad \Leftrightarrow \quad \left(\frac{n}{\gcd(n,j)}\right) \Bigg| \ell \quad \Leftrightarrow \quad \frac{p\ell}{\gcd(p\ell,j)} \Bigg| \ell \quad \Leftrightarrow \quad p|j.$$

Continuing with the notation already established, we now prove part (2). Since f_N is defined using the Reynolds operator, it is $[D_j]$-invariant. To see that it is non-zero, we show that the monomial y^a appears in f_N with a nonzero coefficient. We have $y^a = y^{\sigma^i(a)}$ if and only if i is a multiple of the period ℓ. However, ℓ is divisible by n_j, which is the order of ω^j. So the coefficient of $y^{\sigma^i(a)}$ in the expression for the Reynolds operator is $\omega^{-ij}/n = 1/n$. Therefore, the coefficient of y^a in f_N is the integer $p/n = 1/\ell \neq 0$. It now follows that f_N has degree k.

Let $\mathcal{B}$ be the set of f_N as N varies over m_j-divisible elements of $\mathcal{N}(n,k)$. Distinct elements of $\mathcal{B}$ share no monomials in common, and hence $\mathcal{B}$ is a linearly independent set. To show $\mathcal{B}$ spans the relative invariant module and finish the proof, let $h \in \mathbb{C}[\mathbf{y}]^{\Gamma'}_{[D_j],k}$ be a homogeneous $[D_j]$-invariant of degree k. For the sake of contradiction, suppose $h \notin \mathrm{Span}(\mathcal{B})$. We regard h as a sum of terms where each term is a nonzero constant times a monomial, and the monomials are distinct. Among all elements of $\mathbb{C}[\mathbf{y}]^{\Gamma'}_{[D_j],k}$ that are not in $\mathrm{Span}(\mathcal{B})$, let h be one with the fewest number of terms, and let βy^b be one of these terms. Then $\beta\mathcal{R}_{[D_j]}(y^b)$ is a sum of terms appearing in h. Let $N \in \mathcal{N}(n,k)$ be the necklace with $\mathrm{code}(N) = [b]$. Say $[b]$ has period m and write $n = mq$ for some integer q. The coefficient of y^b in $\mathcal{R}_{[D_j]}(y^b)$ is

$$\frac{1}{n}\sum_{i=1}^{q}(\omega^{-jm})^i,$$

and the order of ω^{-jm} is $q/\gcd(q,j)$, a divisor of q. If $[b]$ is not n_j-divisible, then the order of ω^j does not divide m, and hence, $\omega^{jm} \neq 1$. It would then follow that the above sum is 0, which contradicts the fact that βy^b is a term of h. Therefore, $[b]$ is n_j-divisible and $\mathcal{R}_{[D_j]}(y^b) = f_N \in \mathcal{B}$. However, then the polynomial $h - \beta f_N$ is an element of $\mathbb{C}[\mathbf{y}]^{\Gamma'}_{[D_j],k}$ with fewer terms than h and not in $\mathrm{Span}(\mathcal{B})$, which is a contradiction. So $\mathrm{Span}(\mathcal{B}) = \mathbb{C}[\mathbf{y}]^{\Gamma'}_{[D_j],k}$. $\square$

Corollary 6.4. *With notation as in the theorem,*

$$\begin{aligned}\#|D_j + kq| &= \#\{N \in \mathcal{N}(n,k) : N \textit{ is } m_j\textit{-divisible}\}\\ &= \#\{N \in \mathcal{N}(n,k) : \mathrm{code}(N) \textit{ is } n_j\textit{-divisible}\}.\end{aligned}$$

In particular, $\#|kq| = \#\mathcal{N}(n,k)$.

Proof. The result follows immediately from Corollary 5.2, Theorem 6.3, and the degree-preserving isomorphism (7). $\square$

Example 6.5. Consider the case $(n,k,j) = (4,2,1)$. We have

$$m_1 = \frac{4+2}{\gcd(4,2,1)} = 6 \quad \text{and} \quad n_1 = \frac{4}{\gcd(4,1)} = 4.$$

The three elements in $\mathcal{N}(4,2)$ are pictured in Figure 6, and have codes [2000], [1100], and [1010]. Of these, only, the first two are n_1-divisible.

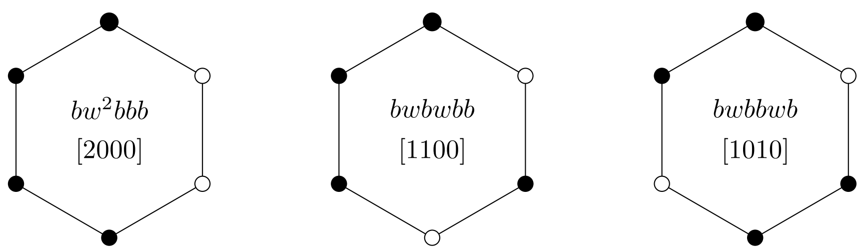


FIGURE 6. Binary necklaces with 4 black beads and 2 white beads with their codes. The first two necklaces are 1-, 2-, 3- and 6-divisible, and the last one is 1- and 3-divisible.

As an instance of Theorem 6.3, apply the Reynold's operator with $\omega = \mathbf{i} = \sqrt{-1}$ to the two 6-divisible necklaces to find a basis for $\mathbb{C}[\mathbf{y}]^{\Gamma'}_{[D_1],2}$:

$$\begin{aligned}4f_{[2000]} &= \mathbf{i}^{-1}y_2^2 + \mathbf{i}^{-2}y_3^2 + \mathbf{i}^{-3}y_4^2 + \mathbf{i}^{-4}y_1^2 = -\mathbf{i}y_2^2 - y_3^2 + \mathbf{i}y_4^2 + y_1^2\\ 4f_{[1100]} &= \mathbf{i}^{-1}y_2y_3 + \mathbf{i}^{-2}y_3y_4 + \mathbf{i}^{-3}y_4y_1 + \mathbf{i}^{-4}y_1y_2 = -\mathbf{i}y_2y_3 - y_3y_4 + \mathbf{i}y_1y_4 + y_1y_2.\end{aligned}$$

To change to the basis corresponding to the diagonal representation, substitute

$$\begin{aligned}y_1 &= x_1 + x_2 + x_3 + x_4\\ y_2 &= \mathbf{i}x_1 - x_2 - \mathbf{i}x_3 + x_4\\ y_3 &= -x_1 + x_2 - x_3 + x_4\\ y_4 &= -\mathbf{i}x_1 - x_2 + \mathbf{i}x_3 + x_4,\end{aligned}$$

to find

$$\begin{aligned}f_{[2000]} &= 2(x_1x_4 + x_2x_3)\\ f_{[1100]} &= (1+\mathbf{i})\,(x_1x_4 - x_2x_3),\end{aligned}$$

which is a basis for $\mathbb{C}[\mathbf{x}]^{\Gamma}_{[D_1],2}$. Note that x_1x_4 and x_2x_3 form a monomial basis for $\mathbb{C}[\mathbf{x}]^{\Gamma}_{[D_1],2}$ whose exponent vectors are exactly the elements of $|D_1+2q| = \{(1,0,0,1),(0,1,1,0)\}$ in accordance with Theorem 5.1.

Next, consider the case $(n,k,j) = (4,2,2)$. We have $m_2 = 3$ and $n_2 = 2$. All three necklaces in $\mathcal{N}(4,2)$ are 3-divisible. The corresponding basis for $\mathbb{C}[\mathbf{y}]^{\Gamma'}_{[D_2],2}$ is

$$4f_{[2000]} = \mathbf{i}^{-2}y_2^2 + \mathbf{i}^{-4}y_3^2 + \mathbf{i}^{-6}y_4^2 + y_1^2 = -y_2^2 + y_3^2 - y_4^2 + y_1^2$$

$$4f_{[1100]} = \mathbf{i}^{-2}y_2y_3 + \mathbf{i}^{-4}y_3y_4 + \mathbf{i}^{-6}y_4y_1 + y_1y_2 = -y_2y_3 + y_3y_4 - y_1y_4 + y_1y_2$$

$$4f_{[1010]} = \mathbf{i}^{-2}y_2y_4 + \mathbf{i}^{-4}y_3y_1 + \mathbf{i}^{-6}y_4y_2 + y_1y_3 = 2y_1y_3 - 2y_2y_4.$$

Substitute to get a basis for $\mathbb{C}[\mathbf{x}]^{\Gamma}_{[D_2],2}$:

$$\begin{aligned} f_{[2000]} &= x_1^2 + 2\,x_2x_4 + x_3^2 \\ f_{[1100]} &= \mathbf{i}(x_1^2 - x_3^2) \\ f_{[1010]} &= -x_1^2 + 2\,x_2x_4 - x_3^2. \end{aligned}$$

The corresponding complete linear system is $|D_2+2q| = \{(2,0,0,0),(0,1,0,1),(0,0,2,0)\}$, which by Theorem 5.1 yields the monomial basis $\{x_1^2, x_2x_4, x_3^2\}$ for $\mathbb{C}[\mathbf{x}]^{\Gamma}_{[D_2],2}$.

6.3. **Combinatorial bijection.** We now give an independent proof of Corollary 6.4 in the case where n and k are relatively prime. Given $D \in \mathrm{Div}(C_n) \simeq \mathbb{Z}^n$ and $v \in \mathbb{Z}^n$, let $D\cdot v$ be the usual dot product of vectors. Extend the rotation operator σ on words to divisors by letting $\sigma(D)(v_i) := D(v_{i+1})$ for all i modulo n.

Lemma 6.6. *Let $n,k\in\mathbb{Z}$ with $n>1$, and let $D\in\mathrm{Div}(C_n)$. Let*

$$\eta := (1,2,\dots,n) \quad \textit{and} \quad \vec{1} = (1,1,\dots,1)\in\mathbb{Z}^n.$$

Then $D\sim D_j + kq$ if and only $D\cdot\vec{1} = k$ and $D\cdot\eta = j \bmod n$.

Proof. First note that if c is a column of the Laplacian matrix for C_n, then $c\cdot\eta = 0 \bmod n$. Given any $D\in\mathrm{Div}(C_n)$, there exists i and m such that $D\sim D_i+mq$. Then $D\cdot\vec{1} = m$, and

$$D\cdot\eta = (D_i+mq)\cdot\eta \bmod n = i \bmod n.$$

The result follows. □

Theorem 6.7. *Given an effective divisor $E = (E(v_1),\dots,E(v_n))\in\mathbb{N}^n\subset\mathrm{Div}^k(C_n)$, define the word*

$$w_E := bw^{E(v_1)}bw^{E(v_2)}\dots bw^{E(v_n)},$$

and the corresponding necklace $N_E := [w_E]\in\mathcal{N}(n,k)$ with $\mathrm{code}(N_E) = [E(v_1)\dots E(v_n)]$. If $\gcd(n,k)=1$, then for each $j\in[n]$,

$$\begin{aligned} \psi\colon |D_j+kq| &\to \mathcal{N}(n,k) \\ E &\mapsto N_E \end{aligned}$$

is a bijection.

Proof. Let $\eta = (1,2,\dots,n)$ as in Lemma 6.6. To show injectivity, suppose $\psi(E) = \psi(E')$ for some pair of effective divisors $E,E'\in\mathrm{Div}^k(C_n)$. It follows that $E' = \sigma^i(E)$ for some i. By Lemma 6.6, working modulo n,

$$j = E'\cdot\eta = \sigma^i(E)\cdot\eta = E\cdot(\eta + i\vec{1}) = j + ik \bmod n.$$

If $\gcd(n,k)=1$, it follows that $i = 0 \bmod n$, and hence $E=E'$.

For surjectivity, let $N\in\mathcal{N}(n,k)$ with $\mathrm{code}(N) = [a_1\dots a_n]$. Let $E := (a_1,\dots,a_n)\in\mathrm{Div}^k(C_n)$, and say $E\cdot\eta = m \bmod n$. Then $\sigma^i(E)\cdot\eta = m+ik \bmod n$ for each i. If $\gcd(n,k)=1$, we can take i so that $m+ik = j \bmod n$ and define $E' := \sigma^i(E)$. Then $E'\in|D_j+kq|$ and $\psi(E') = N$. □

Remark 6.8. If $N\in\mathcal{N}(n,k)$ and $\mathrm{code}(N)$ has period ℓ, then as we saw in the proof of Theorem 6.3, both $\ell|n$ and $\ell|k$. Thus, if $\gcd(n,k)=1$, it follows that $\ell=1$. In other words, each element of $\mathcal{N}(n,k)$ has period $n+k$. Further, by the proof of Theorem 6.7, there is a commutative diagram of isomorphisms of sets:

$$\begin{array}{ccc} |kq| & \xrightarrow{\psi} & \mathcal{N}(n,k) \\ {\scriptstyle \sigma^j}\downarrow & \nearrow_{\psi} & \\ |D_j + kq| & & \end{array}.$$

Remark 6.9. (Duality) Switching colors gives a bijection between $\mathcal{N}(n,k)$ and $\mathcal{N}(k,n)$. Therefore, fixing vertices q on C_n and q' on C_k, Corollary 6.4 says that the cardinality of $|kq|$ on C_n is equal to that of $|nq'|$ on C_k. Further, when n and k are relatively prime, Theorem 6.7 gives a combinatorial bijection between these complete linear systems.

Example 6.10. Figure 7 illustrates the bijection of Theorem 6.7 for the case $n = 3$ and $k = 4$ and for all $j = 0, 1, 2$. The linear systems $|D_j + 4q|$ are the same up to cyclic rotation:

$$\begin{aligned} |4q| &= \{(0,0,4),(0,3,1),(1,1,2),(2,2,0),(3,0,1)\} \\ |D_1 + 4q| &= \{(4,0,0),(1,0,3),(2,1,1),(0,2,2),(1,3,0)\} \\ |D_2 + 4q| &= \{(0,4,0),(3,1,0),(1,2,1),(2,0,2),(0,1,3)\}. \end{aligned}$$

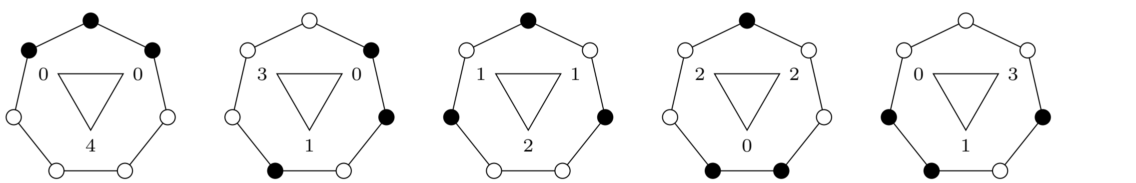


FIGURE 7. The complete linear system $|D_j+4q|$ on C_3 in bijection with the necklaces $\mathcal{N}(3,4)$ according to Theorem 6.7.

7. Extension to M-matrices

In this section, we explain how to extend our results to a broader context and thus indicate possible avenues for further investigation. It has been shown that many aspects of the divisor theory of graphs are retained by a theory in which reduced Laplacians of graphs are replaced by the more general class of matrices called *M-matrices* ([10], [12], [20]). To establish notation: if H and H' are matrices or vectors of the same dimensions, write $H \geq H'$ (resp., $H > H'$) if each entry of $H - H'$ is nonnegative (resp., positive).

Definition 7.1. Let A be an $(n-1) \times (n-1)$ matrix over $\mathbb{R}$ with $A_{ij} \leq 0$ for all $i \neq j$. Then A is a (non-singular) *M-matrix* if any of the following equivalent conditions holds:

(1) $A = sI_{n-1} - B$ for some matrix $B \geq 0$ and some $s > \max\{|\lambda| : \lambda \text{ an eigenvalue of } B\}$.
(2) Each eigenvalue of A has positive real part.
(3) Each principal minor of A is positive.
(4) A^{-1} exists and $A^{-1} \geq 0$.
(5) If $Au \geq 0$, then $u \geq 0$.
(6) There exists $u > 0$ such that $Au > 0$.
(7) There exists $u > 0$ with $Au \geq 0$ and such that if $(Au)_{i_0} = 0$ for some i_0, then there exists indices $i_1, \dots, i_r$ with $A_{i_k i_{k+1}} \neq 0$ for $k = 1, \dots, r-1$ and $(Au)_{i_r} > 0$.

The above seven equivalent conditions come from the list of 40 given by Plemmons ([19]).

From now on, we assume that A is an integer M-matrix. In that case, any integer vector u satisfying property (7) is called a *burning script*. A burning script for A always exists and a unique minimal one (with respect to $\leq$) can be constructed as follows: start with $u = (1, \dots, 1)$, and then as long as $(Au)_i < 0$ for some i, increase u_i by 1 ([18]; [8], Chapter 7). If u is a burning script, then Au is called a *burning configuration*.

Let $u = (u_1, \dots, u_n) > 0$ and $w = (w_1, \dots w_n) > 0$ be any integer vectors such that both $Au \geq 0$ and $wA \geq 0$. Their existence is guaranteed by property (7) and the fact that the transpose A^t of A is also an M-matrix. We do not require that u and w be burning scripts. Next, define the (w,u)-*extension* of A to be the $n \times n$ matrix

$$\widehat{A} := \left(\begin{array}{c|c} wAu & -wA \\ \hline -Au & A \end{array}\right).$$

The following vectors are primitive generators for the left and right kernels, respectively, of $\widehat{A}$:

$$\phi = (1, w_1, w_2, \ldots, w_{n-1}) \quad \text{and} \quad \delta = (1, u_1, u_2, \ldots, u_{n-1}).$$

Example 7.2. Let L be the Laplacian matrix for a connected, undirected graph with respect to some ordering of the vertices, and let $A = \tilde{L}$ be the corrresponding reduced Laplacian with respect to the first vertex. Then $A = A^t$ is an M-matrix ([12]) with minimal burning script $u = w = (1, \ldots, 1)$. The (w,u)-extension of A recovers L, i.e., $\widehat{A} = L$.

We now extend our earlier results on the cardinality of compete linear systems to the setting of M-matrices. A *divisor* is an element $D \in \mathrm{Div}(\widehat{A}) := \mathbb{Z}^n$. The *degree* of a divisor D is given by the dot product $\deg(D) := \phi \cdot D$. Define linear equivalence of divisors by $D \sim D'$ if $D - D' \in \mathrm{im}_{\mathbb{Z}} \widehat{A}$. As before, let $\mathrm{Pic}(\widehat{A}) := \mathbb{Z}^n/\sim$, which is graded by (our new) degree, and $\mathrm{Jac}(\widehat{A}) := \mathrm{Pic}^0(\widehat{A})$, the group of divisor classes of divisors of degree 0.

For notational purposes, define $v_i := e_i$, the i-th standard basis vector, for $i = 1, \ldots, n$. The isomorphisms (1) and (2) of Section 2 hold in this new setting in which $\tilde{L}$ is replaced by A. For each divisor class $[D] \in \mathrm{Pic}(\widehat{A})$, define the *complete linear system* $|D|$, the set $\mathbb{E}_{[D]}$, and the λ-generating function $\Lambda_{[D]}(z)$ as in Section 2. Substituting $\widehat{A}$ and A for the Laplacian and reduced Laplacian, respectively, our main results generalize, with nearly identical proofs, after suitably modifying the statements to take into account our new notion of degree:

Primary and secondary divisors. For each $i = 1, \ldots, n$, the degree of v_i considered as a divisor is $\deg(v_i) = \phi_i$. Redefine $\mathrm{ord}_q(v_i)$ to be the order of $[v_i - \deg(v_i)q] \in \mathrm{Pic}(\widehat{A})$, and let ℓ_i be any positive integer multiple of $\mathrm{ord}_q(v_i)$. Then Theorem 3.1, Corollary 3.2, and Proposition 3.4 hold after replacing each occurrence of ℓ_v with $\phi_i \ell_i$. For instance, in Corollary 3.2, we now have

$$\Lambda_{[D]}(z) = \frac{S(z)}{\prod_{i=1}^{n}(1 - z^{\phi_i \ell_i})} \tag{8}$$

where $S(z) := \sum_{F \in \mathcal{S}_{[D]}} z^{\deg(F)}$.

Polyhedra. The constructions in Section 4 remain valid. One cosmetic change in the exposition is that instead of taking q to be the last vertex of the graph, we now take q to be the *first* standard basis vector.[4] This means, for example, that instead of considering, $(f,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we now consider $(t,f) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Theorem 4.4 then holds as stated, defining $\deg(\omega) := \phi \cdot \omega$ in part (3). Proposition 4.5 holds by again replacing ℓ_i by $\phi_i \ell_i$ and redefining $\mathrm{ord}_q(v_i)$ as discussed above.

Invariant theory. To generalize the results in Section 5, take $\mathbb{C}[\mathbf{x}]$ to have the multigrading determined by $\deg(x_i) := \phi_i$ for $i = 1, \ldots, n$. The representation of ρ, described in (6), becomes

$$\rho(\chi) = (1, \chi([v_2 - \deg(v_2)q]), \chi([v_3 - \deg(v_3)q]), \ldots, \chi([v_n - \deg(v_n)q])). \tag{9}$$

Theorem 5.1 then extends with no changes to its statement. For Corollary 5.2, use a multigraded version of Molien's theorem, for abelian groups, to get

$$\Lambda_{[D]}(z) = \Phi_{\Gamma,\chi}(z^{\phi_1}, \ldots, z^{\phi_n}) := \frac{1}{|\operatorname{Jac}(\widehat{A})|} \sum_{\chi \in \operatorname{Jac}(\widehat{A})^*} \frac{\overline{\chi([D])}}{\det(I_n - \operatorname{diag}(z^{\phi_1}, \ldots, z^{\phi_n})\rho(\chi))} \tag{10}$$

where $\operatorname{diag}(\,\cdot\,)$ denotes the diagonal matrix with the given diagonal entries.

[4] This switch in the placement of q was made in order to conform to the conventions for root systems considered in [6]. See Section 7.1.1, below.

7.1. **Root systems and McKay quivers.** In ([6]), Benkart, Klivans, and Reiner relate two classes of M-matrices to the extended divisor theory described above. For the sake of brevity, we give only a cursory description of some of their work, referring the interested reader to the original paper for definitions and other details.[5]

7.1.1. *Root systems.* Let Φ be a finite, crystallographic, irreducible root system. The Cartan matrix C for Φ is an M-matrix. Its burning configurations are the elements of the root lattice lying in the fundamental chamber (with respect to a choice of simple roots). Making particular natural choices for burning configurations for C and its transpose C^t, the authors define the extended Cartan matrix $\widetilde{C}$, which is the Cartan matrix for the corresponding affine root system. Letting $A = C^t$ and $\widehat{A} = \widetilde{C}^t$, it turns out that $\mathrm{Pic}(\widehat{A})$ is the fundamental group of Φ, i.e., the quotient of the weight lattice by the root lattice. We think of each $D \in \mathrm{Div}(\widehat{A})$ as a divisor on the affine Dynkin diagram for the affine root system corresponding to Φ, and the matrix $\widehat{A}$ can be thought of as defining firing rules (as described for the Laplacian in Section 2).

Example 7.3. Let Φ be the root system B_3. The transpose of its Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$$

The vectors $u = (1, 2, 2)$ and $w = (1, 2, 1)$ are burning scripts for A and A^t, respectively, (though only the latter is minimal). The (w, u)-extension of A is then

$$\widehat{A} = \widetilde{C}^t = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix},$$

with left and right kernel generators:

$$\phi = (1, 1, 2, 1) \quad \text{and} \quad \delta = (1, 1, 2, 2).$$

We have $\mathrm{Jac}(\widehat{A}) \simeq \mathbb{Z}/2\mathbb{Z}$ with generator $D := (-1, 0, 0, 1)$. Follow the procedure in Remark 3.5 to compute primary and secondary divisors:

$$\begin{aligned} \mathcal{P} &= \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,2)\} \\ \mathcal{S}_{[0]} &= \{(0,0,0,0)\} \\ \mathcal{S}_{[D]} &= \{(0,0,0,1)\}. \end{aligned}$$

Note that the primary divisor $(0, 0, 1, 0)$ has degree $\phi \cdot (0, 0, 1, 0) = 2$. By (8),

$$\Lambda_{[0]}(z) = \frac{1}{(1-z)^2(1-z^2)^2} = 1 + 2z + 5z^2 + 8z^3 + 14z^4 + 20z^5 + 30z^6 + 40z^7 + 55z^8 + \cdots$$

$$\Lambda_{[D]}(z) = \frac{z}{(1-z)^2(1-z^2)^2} = z + 2z^2 + 5z^3 + 8z^4 + 14z^5 + 20z^6 + 30z^7 + 40z^8 + 55z^9 + \cdots.$$

There is one non-trivial character χ for $\mathrm{Jac}(\widehat{A})$, determined by $\chi([D]) = -1$. For the modified represention (9) for $\mathrm{Jac}(\widehat{A})^*$, we have $\rho(\chi) = (1, 1, 1, -1)$. Therefore, the new Molien series (10) gives the following forms for the λ-generating functions:

$$\Lambda_{[0]}(z) = \frac{1}{2}\left(\frac{1}{(1-z)^2(1-z^2)(1-z)} + \frac{1}{(1-z)^2(1-z^2)(1+z)}\right)$$

$$\Lambda_{[D]} = \frac{1}{2}\left(\frac{1}{(1-z)^2(1-z^2)(1-z)} + \frac{-1}{(1-z)^2(1-z^2)(1+z)}\right).$$

For example, the coefficient of z^2 in the series expansion of $\Lambda_{[0]}(z)$ indicates there are 5 effective divisors in the complete linear system for the divisor $2q = (2, 0, 0, 0)$. These are pictured in Figure 8.

[5] Note that our convention for the Laplacian of a graph differs from that in [6] by a transpose.

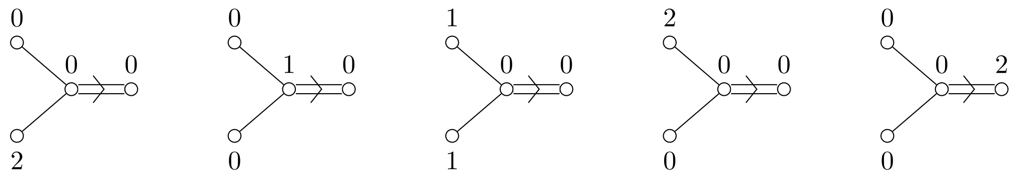


FIGURE 8. The complete linear system of the divisor $2q = (2,0,0,0)$ for the root system B_3.

7.1.2. *McKay quivers.* [6] Let $\rho\colon \Gamma \to \mathrm{GL}(\mathbb{C}^n)$ be a faithful representation of a finite group Γ with character χ_ρ. Let $\rho_1,\dots,\rho_n$ be the irreducible complex representations of Γ, with ρ_1 the trivial representation, and with respective characters $\chi_1,\dots,\chi_n$. For each i, denote the character of the tensor product $\rho\otimes\rho_i$ by $\chi_\rho\cdot\chi_i$, and define integers m_{ij} by

$$\chi_\rho\cdot\chi_i = \sum_{j=1}^n m_{ij}\chi_i.$$

Define the $n\times n$ matrix $M := (m_{ij})$ and the *extended McKay-Cartan* matrix $\widetilde{C} := nI_n - M$. The *McKay-Cartan* matrix is then the submatrix C formed by removing the first row and first column of $\widetilde{C}$. In our notation from above, take $A = C^t$. The vectors $u = w = (\dim\rho_2,\dots,\dim\rho_n)$ are burning scripts for A and A^t, with respect to which $\widehat{A} = \widetilde{C}^t$ with left and right kernel generators

$$\phi = \delta = (\dim\rho_1,\dots,\dim\rho_n).$$

The *McKay quiver* of γ is the directed graph with vertices $\chi_1,\dots,\chi_n$ and m_{ij} directed edges from χ_i to χ_j for each i,j. The matrix $\widehat{A}$ defines firing rules on the McKay quiver (again as described for the Laplacian in Section 2).

Example 7.4. Consider the representation $\rho\colon \mathrm{Jac}(G)^* \to \mathrm{GL}(\mathbb{C}^n)$ defined by (6) of Section 5. When $G = C_n$, the cyclic graph on n-vertices, ρ is the regular representation of the cyclic group $\mathrm{Jac}(G)^* \simeq \mathbb{Z}/n\mathbb{Z}$. Therefore, $m_{ij} = 1$ for all i,j, the McKay quiver may be thought of as the (undirected) complete graph K_n on n vertices, and $\widehat{A}$ is its Laplacian matrix. Thus, $\mathrm{Jac}(\widehat{A}) = \mathrm{Jac}(K_n)$.

More generally ([6], Section 6.2), the McKay quiver for any faithful complex representation γ of an abelian group has (directed) Laplacian matrix equal to the matrix $\widehat{A}$ for γ.

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[6] For this section, in addition to [6], see the work by Gaetz, [11].

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