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ON DISCRETE IDEMPOTENT PATHS

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Abstract. The set of discrete lattice paths from $(0, 0)$ to (n, n) with North and East steps (i.e. words $w \in \{ x, y \}^*$ such that $|w|_x = |w|_y = n$) has a canonical monoid structure inherited from the bijection with the set of join-continuous maps from the chain $\{ 0, 1, \ldots, n \}$ to itself. We explicitly describe this monoid structure and, relying on a general characterization of idempotent join-continuous maps from a complete lattice to itself, we characterize idempotent paths as upper zigzag paths. We argue that these paths are counted by the odd Fibonacci numbers. Our method yields a geometric/combinatorial proof of counting results, due to Howie and to Laradji and Umar, for idempotents in monoids of monotone endomaps on finite chains.

Keywords. discrete path, idempotent, join-continuous map.

1. Introduction

Discrete lattice paths from $(0, 0)$ to (n, m) with North and East steps have a standard representation as words $w \in \{ x, y \}^*$ such that $|w|_x = n$ and $|w|_y = m$. The set $P(n, m)$ of these paths, with the dominance ordering, is a distributive lattice (and therefore of a Heyting algebra), see e.g. [2, 9, 8, 18]. A simple proof that the dominance ordering is a lattice relies on the bijective correspondence between these paths and monotone maps from the chain $\{ 1, \ldots, n \}$ to the chain $\{ 0, 1, \ldots, m \}$, see e.g. [3, 2]. In turn, these maps bijectively correspond to join-continuous maps from $\{ 0, 1, \ldots, n \}$ to $\{ 0, 1, \ldots, m \}$ (those order preserving maps that sends 0 to 0). Join-continuous maps from a complete lattice to itself form, when given the pointwise ordering, a complete lattice in which composition distributes with joins. This kind of algebraic structure combining a monoid operation with a lattice structure is called a quantale [19] or (roughly speaking) a residuated lattice [10]. Therefore, the aforementioned bijection also witnesses a richer structure for $P(n, n)$, that of a quantale and of a residuated lattice. The set $P(n, n)$ is actually a *star-autonomous* quantale or, as a residuated lattice, *involutive*, see [12].

A main aim of this paper is to draw attention to the interplay between the algebraic and enumerative combinatorics of paths and these algebraic structures (lattices, Heyting algebras, quantales, residuated lattices) that, curiously, are all related to logic. We focus in this paper on the monoid structure that corresponds under the bijection to function composition—which, from a logical perspective, can be understood as a sort of non-commutative conjunction. In the literature, the monoid structure appears to be less known than the lattice structure. A notable exception is the work [17] where a different kind of

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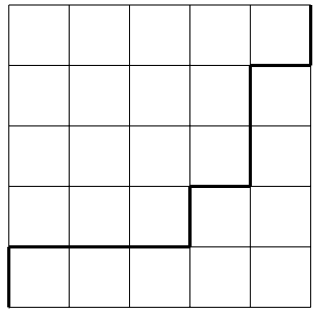

FIGURE 1. The path *yxxxyxyyxy*.

lattice paths, related to Delannoy paths, are considered so to represent monoids of injective order-preserving partial transformations on chains.

We explicitly describe the monoid structure of $P(n,n)$ and characterize those paths that are idempotents. Our characterization relies on a general characterization of idempotent join-continuous maps from a complete lattice to itself. When the complete lattice is the chain $\{0,1,\ldots,n\}$, this characterization yields a description of idempotent paths as those paths whose all North-East turns are above the line $y = x + \frac{1}{2}$ and whose all East-North turns are below this line. We call these paths *upper zigzag*. We use this characterization to provide a geometric/combinatorial proof that upper zigzag paths in $P(n,n)$ are counted by the odd Fibonacci numbers f_{2n+1}. Simple algebraic connections among the monoid structure on $P(n,n)$, the monoid O_n of order preserving maps from $\{1,\ldots,n\}$ to itself, and the submonoid O_n^n of O_n of maps fixing n, yield a geometric/combinatorial proof of counting results due to Howie [13] (the number of idempotents in O_n is the even Fibonacci numbers f_{2n}) and Laradji and Umar [16] (the number of idempotents in O_n^n is the odd Fibonacci numbers f_{2n-1}).

2. A PRODUCT ON PATHS

In the following, $P(n,m)$ shall denote the set of words $w \in \{x,y\}^*$ such that $|w|_x = n$ and $|w|_y = m$. We identify a word $w \in P(n,m)$ with a discrete path from $(0,0)$ to (n,m) which uses only East and North steps of length 1. For example, the word $yxxxyxyyxy \in P(5,5)$ is identified with the path in Figure 1.

Let L_0, L_1 be complete lattices. A map $f : L_0 \to L_1$ is *join-continuous* if $f(\bigvee X) = \bigvee f(X)$, for each subset X of L_0. We use $Q_\vee(L_0, L_1)$ to denote the set of join-continuous maps from L_0 to L_1. If $L_0 = L_1 = L$, then we write $Q_\vee(L)$ for $Q_\vee(L,L)$.

The set $Q_\vee(L_0, L_1)$ can be ordered pointwise (i.e. $f \leq g$ if and only if $f(x) \leq g(x)$, for each $x \in L_0$); with this ordering it is a complete lattice. Function composition distributes over (possibly infinite) joins:

$$(1) \qquad (\bigvee_{j\in J} g_j) \circ (\bigvee_{i\in I} f_i) = \bigvee_{j\in J, i\in I} (g_j \circ f_i),$$

whenever L_0, L_1, L_2 are complete lattices, $\{f_i \mid i \in I\} \subseteq Q_\vee(L_0, L_1)$ and $\{g_j \mid j \in J\} \subseteq Q_\vee(L_1, L_2)$. A *quantale* (see [19]) is a complete lattice endowed with a semigroup operation $\circ$ satisfying the distributive law (1). Thus, $Q_\vee(L)$ is a quantale, for each complete lattice $Q_\vee(L)$.

For $k \geq 0$, we shall use $\mathbb{I}_k$ to denote the chain $\{0,1,\ldots,k\}$. Notice that $f : \mathbb{I}_n \to \mathbb{I}_m$ is join-continuous if and only if it is monotone (or order-preserving) and $f(0) = 0$. For each $n, m \geq 0$, there is a well-known bijective correspondence between paths in $P(n,m)$ and join-continuous maps in $Q_\vee(\mathbb{I}_n, \mathbb{I}_m)$; next, we recall this bijection. If $w \in P(n,m)$, then the occurrences of y in w split w into $m+1$ (possibly empty) blocks of contiguous xs, that we index by the numbers $0, \ldots, m$:

$$w = \mathbf{bl}_0^{w,x} \cdot y \cdot \mathbf{bl}_1^{w,x} \cdot y \ldots \mathbf{bl}_{m-1}^{w,x} \cdot y \cdot \mathbf{bl}_m^{w,x} .$$

We call the words $\mathbf{bl}_0^{w,x}, \mathbf{bl}_1^{w,x}, \ldots, \mathbf{bl}_m^{w,x} \in \{ x \}^*$ the x-blocks of w. Given $i \in \{ 1, \ldots, n \}$, the index of the block of the i-th occurrence of the letter x in w is denoted by $\mathbf{blno}_i^{w,x}$. We have therefore $\mathbf{blno}_i^{w,x} \in \{ 0, \ldots, m \}$. Notice that $\mathbf{blno}_i^{w,x}$ equals the number of ys preceding the i-th occurrence of x in w so, in particular, $\mathbf{blno}_i^{w,x}$ can be interpreted as the height of the i-th occurrence of x when w is considered as a path. Similar definitions, $\mathbf{bl}_j^{w,y}$ and $\mathbf{blno}_j^{w,y}$, for $j = 1, \ldots, m$, are given for the blocks obtained by splitting w by means of the xs:

$$w = \mathbf{bl}_0^{w,y} \cdot x \cdot \mathbf{bl}_1^{w,y} \cdot x \ldots \mathbf{bl}_{n-1}^{w,y} \cdot x \cdot \mathbf{bl}_n^{w,y} .$$

The map $\mathbf{blno}^{w,x}$, sending $i \in \{ 1, \ldots, n \}$ to $\mathbf{blno}_i^{w,x}$, is monotone from the chain $\{ 1, \ldots, n \}$ to the chain $\{ 0, 1, \ldots, m \}$. There is an obvious bijective correspondence from the set of monotone maps from $\{ 1, \ldots, n \}$ to $\mathbb{I}_m = \{ 0, 1, \ldots, m \}$ to the set $Q_\vee(\mathbb{I}_n, \mathbb{I}_m)$ obtained by extending a monotone f by setting $f(0) := 0$. We shall tacitly assume this bijection and, accordingly, we set $\mathbf{blno}_0^{w,x} := 0$. Next, by setting $\mathbf{blno}_{n+1}^{w,x} := m$, we notice that

$$|\mathbf{bl}_i^{w,y}| = \mathbf{blno}_{i+1}^{w,x} - \mathbf{blno}_i^{w,x} ,$$

for $i = 0, \ldots, n$, so w is uniquely determined by the map $\mathbf{blno}^{w,x}$. Therefore, the mapping sending $w \in P(n,m)$ to $\mathbf{blno}^{w,x}$ is a bijection from $P(n,m)$ to the set $Q_\vee(\mathbb{I}_n, \mathbb{I}_m)$. The dominance ordering on $P(n,m)$ arises from the pointwise ordering on $Q_\vee(\mathbb{I}_n, \mathbb{I}_m)$ via the bijection.

For $w \in P(n,m)$ and $u \in P(m,k)$, the product $w \otimes u$ is defined by concatenating the x-blocks of w and the y-blocks of u:

Definition 1. For $w \in P(n,m)$ and $u \in P(m,k)$, we let

$$w \otimes u := \mathbf{bl}_0^{w,x} \cdot \mathbf{bl}_0^{u,y} \cdot \mathbf{bl}_1^{w,x} \cdot \mathbf{bl}_1^{u,y} \ldots \mathbf{bl}_m^{w,x} \cdot \mathbf{bl}_m^{u,y} .$$

Example 1. Let $w = yxxyxy$ and $u = xyxyyx$, so the x-blocks of w are $\epsilon, xx, x, \epsilon$ and the y-blocks of u are $\epsilon, y, yy, \epsilon$; we have $w \otimes u = xxyxyy$. We can trace the original blocks by inserting vertical bars in $w \otimes u$ so to separate $\mathbf{bl}_i^{w,x}\mathbf{bl}_i^{u,y}$ from $\mathbf{bl}_{i+1}^{w,x}\mathbf{bl}_{i+1}^{u,y}$, $i = 0, \ldots, m-1$. That is, we can write $w \otimes_{tr} u = |xxy|xyy|$, so $w \otimes u$ is obtained from $w \otimes_{tr} u$ by deleting vertical bars. Notice that also w and u can be recovered from $w \otimes_{tr} u$, for example w is obtained from $w \otimes_{tr} u$ by deleting the letter y and then renaming the vertical bars to the letter y. Figure 2 suggests that $\otimes$ is a form of synchronisation product, obtained by shuffling the x-blocks of w with the y-blocks of u so to give "priority" to all the xs (that is, the xs precede the ys in each block). It can be argued that there are other similar products, for example, the one where the ys precede the xs in each block, so $w \oplus u = yxxyyx$. It is easy to see that $w \oplus u = (u^\star \otimes w^\star)^\star$, where $w^\star$ is the image of w along the morphism that exchanges the letters x and y.

Proposition 1. *The product $\otimes$ corresponds, under the bijection, to function composition. That is, we have*

$$\mathbf{blno}^{w \otimes u, x} = \mathbf{blno}^{u,x} \circ \mathbf{blno}^{w,x} .$$

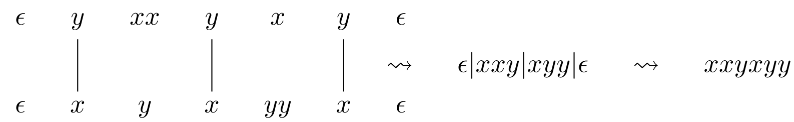

Figure 2. Construction of the product $yxxyxy \otimes xyxyyx$.

Proof. In order to count the number of ys preceding the i-th occurrence of x in $w \otimes u$, it is enough to identify the block number j of this occurrence in w, and then count how many ys precede the j-th occurrence of x in u. That is, we have $\mathbf{blno}_i^{w\otimes u,x} = \mathbf{blno}_j^{u,x}$ with $j = \mathbf{blno}_i^{w,x}$. □

Remark 1. Let us exemplify how the algebraic structure of $Q_\vee(\mathbb{I}_n, \mathbb{I}_m)$ yields combinatorial identities. The product is a function $\otimes : P(n,m) \times P(m,k) \to P(n,k)$, so we study how many preimages a word $w \in P(n,k)$ might have. By reverting the operational description of the product previously given, this amounts to inserting m vertical bars marking the beginning-end of blocks (so to guess a word of the form $u_0 \otimes_{tr} u_1$) under a constraint that we describe next. Each position can be barred more than once, so adding j bars can be done in $\binom{n+k+j}{j}$ ways. The only constraint we need to satisfy is the following. Recall that a position $\ell \in \{0,\ldots,n+k\}$ is a *North-East turn* (or a *descent*), see [15], if $\ell > 0$, $w_{\ell-1} = y$ and $w_\ell = x$. If a position is a North-East turn, then such a position is necessarily barred. Let us illustrate this with the word $xxyxyy$ which has just one descent, which is necessarily barred: $xxy|xyy$. Assuming $m = 3$, we need to add two more vertical bars. For example, for $x|xy|xy|y$ we obtain the following decomposition:

$$x|xy|xy|y \rightsquigarrow (x|x|x|, |y|y|y) \rightsquigarrow (xyxyxy, xyxyxy)\,.$$

Therefore, if w has i descents, then these positions are barred, while the other $m-i$ barred positions can be chosen arbitrarily, and there are $\binom{n+k+m-i}{m-i}$ ways to do this. Recall that there are $\binom{n}{i}\binom{k}{i}$ words $w \in P(n,k)$ with i descents, since such a w is determined by the subsets of $\{1,\ldots,n\}$ and $\{1,\ldots,k\}$ of cardinality i, determining the descents. Summing up w.r.t. the number of descents, we obtain the following formulas:

$$\binom{n+m}{n}\binom{m+k}{k} = \sum_{i=0}^{m}\binom{n+m+k-i}{m-i}\binom{n}{i}\binom{k}{i}, \qquad \binom{2n}{n}^2 = \sum_{i=0}^{n}\binom{3n-i}{n-i}\binom{n}{i}^2.$$

Similar kind of combinatorial transformations and identities appear in [11, 5, 6], yet it is not clear to us at the moment of writing whether these works relate in some way to the product of paths studied here.

Remark 2. The previous remark also shows that if $w \in P(n,k)$ has $m \geq 0$ descents, then there is a canonical factorization $w = w_0 \otimes w_1$ with $w_0 \in P(n,m)$ and $w_1 \in P(m,k)$. It is readily seen that, via the bijection, this is the standard epi-mono factorization in the category of join-semilattices. The word $xxyxyy$, barred at its unique descent as $xxy|xyy$, is decomposed into $xxyx$ and $yxyy$.

Remark 3. As in [17], many semigroup-theoretic properties of the monoid $Q_\vee(\mathbb{I}_n)$ can be read out of (and computed from) the bijection with $P(n,n)$. For example

$$\mathrm{card}(\{ f \in Q_\vee(\mathbb{I}_n) \mid \mathrm{card}(\mathit{Image}(f)) = k+1 \}) = \binom{n}{k}^2$$

since, as in the previous remark, a path with k North-East turns corresponds to a join-continuous map f such that $\mathrm{card}(Image(f)) = k + 1$. Similarly

$$\mathrm{card}(\{ f \in Q_\vee(\mathbb{I}_n) \mid \max(Image(f)) = k \}) = \binom{n+k-1}{k}$$

since a map $f \in Q_\vee(\mathbb{I}_n)$ such that $\max(Image(f)) = k$ (i.e. $f(n) = k$) corresponds to a path in $P(n, k)$ whose last step is an East step, thus to a path in $P(n - 1, k)$. A similar argument can be used to count maps $f \in O_n$ such that $f(n) = k$, cf. [16, Proposition 3.7].

Remark 4. Further properties of the monoid $Q_\vee(\mathbb{I}_n)$ can be easily verified, for example, this monoid is aperiodic. For the next observation, see also [16, Proposition 2.3] and [17, Theorem 3.4]. Recall that $f \in Q_\vee(\mathbb{I}_n)$ is *nilpotent* if, for some $\ell \geq 0$, f^ℓ is the bottom of the lattice, that is, it is the constant map with value 0. It is easily seen that f is nilpotent if and only if $f(x) < x$, for each $x = 1, \ldots, n$. Therefore, a path is nilpotent if and only it lies below the diagonal, that is, it is a Dyck path. Therefore, there are $\frac{1}{n+1}\binom{2n}{n}$ nilpotents in $Q_\vee(\mathbb{I}_n)$.

3. Idempotent join-continuous maps as emmentalers

We provide in this section a characterization of idempotent join-continuous maps from a complete lattice to itself. The characterization originates from the notion of *EA*-duet used to study some elementary subquotients in the category of lattices, see [20, Definition 9.1].

Definition 2. An *emmentaler* of a complete lattice L is a collection $\mathcal{E} = \{ [y_i, x_i] \mid i \in I \}$ of closed intervals of L such that

- $[y_i, x_i] \cap [y_j, x_j] = \emptyset$, for $i, j \in I$ with $i \neq j$,
- $\{ y_i \mid i \in I \}$ is a subset of L closed under arbitrary joins,
- $\{ x_i \mid i \in I \}$ is a subset of L closed under arbitrary meets.

The main result of this section is the following statement.

Theorem 1. *For an arbitrary complete lattice L, there is a bijection between idempotent join-continuous maps from L to L and emmentalers of L.*

For an emmentaler $\mathcal{E} = \{ [y_i, x_i] \mid i \in I \}$ of L, we let

$$J(\mathcal{E}) := \{ y_i \mid i \in I \}, \qquad M(\mathcal{E}) := \{ x_i \mid i \in I \}$$

$$\mathbf{int}_{\mathcal{E}}(z) := \bigvee \{ y \in J(\mathcal{E}) \mid y \leq z \}, \qquad \mathbf{cl}_{\mathcal{E}}(z) := \bigwedge \{ x \in M(\mathcal{E}) \mid z \leq x \}.$$

It is a standard fact that $\mathbf{cl}_{\mathcal{E}}$ is a closure operator on L (that is, it is a monotone inflating idempotent map from L to itself) and that $\mathbf{int}_{\mathcal{E}}$ is an interior operator on L (that is, a monotone, deflating, and idempotent endomap of L). In the following statements an emmentaler $\mathcal{E} = \{ [y_i, x_i] \mid i \in I \}$ is fixed.

Lemma 1. *For each $i \in I$, $x_i = \mathbf{cl}_{\mathcal{E}}(y_i)$ and $\mathbf{int}_{\mathcal{E}}(x_i) = y_i$. Therefore $\mathbf{int}_{\mathcal{E}}$ restricts to an order isomorphism from $M(\mathcal{E})$ to $J(\mathcal{E})$ whose inverse is $\mathbf{cl}_{\mathcal{E}}$.*

Proof. Clearly, $\mathbf{cl}_{\mathcal{E}}(y_i) \leq x_i$. Let us suppose that $y_i \leq x_j$ yet $x_i \not\leq x_j$, then $y_i \leq x_j \wedge x_i < x_i$ and $x_j \wedge x_i = x_\ell$ for some $\ell \in I$ with $\ell \neq i$. But then $x_\ell \in [y_\ell, x_\ell] \cap [y_i, x_i]$, a contradiction. The equality $\mathbf{int}_{\mathcal{E}}(x_i) = y_i$ is proved similarly. □

In view of the following lemma we think of $\mathcal{E}$ as a sublattice of L with prescribed holes/fillings, whence the naming "emmentaler".

Lemma 2. *If $\mathcal{E}$ is an emmentaler of L, then $\bigcup \mathcal{E}$ is a subset of L closed under arbitrary joins and meets. Moreover, the map sending $z \in [y_i, x_i]$ to y_i is a complete lattice homomorphism from $\bigcup \mathcal{E}$ to $J(\mathcal{E})$.*

Proof. Let $\{ z_k \mid k \in K \}$ with $z_k \in [y_k, x_k]$ for each $k \in K$. Then, for some $j \in I$,

$$\begin{aligned} y_j &= \bigvee_{k\in K} y_k \leq \bigvee_{k\in K} z_k \leq \bigvee_{k\in K} x_k \leq \mathbf{cl}_{\mathcal{E}}(\bigvee_{k\in K} x_k) \\ &= \bigvee\nolimits_{M(\mathcal{E})} \{ x_k \mid k \in K \} = \bigvee\nolimits_{M(\mathcal{E})} \{ \mathbf{cl}_{\mathcal{E}}(y_k) \mid k \in K \} = \mathbf{cl}_{\mathcal{E}}(\bigvee_{k\in K} y_k) = \mathbf{cl}_{\mathcal{E}}(y_i) = x_j \,, \end{aligned} \tag{2}$$

where in the second line we have used the fact that $\mathbf{cl}_{\mathcal{E}}(\bigvee_{k\in K} x_k)$ is the join in $M(\mathcal{E})$ of the family $\{ x_k \mid k \in K \}$ and also the fact that $\mathbf{cl}_{\mathcal{E}}$ is an order isomorphism (so it is join-continuous) from $J(\mathcal{E})$ to $M(\mathcal{E})$. Therefore, $\bigvee_{k\in K} z_k \in \bigcup \mathcal{E}$ and, in a similar way, $\bigwedge_{k\in K} z_k \in \bigcup \mathcal{E}$.

Next, let $\pi : \bigcup \mathcal{E} \to J(\mathcal{E})$ be the map sending $z \in [y_i, x_i]$ to $y_i \in J(\mathcal{E})$. The computations in (2) show that π is join-continuous. With similar computations it is seen that $\bigwedge_{k\in K} z_k$ is sent to $\mathbf{int}_{\mathcal{E}}(\bigwedge_{k\in K} y_k)$ which is the meet of the family $\{ y_k \mid k \in K \}$ within $J(\mathcal{E})$. Therefore, π is meet-continuous as well. □

We recall next some facts on adjoint pairs of maps, see e.g. [4, §7]. Two monotone maps $f, g : L \to L$ form an *adjoint pair* if $f(x) \leq y$ if and only if $x \leq g(y)$, for each $x, y \in L$. More precisely, f is *left* (or *lower*) *adjoint* to g, and g is *right* (or *upper*) *adjoint* to f. Each map determines the other: that is, if f is left adjoint to g and g', then $g = g'$; if g is right adjoint to f and f', then $f = f'$. If L is a complete lattice, then a monotone $f : L \to L$ is a left adjoint (that is, there exists g for which f is left adjoint to g) if and only if it is join-continuous; under the same assumption, a monotone $g : L \to L$ is a right adjoint if and only if it is meet-continuous.

Proposition 2. *If $\mathcal{E}$ is an emmentaler of L, then the maps $f_{\mathcal{E}}$ and $g_{\mathcal{E}}$ defined by*

$$f_{\mathcal{E}}(z) := \mathbf{int}_{\mathcal{E}}(\mathbf{cl}_{\mathcal{E}}(z)) \,, \qquad\qquad g_{\mathcal{E}}(z) := \mathbf{cl}_{\mathcal{E}}(\mathbf{int}_{\mathcal{E}}(z)) \,,$$

are idempotent and adjoint to each other. In particular, $f_{\mathcal{E}}$ is join-continuous, so it belongs to $Q_{\vee}(L)$.

Proof. Clearly, $f_{\mathcal{E}}$ is idempotent:

$$\mathbf{int}_{\mathcal{E}}(\mathbf{cl}_{\mathcal{E}}(\mathbf{int}_{\mathcal{E}}(\mathbf{cl}_{\mathcal{E}}(z)))) = \mathbf{int}_{\mathcal{E}}(\mathbf{cl}_{\mathcal{E}}(z)) \,,$$

since $\mathbf{cl}_{\mathcal{E}}(z) = x_i$ for some $i \in I$ and $\mathbf{cl}_{\mathcal{E}}(\mathbf{int}_{\mathcal{E}}(x_i)) = x_i$. In a similar way, $g_{\mathcal{E}}$ is idempotent. Let us argue that $f_{\mathcal{E}}$ and $g_{\mathcal{E}}$ are adjoint. If $z_0 \leq \mathbf{cl}_{\mathcal{E}}(\mathbf{int}_{\mathcal{E}}(z_1))$, then $\mathbf{cl}_{\mathcal{E}}(z_0) \leq \mathbf{cl}_{\mathcal{E}}(\mathbf{cl}_{\mathcal{E}}(\mathbf{int}_{\mathcal{E}}(z_1))) = \mathbf{cl}_{\mathcal{E}}(\mathbf{int}_{\mathcal{E}}(z_1))$ and $\mathbf{int}_{\mathcal{E}}(\mathbf{cl}_{\mathcal{E}}(z_0)) \leq \mathbf{int}_{\mathcal{E}}(\mathbf{cl}_{\mathcal{E}}(\mathbf{int}_{\mathcal{E}}(z_1))) = \mathbf{int}_{\mathcal{E}}(z_1) \leq z_1$. Similarly, if $\mathbf{int}_{\mathcal{E}}(\mathbf{cl}_{\mathcal{E}}(z_0)) \leq z_1$, then $z_0 \leq \mathbf{cl}_{\mathcal{E}}(\mathbf{int}_{\mathcal{E}}(z_1))$. □

Lemma 3. *$J(\mathcal{E}) = Image(f_{\mathcal{E}})$ and $M(\mathcal{E}) = Image(g_{\mathcal{E}})$.*

Proof. Clearly, if $y = \mathbf{int}_{\mathcal{E}}(\mathbf{cl}_{\mathcal{E}}(z))$ for some $z \in L$, then $y \in J(\mathcal{E})$. Conversely, if $y \in J(\mathcal{E})$, then $y = \mathbf{int}_{\mathcal{E}}(\mathbf{cl}_{\mathcal{E}}(y))$, so $y \in Image(f_{\mathcal{E}})$. The other equality is proved similarly. □

For the next proposition, recall that if f, g are adjoint, then $f \circ g \circ f = f$ and $g \circ f \circ g = g$.

Proposition 3. *Let $f \in Q_{\vee}(L)$ be idempotent and let g be its right adjoint. Then*

1. *$y \leq g(y)$, for each $y \in Image(f)$,*
2. *the collection of intervals $\mathcal{E}_f := \{ [y, g(y)] \mid y \in Image(f) \}$ is an emmentaler of L,*
3. *$J(\mathcal{E}_f) = Image(f)$ and $M(\mathcal{E}_f) = Image(g)$.*

Proof. If $y \in Image(f)$, then $y = f(y)$ and therefore the relation $y \leq g(y)$ follows from $f(y) \leq y$. The subset $Image(f)$ is closed under arbitrary joins since f is join-continuous. Similarly, $Image(g)$ is closed under arbitrary meets, since g is meet-continuous. Let us show that $\{ g(y) \mid y \in Image(f) \} = Image(g)$. To this end, observe that if $x = g(z)$ for some $z \in L$, then $x = g(z) = g(f(g(z)))$, so $x = g(y)$ with $y = f(g(z))$.

Finally, let $z \in [y_1, g(y_1)] \cap [y_2, g(y_2)]$. Then $y_i = f(y_i) \leq f(z) \leq f(g(y_i))$. We already observed that $f(g(y_i)) = y_i$, so $y_i = f(z)$, for $i = 1, 2$. We have therefore $y_1 = y_2$ and $g(y_1) = g(y_2)$. □

Lemma 4. *If $f \in Q_\vee(L)$ is idempotent then, for each $x \in L$,*

(1) $\mathbf{int}_{\mathcal{E}_f}(x) \leq f(x)$,
(2) *if* $f(x) \leq x$, *then* $f(x) = \mathbf{int}_{\mathcal{E}_f}(x)$,
(3) *if* $x \in M(\mathcal{E}_f)$, *then* $f(x) \leq x$, *and so* $f(x) = \mathbf{int}_{\mathcal{E}_f}(x)$.

Proof. 1. Recall that $\mathbf{int}_{\mathcal{E}_f}(x) \leq x$ and $\mathbf{int}_{\mathcal{E}_f}(x) \in J(\mathcal{E}_f) = Image(f)$, so $\mathbf{int}_{\mathcal{E}_f}(x)$ is a fixed point of f. Then, using monotonicity, $\mathbf{int}_{\mathcal{E}_f}(x) = f(\mathbf{int}_{\mathcal{E}_f}(x)) \leq f(x)$.

2. From $f(x) \leq x$ and recalling that $\mathbf{int}_{\mathcal{E}_f}(x)$ is the greatest element of $J(\mathcal{E}_f) = Image(f)$ below x, it immediately follows that $f(x) \leq \mathbf{int}_{\mathcal{E}_f}(x)$.

3. Recall that $M(\mathcal{E}_f) = Image(g)$, where g is right adjoint to f. Let z be such that $x = g(z)$, so we aim at proving that $f(g(z)) \leq g(z)$. This is follows from $f(f(g(z))) = f(g(z)) \leq z$ and adjointness. □

Proposition 4. *For each idempotent $f \in Q_\vee(L)$, we have $f = \mathbf{int}_{\mathcal{E}_f} \circ \mathbf{cl}_{\mathcal{E}_f} = f_{\mathcal{E}_f}$.*

Proof. Since $\mathbf{cl}_{\mathcal{E}_f}(z) \in M(\mathcal{E}_f)$, then $f(\mathbf{cl}_{\mathcal{E}_f}(z)) = \mathbf{int}_{\mathcal{E}_f}(\mathbf{cl}_{\mathcal{E}_f}(z))$, by the previous Lemma. Therefore we need to prove that $f(\mathbf{cl}_{\mathcal{E}_f}(z)) = f(z)$. This immediately follows from the relation $\mathbf{cl}_{\mathcal{E}_f} = g \circ f$ that we prove next.

We show that $g(f(z))$ is the least element of $Image(g)$ above z. We have $z \leq g(f(z)) \in Image(g)$ by adjointness. Suppose now that $x \in Image(g)$ and $z \leq x$. If $y \in L$ is such that $x = g(y)$, then $z \leq g(y)$ yields $f(z) \leq y$ and $g(f(z)) \leq g(y) = x$. □

We can now give a proof of the main result of this section, Theorem 1.

Proof of Theorem 1. We argue that the mappings $\mathcal{E} \mapsto f_{\mathcal{E}}$ and $f \mapsto \mathcal{E}_f$ are inverse to each other.

We have seen (Proposition 4) that, for an idempotent $f \in Q_\vee(L)$, $f_{\mathcal{E}_f} = f$. Given an emmentaler $\mathcal{E}$, we have $J(\mathcal{E}) = Image(f_{\mathcal{E}})$ by Lemma 3, and $J(\mathcal{E}_{f_{\mathcal{E}}}) = Image(f_{\mathcal{E}})$, by Proposition 3. Therefore, $J(\mathcal{E}) = J(\mathcal{E}_{f_{\mathcal{E}}})$ and, similarly, $M(\mathcal{E}) = M(\mathcal{E}_{f_{\mathcal{E}}})$. Since the two sets $J(\mathcal{E})$ and $M(\mathcal{E})$ completely determine an emmentaler, we have $\mathcal{E} = \mathcal{E}_{f_{\mathcal{E}}}$. □

4. Idempotent discrete paths

It is easily seen that an emmentaler of the chain $\mathbb{I}_n$ can be described by an alternating sequence of the form

$$0 = y_0 \leq x_0 < y_1 \leq x_1 < y_2 \leq \ldots < y_k \leq x_k = n\,,$$

so $J(\mathcal{E}) = \{ 0, y_1, \ldots, y_k \}$ and $M(\mathcal{E}) = \{ x_1, x_2, \ldots, x_{k-1}, n \}$. Indeed, $J(\mathcal{E})$ is closed under arbitrary joins if and only if $0 \in J(\mathcal{E})$, while $M(\mathcal{E})$ is closed under arbitrary meets if and only if $n \in M(\mathcal{E})$.

The correspondences between idempotents of $Q_\vee(\mathbb{I}_n)$, their paths, and emmentalers can be made explicit as follows: for $y \in J(\mathcal{E})$ such that $y \neq 0$, the path corresponding to $f_{\mathcal{E}}$ touches the point (y, y) coming from the left of the diagonal; for $x \in M(\mathcal{E}) \setminus J(\mathcal{E})$, the path

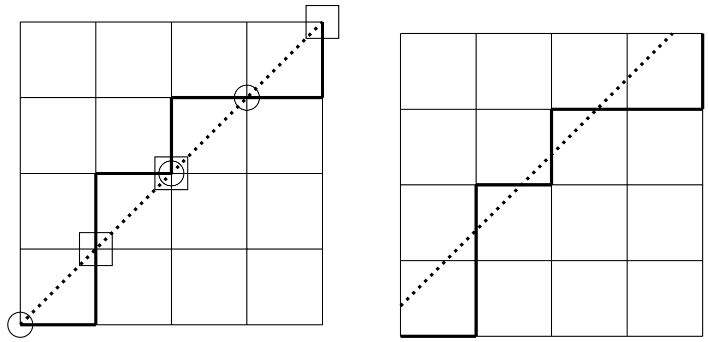

FIGURE 3. Idempotent path corresponding to $\{ 0 < 1 < 2 \leq 2 < 3 < 4 \}$.

corresponding to $f_{\mathcal{E}}$ touches (x, x) coming from below the diagonal. For $\mathcal{E} = \{ 0 < 1 < 2 \leq 2 < 3 < 4 \}$, with $J(\mathcal{E}) = \{ 0, 2, 3 \}$ and $M(\mathcal{E}) = \{ 1, 2, 4 \}$, the path corresponding to $f_{\mathcal{E}}$ is illustrated in Figure 3. On the left of the figure, points of the form (x, x) with $x \in M(\mathcal{E})$ are squared, while points of the form (y, y) with $y \in J(\mathcal{E})$ are circled.

Our next goal is to give a geometric characterization of idempotent paths using their North-East and East-North turns. To this end, observe that we can describe North-East turns of a path $w \in P(n, m)$ by discrete points in the plane. Namely, if if $w = w_0 w_1 \in P(n, m)$ with $w_0 = u_0 y$, $w_1 = x u_1$, and $|w_0| = \ell$ (so w has a North-East turn at position ℓ), then we can denote this North-East turn with the point $(|w_0|_x, |w_0|_y)$. In a similar way, we can describe East-North turns by discrete points in the plane.

Let us call a path an *upper zigzag* if every of its North-East turns is above the line $y = x + \frac{1}{2}$ while every of its East-North turns is below this line. Notice that a path is an upper zigzag if and only every North-East turn is of the form (x, y) with $x < y$ and every East-North turn is of the form (x, y) with $y \leq x$. This property is illustrated on the right of Figure 3.

Theorem 2. *A path $w \in P(n, n)$ is idempotent if and only if it is an upper zigzag.*

The proof of the theorem is scattered into the next three lemmas.

Lemma 5. *An upper zigzag path is idempotent.*

Proof. Let w be an upper zigzag with $\{ (x_i, y_i) \mid i = 1, \ldots, k \}$ the set of its North-East turns. For $(i, j) \in \{ 0, \ldots, n-1 \} \times \{ 1, \ldots, n \}$, let $e_{i,j} := x^i y^j x^{n-i} y^{n-j}$ be the path that has a unique North-East turn at (i, j). Notice that $w = \bigvee_{i=1,\ldots,k} e_{x_i, y_i}$. By equation (1),

$$w \otimes w = (\bigvee_{i=1,\ldots,k} e_{x_i,y_i}) \otimes (\bigvee_{j=1,\ldots,k} e_{x_j,y_j}) = \bigvee_{i,j=1,\ldots,k} e_{x_i,y_i} \otimes e_{x_j,y_j} \,. \tag{3}$$

It is now enough to observe that $e_{a,b} \otimes e_{c,d} = e_{a,d}$ if $c < b$ and, otherwise, $e_{a,b} \otimes e_{c,d} = \bot$, where $\bot = x^n y^n$ is the least element of $P(n, n)$. Therefore, we have: (i) $e_{x_i,y_i} \otimes e_{x_i,y_i} = e_{x_i,y_i}$, since $x_i < y_i$, (ii) if $i < j$, then $e_{x_i,y_i} \otimes e_{x_j,y_j} = \bot$, since $y_i \leq x_j$, (iii) if $j < i$, then $e_{x_i,y_i} \otimes e_{x_j,y_j} = e_{x_i,y_j}$, since $x_j < y_j \leq y_i$; in the latter case, we also have $e_{x_i,y_j} \leq e_{x_i,y_i}$, since $y_j \leq y_i$. Consequently, the expression on the right of (3) evaluates to $\bigvee_{i=1,\ldots,k} e_{x_i,y_i} = w$. □

Next, let us say that $i \in \mathbb{I}_n \setminus \{ n \}$ is an *increase* of $f \in Q_\vee(\mathbb{I}_n, \mathbb{I}_m)$ if $f(i) < f(i+1)$. It is easy to see that the set of North-East turns of w is the set $\{ (i, \mathbf{blno}^{w,x}_{i+1}) \mid i$ is an increase of $\mathbf{blno}^{w,x} \}$.

Lemma 6. *Let $f \in Q_\vee(\mathbb{I}_n, \mathbb{I}_n)$ and let g be its right adjoint. Then $i \in \mathbb{I}_n \setminus \{ n \}$ is an increase of f if and only if $i \in Image(g) \setminus \{ n \}$.*

Proof. Suppose $i = g(j)$ for some $j \in \mathbb{I}_n$. If $f(i+1) \leq f(i)$, then $i + 1 \leq g(f(i)) = g(f(g(j))) = g(j) = i$, a contradiction. Therefore $f(i) < f(i+1)$.

Conversely, if $f(i) < f(i+1)$, then $f(i+1) \not\leq f(i)$, $i+1 \not\leq g(f(i))$, and $g(f(i)) < i+1$. Since $i \leq g(f(i))$, then $g(f(i)) = i$, so $i \in Image(g)$. □

Lemma 7. *The North-East turns of an idempotent path $w \in P(n,n)$ corresponding to the emmentaler $\{ 0 = y_0 \leq x_0 < y_1, \ldots y_k \leq x_k = n \}$ of $\mathbb{I}_n$ are of the form $(x_\ell, y_{\ell+1})$, for $\ell = 0, \ldots, k-1$. Its East-North turns are of the form (x_ℓ, y_ℓ), for $\ell = 0, \ldots, k$. Therefore w is an upper zigzag.*

Proof. For the first statement, since $Image(g_\mathcal{E}) = \{ x_0, \ldots, x_{k-1}, n \}$ and using Lemma 6, we need to verify that $f_\mathcal{E}(x_\ell) = y_\ell$: this is Lemma 4, point 3. The last statement is a consequence of the fact that East-North turns are computable from North-East turns: if (x_i, y_i), $i = 1, \ldots, k$, are the North-East turns of w, with $x_i < x_j$ and $y_i < y_j$ for $i < j$, then East-North turns of w are of the form $(x_1, 0)$ (if $x_1 > 0$), (x_{i+1}, y_i), $i = 1, \ldots, k-1$, and (n, y_k) (if $y_k < n$). □

5. Counting idempotent discrete paths

The goal of this section is to exemplify how the characterizations of idempotent discrete paths given in Section 4 can be of use. It is immediate to establish a bijective correspondence between emmentalers of the chain $\mathbb{I}_n$ and words $w = w_0 \ldots w_n$ on the alphabet $\{ \underline{1}, 0, 1 \}$ that avoid the pattern $\underline{1}0^*\underline{1}$ and such that $w_0 = 1$ and $w_n \in \{ 1, \underline{1} \}$; this bijection can be exploited for the sake of counting. We prefer to count idempotents using the characterization given in Theorem 2. In the following, we provide a geometric/combinatorial proof of counting results [16, 13] for the number of idempotent elements in the monoid $Q_\vee(\mathbb{I}_n)$ and, also, in the monoid O_n of order preserving maps from $\{ 1, \ldots, n \}$ to itself. Let us recall that the Fibonacci sequence is defined by $f_0 := 0$, $f_1 := 1$, and $f_{n+2} := f_{n+1} + f_n$. Howie [13] proved that $\phi_n = f_{2n}$ (for $n \geq 1$), where ϕ_n is the number of idempotents in the monoid O_n. Laradji and Umar [16] proved that $\gamma_n = f_{2n-1}$ ($n \geq 1$), where now γ_n is the number of idempotent elements of O_n^n, the submonoid in O_n of maps fixing n. Clearly, O_n^n is a monoid isomorphic (and anti-isomorphic as well) to $Q_\vee(\mathbb{I}_{n-1})$. We infer that the number ψ_n of idempotents in the monoid $Q_\vee(\mathbb{I}_n)$ equals f_{2n+1} (for $n \geq 0$).

Remark 5. It is argued in [13] that $\phi_n = \frac{1}{2^n\sqrt{5}}\{(3+\sqrt{5})^n - (3-\sqrt{5})^n\}$, which can easily be verified using the fact that $f_n = \frac{\theta_0^n - \theta_1^n}{\theta_0 - \theta_1}$ with $\theta_0 = \frac{1+\sqrt{5}}{2}$ and $\theta_1 = \frac{1-\sqrt{5}}{2}$, see [7]. In a similar way, we derive the following explicit formula:

$$\psi_n = \frac{1}{2^{n+1}\sqrt{5}}\{(3+\sqrt{5})^n(1+\sqrt{5}) - (3-\sqrt{5})^n(1-\sqrt{5})\}\,.$$

Let us observe that the monoid O_n can be identified with the submonoid of $Q_\vee(\mathbb{I}_n)$ of join-continuous maps f such that $1 \leq f(1)$. A path corresponds to such an f if and only if its first step is a North step. Having observed that $\psi_0 = \phi_1 = 1$, the following proposition suffices to assert that $\phi_n = f_{2n}$ and $\psi_n = f_{2n+1}$.

Proposition 5. *The following recursive relations hold:*

$$\phi_{n+1} = \psi_n + \phi_n\,, \qquad\qquad \psi_{n+1} = \phi_{n+1} + \psi_n\,.$$

Proof. Every discrete path from $(0,0)$ to $(n+1, n+1)$ ends with y—that is, it visits the point $(n+1, n)$—or ends with x—that is, it visits the point $(n, n+1)$. Consider now an upper zigzag path π from $(0,0)$ to $(n+1, n+1)$ that visits $(n+1, n)$, see Figure 4. By clipping on

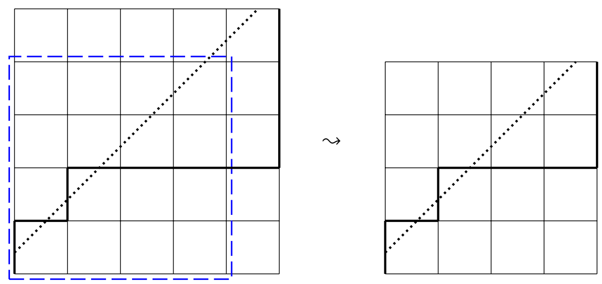

Figure 4. An upper zigzag path to $(5,5)$ ending with y.

the rectangle with left-bottom corner $(0,0)$ and right-up corner (n,n), we obtain an upper zigzag path π' from $(0,0)$ to (n,n). If π starts with y, then π' does as well. This proves the right part of the recurrences above, i.e. $\phi_{n+1} = \ldots + \phi_n$ and $\psi_{n+1} = \ldots + \psi_n$.

Consider now an upper zigzag path π ending with x, see Figure 5. The reflection along

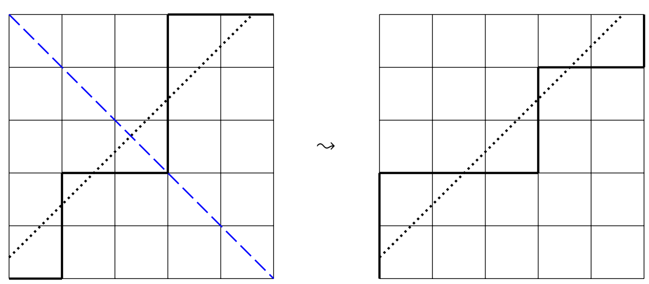

Figure 5. An upper zigzag path to $(5,5)$ ending with x.

the line $y = n - x$ sends (x,y) to $(n-y, n-x)$, so it preserves upper zigzag paths. Applying this reflection to π, we obtain an upper zigzag path from $(0,0)$ to $(n+1, n+1)$ whose first step is y. This proves the $\psi_{n+1} = \phi_{n+1} + \ldots$ part of the recurrences above.

Consider now an upper zigzag path π ending with x and beginning with y, see Figure 6. By clipping on the rectangle with left-bottom corner $(0,1)$ and right-up corner $(n, n+1)$ and then by applying the translation $x \mapsto x-1$, we obtain a path whose all North-East turns are above the line $y = x - \frac{1}{2}$ and whose all East-North turns are below this line. By reflecting along diagonal, we obtain an upper zigzag path from $(0,0)$ to (n,n). This proves the $\phi_{n+1} = \psi_n + \ldots$ part of the recurrences above. □

The geometric ideas used in the proof of Proposition 5 can be exploited further, so to show that the number of idempotent maps $f \in Q_\vee(\mathbb{I}_n)$ such that $f(n) = k$ equals f_{2k}, see the analogous statement in [16, Corollary 4.5]. Indeed, if $f(n) = k$, then the path corresponding to f visits the points $(n-1, k)$ and (n,k); therefore, since it is an upper zigzag, also the points $(k-1,k)$ and (k,k). By clipping on the rectangle from $(0,0)$ to (k,k), we obtain an upper zigzag path in $P(k,k)$ ending in x. As seen in the proof of Proposition 5, these paths bijectively correspond to upper zigzag paths in $P(k,k)$ beginning with y.

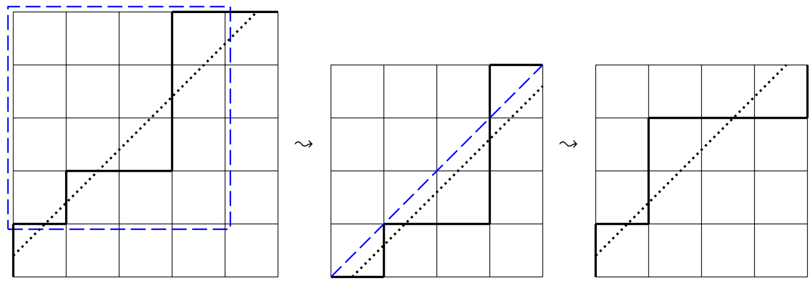

FIGURE 6. An upper zigzag path to $(5, 5)$ ending with x and beginning with y.

6. CONCLUSIONS

We have presented the monoid structure on the set $P(n, n)$ of discrete lattice paths (with North and East steps) that corresponds, under a well-known bijection, to the monoid $Q_\vee(\mathbb{I}_n)$ of join-continuous functions from the chain $\{ 0, 1, \ldots n \}$ to itself. In particular, we have studied the idempotents of this monoid, relying on a general characterization of idempotent join-continuous functions from a complete lattice to itself. This general characterization yields a bijection with a language of words on a three letter alphabet and a geometric description of idempotent paths. Using this characterization, we have given a geometric/combinatorial proof of counting results for idempotents in monoids of monotone endomaps of a chain [13, 16].

Our initial motivations for studying idempotents in $Q_\vee(\mathbb{I}_n)$ originates from the algebra of logic, see e.g. [14]. Willing to investigate congruences of $Q_\vee(\mathbb{I}_n)$ as a residuated lattice [10], it can be shown, using idempotents, that every subalgebra of a residuated lattice $Q_\vee(\mathbb{I}_n)$ is simple. This property does not generalize to infinite complete chains: if $\mathbb{I}$ is the interval $[0, 1] \subseteq \mathbb{R}$, then $Q_\vee(\mathbb{I})$ is simple but has subalgebras that are not simple [1]. Despite the results we presented are not related to our original motivations, we aimed at exemplifying how a combinatorial approach based on paths might be fruitful when investigating various kinds of monotone maps and the multiple algebraic structures these maps may carry.

We used the Online Encyclopedia of Integer Sequences to trace related research. In particular, we discovered Howie's work [13] on the monoid O_n through the OEIS sequences A001906 and A088305. The sequence ψ_n is a shift of the sequence A001519. Related to this sequence is the doubly parametrized sequence A144224 collecting some counting results from [16] on idempotents. Relations with other kind of combinatorial objects counted by the sequence ψ_n still need to be understood.

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