# A Study of @-numbers.

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**Abstract**

This paper deals more generally with @-numbers defined as follows. Call '*alpha number* of order $k$', (denote its family by $@_{k;\mathbb{N}}$) any positive integer $n$ satisfying $f_k(n) := (\alpha_1/\alpha_2)n$ with $f_k(n) := \lfloor|\sigma_k(n)|\rfloor$, arbitrary pair integers $\alpha_1, \alpha_2$ is such that $1 < \alpha_1, \alpha_2 \leq \tau(n)$ where $\tau(n)$ is the number of factors of $n$, and $\sigma_k(n)$ is the sum of divisors function of $n$. We give some examples and conjecture that there is no odd alpha number of integral order above 1, which implies that there is no odd perfect, multiperfect or Ore's harmonic number greater than 1. In this paper, using Rossen, Schonfield and Sandor's inequalities, in addition to the aforementioned definition, we also provide a form for odd @-numbers, and remark that this form can be improved towards solving the conjectures of this paper. Some areas for future research are also pointed out as recommendations.

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## 1 Introduction

Throughout, let $\sigma_x(n)$, $\omega(n)$. $\Omega(n)$ and $\tau(n)$ represent the sum of positive divisors function of $n$, the number of distinct prime factors of $n$, number of prime divisors of $n$, and the number of distinct divisors of $n$ respectively. By definition, for every positive integer $n$ and complex number $x$,

$$\sigma_x(n) = \sum_{d|n} d^x, \quad \Omega(n) = \sum_{p|n} 1 \quad and \quad \tau(n) = \sum_{d|n} 1$$

Note that traditionally when $x = 1$, we drop the subscript and simply write $\sigma(n)$ (called the sigma function) which now represents the sum of the factors of $n$, including $n$ itself. For example, the sum of positive divisors of $n = p^\alpha$ is $\sigma(p^\alpha) = 1 + p + ... + p^\alpha$ while it has $\Omega(n) = \alpha$ prime factors, $\omega(n) = 1$ distinct prime factor, and $\tau(n) = \alpha + 1$ number of distinct factors. Then, we call a positive integer $n \in \mathbb{N}$, *alpha number* of order $k$, and denote its family by $@_{k;\mathbb{N}}$ if it satisfies

$$f_k(n) := \frac{\alpha_1}{\alpha_2} n \tag{1}$$

where $f_k(n) := \lfloor|\sigma_k(n)|\rfloor$ represents the 'integral sum function of order $k$' of positive divisors of $n$, $\alpha_1, \alpha_2$ are arbitrary positive integers such that $1 < \alpha_1, \alpha_2 \leq \tau(n)$, $|\ |$ is the modulus function and $\lfloor\ \rfloor$ represents the floor function. Meaning that $n \in @_{k;\mathbb{N}}$ reads "an integral alpha number $n$ of order $k$", and for completion sake, concerning those $n$s which do not satisfy (1), when the choice of $\alpha_1$ (and $\alpha_2$) is strictly bounded by 1 and $\tau(n)$ implies $f_k(n) < \frac{\alpha_1}{\alpha_2}n$ we say that $n$ is being 'alpha deficient' and when the choice of $\alpha_1$ (and even $\alpha_2$) changes but is again bounded in the same manner implies $f_k(n) > \frac{\alpha_1}{\alpha_2}n$, we say at that point that $n$ is 'alpha abundant'. Moreover, we relax the strict condition $1 < \alpha_1, \alpha_2 \leq \tau(n)$ attached to alpha numbers and refer to any positive integer $n$ which satisfy (1) when $\alpha_1 \in \mathbb{N}$ and $\alpha_2 = 1$ as a 'partial alpha number of order $k$' and denote its set by $@^p_{k;\mathbb{N}}$. We shall return to this later. The major interest yet in number theory is the existence of even and odd of such numbers, as the application of special numbers is still a puzzle, though a golden challenge in Mathematics. Truly, even alpha numbers exist as we shall soon see various examples in this paper, but no odd alpha number above 1, in particular, of order 1 is known. In what follows, we begin with some examples of @-numbers.

## 2 Examples of @-numbers

Note that we shall extract some @-numbers from the table of the sum of positive divisors function given below:

| $n$ | $\sigma_0(n) = \tau(n)$ | $\sigma_1(n)$ | $\sigma_2(n)$ | $\sigma_{0.5}(n)$ | $\sigma_{\sqrt{-1}}(n)$ | $\lfloor\lvert\sigma_{0.5}(n)\rvert\rfloor$ | $\lfloor\lvert\sigma_{\sqrt{-1}}(n)\rvert\rfloor$ |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 2 | 3 | 5 | 2.4142 | 1.7692+ 0.6390i | 2 | 1 |
| 3 | 2 | 4 | 10 | 2.7321 | 1.4548+ 0.8906i | 2 | 1 |
| 4 | 3 | 7 | 21 | 4.4142 | 1.9527+ 1.6220i | 4 | 2 |
| 5 | 2 | 6 | 26 | 3.2361 | 0.9614+ 0.9993i | 3 | 1 |
| 6 | 4 | 12 | 50 | 6.5959 | 2.0049+ 2.5052i | 6 | 3 |
| 7 | 2 | 8 | 50 | 3.6458 | 0.6336+ 0.9305i | 3 | 1 |
| 8 | 4 | 15 | 85 | 7.2426 | 1.466+ 2.4954i | 7 | 2 |
| 9 | 3 | 13 | 91 | 5.7321 | 0.8686+ 1.7007i | 5 | 1 |
| 10 | 4 | 18 | 130 | 7.8126 | 1.0624+ 2.3822i | 7 | 2 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 24 | 8 | 60 | 850 | 19.787 | -0.0899+ 4.936i | 19 | 4 |
| 25 | 3 | 31 | 651 | 8.236 | -0.0356+ 0.922i | 8 | 0 |
| 26 | 4 | 42 | 850 | 11.118 | -0.0623+1.068i | 11 | 1 |
| 27 | 4 | 40 | 820 | 10.928 | -0.1200+ 1.547i | 10 | 1 |
| 28 | 6 | 56 | 1050 | 16.03 | -0.2719+ 2.845i | 16 | 2 |
| 29 | 2 | 30 | 842 | 6.385 | 0.0025-0.224i | 6 | 0 |
| 30 | 8 | 72 | 1300 | 21.344 | -0.5759+ 4.412i | 21 | 4 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

Table 1.

From the above table we give some practical examples of alpha numbers.

**Examples of @-numbers of order 1**:

| @-numbers | order $k$ | $\alpha_1$ | $\alpha_2$ |
|---|---|---|---|
| $n = 1$ | 1 | 1 | 1 |
| $n = 6$ | 1 | 2 | 1 |
| $n = 24^*$ | 1 | 5 | 2 |
| $n = 28$ | 1 | 2 | 1 |

Table 2.

Note that the asterisk on Table 2 refers to the alpha numbers that are neither multiperfect nor Ores harmonic number.

Note that every $n$ in Table 2. is alpha number of order $1$ since they all satisfy relationship (1) for their respective values of $\alpha_1$ and $\alpha_2$ given in the table when $k = 1$.

**Examples of @-numbers of order 2**:

| @-numbers | order $k$ | $\alpha_1$ | $\alpha_2$ |
|---|---|---|---|
| $n = 1$ | 2 | 1 | 1 |

Table 3.

Note that in Table 3. only $n = 1$ is the alpha number of order 2.

**Examples of @-numbers of order 0.5**:

| @-numbers | order $k$ | $\alpha_1$ | $\alpha_2$ |
|---|---|---|---|
| $n = 1$ | 0.5 | 1 | 1 |
| $n = 2^*$ | 0.5 | 1,2 (in order) | 1,2 (in order) |
| $n = 4^*$ | 0.5 | 1,2,3 (in order) | 1,2,3 (in order) |
| $n = 6$ | 0.5 | 1,2,3,4 (in order) | 1,2,3,4 (in order) |

Table 4.

Note that the asterisk on Table 4 refers to the alpha numbers that are neither multiperfect nor Ores harmonic number.

Note that every n in Table 4. is alpha number of order $0.5$ since they all satisfy relationship (1) for their respective values of $\alpha_1$ and $\alpha_2$ when $k = 0.5$:

**Examples of @-numbers of order** $i = \sqrt{-1}$:

| @-numbers | order $k$ | $\alpha_1$ | $\alpha_2$ |
|---|---|---|---|
| $n = 1$ | $i$ | 1 | 1 |
| $n = 2^*$ | $i$ | 1 | 2 |
| $n = 4^*$ | $i$ | 1 | 2 |
| $n = 6$ | $i$ | 1 | 2 |

Table 5.

Note that the asterisk on Table 5 refers to the alpha numbers that are neither multiperfect nor Ores harmonic number.

Note that every $n$ in Table 5 above is alpha number of order $i$ since they all satisfy relationship (1) for their respective values of $\alpha_1$ and $\alpha_2$ when $k = i$.

**Examples of @$^p$-numbers of integral order** $k = 2$:

| @-numbers | order $k$ | $\alpha_1$ | $\alpha_2$ |
|---|---|---|---|
| $n = 1$ | 2 | 1 | 1 |
| $n = 10^*$ | 2 | 13 | 1 |

Table 6.

Note that the asterisk on Table 6 refers to the alpha numbers that are neither multiperfect nor Ores harmonic number.

Note that 1 and 10 are the @-numbers of integral order $k = 2$ as well as the above examples for which $\alpha_2 = 1$.

**Remark 1.** *From the above examples, one can easily observe that there is no singe odd alpha number of integral order strictly between* 1 *and* 30*, and so, we formally state the following:*

**Conjecture 1.** $@_{1;2\mathbb{N}+1} = \emptyset$.

**Conjecture 2.** $@^p_{1;2\mathbb{N}+1} = \emptyset$

**Conjecture 3.** $@_{k;2\mathbb{N}}$ *has infinite cardinality for all* $k \in \mathbb{C}$

**Remark 2.** *In passing, we mention that not much has been done on the sum of the divisors function* $\sigma_k(n)$ *defined at the onset of this paper, especially when order* $k$ *is a non-real complex number. At most, few results found in the literature are akin to the following identities on* Dirichilet's *series due to* Hardy and Ramanujan [3]

$$\zeta(s)\zeta(s-a) = \sum_{i=1}^{\infty} \frac{\sigma_a(n)}{n^s}, \quad s - a > 1$$

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{i=1}^{\infty} \frac{\sigma_a(n)\sigma_a(n)}{n^s}, \quad s-a, s-b, s-a-b > 1$$

*(which are* Theorem 291 *and* Theorem 305 *of* [3] *respectively) and* Lambart's series [6] *which holds that*

$$\sum_{i=1}^{\infty} q^n \sigma_\alpha(n) = \sum_{i=1}^{\infty} \frac{n^\alpha q^n}{1-q^n}, \quad \alpha, q \in \mathbb{C}, \ |q| \leq 1.$$

*Now that the function* $\sigma_k(n)$ *is connected to function* $f_k(n)$ *the function for the integral sum of divisors function defined in this paper, it is therefore imperative to further study the sum of positive divisors function.*

**Remark 3.** *We also note that the function $f_k(n)$ defined for @-numbers shares attributes with $\sigma_k(n)$ when $n$ is a positive integer, thus the identities obtained for $\sigma_k(n)$ in* Remark 2 *also hold for $f_k(n)$ when $n$ is a positive integer; moreover, $f_k(n)$ is multiplicative when $k \in \mathbb{N}$ since $\sigma_k(n)$ is also multiplicative, so in essence*

$$f_k(\prod_{i=1}^{\omega(n)} p_i^{n_i}) = \prod_{i=1}^{\omega(n)} f_k(p_i^{n_i})$$

Remarkably, the study of integers with special properties defined above is of antiquity, as it dates back to the time of Euclid (≈300BC), Nicomachus (≈100 AD), Descartes (≈1600AD) and Sylvester (≈1800AD), according to Dickson [2] and Sándor [12]. The case $k = 1, \alpha_1 = 2,\ \alpha_2 = 1$ in (1) corresponds to perfect numbers; the case $k = 1, \alpha_1 \geq 2, \alpha_2 = 1$ in (1) refers to multiply perfect numbers ([12]) and also; the case $k = 1, \alpha_1 = \tau(n)$, and $\alpha_2$ is the harmonic mean of divisors of $n$ in (1) clearly defines the (Ore's) harmonic numbers, since Ore's definition of harmonic mean $k$ of positive integer $n$ is given by

$$k = \frac{\tau(n)}{\sum_{q_i|n} \frac{1}{q_i}} \Rightarrow \frac{\sigma(n)}{n} = \frac{\tau(n)}{k}$$

(see [8] & [12]). Thus, by the definition, every perfect number, every $k$- multi perfect number for which $k \leq \tau(n)$ and every Ore's number are @-numbers.

By implication, the even perfect numbers (OEIS A000396), even multiply perfect numbers (OEIS A007539, A005820, A027687) and even Ore's harmonic numbers (OEIS A001599) are special cases of even @-numbers. However, odd perfect number conjecture suggests that there is no odd perfect number; likewise, it is not known whether odd multiply perfect number exist or not; and in 1948, Ore conjectured that no odd harmonic divisor number exists except 1 (see [7], [8] & [12]).

Towards solving these problems, Euler formally presented a form for such odd perfect numbers which holds that every perfect odd $n$ should have the representation

$$n = p^{\alpha} \prod_{i=1}^{r} q_i^{2\beta_i}$$

where $p \equiv \alpha \equiv 1(mod4)$, $p$ and $q_i$ are distinct prime numbers, see [12]. This restriction for perfect numbers enabled Steurwald to establish that $n$ cannot be perfect if $\beta_1 = ... = \beta_r = 1$. This result was later improved by Brauer, Kanold, Hagis, McDaniel, Iannuci, Kanold, McCarthy, Robbins, Pomerance, Chen, and Condict via the number, the size, the bounds and the density of the factors of $n$ and $n$ itself (as recorded in [12]).

Moreover, the computer verification by mathematicians like Ochem, Rao, te Riele and others has established that up to $10^{1500}$ no such odd number([5]) and by building on the computational evidence that any perfect number $\geq 10^{300}$, Pomerance (*OddPerfect.org*), has presented a heuristic argument supporting this assertion. In the next session, we shall investigate the square-free, and prime-power form of odd integers with respect to the conditions of alpha number.

# 3 Odd @-numbers:

Recall that

$$\sigma(n) = \sum_{d|n} d, \ and \ \ \tau(n) = \sum_{d|n} 1$$

Obviously, $\sigma(.)$ is a multiplicative function and the implication of this arithmetic property is the following result in [1], [3], [7] and [13]: For every $n = \prod_{i=1}^{\omega(n)} p_i^{n_i},\ n_i \in \mathbb{N}$ and $p_i$ is prime, $\sigma(n) = \sigma(p_1^{n_1})\sigma(p_2^{n_2}) \cdots \sigma(p_{\omega(n)}^{n_{\omega(n)}})$ and $\tau(n) = \sigma_0(n) = \prod_{i=1}^{\omega(n)}(n_i+1)$. Thus, $\tau(p_1 \cdot p_2 \cdots p_{\omega(n)}) = 2^{\omega(n)}$. Also, note that some important consequences of arithmetic functions such $\phi(n)$ and $\sigma(n)$ which are germain in this paper are the following:

**Lemma 3.1.** (J.B. Rosser and L. Schenfield Theorem [9])
*If $n \geq 3$, then $\frac{n}{\phi(n)} < e^{\gamma} \log\log n + \frac{0.6483}{\log\log n}$ where $\gamma$ is the* Euler constant.

**Lemma 3.2.** (J. sándor [10])
*There is a constant $C > 0$ such that $\frac{n}{\phi(n)} < C \cdot \log\log \phi(n)\ \forall\ n > 3$.*

In this session we keep to the standard notations of the set of numbers such as $\mathbb{N}$ for the set of natural numbers, $\mathbb{Z}_{>0}$ for the set of integers above zero, $\mathbb{C}$ for the set of complex numbers. Then, we proceed to the main results of this paper.

**Theorem 3.3.** *Let $@_{1;2\mathbb{N}+1} \neq \emptyset$ such that $n \in @_{1;2\mathbb{N}+1}$, then $n = \prod_{i=1}^{\omega(n)} p_i^{n_i}$ and $n$ satisfies* (1) *where*

*(i)* *distinct prime $p_i \geq 3\ \forall\ i$ and $n_i \geq 2$ for some or all $i \in \{1, 2, ..., \omega(n)$,*

*(ii)* $\omega(n) \geq 2$,

*(iii)* *the arbitrary $\alpha_1, \alpha_2 \in \mathbb{Z}_{>0}$, $1 < \alpha_2 < \alpha_1 \leq \tau(n)$ and implies $1 \leq \alpha_2' < \alpha_1' \leq \tau(n)$ with $\gcd(\alpha_2', \alpha_1') = 1$, $\alpha_1' \mid \alpha_1$ and $\alpha_2' \mid \alpha_2$,*

*(iv)* *at least a point in $\{\sigma(p_1^{n_1}), \sigma(p_2^{n_2}), ..., \sigma(p_{\omega(n)}^{n_{\omega(n)}})\}$ is not prime whenever $\Omega(n) - \omega(n) \geq \omega(n)$, and*

*(v)* *$\frac{lin\ \alpha_1 - lin\ \alpha_2}{\omega(n)} < lin\ 2$ and it implies that $\alpha_1 < 2^{\omega(n)}\alpha_2 \leq \tau(n)$ or $\alpha_1 \leq \tau(n) \leq 2^{\omega(n)}\alpha_2$, furthermore, $\alpha_1 < \alpha_2(e^{\gamma} \log\log n + \frac{0.6483}{\log\log n})$, $\gamma$ is the* Euler constant,
*and*

*(vi)* *$n \leq C \log\log \phi(n)$, $C > 0$ is a constant.*

To establish the above Theorem, we shall investigate each of the cases independently as follows:

*Proof.* For *Case (i)*: The condition $p_i \geq 3 \ \forall \ i$ is a direct consequence of the definition of odd integers and the unique prime factorization theorem. Now, consider an odd square-free $n = \prod_{i=1}^{\omega(n)} p_i$ which satisfies equation (1), where $p_i$ is a distinct odd prime number for every $i$ and pair $\alpha_1, \alpha_2 \in \mathbb{Z}^+$ such that $\alpha_1 \leq \tau(n)$. From (1), we obtain

$$\alpha_1 = \frac{\alpha_2 \prod_{i=1}^{\omega(n)} \sigma(p_i)}{\prod_{i=1}^{\omega(n)} p_i} \tag{2}$$

which implies that $\prod_{i=1}^{\omega(n)} p_i \mid \alpha_2 \prod_{i=1}^{\omega(n)} \sigma(p_i)$, and also implies that $p_i \mid \alpha_2 \prod_{i=1}^{\omega(n)} \sigma(p_i) \ \forall \ i$, since $\alpha_1 \in \mathbb{Z}^+$. Then, observe that $\sigma(p_i) \in 2\mathbb{Z}^+$ with $\gcd(p_i, \sigma(p_i)) = 1 \ \forall \ i$ and set $p_i < p_{i+1} \ \forall \ i \in \{1, 2, ..., \omega(n) - 1\}$, thus follows that $p_{\omega(n)} \nmid \sigma(p_i) \ \forall \ i \in \{1, 2, ...\omega(n)\}$.
Consequently, $\alpha_1 > 2^{\omega(n)}$, since $p_{\omega(n)}$ must now divide $\alpha_2$ (an implication of Euclid's Lemma which asserts that if an integral prime $p \mid ab$ where $a, b \in \mathbb{Z}^+$, then $p \mid a \ or \ p \mid b$). Now that our initial claim is $\alpha_1 \leq \tau(n) = 2^{\omega(n)}$, a contradiction is thus obtained. Hence, Case (i) follows.

□

*Proof.* Case (ii): The proof is direct, since if such odd number $n = p^\alpha$ with $\dfrac{\sigma(n)}{n} = \dfrac{\alpha_1}{\alpha_2}$ and $\alpha_1 \leq \tau(n)$ exists, it would inevitably lead to a contradiction. Bluntly, $\sigma(p^\alpha) = 1 + p + ... + p^\alpha$ is co-prime to $p$, so to $p^\alpha$, therefore the fraction $\sigma(n)/n$ is reduced. Should this equal $\alpha_1/\alpha_2$, then $\sigma(n)$ divides $\alpha_1$. In turn, this leads to $\sigma(n) \leq \alpha_1 \leq \tau(n)$, so $1 + p + ... + p^\alpha \leq \alpha + 1$, a contradiction. □

*Proof.* *Case (iii)* follows from the definition of alpha numbers (since $\sigma(n) > n \ \forall \ n > 1$ such that integral $n$ satisfies (1)). □

To establish case (iv), we quickly recall that the total number of prime factors of $n = \prod_{i=1}^{\omega(n)} p_i^{n_i}$ is given as $\Omega(n) = \sum_{i=1}^{\omega(n)} n_i$ (see the notations in [11]).

*Proof.* For Case (iv): On the contrary, let $\sigma(p_i^{n_i})$ be prime for all $i \in \{1, 2, ..., \omega(n)\}$ when $\Omega(n) - \omega(n) \geq \omega(n)$ and (1), (i), and (ii) also hold. Then, its consequence which is

$$\alpha_2 = \frac{\alpha_1 \prod_{i=1}^{\omega(n)} p_i^{n_i}}{\prod_{i=1}^{\omega(n)} \sigma(p_i^{n_i})} \geq 3^{\frac{\Omega(n)}{2}} \geq \prod_{i=1}^{\omega(n)} (n_i + 1) = \tau(n) \tag{3}$$

clearly contradicts assertion (iii) above, and thus concludes case (iv). □

*Proof.* For *Case (v)*, it is sufficient to claim and show that $\frac{\alpha_1}{\alpha_2} < 2^{\omega(n)}$, and then applying Lemma 3.1. So, to achieve this goal, we first recall that if $n = \prod_{i=1}^{\omega(n)} p_i^{n_i}$, then $\sigma(n) = \prod_{i=1}^{\omega(n)} \frac{p^{n_i+1}-1}{p-1}$, thus from (1)

$$\frac{\alpha_1}{\alpha_2} = \frac{1}{n} \prod_{i=1}^{\omega(n)} \frac{p_i^{n_i+1} - 1}{p_i - 1} < \prod_{i=1}^{\omega(n)} \frac{p_i}{p_i - 1} < 2^{\omega(n)} \tag{4}$$

as claimed. □

*Proof.* For *Case (vi)*: From the proof of *Case (v)*, it is obvious that $\frac{\sigma(n)}{n} < \frac{n}{\phi(n)}$ since Euler product formular holds that $\phi(n) = n\prod_{p|n}(1-\frac{1}{p})$. Thus *Case (vi)* follows by applying Lemma 3.2. Hence, Theorem 3.3 completely holds. □

**Remark 4.** *Alternatively, Case (i) can be established thus: Since $p_i$ are all odd, the product $\prod_i^{\omega(n)} p_i$ must be odd. Moreover, no integral prime, say $p_\star$, dividing $p_1+1$ can divide the product $\prod_i^{\omega(n)} p_i$, provided $p_1 < p_2 < ... < p_{\omega(n)}$. Thus, recalling that $n$ is square-free, $\alpha_1$ must be divisible by $p_\star \cdot 2^{\omega(n)} > 2^{\omega(n)} = \tau(n)$, contrary to $\alpha_1 \leq \tau(n)$. Therefore, Case (ii) follows.*

**Remark 5.** *Note that* Theorem 3.3 *implies that there is no square-free alpha number and prime-power alpha number via its assertions (i) and (ii) respectively.*

In what follows, we extend the above results to odd partial @-numbers of natural order $k$, examples of which are multiperfect numbers given in Flammenkamp' s record of multiperfect numbers in OEIS A007691 (see [8] and [9]). In the light of Theorem 3.3 above, we establish the following result:

**Theorem 3.4.** *Every partial @-number $n > 1$ of order $k \in \mathbb{N}$ satisfies* (1) *when $\alpha_2 = 1$, $2 \leq \alpha_1 < n^{k-1}\tau(n)$ and $\alpha_1 \in \mathbb{N}$; and hence, there is neither odd prime-power partial @-number of order $k \in \mathbb{N}$ nor square-free partial @-number of order $k = 1$.*

*Proof.* The first part of Theorem 3.4 is a consequence of the definition of partial @-numbers, since $\sigma_k(n) < n^k\tau(n)$ for every partial @-number $n > 1$ of order $k \in \mathbb{N}$ that satisfies (1). Now, note that for any prime-power partial @-number $p^\alpha$ satisfying (1), $\gcd(p^\alpha, \sigma_k(p^\alpha)) = 1 \ \forall \ k \in \mathbb{N}$ and it implies that $\alpha_1 \notin \mathbb{N}$, and also setting $p_1 < p_2 < ... < p_{\omega(n)}$ for such odd square-free $n = p_1 \cdot p_2 \cdots p_{\omega(n)}$ in (1) where $\alpha_2 = 1$ implies $p_{\omega(n)} \nmid \sigma(p_i) \ \forall \ i \in \{1, 2, ..., \omega(n)\}$ and further implies $\alpha_1 = \sigma(n)/n \notin \mathbb{N}$, both contradicting our initial condition for partial @- numbers. Hence, such numbers exist with a contradiction.

□

**Remark 6.** *Note that we can as well easily see that there is no square free multiperfect number, since if $n = \prod_{i=1}^{\omega(n)} p_i$ is multiperfect, by the definition of multiperfect numbers, it must divide $\prod_{i=1}^{\omega(n)} \frac{p_i+1}{2} < n$, a contradiction.*

**Remark 7.** *Note also that the upper bound for $\alpha_1$ in* Theorem 3.4 *is very weak, and so, it can be improved towards attacking Conjecture 2 of this paper.*

In continuation, we emphasize that among other arithmetic properties of $\sigma(n)$ that is important and that has been investigated is abundancy index, at least, according to the following conclusion by Laatch in [4]:

> *"The abundancy index as a hierarchical classification of numbers is an interesting concept in its own right-at least in parts of its recreational value when used to investigate the general topic of abundant and deficient numbers. In addition, it has growth and density properties to intrigue both the serious and the recreational students of number theory. Its analysis provides a vehicle for unifying several parts of the theory; in so doing it suggests new unsolved problems and illuminates old ones".*

By definition, abundancy index $I(n) := \sigma(n)/n$. A number $n$ is perfect if and only if its abundancy index is 2. Numbers for which this ratio is greater than (less than) 2 are called abundant (deficient) numbers. It was observed that the abundancy index is a multiplicative number-theoretic function because $\sigma$-function is multiplicative and it could take on arbitrarily large value as well as value so close to 1 as possible. For instance, Laatch showed in [4] that the set of abundancy indices $I(n)$ for $n > 1$ is dense in the interval $\gcd(1, \infty)$ and Weiner established in [14] that there are infinite outlaws in the distribution i.e not all rationals in the interval $(1, \infty)$ are abundancy ratios. It is remarkable that all of this approach though has not yet settled the problem of odd perfect number conjecture, but can surely be extended to study alpha numbers as follows:

**Theorem 3.5.** *Let the classified abundancy index be* $I_\alpha(n) := \frac{\alpha\, f_k(n)}{n}$, $f_k(n) := \lfloor|\sigma_k(n)|\rfloor$ *with class index* $\alpha \in \mathbb{N}$ *and its collections* $\mathbb{I}_\alpha(n) := \{I_1(n), I_2(n), I_3(n), ..., I_\alpha(n) : I_\alpha(n) = \frac{\alpha\sigma(n)}{n}\}$, *then* $n \in @_{1;2\mathbb{N}+1} \neq \emptyset$ *if and only if there exist at least a* $\lambda \in \{1, 2, 3, ..., \tau(n)\}$ *such that* $I_\lambda(n) \in \mathbb{I}_{\tau(n)}(n) \cap \mathbb{N}$ *with* $I_\lambda(n) < \max\{I_1(n), I_2(n), I_3(n), ..., I_{\tau(n)}\}$.

*Proof.* To establish Theorem 3.5, it suffices to show that $\mathbb{I}_{\tau(n)}(n) \cap \mathbb{N} \neq \emptyset$ for every supposed @-number, and this is a direct implication of the definition of @-numbers. □

**Remark 8.** *Note that among several properties of function* $I_\alpha(n)$ *that can be established is the simple assertion:* $I_\alpha(kN) > kI_\alpha(N)\ \forall\ k > 1$ *(see* [4]*), but by* Theorem 3.6 *and* Remark 4*, the condition* $I_\alpha(n) \notin \mathbb{N}\ \forall\ \alpha \leq \tau(n)$ *is suffices for* Conjecture 1 *of this paper to be solved. We also realised that the sequence* $\mathbb{I}_\alpha(n)$ *forms an arithmetic progression of* $\alpha$ *terms, with first term and common difference of* $I_1(n)$*, thereby given rise to a sequence of rational numbers in an AP (Arithmetic Progression) which can be studied further.*

**Theorem 3.6.** *Let the classified abundancy index* $I_\alpha(n)$ *be as defined in* Theorem 3.3*, then* $n \in @^p_{1;2\mathbb{N}+1} \neq \emptyset$ *if and only if* $\mathbb{I}_1(n) \cap \mathbb{N} \neq \emptyset$ *such that* $I_1(n) \leq n^{k-1}\tau(n)$.

*Proof.* By Theorem 3.5, note that every $@^p$-number $n$ of natural order $k$ satisfies (1) when $\alpha_1 \leq n^{k-1}\tau(n)$ and $\alpha_2 = 1$, so in order to establish Theorem 3.6, it suffices to show that $\mathbb{I}_1(n) \cap \mathbb{N} \neq \emptyset$ for every supposed odd $@^p$-number $n$ of natural order $k = 1$, and this is a direct implication of the definition of $@^p$-numbers. □

**Remark 9.** *Note that Conjecture 2 of this paper is solved the moment it is shown that* $I_1(n) \notin \mathbb{N}$ *for every non-square-free, non-prime-power odd* $n$.

# 4 Conclusion and Recommendation:

The results of this paper, particularly, Theorem 3.3 posits a form for odd @-numbers of order 1, if at all exist. So, in order to fully establish Conjectures (1) and (2) of this paper, by Remark 5 and Theorem 3.4, it suffices to establish the case of non-square-free, non-prime-power odd n i.e there is no non-square-free, non-prime-power odd n which satisfies equation (1) when pair integers $1 < \alpha_1, \alpha_2 \leq \tau(n)$ (for every full @-number) and when $\alpha_1 \leq n^{k-1}\tau(n)$, $\alpha_2 = 1$ (for every partial @-number). This is recommended for further study (see [11] & [12] for motivation). Also note that an in-depth study of alpha numbers can be pursued further as follows:

(1) Are alpha numbers infinitely many and are they applicable, in particular, in RSA encrypting and decrypting, taking a clue from the definition of alpha number which implies that each key $\alpha_1$ take on a unique key $\alpha_2$ for every public alpha number $n$, and secret order $k \in \mathbb{C}$?

(2) Is there a general form for even alpha numbers analogous to Euclid-Euler form for even perfect numbers?

(3) What is the congruent form (properties) of odd alpha numbers analogous to Euler form for odd perfect numbers?

(4) Is every alpha number a practical number?

(5) What are the properties of odd alpha number $n$ and its factors in terms of size, the bounds (lower and upper), abundancy and etc.

(6) Is there any applicable relationship between function $f_k(n)$ and the Riemann zeta function $\zeta(.)$?

(7) Is there any efficient and effective algorithm to generate alpha numbers?

(8) Can there be a counting function, say $C(x)$ generating the number of alpha number up to a desired bound $x$ such that the number of alpha numbers in regular intervals, say $10^0 - 10^3$, $10^3 - 2 \cdot 10^3$, $2 \cdot 10^3 - 3 \cdot 10^3$ and etc can be determined?

(9) What other hidden properties (results) of alpha numbers can be obtained from Table 1, especially of those alpha numbers with non-rational complex order?

(10) Are the zeros of $f_k(n)$ of complex order $k$ significant in any way?

(11) What other hidden identities of sum of the positive divisors function can be derived to solve the problem of existence of alpha numbers?

(12) What formidable results can come forth from the following certain generalizations of alpha numbers (see [11] & [12] for definitions, notations and motivation)?

(I) Let any positive integer $n$ satisfying $f_k^m(n) := (\alpha_1/\alpha_2)n$ be called $m$-super @-number, where $f_k(n) := \lfloor|\sigma_k(n)|\rfloor$ is the integral sum of positive divisor function, $f_k^m$ denotes the $m$th iterate of $f$-function and integral $\alpha_1, \alpha_2$ is such that $1 < \alpha_1, \alpha_2 \leq \tau(n)$, with a complementing classified abundancy index $I_\alpha(n) := \frac{\alpha f_k^m(n)}{n}$.

(II) Let any positive integer $n$ satisfying $f_k^m(n) := (\alpha_1/\alpha_2)n^s$ be called $m$-supra @-number, where $f_k(n) := \lfloor|\sigma_k(n)|\rfloor$ is the integral sum of positive divisor function, $f_k^m$ denotes the $m$th iterate of $f$-function and integral $\alpha_1, \alpha_2$ is such that $1 < \alpha_1, \alpha_2 \leq \tau(n)$, with a complementing classified abundancy index $I_\alpha(n) := \frac{\alpha f_k^m(n)}{n^s}$.

(III) Let any positive integer $n$ satisfying $f_k^{m(\star)}(n) := (\alpha_1/\alpha_2)n$ be called $m$-unitary @-number, where $f_k^\star(n) := \lfloor|\sigma_k^\star(n)|\rfloor$ is the unitary integral divisor function such that unitary divisors of $n$ are used instead of positive divisors of $n$ in the computation of $\sigma_k(n)$, $f_k^m$ denotes the $m$th iterate of $f$-function and integral $\alpha_1, \alpha_2$ is such that $1 < \alpha_1, \alpha_2 \leq$ the number of unitary factors of $n$, with a complementing classified abundancy index $I_\alpha(n) := \frac{\alpha f_k^{m(\star)}(n)}{n}$.

(IV) Let any positive integer $n$ satisfying $f_{k,\infty}^m(n) := (\alpha_1/\alpha_2)n$ be called $m$-infinitary @-number, where $f_{k,\infty}(n) := \lfloor|\sigma_{k,\infty}(n)|\rfloor$ is infinitary integral divisor function such that infinitary divisors of $n$ are used instead of positive divisors of $n$ in the computation of $\sigma_k(n)$, $f_k^m$ denotes the $m$th iterate of $f$-function and integral $\alpha_1, \alpha_2$ is such that $1 < \alpha_1, \alpha_2 \leq$ the number of infinitary factors of $n$, with a complementing classified abundancy index $I_\alpha(n) := \frac{\alpha f_{k,\infty}^m(n)}{n}$.

(V) Let any positive integer $n$ satisfying $f_{k,e}^m(n) := (\alpha_1/\alpha_2)n$ be called m-exponential @-number, where $f_{k,e}(n) := \lfloor|\sigma_{k,e}(n)|\rfloor$ is the integral exponential divisor function such that exponential divisors of $n$ are used instead of positive divisors of $n$ in the computation of $\sigma_k(n)$, $f_k^m$ denotes the $m$th iterate of $f$-function and integral $\alpha_1, \alpha_2$ is such that $1 < \alpha_1, \alpha_2 \leq$ the number of exponential factors of $n$, with a complementing classified abundancy index $I_\alpha(n) := \frac{\alpha f_{k,e}^m(n)}{n}$.

(VI) Consider a positive integer $n$ m-prime-@-number of order $k$ if and only if $n$ and $f_k^m(n) := \lfloor|\sigma_k(n)|\rfloor$ share the same set of distinct prime divisors.

(VII) Consider any complex number $n$ m-complex @-number if and only if $f_k^m(n) := (\alpha_1/\alpha_2)\lfloor|n|\rfloor$ where $f_k(n) := \lfloor|\sigma_k(n)|\rfloor$, $f_k^m$ denotes the $m$th iterate of $f$-function and integral $\alpha_1, \alpha_2$ is such that $1 < \alpha_1, \alpha_2 \leq \tau(n)$, with a complementing classified abundancy index $I_\alpha(n) := \frac{\alpha f_k^m(n)}{\lfloor|n|\rfloor}$.

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# References

[1] D. M. Burton, *Elementary Number Theory*, 7th edition, McGraw Hill Publishers, 2011, pp 102-215.

[2] L. E. Dickson, *History of the Theory of Numbers: Divisibility and Primality*. New York: Dover, Vol. 1, 2005, pp 3-33.

[3] G. H. Hardy & E. M. Wright, *An Introduction to The Theory of Numbers*, 4th edition, Oxford at the Clarendon Press, 1960, pp 1-4, 239-241.

[4] R. Laatch, *Measuring the Abundancy of Integers, Mathematics Magazine*, vol 59, 1986, pp 84-92.

[5] P. Ochem and M. Rao, *Odd Perfect Numbers Are Greater than* $10^{15000}$, Math. of Computation, 81, 2012, pp 1869-1877.

[6] Olver F. W. J., Lozier D.W., *Handbook of Mathematical Functions-Lambert Series and Generating functions* National Institute of Standards and Technology, 2010. pp 641.

[7] O. Ore, *Number Theory and its History*, Mc Graw Hill, first edition, 1948 pp 86-106.

[8] O. Ore, *On the Averages of the Divisors of a number.* Amer. Math. Monthly 55(10), 1948, pp 615-619. doi:2307/2004277 JSTOR 2004277.

[9] J. B. Rosser and L. Schoenfield, *Approximate formulas for some functions of prime numbers*, Illinious Journal of Math. 6, 1962, pp 64-94.

[10] J. S$\acute{a}$ndor, *Remarks on the function* $\phi(n)$ *and* $\sigma(n)$, Seminar on Math Analysis, Babes-Bolyai, Univ, preprint No 7, 1989, pp 7-12.

[11] J. S$\acute{a}$ndor J. & B. Crstici, *Handbook of Number Theory II*, vol 2, Kluwer Academic Publishers, 2004, pp 15-43.

[12] J. S$\acute{a}$ndor, B. Crstici & D. S. Mitrinovic, *Handbook of Number Theory I*, Springer, 2006, pp 100-119.

[13] J. J. Tattersall, *Elementary Number Theory in Nine Chapters*, Cambridge University press, 1999. pp 79-146.

[14] P. A. Weiner, *The Abundancy Ratio, A Measure of Perfection*, MathematicsMagazine, vol 73, 2000, pp 307-310.