ON THE x–COORDINATES OF PELL EQUATIONS WHICH ARE PRODUCTS OF TWO LUCAS NUMBERS

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Abstract. Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers given by $L_0 = 2,\ L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. In this paper, for an integer $d \geq 2$ which is square-free, we show that there is at most one value of the positive integer x participating in the Pell equation $x^2 - dy^2 = \pm 1$ which is a product of two Lucas numbers, with a few exceptions that we completely characterize.

1. Introduction

Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers given by $L_0 = 2,\ L_1 = 1$ and

$$L_{n+2} = L_{n+1} + L_n$$

for all $n \geq 0$. This is sequence A000032 on the Online Encyclopedia of Integer Sequences (OEIS). The first few terms of this sequence are

$$\{L_n\}_{n\geq 0} \quad = \quad 2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, \ldots.$$

Putting $(\alpha, \beta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ for the roots of the characteristic equation $r^2 - r - 1 = 0$ of the Lucas sequence, the Binet formula for its general terms is given by

$$L_n = \alpha^n + \beta^n, \qquad \text{for all} \quad n \geq 0. \tag{1.1}$$

Furthermore, we can prove by induction that the inequality

$$\alpha^{n-1} \leq L_n \leq \alpha^{n+2}, \tag{1.2}$$

holds for all $n \geq 0$.

Let $d \geq 2$ be a positive integer which is not a perfect square. It is well known that the Pell equation

$$x^2 - dy^2 = \pm 1 \tag{1.3}$$

has infinitely many positive integer solutions (x, y). By putting (x_1, y_1) for the smallest positive solution, all solutions are of the form (x_k, y_k) for some positive integer k, where

$$x_k + y_k\sqrt{d} = (x_1 + y_1\sqrt{d})^n \qquad \text{for all} \quad k \geq 1. \tag{1.4}$$

Furthermore, the sequence $\{x_k\}_{k\geq 1}$ is binary recurrent. In fact, the following formula

$$x_k = \frac{(x_1 + y_1\sqrt{d})^k + (x_1 - y_1\sqrt{d})^k}{2},$$

holds for all positive integers k.

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Recently, Kafle et al. [11] considered the Diophantine equation

$$x_n = F_\ell F_m, \tag{1.5}$$

where $\{F_m\}_{m\geq 0}$ is the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 0$. They proved that equation (1.5) has at most one solution n in positive integers except for $d = 2, 3, 5$, for which case equation (1.5) has the solutions $x_1 = 1$ and $x_2 = 3$, $x_1 = 2$ and $x_2 = 26$, $x_1 = 2$ and $x_2 = 9$, respectively.

There are many other researchers who have studied related problems involving the intersection sequence $\{x_n\}_{n\geq 1}$ with linear recurrence sequences of interest. For example, see [4, 7, 8, 9, 12, 13, 14, 16, 17, 19].

2. Main Result

In this paper, we study a similar problem to that of Kafle et al. [11], but with the Lucas numbers instead of the Fibonacci numbers. That is, we show that there is at most one value of the positive integer x participating in (1.3) which is a product of two Lucas numbers, with a few exceptions that we completely cahracterize. This can be interpreted as solving the Diophantine equation

$$x_k = L_n L_m, \tag{2.1}$$

in nonnegative integers (k, n, m) with $k \geq 1$ and $0 \leq m \leq n$.

Theorem 2.1. *For each square-free integer $d \geq 2$ there is at most one integer k such that the equation (2.1) holds, except for $d \in \{2, 3, 5, 15, 17, 35\}$ for which $x_1 = 1,\ x_2 = 3, x_3 = 7, x_9 = 1393$ (for $d = 2$), $x_1 = 2,\ x_2 = 7$ (for $d = 3$), $x_1 = 2,\ x_2 = 9$ (for $d = 5$), $x_1 = 4,\ x_5 = 15124$ (for $d = 15$), $x_1 = 4,\ x_2 = 33$ (for $d = 17$) and $x_1 = 6,\ x_3 = 846$ (for $d = 35$).*

3. Preliminary Results

3.1. Notations and terminology from algebraic number theory.

We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned} \tag{3.1}$$

3.2. **Linear forms in logarithms.** In order to prove our main result Theorem 2.1, we need to use several times a Baker–type lower bound for a nonzero linear form in logarithms of algebraic numbers. There are many such in the literature like that of Baker and Wüstholz from [2]. We start by recalling the result of Bugeaud, Mignotte and Siksek ([5], Theorem 9.4, pp. 989), which is a modified version of the result of Matveev [18], which is one of our main tools in this paper.

Theorem 3.1. *Let $\gamma_1, \ldots, \gamma_t$ be positive real numbers in a number field $\mathbb{K} \subseteq \mathbb{R}$ of degree D, $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1, \tag{3.2}$$

is nonzero. Then

$$\log |\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}, \qquad \textit{for all} \qquad i = 1, \ldots, t.$$

When $t = 2$ and $\gamma_1,\ \gamma_2$ are positive and multiplicatively independent, we can use a result of Laurent, Mignotte and Nesterenko [15]. Namely, let in this case $B_1,\ B_2$ be real numbers larger than 1 such that

$$\log B_i \geq \max\left\{h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D}\right\}, \qquad \text{for} \quad i = 1, 2,$$

and put

$$b' := \frac{|b_1|}{D \log B_2} + \frac{|b_2|}{D \log B_1}.$$

Put

$$\Gamma := b_1 \log \gamma_1 + b_2 \log \gamma_2. \tag{3.3}$$

We note that $\Gamma \neq 0$ because γ_1 and γ_2 are multiplicatively independent. The following result is Corollary 2 in [15].

Theorem 3.2. *With the above notations, assuming that η_1, η_2 are positive and multiplicatively independent, then*

$$\log |\Gamma| > -24.34 D^4 \left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log B_1 \log B_2. \tag{3.4}$$

Note that with Γ given by (3.3), we have $e^{\Gamma} - 1 = \Lambda$, where Λ is given by (3.2) in case $t = 2$, which explains the connection between Theorem 3.1 and Theorem 3.2.

3.3. **Reduction procedure.** During the calculations, we get upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some results from the theory of continued fractions.

For the treatment of linear forms homogeneous in two integer variables, we use the well-known classical result in the theory of Diophantine approximation.

Lemma 3.3. *Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction of τ and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [10], Lemma 5a). For a real number X, we write $||X|| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 3.4. *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let further $\varepsilon := ||\mu q|| - M||\tau q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

At various occasions, we need to find a lower bound for linear forms in logarithms with bounded integer coefficients in three and four variables. In this case we use the LLL algorithm that we describe below. Let $\tau_1, \tau_2, \ldots \tau_t \in \mathbb{R}$ and the linear form

$$x_1\tau_1 + x_2\tau_2 + \cdots + x_t\tau_t \quad \text{with} \quad |x_i| \leq X_i. \tag{3.5}$$

We put $X := \max\{X_i\}$, $C > (tX)^t$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\tau_j \rceil \quad \text{for} \quad 1 \leq j \leq t-1 \quad \text{and} \quad \mathbf{b}_t := \lfloor C\tau_t \rceil \mathbf{e}_t,$$

where C is a sufficiently large positive constant.

Lemma 3.5. *Let $X_1, X_2, \ldots, X_t$ be positive integers such that $X := \max\{X_i\}$ and $C > (tX)^t$ is a fixed sufficiently large constant. With the above notation on the lattice Ω, we consider a reduced base $\{\boldsymbol{b}_i\}$ to Ω and its associated Gram-Schmidt orthogonalization base $\{\boldsymbol{b}_i^*\}$. We set*

$$c_1 := \max_{1 \leq i \leq t} \frac{||\boldsymbol{b}_1||}{||\boldsymbol{b}_i^*||}, \quad \theta := \frac{||\boldsymbol{b}_1||}{c_1}, \quad Q := \sum_{i=1}^{t-1} X_i^2 \quad \text{and} \quad R := \left(1 + \sum_{i=1}^{t} X_i\right)/2.$$

If the integers x_i are such that $|x_i| \leq X_i$, for $1 \leq i \leq t$ and $\theta^2 \geq Q + R^2$, then we have

$$\left|\sum_{i=1}^{t} x_i\tau_i\right| \geq \frac{\sqrt{\theta^2 - Q} - R}{C}.$$

For the proof and further details, we refer the reader to the book of Cohen. (Proposition 2.3.20 in [6], pp. 58–63).

3.4. **Pell equations and Dickson polynomials.** Here we give some relations about Pell equations and Dickson polynomials that will be useful in the next section of this paper.

Let $d \geq 2$ be a squarefree integer. We put $\delta := x_1 + \sqrt{x_1^2 - \epsilon}$ for the smallest positive integer x_1 such that

$$x_1^2 - dy_1^2 = \epsilon, \qquad \epsilon \in \{\pm 1\}$$

for some positive integer y_1. Then,

$$x_k + y_k\sqrt{d} = \delta^k \qquad \text{and} \qquad x_k - y_k\sqrt{d} = \eta^k, \qquad \text{where} \qquad \eta := \epsilon\delta^{-1}.$$

From the above, we get

$$2x_k = \delta^k + (\epsilon\delta^{-1})^k \qquad \text{for all} \qquad k \geq 1. \tag{3.6}$$

There is a formula expressing $2x_k$ in terms of $2x_1$ by means of the Dickson polynomial $D_k(2x_1, \epsilon)$, where

$$D_k(x, y) = \sum_{i=0}^{\lfloor k/2 \rfloor} \frac{k}{k-i}\binom{k-i}{i}(-y)^i x^{k-2i}.$$

These polynomials appear naturally in many number theory problems and results, for example in a result of Bilu and Tichy [3] concerning polynomials $f(X), g(X) \in \mathbb{Z}[X]$ such that the Diophantine equation $f(x) = g(y)$ has infinitely many integer solutions (x, y).

Example 3.6. (i) $k = 2$. *We have*

$$2x_2 = \sum_{i=0}^{1} \frac{2}{2-i}\binom{2-i}{i}(-\epsilon)^i(2x_1)^{2-2i} = 4x_1^2 - 2\epsilon, \quad \textit{so} \quad x_2 = 2x_1^2 - \epsilon.$$

(ii) $k = 3$. *We have*

$$2x_3 = \sum_{i=0}^{1} \frac{3}{3-i}\binom{3-i}{i}(-\epsilon)^i(2x_1)^{3-2i} = (2x_1)^3 - 3\epsilon(2x_1), \quad \textit{so} \quad x_3 = 4x_1^3 - 3\epsilon x_1.$$

4. Bounding the variables

We assume that (x_1, y_1) is the smallest positive solution of the Pell equation (1.3). As in Subsection 3.4, we set

$$x_1^2 - dy_1^2 =: \epsilon, \qquad \epsilon \in \{\pm 1\},$$

and put

$$\delta := x_1 + \sqrt{d}y_1 \qquad \text{and} \qquad \eta := x_1 - \sqrt{d}y_1 = \epsilon\delta^{-1}.$$

From (1.4), we get

$$x_k = \frac{1}{2}\left(\delta^k + \eta^k\right). \tag{4.1}$$

Since $\delta \geq 1 + \sqrt{2} > \alpha^{3/2}$, it follows that the estimate

$$\frac{\delta^k}{\alpha^2} \leq x_k < \frac{\delta^k}{\alpha} \quad \text{holds for all} \quad k \geq 1. \tag{4.2}$$

We let $(k, n, m) := (k_i, n_i, m_i)$ for $i = 1, 2$ be the solutions of (2.1). By (1.2) and (4.2), we get

$$\alpha^{n+m-2} \leq L_n L_m = x_k < \frac{\delta^k}{\alpha} \quad \text{and} \quad \frac{\delta^k}{\alpha^2} \leq x_k = L_n L_m \leq \alpha^{n+m+4}, \tag{4.3}$$

so

$$kc_1 \log\delta - 6 < n + m < kc_1 \log\delta + 1 \quad \text{where} \quad c_1 := \frac{1}{\log\alpha}. \tag{4.4}$$

To fix ideas, we assume that

$$n \geq m \qquad \text{and} \qquad k_1 < k_2.$$

We also put

$$m_3 := \min\{m_1, m_2\}, \quad m_4 := \max\{m_1, m_2\}, \quad n_3 := \min\{n_1, n_2\}, \quad n_4 := \max\{n_1, n_2\}.$$

Using the inequality (4.4) together with the fact that $\delta \geq 1 + \sqrt{2} = \alpha^{3/2}$ (so, $c_1 \log \delta > 3/2$), gives us that

$$\frac{3}{2} k_2 < k_2 c_1 \log \delta < 2n_2 + 6 \leq 2n_4 + 6,$$

so

$$k_1 < k_2 < \frac{4}{3} n_4 + 4. \tag{4.5}$$

Thus, it is enough to find an upper bound on n_4. Substituting (1.1) and (4.1) in (2.1) we get

$$\frac{1}{2}(\delta^k + \eta^k) = (\alpha^n + \beta^n)(\alpha^m + \beta^m). \tag{4.6}$$

This can be regrouped as

$$\delta^k 2^{-1} \alpha^{-n-m} - 1 = -2^{-1} \eta^k \alpha^{-n-m} + (\beta\alpha^{-1})^n + (\beta\alpha^{-1})^m + (\beta\alpha^{-1})^{n+m}.$$

Since $\beta = -\alpha^{-1}$, $\eta = \varepsilon\delta^{-1}$ and using the fact that $\delta^k \geq \alpha^{n+m-1}$ (by (4.3)), we get

$$\begin{aligned} \left|\delta^k 2^{-1} \alpha^{-n-m} - 1\right| &\leq \frac{1}{2\delta^k \alpha^{n+m}} + \frac{1}{\alpha^{2n}} + \frac{1}{\alpha^{2m}} + \frac{1}{\alpha^{2(n+m)}} \\ &\leq \frac{\alpha}{2\alpha^{2(n+m)}} + \frac{3}{\alpha^{2m}} \quad < \quad \frac{6}{\alpha^{2m}}, \end{aligned}$$

In the above, we have also used the facts that $n \geq m$ and $(1/2)\alpha + 3 < 6$. Hence,

$$\left|\delta^k 2^{-1} \alpha^{-n-m} - 1\right| < \frac{6}{\alpha^{2m}}. \tag{4.7}$$

We let $\Lambda_1 := \delta^k 2^{-1} \alpha^{-n-m} - 1$. We put

$$\Gamma_1 := k \log \delta - \log 2 - (n+m) \log \alpha. \tag{4.8}$$

Note that $e^{\Gamma_1} - 1 = \Lambda_1$. If $m > 100$, then $\frac{6}{\alpha^{2m}} < \frac{1}{2}$. Since $|e^{\Gamma_1} - 1| < 1/2$, it follows that

$$|\Gamma_1| < 2|e^{\Gamma_1} - 1| < \frac{12}{\alpha^{2m}}. \tag{4.9}$$

By recalling that $(k, n, m) = (k_i, n_i, m_i)$ for $i = 1, 2$, we get that

$$|k_i \log \delta - \log 2 - (n_i + m_i) \log \alpha| < \frac{12}{\alpha^{2m_i}} \tag{4.10}$$

holds for both $i = 1, 2$ provided $m_3 > 100$.

We apply Theorem 3.1 on the left-hand side of (4.7). First, we need to check that $\Lambda_1 \neq 0$. Well, if it were, then $\delta^k \alpha^{-n-m} = 2$. However, this is impossible since $\delta^k \alpha^{-n-m}$ is a unit while 2 is not. Thus, $\Lambda_1 \neq 0$, and we can apply Theorem 3.1. We take the data

$$t := 3, \quad \gamma_1 := \delta, \quad \gamma_2 := 2, \quad \gamma_3 := \alpha, \quad b_1 := k, \quad b_2 := -1, \quad b_3 := -n-m.$$

We take $\mathbb{K} := \mathbb{Q}(\sqrt{d}, \alpha)$ which has degree $D \leq 4$ (it could be that $d = 5$ in which case $D = 2$; otherwise, $D = 4$). Since $\delta \geq 1 + \sqrt{2} > \alpha$, the second inequality in (4.4) tells us that $k < n + m$, so we take $B := 2n$. We have $h(\gamma_1) = h(\delta) = \frac{1}{2} \log \delta$, $h(\gamma_2) = h(2) = \log 2$ and

$h(\gamma_3) = h(\alpha) = \frac{1}{2}\log\alpha$. Thus, we can take $A_1 := 2\log\delta$, $A_2 := 4\log 2$ and $A_3 := 2\log\alpha$. Now, Theorem 3.1 tells us that

$$\begin{aligned}\log|\Lambda_1| &> -1.4\times 30^6\times 3^{4.5}\times 4^2(1+\log 4)(1+\log(2n))(2\log\delta)(4\log 2)(2\log\alpha)\\ &> -2.92\times 10^{13}\log\delta(1+\log(2n)).\end{aligned}$$

By comparing the above inequality with (4.7), we get

$$2m\log\alpha - \log 6 < 2.92\times 10^{13}\log\delta(1+\log(2n)). \tag{4.11}$$

Thus

$$m < 6.06\times 10^{13}\log\delta(1+\log(2n)). \tag{4.12}$$

Since, $\delta^k < \alpha^{n+m+6}$, we get that

$$k\log\delta < (n+m+6)\log\alpha \le (2n+6)\log\alpha, \tag{4.13}$$

which together with the estimate (4.12) gives

$$km < 5.84\times 10^{13} n(1+\log(2n)). \tag{4.14}$$

Let us record what we have proved, since this will be important later-on.

Lemma 4.1. *If $x_k = L_n L_m$ and $n \ge m$, then*

$$m < 6.06\times 10^{13}\log\delta(1+\log(2n)), \quad km < 5.84\times 10^{13} n(1+\log(2n)), \quad k\log\delta < 4n\log\alpha.$$

Note that we did not assume that $m_3 > 100$ for Lemma 4.1 since we have worked with the inequality (4.7) and not with (4.9). We now again assume that $m_3 > 100$. Then the two inequalities (4.10) hold. We eliminate the term involving $\log\delta$ by multiplying the inequality for $i = 1$ with k_2 and the one for $i = 2$ with k_1, subtract them and apply the triangle inequality as follows

$$\begin{aligned}&|(k_2-k_1)\log 2 - (k_2(n_1+m_1) - k_1(n_2+m_2))\log\alpha|\\ &= |k_2(k_1\log\delta - \log 2 - (n_1+m_1)\log\alpha) - k_1(k_2\log\delta - \log 2 - (n_2+m_2)\log\alpha)|\\ &\le k_2\,|k_1\log\delta - \log 2 - (n_1+m_1)\log\alpha| + k_1\,|k_2\log\delta - \log 2 - (n_2+m_2)\log\alpha|\\ &\le \frac{12k_2}{\alpha^{2m_1}} + \frac{12k_1}{\alpha^{2k_2}} < \frac{24k_2}{\alpha^{2m_3}}.\end{aligned}$$

Thus,

$$|\Gamma_2| := |(k_2-k_1)\log 2 - (k_2(n_1+m_1) - k_1(n_2+m_2))\log\alpha| < \frac{24k_2}{\alpha^{2m_3}}. \tag{4.15}$$

We are now set to apply Theorem 3.2 with the data

$$t := 2,\quad \gamma_1 := 2,\quad \gamma_2 := \alpha,\quad b_1 := k_2 - k_1,\quad b_2 := k_2(n_1+m_1) - k_1(n_2+m_2).$$

The fact that $\gamma_1 = 2$ and $\gamma_2 = \alpha$ are multiplicatively independent follows because α is a unit while 2 is not. We observe that $k_2 - k_1 < k_2$, whereas by the absolute value of the inequality in (4.15), we have

$$|k_2(n_1+m_1) - k_1(n_2+m_2)| \le (k_2-k_1)\frac{\log 2}{\log\alpha} + \frac{24k_2}{\alpha^{2m_3}\log\alpha} < 2k_2,$$

because $m_3 > 10$. We have that $\mathbb{K} := \mathbb{Q}(\alpha)$, which has $D = 2$. So we can take

$$\log B_1 = \max\left\{h(\gamma_1), \frac{|\log\gamma_1|}{2}, \frac{1}{2}\right\} = \log 2,$$

and
$$\log B_2 = \max\left\{h(\gamma_2), \frac{|\log\gamma_2|}{2}, \frac{1}{2}\right\} = \frac{1}{2}.$$
Thus,
$$b' = \frac{|k_2-k_1|}{2\log B_2} + \frac{|k_2(n_1+m_1)-k_1(n_2+m_2)|}{2\log B_1} \le k_2 + \frac{k_2}{\log 2} < 3k_2.$$
Now Theorem 3.2 tells us that with
$$\Gamma_2 = (k_2-k_1)\log 2 - (k_2(n_1+m_1)-k_1(n_2+m_2))\log\alpha,$$
we have
$$\log|\Gamma_2| > -24.34\times 2^4\left(\max\{\log(3k_2)+0.14, 10.5\}\right)^2\cdot(2\log 2)\cdot(1/2).$$
Thus,
$$\log|\Gamma_2| > -270\left(\max\{\log(3k_2)+0.14, 10.5\}\right)^2.$$
By comparing the above inequality with (4.15), we get
$$2m_3\log\alpha - \log(24k_2) < 270\left(\max\{\log(3k_2)+0.14, 10.5\}\right)^2.$$
If $k_2 \le 10523$, then $\log(3k_2)+0.14 < 10.5$. Thus, the last inequality above gives
$$2m_3\log\alpha < 270\times 10.5^2 + \log(24\times 10523),$$
giving $m_3 < 30942$ in this case. Otherwise, $k_2 > 10523$, and we get
$$2m_3\log\alpha < 272(1+\log k_2)^2 + \log(24k_2) < 280(1+\log k_2)^2,$$
which gives
$$m_3 < 160(1+\log k_2)^2.$$
We record what we have proved

Lemma 4.2. *If $m_3 > 100$, then either*
(i) $k_2 \le 10523$ *and* $m_3 < 30942$ *or*
(ii) $k_2 > 10523$, *in which case* $m_3 < 160(1+\log k_2)^2$.

Now suppose that some m is fixed in (2.1), or at least we have some good upper bounds on it. We rewrite (2.1) using (1.1) and (4.1) as
$$\frac{1}{2}(\delta^k + \eta^k) = L_m(\alpha^n + \beta^n),$$
so
$$\delta^k(2L_m)^{-1}\alpha^{-n} - 1 = -\frac{1}{2L_m}\eta^k\alpha^{-n} + (\beta\alpha^{-1})^n.$$
Since $m \ge 1$, $\beta = -\alpha^{-1}$, $\eta = \varepsilon\delta^{-1}$ and $\delta^k > \alpha^{n+m-1}$, we get
$$\begin{aligned}\left|\delta^k(2L_m)^{-1}\alpha^{-n} - 1\right| &\le \frac{1}{2L_m\delta^k\alpha^n} + \frac{1}{\alpha^{2n}} \le \frac{\alpha}{\alpha^{2(n+m)}} + \frac{1}{\alpha^{2n}}\\ &\le \frac{\alpha+1}{\alpha^{2n}} < \frac{6}{\alpha^{2n}},\end{aligned}$$
where we have used the fact that $n \ge m \ge 0$ and $\alpha + 1 < 6$. Hence,
$$|\Lambda_3| := \left|\delta^k(2L_m)^{-1}\alpha^{-n} - 1\right| < \frac{6}{\alpha^{2n}}. \tag{4.16}$$

We assume that $n_3 > 100$. In particular, $\frac{6}{\alpha^{2n}} < \frac{1}{2}$ for $n \in \{n_1, n_2\}$, so we get by the previous argument that

$$|\Gamma_3| := |k \log \delta - \log(2L_m) - n \log \alpha| < \frac{12}{\alpha^{2n}}. \tag{4.17}$$

We are now set to apply Theorem 3.1 on the left-hand side of (4.16) with the data

$$t := 3, \quad \gamma_1 := \delta, \quad \gamma_2 := 2L_m, \quad \gamma_3 := \alpha, \quad b_1 := k, \quad b_2 := -1, \quad b_3 := -n.$$

First, we need to check that $\Lambda_3 := \delta^k (2L_m)^{-1}\alpha^{-n} - 1 \neq 0$. If not, then $\delta^k = 2L_m\alpha^m$. The left-hand side belongs to the field $\mathbb{Q}(\sqrt{d})$ but not rational while the right-hand side belongs to the field $\mathbb{Q}(\sqrt{5})$. This is not possible unless $d = 5$. In this last case, δ is a unit in $\mathbb{Q}(\sqrt{5})$ while $2L_m$ is not a unit in $\mathbb{Q}(\sqrt{5})$ since the norm of this first element is $4L_m^2 \neq \pm 1$. So, $\Lambda_3 \neq 0$. Thus, we can apply Theorem 3.1. We have the field $\mathbb{K} := \mathbb{Q}(\sqrt{d}, \sqrt{5})$ which has degree $D \leq 4$. We also have

$$\begin{aligned} h(\gamma_2) &= h(2L_m) = h(2) + h(L_m) \\ &\leq \log 2 + (m+1)\log\alpha < 2 + m\log\alpha \\ &\leq 2.92 \times 10^{13} \log\delta(1 + \log(2n)) \quad \text{by (4.12)}. \end{aligned}$$

So, we take

$$h(\gamma_1) = \frac{1}{2}\log\delta, \quad h(\gamma_2) = 2.92 \times 10^{13}\log\delta(1+\log(2n)) \quad \text{and} \quad h(\gamma_3) = \frac{1}{2}\log\alpha.$$

Then,

$$A_1 := 2\log\delta, \quad A_2 := 1.18 \times 10^{14}\log\delta(1+\log(2n)) \quad \text{and} \quad A_3 := 2\log\alpha.$$

Then, by Theorem 3.1 we get

$$\begin{aligned} \log|\Lambda_3| &> -1.4 \times 30^6 \times 3^{4.5} \times 4^2(1+\log 4)(1+\log n)(2\log\delta) \\ &\quad \times (1.18 \times 10^{14}\log\delta(1+\log(2n)))(2\log\alpha) \\ &> -8.6 \times 10^{26}(1+\log(2n))^2(\log\delta)^2\log\alpha. \end{aligned}$$

Comparing the above inequality with (4.16), we get

$$2n\log\alpha - \log 6 < 8.6 \times 10^{26}(1+\log(2n))^2(\log\delta)^2\log\alpha,$$

which implies that

$$n < 4.3 \times 10^{26}(1+\log(2n))^2(\log\delta)^2. \tag{4.18}$$

We record what we have proved.

Lemma 4.3. *If $x_k = L_nL_m$ with $n \geq m \geq 1$, then we have*

$$n < 4.3 \times 10^{26}(1+\log(2n))^2(\log\delta)^2.$$

Note that we did not use the assumption that $m_3 > 100$ of that $n_3 > 100$ for Lemma 4.3 since we worked with the inequality (4.16) not with the inequality (4.17). We now assume that $n_3 > 100$ and in particular (4.17) holds for $(k, n, m) = (k_i, n_i, m_i)$ for both $i = 1, 2$. By the previous procedure, we also eliminate the term involving $\log\delta$ as follows

$$|k_2\log(2L_{m_1}) - k_1\log(2L_{m_2}) - (k_2n_1 - k_1n_2)\log\alpha| < \frac{12k_2}{\alpha^{2n_1}} + \frac{12k_1}{\alpha^{2n_2}} < \frac{24k_2}{\alpha^{2n_3}}. \tag{4.19}$$

We assume that $\alpha^{2n_3} > 48k_2$. If we put

$$\Gamma_4 := k_2\log(2L_{m_1}) - k_1\log(2L_{m_2}) - (k_2n_1 - k_1n_2)\log\alpha,$$

we have that $|\Gamma_4| < 1/2$. We then get that

$$|\Lambda_4| := |e^{\Gamma_4} - 1| < 2|\Gamma_4| < \frac{48k_2}{\alpha^{2n_3}}. \tag{4.20}$$

We apply Theorem 3.1 to

$$\Lambda_4 := (2L_{m_1})^{k_2}(2L_{m_2})^{-k_1}\alpha^{-(k_2n_1-k_1n_2)} - 1.$$

First, we need to check that $\Lambda_4 \neq 0$. Well, if it were, then it would follow that

$$\frac{L_{m_1}^{k_2}}{L_{m_2}^{k_1}} = 2^{k_1-k_2}\alpha^{k_2n_1-k_1n_2}. \tag{4.21}$$

We consider the following Lemma.

Lemma 4.4. *The equation* (4.21) *has only many small positive integer solutions* (k_i, n_i, m_i) *for* $i = \{1, 2\}$ *with* $k_1 < k_2$ *and* $m_1 \le m_2 \le 6$. *Futhermore, none of these solutions lead to a valid solution to the original Diophantine equation* (2.1).

Proof. We suppose that (4.21) holds and assume that $\gcd(k_1, k_2) = 1$. Since $\alpha^{k_2n_1-k_1n_2} \in \mathbb{Q}$, it follows $k_2n_1 = k_1n_2$. Thus, if one of the n_1, n_2 is zero, so is the other. Since $n_i \ge m_i$ for $i \in \{1, 2\}$, it follows that $n_1 = n_2 = 0$, $m_1 = m_2 = 0$, so $x_{k_1} = x_{k_2}$, therefore $k_1 = k_2$ a contradiction. Thus, n_1 and n_2 are both positive integers. Next $L_{m_1}^{k_2}/L_{m_2}^{k_1} = 2^{k_1-k_2} < 1$. Thus, $L_{m_1}^{k_2} < L_{m_2}^{k_1} < L_{m_2}^{k_2}$, so $L_{m_1} < L_{m_2}$. This implies that either $(m_1, m_2) = (1, 0)$ or $m_1 < m_2$. The case $(m_1, m_2) = (1, 0)$ gives $1/2^{k_1} = 2^{k_1-k_2}$. Thus, $k_2 = 2k_1$ and since $\gcd(k_1, k_2) = 1$, we get $k_1 = 1$, $k_2 = 2$, so $n_2 = 2n_1$. But then $x_2 = x_{k_2} = L_{n_2}L_{m_2} = L_{2n_1}L_0 = 2L_{2n_1}$ is even, a contradiction since $x_2 = 2x_1 \pm 1$ (by Example 3.6 (i)) is odd. Thus, $m_1 < m_2$. If $m_2 > 6$, the Carmichael Primitive Divisor Theorem for Lucas numbers shows that L_{m_2} is divisible by a prime $p > 7$ which does not divide L_{m_1}. This is impossible since it contradicts the assumption that (4.21) holds. Thus, $m_2 \le 6$. Further since $L_{m_1}^{k_2}/L_{m_2}^{k_1} = 1/2^{k_2-k_1}$ it follows that $L_{m_1}^{k_1} \mid L_{m_1}^{k_2} \mid L_{m_2}^{k_1}$, so $L_{m_1} \mid L_{m_2}$. So, there are three cases that we analyse:

Case 1. $m_1 = 0$, $m_2 \in \{3, 6\}$. If $(m_1, m_2) = (0, 3)$, then $2^{k_2}/4^{k_1} = 1/2^{2k_1-k_2} = 1/2^{k_2-k_1}$. This gives $2k_2 = 3k_1$ and since k_1 and k_2 are coprime, it follows that $k_1 = 2$ and $k_2 = 3$. Then $x_2 = x_{k_1} = L_{n_1}L_{m_1} = L_{n_1}L_0 = 2L_{n_1}$ is even, a contradiction since $x_2 = 2x_1 \pm 1$ is odd. If $(m_1, m_2) = (0, 6)$, then $2^{k_2}/18^{k_1} = 1/2^{k_2-k_1}$, which is impossible since by looking at the exponent of 3 we would get $k_1 = 0$, a contradiction.

Case 2. $m_1 = 2$ and L_{m_2} is a power of 2. The case $m_2 = 0$ has been treated so the only other case left is $m_2 = 3$. In this case, $1/4^{k_1} = 1/2^{k_2-k_1}$, giving $k_2 = 3k_1$. Thus, since $\gcd(k_1, k_2) = 1$, then $k_1 = 1$ and $k_2 = 3$. Since $k_2n_1 = k_1n_2$, we get $n_2 = 3n_1$. Thus, $x_1 = L_{n_1}L_1 = L_{n_1}$ and $x_3 = L_{3n_1}L_3 = 4L_{3n_1}$. Now $x_3 = x_1(4x_1^2 \pm 3)$ (by Example 3.6 (ii)) and the second factor is odd, so the power of 2 dividing $4L_{3n_1}$ divides $x_1 = L_{n_1}$. But $4L_{3n_1}$ is a multiple of 8 since L_{3n_1} is even. Thus, $8 \mid L_{n_1}$, which is false.

Case 3. $m_1 = 2$ and $m_2 = 6$. We get $3^{k_2}/(2.3^2)^{k_1} = 1/2^{k_2-k_1}$. Looking at the exponent of 3, we get $k_2 = 2k_1$ and loking at the exponent of 2 we also get $k_2 = 2k_1$, so $k_1 = 1$ and $k_2 = 2$. Also, $n_2 = 2n_1$. Thus, $x_1 = L_{n_1}L_{m_1} = 3L_{n_1}$ and $x_2 = L_{n_2}L_{m_2} = 18L_{2n_1}$ is even, a contradiction with the fact that $x_2 = 2x_1^2 \pm 1$ is odd. □

So, by Lemma 4.4 we have $\Lambda_4 \neq 0$. Thus, we can now apply Theorem 3.1 with the data

$$t := 3, \quad \gamma_1 := 2L_{m_1}, \quad \gamma_2 := 2L_{m_2}, \quad \gamma_3 := \alpha, \quad b_1 = k_2,$$
$$b_2 := -k_1, \quad b_3 := -(k_2n_1 - k_1n_2).$$

We have $\mathbb{K} := \mathbb{Q}(\sqrt{5})$ which has degree $D := 2$. Also, using (4.5), we can take $B := 4n_4^2$. We can also take $A_1 := 2(2 + m_1 \log \alpha) \leq 4m_1 \log \alpha$, $A_2 := 2(2 + m_2 \log \alpha) \leq 4m_2 \log \alpha$ and $A_3 := \log \alpha$. Theorem 3.1 gives that

$$\begin{aligned} \log|\Lambda_4| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2 (1 + \log 2)(1 + \log(4n_4^2))(4m_1 \log \alpha)(4m_2 \log \alpha) \log \alpha, \\ &> -3.44 \times 10^{12} m_1 m_2 (1 + \log(2n_4)). \end{aligned}$$

By comparing this with the inequality (4.20), we get

$$2n_3 \log \alpha - \log(48k_2) < 3.44 \times 10^{12} m_1 m_2 (1 + \log(2n_4)).$$

Since $k_2 < 4n_4$ and $n_4 > 10$, we get that $\log(48k_2) < 2(1 + \log(2n_4))$. Thus,

$$n_3 < 3.58 \times 10^{12} m_1 m_2 (1 + \log(2n_4)). \tag{4.22}$$

All this was done under the assumption that $\alpha^{2n_3} > 48k_2$. But if that inequality fails, then

$$n_3 < c_1 \log(48k_2) < 12(1 + \log(2n_4)),$$

which is much better than (4.22). Thus, (4.22) holds in all cases. Next, we record what we have proved.

Lemma 4.5. *Assuming that $n_3 > 100$, then we have*

$$n_3 < 3.58 \times 10^{12} m_1 m_2 (1 + \log(2n_4)).$$

We now start finding effective bounds for our variables.

Case 1. $m_4 \leq 100$.

Then $m_1 < 100$ and $m_2 < 100$. By Lemma 4.5, we get that

$$n_3 < 3.58 \times 10^{16} (1 + \log(2n_4)).$$

By Lemma 4.1, we get

$$\log \delta < 4n_3 \log \alpha < 6.89 \times 10^{16} (1 + \log(2n_4)).$$

By the inequality (4.4), we have that

$$\begin{aligned} n_4 &\leq n_4 + m_4 - 1 \\ &< k_2 c_1 \log \delta \\ &< 1.72 \times 10^{27} c_1 (1 + \log(2n_4))^2 (\log \delta)^3 \quad \text{(by (4.5) and Lemma 4.3)} \\ &< \frac{1}{\log \alpha} (1.72 \times 10^{27} (1 + \log(2n_4))^2)(6.89 \times 10^{16} (1 + \log(2n_4)))^3 \\ &< 1.17 \times 10^{78} \log(1 + \log(2n_4))^5. \end{aligned}$$

With the help of *Mathematica*, we get that $n_4 < 4.6 \times 10^{89}$. Thus, using (4.5), we get

$$\max\{k_2, n_4\} < 4.6 \times 10^{89}.$$

We record what we have proved.

Lemma 4.6. *If $m_4 := \max\{m_1, m_2\} \leq 100$, then*

$$\max\{k_2, n_4\} < 4.6 \times 10^{89}.$$

Case 2. $m_4 > 100$.

Note that either $m_3 \leq 100$ or $m_3 > 100$ case in which by Lemma 4.2 and the inequality (4.5), we have $m_3 \leq 160(1 + \log(4n_4))^2$ provided that $m_4 > 10000$, which we now assume.

We let $i \in \{1,2\}$ be such that $m_i = m_3$ and j be such that $\{i,j\} = \{1,2\}$. We assume that $n_3 > 100$. We work with (4.17) for i and (4.10) for j and noting the conditions $n_i > 100$ and $m_j = m_4 > 100$ are fullfilled. That is,

$$\begin{aligned} |k_i \log \delta + \log(2L_{m_i}) - n_i \log \alpha| &< \frac{12}{\alpha^{2n_i}}, \\ |k_j \log \delta - \log 2 - (n_j + m_j) \log \alpha| &< \frac{12}{\alpha^{2m_j}}. \end{aligned}$$

By a similar procedure as before, we eliminate the term involving $\log \delta$. We multiply the first inequality by k_j, the second inequality by k_i, subtract the resulting inequalities and apply the triangle inequalty to get

$$\begin{aligned} |k_j \log(2L_{m_i}) - k_i \log 2 - (k_j n_i - k_i(n_j + m_j)) \log \alpha| &< \frac{12k_j}{\alpha^{2m_i}} + \frac{12k_i}{\alpha^{2l_j}} \\ &< \frac{24k_2}{\alpha^{2\min\{n_i,m_j\}}}. \end{aligned} \tag{4.23}$$

Assume that $\alpha^{2\min\{n_i,m_j\}} > 48k_2$. We put

$$\Gamma_5 := k_j \log(2L_{m_i}) - k_i \log 2 - (k_j n_i - k_i(n_j + m_j)) \log \alpha.$$

We can write $\Lambda_5 := (2L_{m_i})^{k_j} 2^{-k_i} \alpha^{(k_j n_i - k_i(n_j + m_j))} - 1$. Under the above assumption and using (4.23), we get that

$$|\Lambda_5| = |e^{\Gamma_5} - 1| < 2|\Gamma_5| < \frac{48k_2}{\alpha^{2\min\{n_i,m_j\}}}. \tag{4.24}$$

We are now set to apply Theorem 3.1 on Λ_5. First, we need to check that $\Lambda_5 \neq 0$. Well, if it were, then we would get that

$$L_{m_i}^{k_j} = 2^{k_i - k_j} \alpha^{(k_j n_i - k_i(n_j + m_j))}. \tag{4.25}$$

We consider the following lemma.

Lemma 4.7. *The equation* (4.25) *has only many small positive integer solutions* $(k_i, k_j, n_i, n_j, m_i, m_j)$ *for* $i, j = \{1,2\}$ *with* $k_1 < k_2$ *and* $m_1 \le m_2 \le 6$. *Futhermore, none of these solutions lead to a valid solution to the original Diophantine equation* (2.1).

Proof. Suppose that (4.25) holds and assume that $\gcd(k_1, k_2) = 1$. Since $\alpha^{(k_j n_i - k_i(n_j + m_j))} \in \mathbb{Q}$, then $k_j n_i = k_i(n_j + m_j)$. Next $L_{m_i}^{k_j} = 2^{k_i - k_j}$. Thus, $k_i \ge k_j$, so $i = 2$, $j = 1$, $k_2 > k_1$ and $m_2 \neq 1$. Since $L_{m_2} > 1$ is a power of 2, it follows that $m_2 \in \{0, 3\}$. Suppose $m_2 = 0$. Then $L_{m_2}^{k_1} = 2^{k_1} = 2^{k_2 - k_1}$, so $k_2 = 2k_1$. Hence, $k_1 = 1$ and $k_2 = 2$. Further, $n_2 = 2(n_1 + m_1)$. Thus, $x_2 = x_{k_2} = L_{n_2} L_{m_2} = 2L_{2(n_1+m_1)}$ is even, which false because $x_2 = 2x_1^2 \pm 1$ is odd. Suppose next that $m_2 = 3$. Then $4^{k_1} = 2^{k_2 - k_1}$. Thus, $k_2 = 3k_1$, so $k_1 = 1$ and $k_2 = 3$. Next, $n_2 = 3(n_1 + m_1)$. Hence, $x_1 = x_{k_1} = L_{n_1} L_{m_1}$ and $x_3 = x_{k_2} = L_{n_2} L_{m_2} = 4L_{3(n_1+m_1)}$. By the previous argument in the proof of Lemma 4.4, 8 divides $x_3 = x_1(4x_1^2 \pm 1)$, so $8 \mid x_1$. Since $x_1 = L_{n_1} L_{m_1}$ and $8 \nmid L_n$ for any n, it follows that L_{n_1} and L_{m_1} are both even. Thus, $3 \mid n_1$, $3 \mid m_1$. Further, one of L_{n_1}, L_{m_1} is a multiple of 4, so one of n_1, m_1 is odd. Suppose both are odd. Then $4 \mid L_{n_1}$, $4 \mid L_{m_1}$ so $16 \mid x_1 \mid x_3 \mid 4L_{3(n_1+m_1)}$. This implies that $4 \mid L_{3(n_1+m_1)}$, which is false because $3(n_1 + m_1)$ is an even multiple of 3, and $2 \| L_{6m}$ for any m. Suppose now that one of n_1, m_1 is an even multiple of 3, and the other is odd. Then $\mathrm{ord}_2(x_1) = 3$, where $\mathrm{ord}_2(x)$ is the exponent at which 2 appears in the factorization of x. Hence,

$$3 = \mathrm{ord}_2(x_3) = \mathrm{ord}_2(4L_{3(n_1+m_1)}) = 2 + \mathrm{ord}_2(L_{3(n_1+m_1)}),$$

giving $\text{ord}_2(L_{3(n_1+m_1)}) = 1$, which is again false since $3(n_1+m_1)$ is an odd multiple 3, so a number of the form $3+6m$, and for such numbers we have $4\|L_{3+6m}$. Hence, in all instances we have gotten a contradiction. □

Thus, by Lemma 4.7 we have that1 $\Lambda_5 \neq 0$. So, we can apply Theorem 3.1 with the data

$$t := 3, \quad \gamma_1 := 2L_{m_i}, \quad \gamma_2 := 2 \quad \gamma_3 := \alpha \quad b_1 := k_j,$$
$$b_2 := -k_i, \quad b_3 := -(k_j n_i - k_i(n_j + m_j)).$$

From the previous calculations, we know that $\mathbb{K} := \mathbb{Q}(\sqrt{2})$ which has degree $D = 2$ and $A_1 := 4m_i \log\alpha$, $A_2 := 2\log 2$ and $A_3 := \log\alpha$. We also take $B := 4n_4^2$. By Theorem 3.1, we get that

$$\begin{aligned}\log|\Lambda_5| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1+\log 2)(1+\log(4n_4^2))(4m_i\log\alpha)(2\log 2)\log\alpha,\\ &> -5.18 \times 10^{12} m_i(1+\log(2n_4)).\end{aligned}$$

Comparing the above inequality with (4.24), we get

$$2\min\{n_i, m_j\}\log\alpha - \log(48k_2) < 5.12 \times 10^{12} m_i(1+\log(2n_4)).$$

Since $m_4 > 100$, we get using (4.5) ($k_2 < 4n_4$) that,

$$\min\{n_i, n_j\} < 5.38 \times 10^{12}(160(1+\log(4n_4))^2)(1+\log(2n_4)) + \frac{c_1}{2}\log(192n_4),$$

which implies that

$$\min\{n_i, m_j\} < 1.72 \times 10^{15}(1+\log(2n_4))^3. \tag{4.26}$$

All this was under the assumptions that $n_4 > 10000$, and that $\alpha^{2\min\{n_i,m_j\}} > 48k_2$. But, still under the condition that $n_4 > 10000$, if $\alpha^{2\min\{n_i,m_j\}} < 48k_2$, then we get an inequality for $\min\{n_i, n_j\}$ which is even much better than (4.26). So, (4.26) holds provided that $n_4 > 10000$. Suppose say that $\min\{n_i, m_j\} = m_j$. Then we get that

$$m_3 < 160(1+\log(4n_4))^2, \quad m_4 < 1.72 \times 10^{15}(1+\log(2n_4))^3.$$

By Lemma 4.5, since $m_3 > 100$, we get

$$\begin{aligned}n_3 &< (3.58 \times 10^{12})(160(1+\log(4n_4))^2)(1+\log(2n_4))\\ &\quad \times 1.72 \times 10^{15}(1+\log(2n_4))^3\\ &< 1.98 \times 10^{30}(1+\log(2n_4))^6.\end{aligned}$$

Together with Lemma 4.1, we get

$$\log\delta < 3.80 \times 10^{30}(1+\log(2n_4))^6,$$

which together with Lemma 4.3 gives

$$n_4 < 4.30 \times 10^{26}(1+\log(2n_4))^2(3.80 \times 10^{30}(1+\log(2n_4))^6)^2,$$

which implies that

$$n_4 < 6.21 \times 10^{87}(1+\log(2n_4))^{14}. \tag{4.27}$$

With the help of *Mathematica* we get that $n_4 < 1.30 \times 10^{122}$. This was proved under the assumption that $n_4 > 10000$, but the situation $n_4 \leq 10000$ already provides a better bound than $n_4 < 1.30 \times 10^{122}$. Hence,

$$\max\{k_2, n_1, n_2\} < 1.30 \times 10^{122}. \tag{4.28}$$

This was when $m_j = \min\{n_i, m_j\}$. Now we assume that $n_i = \min\{n_i, m_j\}$. Then we get

$$n_i < 1.72 \times 10^{15}(1 + \log(2n_4))^3.$$

By Lemma 4.1, we get that

$$\log \delta < 3.31 \times 10^{15}(1 + \log(2n_4))^3.$$

Now by Lemma 4.3 together with Lemma 4.1 to bound l_4 give

$$\begin{aligned} n_4 &< 4.30 \times 10^{26}(1 + \log(2n_4)))^2(3.31 \times 10^{15}(1 + \log(2n_4))^3)^2 \\ &< 4.72 \times 10^{57}(1 + \log(2n_4))^{10}. \end{aligned}$$

This gives, $n_4 < 2.44 \times 10^{80}$ which is a better bound than 1.30×10^{122}. We record what we have proved.

Lemma 4.8. *If $m_4 := \max\{m_1, m_2\} > 100$ and $n_3 := \min\{n_1, n_2\} > 100$, then*

$$\max\{k_2, n_1, n_2\} < 1.30 \times 10^{122}.$$

It now remains the case when $m_4 > 100$ and $n_3 \le 100$. But then, by Lemma 4.1, we get $\log \delta < 192$ and now Lemma 4.1 together with Lemma 4.3 give

$$n_4 < 1.56 \times 10^{31}(1 + \log(2n_4))^2,$$

which implies that $n_4 < 10^{36}$ and further $\max\{k_1, n_1, n_2\} < 10^{40}$. We record what we have proved.

Lemma 4.9. *If $m_4 > 100$ and $n_3 \le 100$, then*

$$\max\{k_1, n_1, n_2\} < 10^{40}.$$

5. The final computations

5.1. **The first reduction.** In this subsection we reduce the bounds for k_1, m_1, n_1 and k_2, m_2, n_2 to cases that can be computationally treated. For this we return to the inequalities for Γ_2, Γ_4 and Γ_5.

We return to (4.15) and we set $s := k_2 - k_1$ and $r := k_2(n_1 + m_1) - k_1(n_2 + m_2)$ and divide both sides by $s \log \alpha$ to get

$$\left| \frac{\log 2}{\log \alpha} - \frac{r}{s} \right| < \frac{24k_2}{\alpha^{2m_3} s \log \alpha}. \tag{5.1}$$

We assume that l_3 is so large that the right-hand side of the inequality in (5.1) is smaller than $1/(2s^2)$. This certainly holds if

$$\alpha^{2m_3} > 48k_2^2/\log \alpha. \tag{5.2}$$

Since $k_2 < 1.3 \times 10^{122}$, it follows that the last inequality (5.2) holds provided that $m_3 \ge 589$, which we now assume. In this case r/s is a convergent of the continued fraction of $\tau := \frac{\log 2}{\log \alpha}$ and $s < 1.30 \times 10^{122}$. We are now set to apply Lemma 3.3.

We write $\tau := [a_0; a_1, a_2, a_3, \ldots] = [1, 2, 3, 1, 2, 3, 2, 4, 2, 1, 2, 11, 2, 1, 11, 1, 1, 134, 2, 2, \ldots]$ for the continued fraction of τ and p_k/q_k for the $k-$th convergent. We get that $r/s = p_j/q_j$ for some $j \le 237$. Furthermore, putting $a(M) := \max\{a_j : j = 0, 1, \ldots, 237\}$, we get $a(M) := 880$. By Lemma 3.3, we get

$$\frac{1}{882s^2} = \frac{1}{(a(M)+2)s^2} \le \left|\tau - \frac{r}{s}\right| < \frac{24k_2}{\alpha^{2m_3} s \log \alpha},$$

giving

$$\alpha^{2m_3} < \frac{882 \times 24k_2^2}{\log \alpha} < \frac{882 \times 24 \times (1.30 \times 10^{122})^2}{\log \alpha},$$

leading to $m_3 \leq 1190$. We record what we have just proved.

Lemma 5.1. *We have* $m_3 := \min\{m_1, m_2\} \leq 1190$.

If $m_1 = m_3$, then we have $i = 1$ and $j = 2$, otherwise $m_2 = m_3$ implying that we have $i = 2$ and $j = 1$. In both cases, the next step is the application of Lemma 3.5 (LLL algorithm) for (4.23), where $n_i < 1.30 \times 10^{112}$ and $|k_j n_i - k_i(n_j + m_j)| < 10^{116}$. For each $m_j \in [1, 1190]$ and

$$\Gamma_5 := k_j \log(2L_{m_i}) - k_i \log 2 - (k_j n_i - k_i(n_j + m_j)) \log \alpha, \tag{5.3}$$

we apply the LLL-algorithm on Γ_3 with the data

$$t := 3, \quad \tau_1 := \log(2L_{m_i}), \quad \tau_2 := \log 2, \quad \tau_3 := \log \alpha$$
$$x_1 := k_j, \quad x_2 := -k_i, \quad x_3 := k_j n_i - k_i(n_j + m_j).$$

Further, we set $X := 10^{116}$ as an upper bound to $|x_i|$ for $i = 1, 2, 3$, and $C := (5X)^5$. A computer search in *Mathematica* allows us to conclude, together with the inequality (4.23), that

$$2 \times 10^{-480} < \min_{1 \leq \min\{n_i, m_j\} \leq 1190} |\Gamma_5| < \frac{24k_2}{\alpha^{2 \min\{n_i, m_j\}}}. \tag{5.4}$$

Thus, $\min\{n_i, m_j\} \leq 1419$. We assume first that $i = 1$, $j = 2$. Thus, $n_1 \leq 1419$ or $m_j = \min\{n_i, m_j\} \leq 1419$.

Next, we suppose that $m_j = \min\{n_i, m_j\} \leq 1419$. Since $m_1 := m_3 \leq 1190$, we have

$$m_3 := \min\{m_1, m_2\} \leq 1190 \quad \text{and} \quad m_4 := \max\{m_1, m_2\} \leq 1419.$$

Now, returning to the inequality (4.19) which involves

$$\Gamma_4 := k_2 \log(2L_{m_1}) - k_1 \log(2L_{m_2}) - (k_2 n_1 - k_1 n_2) \log \alpha, \tag{5.5}$$

we use again the LLL algorithm to estimate the lower bound for $|\Gamma_4|$ and thus, find a bound for n_1 that is better than the one given in Lemma 4.8. We distinguish the cases $m_3 < m_4$ and $m_3 = m_4$.

5.1.1. *The case* $m_3 < m_4$. We take $m_1 := m_3 \in [1, 1190]$ and $m_2 := m_4 \in [m_3 + 1, 1419]$ and apply Lemma 3.5 with the data:

$$t := 3, \quad \tau_1 := 2L_{m_1}, \quad \tau_2 := 2L_{m_2}, \quad \tau_3 := \log \alpha,$$
$$x_1 := k_2, \quad x_2 := -k_1, \quad x_2 := k_1 n_2 - k_2 n_1.$$

We also put $X := 10^{116}$ and $C := (20X)^9$. After a computer search in *Mathematica* together with the inequality (4.19), we can confirm that

$$2 \times 10^{-1120} \leq \min_{\substack{1 \leq m_3 \leq 1190 \\ m_3 + 1 \leq m_4 \leq 1419}} |\Gamma_4| < 24k_2\alpha^{-2n_3}.$$

This leads to the inequality

$$\alpha^{2n_3} < 12 \times 10^{1120} k_2.$$

Sustituting for the bound k_2 given in Lemma 4.8, we get that $n_1 := n_3 \leq 2950$.

5.1.2. *The case $m_3 = m_4$.* . In this case $m_1 = m_2 \leq 1419$ and we have

$$\Gamma_4 := (k_2 - k_1)\log(2L_{m_1}) - (k_2 n_1 - k_1 n_2)\log\alpha \neq 0.$$

This is similar to the case we have handled in the previous steps and yields the bound on n_1 which is less than 2950. So in both cases we have $n_1 \leq 2950$. From the fact that

$$\log\delta \leq k_1 \log\delta \leq 4n_1 \log\alpha < 5678,$$

and by considering the inequality given in Lemma 4.3, we conclude that

$$n_2 < 1.4 \times 10^{34}(1 + \log(2n_2))^2,$$

which with the help of *Mathematica* yields $n_2 < 1.12 \times 10^{38}$. We summarise the first cycle of our reductions.

$$\max\{k_1, m_1\} \leq n_1 < 2950 \quad \text{and} \quad \max\{k_2, m_2\} \leq n_2 < 1.12 \times 10^{38}. \tag{5.6}$$

From (5.6), we note that the upper bound on n_2 represents a very good reduction of the bound given in Lemma 4.8. Hence, we expect that if we restart our reduction cycle with the new bound on n_2, then we get better bounds on n_1 and n_2. Thus, we return to the inequality (5.1) and take $M := 1.12 \times 10^{38}$. A computer seach in *Mathematica* reveals that

$$q_{82} > M > n_2 > k_2 - k_1 \quad \text{and} \quad a(M) := \max\{a_i : 0 \leq i \leq 82\} = a_{12} = 134,$$

from which it follows that $m_3 \leq 100$. We now return to (5.3) and we put $X := 1.12 \times 10^{40}$ and $C := (20X)^5$ and then apply the LLL algorithm in Lemma 3.5 to $m_3 \in [1, 100]$. After a computer search in *Mathematica*, we get

$$1.04 \times 10^{-139} < \min_{1 \leq m_3 \leq 100} |\Gamma_4| < 24k_2\alpha^{-2\min\{n_i, m_j\}},$$

then $\min\{n_i, m_j\} \leq 410$. By continuing under the assumption that $m_j := \min\{n_i, m_j\} \leq 426$, we return to (5.5) and put $X := 1.12 \times 10^{40}$, $C := (20X)^5$ and $M := 1.12 \times 10^{38}$ for the case $m_3 < m_4$ and the case $m_3 = m_4$. After a computer search, we confirm that

$$4.39 \times 10^{-168} < \min_{\substack{1 \leq m_3 \leq 100 \\ m_3+1 \leq m_4 \leq 426}} |\Gamma_4| < 24k_2\alpha^{-2n_3}. \tag{5.7}$$

This gives $n_1 \leq 494$ which holds in both cases. Hence, by a similar procedure given in the first cycle, we get that $n_2 < 3 \times 10^{36}$.

We record what we have proved.

Lemma 5.2. *Let $(k_i, n_i, m_i\}$ be a solution to the Diophantine equation $x_{k_i} = L_{n_i}L_{m_i}$, with $0 \leq m_i \leq n_i$ for $i = 1, 2$ and $1 \leq k_1 \leq k_2$, then*

$$\max\{k_1, m_1\} \leq n_1 \leq 494 \quad \textit{and} \quad \max\{k_2, m_2\} \leq n_2 < 3 \times 10^{36}.$$

5.2. **The final reduction.** Returning back to (4.9) and (4.17) and using the fact that (x_1, y_1) is the smallest positive solution to the Pell equation (1.3), we obtain

$$\begin{aligned} x_k &= \frac{1}{2}(\delta^k + \eta^k) = \frac{1}{2}\left(\left(x_1 + y_1\sqrt{d}\right)^k + \left(x_1 - y_1\sqrt{d}\right)^k\right) \\ &= \frac{1}{2}\left(\left(x_1 + \sqrt{x_1^2 \mp 1}\right)^k + \left(x_1 - \sqrt{x_1^2 \mp 1}\right)^k\right) := P_k^{\pm}(x_1). \end{aligned}$$

Thus, we return to the Diophantine equation $x_{k_1} = L_{n_1}L_{m_1}$ and consider the equations

$$P_{k_1}^{+}(x_1) = L_{n_1}L_{m_1} \quad \text{and} \quad P_{k_1}^{-}(x_1) = L_{n_1}L_{m_1}, \tag{5.8}$$

with $k_1 \in [1, 500]$, $m_1 \in [0, 500]$ and $n_1 \in [m_1 + 1, 500]$.

Besides the trivial case $k_1 = 1$, with the help of a computer search in *Mathematica* on the above equations in (5.8), we list the only nontrivial solutions in Table 1 below. We also note that

$$7 + 5\sqrt{2} = (1 + \sqrt{2})^3,$$

so these solutions come from the same Pell equation with $d = 2$.

$Q^+_{k_1}(x_1)$				
k_1	x_1	y_1	d	δ
2	2	1	3	$2+\sqrt{3}$
2	5	2	6	$5+2\sqrt{6}$
2	10	3	11	$10+3\sqrt{11}$
2	4	1	15	$4+\sqrt{15}$
2	6	1	35	$6+\sqrt{35}$

$Q^-_{k_1}(x_1)$				
k_1	x_1	y_1	d	δ
2	1	1	2	$1+\sqrt{2}$
2	2	1	5	$2+\sqrt{5}$
2	7	5	2	$7+5\sqrt{2}$
2	4	1	17	$4+\sqrt{17}$
2	26	1	677	$26+\sqrt{677}$
2	179	1	32042	$179+\sqrt{32042}$

TABLE 1. Solutions to $P^{\pm}_{k_1}(x_1) = L_{n_1}L_{m_1}$

From the above tables, we set each $\delta := \delta_t$ for $t = 1, 2, \ldots 10$. We then work on the linear forms in logarithms Γ_1 and Γ_2, in order to reduce the bound on n_2 given in Lemma 5.2. From the inequality (4.10), for $(k, n, m) := (k_2, n_2, m_2)$, we write

$$\left| k_2 \frac{\log \delta_t}{\log \alpha} - (n_2 + m_2) + \frac{\log 2}{\log(\alpha^{-1})} \right| < \left(\frac{12}{\log \alpha} \right) \alpha^{-2m_2}, \tag{5.9}$$

for $t = 1, 2, \ldots 10$.

We put

$$\tau_t := \frac{\log \delta_t}{\log \alpha}, \qquad \mu_t := \frac{\log 2}{\log(\alpha^{-1})} \qquad \text{and} \quad (A_t, B_t) := \left(\frac{12}{\log \alpha}, \alpha \right).$$

We note that τ_t is transcendental by the Gelfond-Schneider's Theorem and thus, τ_t is irrational. We can rewrite the above inequality, (5.9) as

$$0 < |k_2\tau_t - (n_2 + m_2) + \mu_t| < A_t B_t^{-2m_2}, \quad \text{for} \quad t = 1, 2, \ldots, 10. \tag{5.10}$$

We take $M := 3 \times 10^{36}$ which is the upper bound on n_2 according to Lemma 5.2 and apply Lemma 3.4 to the inequality (5.10). As before, for each τ_t with $t = 1, 2, \ldots, 10$, we compute its continued fraction $[a_0^{(t)}, a_1^{(t)}, a_2^{(t)}, \ldots]$ and its convergents $p_0^{(t)}/q_0^{(t)}, p_1^{(t)}/q_1^{(t)}, p_2^{(t)}/q_2^{(t)}, \ldots$. For each case, by means of a computer search in *Mathematica*, we find and integer s_t such that

$$q_{s_t}^{(t)} > 18 \times 10^{36} = 6M \qquad \text{and} \qquad \varepsilon_t := ||\mu_t q^{(t)}|| - M||\tau_t q^{(t)}| > 0.$$

We finally compute all the values of $b_t := \lfloor \log(A_t q_{s_t}^{(t)}/\epsilon_t)/\log B_t \rfloor/2$. The values of b_t correspond to the upper bounds on m_2, for each $t = 1, 2, \ldots, 10$, according to Lemma 3.4.

Note that we have a problem at $\delta_7 := 2 + \sqrt{5}$. This is because

$$2 + \sqrt{5} = 2\left(\frac{1 + \sqrt{5}}{2} \right)^2 = 2\alpha^2.$$

So in this case we have $\Gamma_1 := (k_2 - 1)\log 2 - (n_2 + m_2 - 2k_2)\log \alpha$. Thus,

$$\left| \frac{\log 2}{\log \alpha} - \frac{n_2 + m_2 - 2k_2}{k_2 - 1} \right| < \frac{12}{(k_2 - 1)\alpha^{2m_2} \log \alpha}$$

By a similar procedure given in Subsection 5.1 with $M := 3 \times 10^{36}$, we get that $q_{77} > M$ and $a(M) := \max\{a_i : 0 \le i \le 77\} = 134$. From this we can conclude that $m_2 \le 96$.

The results of the computation for each t are recorded in Table 2 below.

t	δ_t	s_t	q_{s_t}	$\varepsilon_t >$	b_t
1	$2+\sqrt{3}$	68	2.07577×10^{37}	0.319062	94
2	$5+2\sqrt{6}$	91	8.19593×10^{37}	0.087591	97
3	$10+3\sqrt{11}$	67	2.25831×10^{38}	0.316767	96
4	$4+\sqrt{15}$	70	2.78896×10^{37}	0.329388	94
5	$6+\sqrt{35}$	74	1.75745×10^{38}	0.409752	96
6	$1+\sqrt{2}$	76	2.02409×10^{37}	0.263855	94
7	$2+\sqrt{5}$	–	–	–	96
8	$4+\sqrt{17}$	78	4.76137×10^{37}	0.131771	96
9	$26+\sqrt{677}$	65	3.17521×10^{37}	0.356148	94
10	$179+\sqrt{32042}$	77	3.45317×10^{37}	0.384127	94

Table 2. First reduction computation results

By replacing $(k, n, m) := (k_2, n_2, m_2)$ in the inequality (4.17), we can write

$$\left| k_2 \frac{\log \delta_t}{\log \alpha} - n_2 + \frac{\log(2L_{m_2})}{\log(\alpha^{-1})} \right| < \left(\frac{12}{\log \alpha} \right) \alpha^{-2n_2}, \tag{5.11}$$

for $t = 1, 2, \ldots, 10$.

We now put

$$\tau_t := \frac{\log \delta_t}{\log \alpha}, \quad \mu_{t,m_2} := \frac{\log(2L_{m_2})}{\log(\alpha^{-1})} \quad \text{and} \quad (A_t, B_t) := \left(\frac{12}{\log \alpha}, \alpha \right).$$

With the above notations, we can rewrite (5.11) as

$$0 < |k_2 \tau_t - n_2 + \mu_{t,m_2}| < A_t B_t^{-2n_2}, \quad \text{for} \quad t = 1, 2, \ldots 10. \tag{5.12}$$

We again apply Lemma 3.4 to the above inequality (5.12), for

$$t = 1, 2, \ldots, 10, \quad m_2 = 1, 2, \ldots, b_t, \quad \text{with} \quad M := 3 \times 10^{36}.$$

We take

$$\varepsilon_{t,m_2} := ||\mu_t q^{(t,m_2)}|| - M||\tau_t q^{(t,m_2)}|| > 0,$$

and

$$b_t = b_{t,m_2} := \lfloor \log(A_t q_{s_t}^{(t,m_2)} / \epsilon_{t,m_2}) / \log B_t \rfloor / 2.$$

The case $\delta_7 = 2 + \sqrt{5}$ is again treated individually by a similar procedure as in the previous step. With the help of Mathematica, we record the results of the computation in Table 3 below.

t	1	2	3	4	5	6	7	8	9	10
$\varepsilon_{t,m_2} >$	0.0145	0.0002	0.0006	0.0034	0.0106	0.0005	–	0.0009	0.0019	0.0010
b_{t,m_2}	97	103	102	99	99	100	102	100	99	100

Table 3. Final reduction computation results

$$\text{Therefore,}\quad \max\{b_{t,m_2} : t = 1, 2, \dots, 10 \quad \text{and} \quad m_2 = 1, 2, \dots b_t\} \leq 103.$$

Thus, by Lemma 3.4, we have that $n_2 \leq 103$, for all $t = 1, 2, \dots, 10$. From the fact that $\delta^k \leq \alpha^{n+m+6}$, we can conclude that $k_1 < k_2 \leq 198$. Collecting everything together, our problem is reduced to search for the solutions for (2.1) in the following ranges

$$1 \leq k_1 < k_2 \leq 200, \quad 0 \leq m_1 \leq n_1 \leq 200 \quad \text{and} \quad 0 \leq m_2 \leq n_2 \leq 200.$$

After a computer search on the equation (2.1) on the above ranges, we obtained the following solutions, which are the only solutions for the exceptional d cases we have stated in Theorem 2.1:

For the $+1$ case:

$$\begin{aligned} (d = 3) &\qquad x_1 = 2 = L_1L_0, \quad x_2 = 7 = L_4L_1; \\ (d = 15) &\qquad x_1 = 4 = L_3L_1 = L_0L_0, \quad x_5 = 15124 = L_{11}L_9; \\ (d = 35) &\qquad x_1 = 6 = L_2L_0, \quad x_3 = 846 = L_8L_6. \end{aligned}$$

For the -1 case:

$$\begin{aligned} (d = 2) &\qquad x_1 = 1 = L_3L_3, \quad x_2 = 3 = L_2L_1, \quad x_3 = 7 = L_4L_1, \quad x_9 = 1393 = L_{11}L_4; \\ (d = 5) &\qquad x_1 = 2 = L_1L_0, \quad x_2 = 9 = L_2L_2; \\ (d = 17) &\qquad x_1 = 4 = L_3L_1 = L_0L_0, \quad x_2 = 33 = L_5L_2. \end{aligned}$$

This completes the proof of Theorem 2.1. □

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ON THE x–COORDINATES OF PELL EQUATIONS WHICH ARE PRODUCTS OF TWO PELL NUMBERS

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Abstract. Let $\{P_m\}_{m\geq 0}$ be the sequence of Pell numbers given by $P_0 = 0,\ P_1 = 1$ and $P_{m+2} = 2P_{m+1} + P_m$ for all $m \geq 0$. In this paper, for an integer $d \geq 2$ which is square free, we show that there is at most one value of the positive integer x participating in the Pell equation $x^2 - dy^2 = \pm 1$ which is a product of two Pell numbers.

1. Introduction

Let $\{P_m\}_{m\geq 0}$ be the sequence of Pell numbers given by $P_0 = 0,\ P_1 = 1$ and

$$P_{m+2} = 2P_{m+1} + P_m$$

for all $m \geq 0$. This is sequence A000129 on the Online Encyclopedia of Integer Sequences (OEIS). The first few terms of this sequence are

$$\{P_m\}_{m\geq 0} = 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, \ldots.$$

Putting $(\alpha, \beta) = (1+\sqrt{2}, 1-\sqrt{2})$ for the roots of the characteristic equation $r^2-2r-1 = 0$ of the Pell sequence, the Binet formula for its general terms is given by

$$P_m = \frac{\alpha^m - \beta^m}{2\sqrt{2}}, \qquad \text{for all} \quad m \geq 0. \tag{1}$$

Furthermore, we can prove by induction that the inequality

$$\alpha^{m-2} \leq P_m \leq \alpha^{m-1}, \tag{2}$$

holds for all $m \geq 1$.

Let $d \geq 2$ be a positive integer which is not a perfect square. It is well known that the Pell equation

$$x^2 - dy^2 = \pm 1 \tag{3}$$

has infinitely many positive integer solutions (x, y). By putting (x_1, y_1) for the smallest positive solution, all solutions are of the form (x_n, y_n) for some positive integer n, where

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n \qquad \text{for all} \quad n \geq 1. \tag{4}$$

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Furthermore, the sequence $\{x_k\}_{k\geq 1}$ is binary recurrent. In fact, the following formula

$$x_n = \frac{(x_1 + y_1\sqrt{d})^n + (x_1 - y_1\sqrt{d})^n}{2},$$

holds for all positive integers n.

Recently, Kafle et al. [8] considered the Diophantine equation

$$x_n = F_\ell F_m, \tag{5}$$

where $\{F_m\}_{m\geq 0}$ is the sequence of Fibonacci numbers given by $F_0 = 0$, $F_1 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for all $m \geq 0$. They proved that equation (5) has at most one solution n in positive integers except for $d = 2,\ 3, 5$, for which case equation (5) has the solutions $x_1 = 1$ and $x_2 = 3$, $x_1 = 2$ and $x_2 = 26$, $x_1 = 2$ and $x_2 = 9$, respectively.

There are many other researchers who have studied related problems involving the intersection sequence $\{x_n\}_{n\geq 1}$ with linear recurrence sequences of interest. For example, [4, 6, 9, 10, 11, 13, 14, 16].

2. Main Result

In [5], together with Luca and Rakotomala we studied a problem involving the intersection of Fibonacci numbers with a product of two Pell numbers, so it is natural to study the intersection of the $x-$coordinates of Pell equations with a product of two Pell numbers. In this paper, we study a similar problem to that of Kafle et al. [8], but with the Pell numbers instead of the Fibonacci numbers. That is, we show that there is at most one value of the positive integer x participating in (3) which is a product of two Pell numbers. This can be interpreted as solving the Diophantine equation

$$x_n = P_\ell P_m. \tag{6}$$

Theorem 1. *For each square-free integer $d \geq 2$ there is at most one n such that the equation* (6) *holds.*

3. Preliminary Results

3.1. **Notations and terminology from algebraic number theory.**

We begin by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with minimal primitive polynomial over the integers

$$a_0x^d + a_1x^{d-1} + \cdots + a_d = a_0\prod_{i=1}^{d}(x - \eta^{(i)}),$$

where the leading coefficient a_0 is positive and the $\eta^{(i)}$'s are the conjugates of η. Then the *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d}\log\left(\max\{|\eta^{(i)}|, 1\}\right)\right).$$

In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log\max\{|p|, q\}$. The following are some of the properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections of this paper without reference:

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta) \qquad (s \in \mathbb{Z}). \end{aligned} \tag{7}$$

3.2. **Linear forms in logarithms.** In order to prove our main result Theorem 1, we need to use several times a Baker–type lower bound for a nonzero linear form in logarithms of algebraic numbers. There are many such in the literature like that of Baker and Wüstholz from [2]. We use the one of Matveev from [15]. Matveev [15] proved the following theorem, which is one of our main tools in this paper.

Theorem 2. *Let $\gamma_1, \ldots, \gamma_t$ be positive real algebraic numbers in a real algebraic number field $\mathbb{K}$ of degree D, $b_1, \ldots, b_t$ be nonzero integers, and assume that*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1, \tag{8}$$

is nonzero. Then

$$\log |\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{D h(\gamma_i), |\log \gamma_i|, 0.16\}, \qquad \text{for all} \qquad i = 1, \ldots, t.$$

When $t = 2$ and γ_1, γ_2 are positive and multiplicatively independent, we can use a result of Laurent, Mignotte and Nesterenko [12]. Namely, let in this case B_1, B_2 be real numbers larger than 1 such that

$$\log B_i \geq \max\left\{h(\gamma_i), \frac{|\log \gamma_i|}{D}, \frac{1}{D}\right\}, \qquad \text{for} \quad i = 1, 2,$$

and put

$$b' := \frac{|b_1|}{D \log B_2} + \frac{|b_2|}{D \log B_1}.$$

Put

$$\Gamma := b_1 \log \gamma_1 + b_2 \log \gamma_2. \tag{9}$$

We note that $\Gamma \neq 0$ because $\gamma_1$and γ_2 are multiplicatively independent. The following result is Corollary 2 in [12].

Theorem 3. *With the above notations, assuming that η_1, η_2 are positive and multiplicatively independent, then*

$$\log |\Gamma| > -24.34 D^4 \left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log B_1 \log B_2. \tag{10}$$

Note that with Γ given by (9), we have $e^{\Gamma} - 1 = \Lambda$, where Λ is given by (8) in case $t = 2$, which explains the connection between Theorems 2 and 3.

3.3. **Reduction procedure.** During the calculations, we get upper bounds on our variables which are too large, thus we need to reduce them. To do so, we use some results from the theory of continued fractions.

For the treatment of linear forms homogeneous in two integer variables, we use the well-known classical result in the theory of Diophantine approximation.

Lemma 1. *Let τ be an irrational number, $\frac{p_0}{q_0}, \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots$ be all the convergents of the continued fraction of τ and M be a positive integer. Let N be a nonnegative integer such that $q_N > M$. Then putting $a(M) := \max\{a_i : i = 0, 1, 2, \ldots, N\}$, the inequality*

$$\left|\tau - \frac{r}{s}\right| > \frac{1}{(a(M)+2)s^2},$$

holds for all pairs (r, s) of positive integers with $0 < s < M$.

For a nonhomogeneous linear form in two integer variables, we use a slight variation of a result due to Dujella and Pethő (see [7], Lemma 5a). For a real number X, we write $||X|| := \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 2. *Let M be a positive integer, $\frac{p}{q}$ be a convergent of the continued fraction of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let further $\varepsilon := ||\mu q|| - M||\tau q||$. If $\varepsilon > 0$, then there is no solution to the inequality*

$$0 < |u\tau - v + \mu| < AB^{-w},$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

At various occasions, we need to find a lower bound for linear forms in logarithms with bounded integer coefficients in three and four variables. In this case we use the LLL algorithm that we describe below. Let $\tau_1, \tau_2, \ldots \tau_t \in \mathbb{R}$ and the linear form

$$x_1\tau_1 + x_2\tau_2 + \cdots + x_t\tau_t \quad \text{with} \quad |x_i| \leq X_i. \tag{11}$$

We put $X := \max\{X_i\}$, $C > (tX)^t$ and consider the integer lattice Ω generated by

$$\mathbf{b}_j := \mathbf{e}_j + \lfloor C\tau_j \rceil \quad \text{for} \quad 1 \leq j \leq t-1 \quad \text{and} \quad \mathbf{b}_t := \lfloor C\tau_t \rceil \mathbf{e}_t,$$

where C is a sufficiently large positive constant.

Lemma 3. *Let $X_1, X_2, \ldots, X_t$ be positive integers such that $X := \max\{X_i\}$ and $C > (tX)^t$ is a fixed sufficiently large constant. With the above notation on the lattice Ω, we consider a reduced base $\{\boldsymbol{b}_i\}$ to Ω and its associated Gram-Schmidt orthogonalization base $\{\boldsymbol{b}_i^*\}$. We set*

$$c_1 := \max_{1 \leq i \leq t} \frac{||\boldsymbol{b}_1||}{||\boldsymbol{b}_i^*||}, \quad \theta := \frac{||\boldsymbol{b}_1||}{c_1}, \quad Q := \sum_{i=1}^{t-1} X_i^2 \quad \text{and} \quad R := \frac{1}{2}\left(1 + \sum_{i=1}^{t} X_i\right).$$

If the integers x_i are such that $|x_i| \leq X_i$, for $1 \leq i \leq t$ and $\theta^2 \geq Q + R^2$, then we have

$$\left|\sum_{i=1}^{t} x_i\tau_i\right| \geq \frac{\sqrt{\theta^2 - Q} - R}{C}.$$

For the proof and further details, we refer the reader to the book of Cohen. (Proposition 2.3.20 in [3], pp. 58–63).

4. Bounding the variables

We assume that (x_1, y_1) is the smallest positive solution of the Pell equation (3). We set

$$x_1^2 - dy_1^2 =: \epsilon, \qquad \epsilon \in \{\pm 1\},$$

and put

$$\delta := x_1 + \sqrt{d}y_1 \qquad \text{and} \qquad \eta := x_1 - \sqrt{d}y_1 = \epsilon\delta^{-1}.$$

From (4), we get

$$x_n = \frac{1}{2}\left(\delta^n + \eta^n\right). \tag{12}$$

Since $\delta \geq 1 + \sqrt{2} = \alpha$, it follows that the estimate

$$\frac{\delta^n}{\alpha^2} \leq x_n < \frac{\delta^n}{\alpha} \quad \text{holds for all} \quad n \geq 1. \tag{13}$$

We let $(n, \ell, m) := (n_i, \ell_i, m_i)$ for $i = 1, 2$ be the solutions of (6). By (2) and (13), we get

$$\alpha^{\ell+m-4} \leq P_\ell P_m = x_n < \frac{\delta^n}{\alpha} \quad \text{and} \quad \frac{\delta^n}{\alpha^2} \leq x_n = P_\ell P_m \leq \alpha^{\ell+m-2}, \tag{14}$$

so

$$nc_1 \log\delta < \ell + m < nc_1\log\delta + 3 \quad \text{where} \quad c_1 := \frac{1}{\log\alpha}. \tag{15}$$

To fix ideas, we assume that

$$m \geq \ell \qquad \text{and} \qquad n_1 < n_2.$$

We also put

$$\ell_3 := \min\{\ell_1, \ell_2\}, \quad \ell_4 := \max\{\ell_1, \ell_2\}, \quad m_3 := \min\{m_1, m_2\}, \quad m_4 := \max\{m_1, m_2\}.$$

Using the inequality (15) together with the fact that $\delta \geq 1 + \sqrt{2} = \alpha$ (so, $c_1 \log\delta > 1$), gives us that

$$n_2 < n_2 c_1 \log\delta < 2m_2 \leq 2m_4,$$

so

$$n_1 < n_2 < 2m_4. \tag{16}$$

Thus, it is enough to find an upper bound on m_4. Substituting (1) and (12) in (6) we get

$$\frac{1}{2}(\delta^n + \eta^n) = \frac{1}{8}(\alpha^\ell - \beta^\ell)(\alpha^m - \beta^m). \tag{17}$$

This can be regrouped as

$$\delta^n(2^2)\alpha^{-\ell-m} - 1 = -4\eta^n\alpha^{-\ell-m} - (\beta\alpha^{-1})^\ell - (\beta\alpha^{-1})^m + (\beta\alpha^{-1})^{\ell+m}.$$

Since $\beta = -\alpha^{-1}$, $\eta = \varepsilon\delta^{-1}$ and using the fact that $\delta^n \geq \alpha^{l+m-3}$ (by (14)), we get

$$\begin{aligned}\left|\delta^n(2^2)\alpha^{-\ell-m} - 1\right| &\leq \frac{4}{\delta^n\alpha^{\ell+m}} + \frac{1}{\alpha^{2l}} + \frac{1}{\alpha^{2m}} + \frac{1}{\alpha^{2(\ell+m)}}\\ &\leq \frac{4\alpha^3}{\alpha^{2(\ell+m)}} + \frac{3}{\alpha^{2\ell}} \quad < \quad \frac{60}{\alpha^{2\ell}},\end{aligned}$$

In the above, we have also used the facts that $m \geq \ell$ and $4\alpha^3 + 3 < 60$. Hence,

$$\left|\delta^n(2^2)\alpha^{-\ell-m} - 1\right| < \frac{60}{\alpha^{2\ell}}. \tag{18}$$

We let $\Lambda := \delta^n(2^2)\alpha^{-\ell-m} - 1$. We put

$$\Gamma := n\log\delta - 2\log 2 - (\ell+m)\log\alpha. \tag{19}$$

Note that $e^\Gamma - 1 = \Lambda$. If $\ell > 100$, then $\frac{60}{\alpha^{2\ell}} < \frac{1}{2}$. Since $|e^\Gamma - 1| < 1/2$, it follows that

$$|\Gamma| < 2|e^\Gamma - 1| < \frac{120}{\alpha^{2l}}. \tag{20}$$

By recalling that $(n, \ell, m) = (n_i, \ell_i, m_i)$ for $i = 1, 2$, we get that

$$|n_i\log\delta - 2\log 2 - (\ell_i + m_i)\log\alpha| < \frac{120}{\alpha^{2\ell_i}} \tag{21}$$

holds for both $i = 1, 2$ provided $\ell_3 > 100$.

We apply Theorem 2 on the left-hand side of (18). First, we need to check that $\Lambda \neq 0$. Well, if it were, then $\delta^n\alpha^{-\ell-m} = \frac{1}{4}$. However, this is impossible since $\delta^n\alpha^{-\ell-m}$ is a unit while $1/4$ is not. Thus, $\Lambda \neq 0$, and we can apply Theorem 2. We take the data

$$t := 3, \quad \gamma_1 := \delta, \quad \gamma_2 := 2, \quad \gamma_3 := \alpha, \quad b_1 := n, \quad b_2 := 2, \quad b_3 := -\ell - m.$$

We take $\mathbb{K} := \mathbb{Q}(\sqrt{d}, \alpha)$ which has degree $D \leq 4$ (it could be that $d = 2$ in which case $D = 2$; otherwise, $D = 4$). Since $\delta \geq 1 + \sqrt{2} = \alpha$, the second inequality in (14) tells us that $n \leq \ell + m$, so we take $B := 2m$. We have $h(\gamma_1) = h(\delta) = \frac{1}{2}\log\delta$, $h(\gamma_2) = h(2) = \log 2$ and $h(\gamma_3) = h(\alpha) = \frac{1}{2}\log\alpha$. Thus, we can take $A_1 := 2\log\delta$, $A_2 := 4\log 2$ and $A_3 := 2\log\alpha$. Now, Theorem 2 tells us that

$$\begin{aligned}\log|\Lambda| &> -1.4 \times 30^6 \times 3^{4.5} \times 4^2(1+\log 4)(1+\log(2m))(2\log\delta)(4\log 2)(2\log\alpha)\\ &> -5.34 \times 10^{13}\log\delta(1+\log(2m)).\end{aligned}$$

By comparing the above inequality with (18), we get

$$2\ell\log\alpha - \log 60 < 5.34 \times 10^{13}\log\delta(1+\log(2m)). \tag{22}$$

Thus

$$\ell < 5.36 \times 10^{13}\log\delta(1+\log(2m)). \tag{23}$$

Since, $\delta^n < \alpha^{\ell+m}$, we get that

$$n\log\delta < (\ell+m)\log\alpha \leq 2m\log\alpha, \tag{24}$$

which together with the estimate (23) gives

$$n\ell < 5.35 \times 10^{13} m(1+\log(2m)). \tag{25}$$

Let us record what we have proved, since this will be important later-on.

Lemma 4. *If $x_n = P_\ell P_m$ and $m \ge \ell$, then*

$$\ell < 5.36\times 10^{13}\log\delta(1+\log(2m)), \quad n\ell < 5.35\times 10^{13} m(1+\log(2m)), \quad n\log\delta < 2m\log\alpha.$$

Note that we did not assume that $\ell_3 > 100$ for Lemma 4 since we have worked with the inequality (18) and not with (20). We now again assume that $l_3 > 100$. Then the two inequalities (21) hold. We eliminate the term involving $\log\delta$ by multiplying the inequality for $i = 1$ with n_2 and the one for $i = 2$ with n_1, subtract them and apply the triangle inequality as follows

$$\begin{aligned}
&|2(n_2-n_1)\log 2-(n_2(\ell_1+m_1)-n_1(\ell_2+m_2))\log\alpha|\\
&=|n_2(n_1\log\delta+2\log 2-(\ell_1+m_1)\log\alpha)-n_1(n_2\log\delta+2\log 2-(\ell_2+m_2)\log\alpha)|\\
&\le n_2\,|n_1\log\delta+2\log 2-(\ell_1+m_1)\log\alpha|+n_1\,|n_2\log\delta+2\log 2-(\ell_2+m_2)\log\alpha|\\
&\le \frac{120n_2}{\alpha^{2\ell_1}}+\frac{120n_1}{\alpha^{2\ell_2}} \quad < \quad \frac{240n_2}{\alpha^{2\ell_3}}.
\end{aligned}$$

Thus,

$$|\Gamma| := |(n_2-n_1)\log 4-(n_2(\ell_1+m_1)-n_1(\ell_2+m_2))\log\alpha| < \frac{240n_2}{\alpha^{2\ell_3}}. \tag{26}$$

We are now set to apply Theorem 3 with the data

$$t:=2,\quad \gamma_1:=4,\quad \gamma_2:=\alpha,\quad b_1:=n_2-n_1,\quad b_2:=n_2(\ell_1+m_1)-n_1(\ell_2+m_2).$$

The fact that $\gamma_1 = 2$ and $\gamma_2 = \alpha$ are multiplicatively independent follows because α is a unit while 2 is not. We observe that $n_2 - n_1 < n_2$, whereas by the absolute value of the inequality in (26), we have

$$|n_2(\ell_1+m_1)-n_1(\ell_2+m_2)| \le (n_2-n_1)\frac{2\log 2}{\log\alpha}+\frac{240n_2}{\alpha^{2\ell_3}\log\alpha} < 2n_2,$$

because $\ell_3 > 100$. We have that $\mathbb{K} := \mathbb{Q}(\alpha)$, which has $D := 2$. So we can take

$$\log B_1 = \max\left\{h(\gamma_1), \frac{|\log\gamma_1|}{2}, \frac{1}{2}\right\} = 2\log 2,$$

and

$$\log B_2 = \max\left\{h(\gamma_2), \frac{|\log\gamma_2|}{2}, \frac{1}{2}\right\} = \frac{1}{2}.$$

Thus,

$$b' = \frac{|n_2-n_1|}{2\log B_2}+\frac{|n_2(\ell_1+m_1)-n_1(\ell_2+m_2)|}{2\log B_1} \le n_2+\frac{n_2}{2\log 2} < 2n_2.$$

Now Theorem 3 tells us that with

$$\Gamma = 2(n_2-n_1)\log 2-(n_2(\ell_1+m_1)-n_1(\ell_2+m_2))\log\alpha,$$

we have

$$\log|\Gamma| > -24.34\times 2^4\left(\max\{\log(2n_2)+0.14, 10.5\}\right)^2\cdot(2\log 2)\cdot(1/2).$$

Thus,

$$\log|\Gamma| > -270\left(\max\{\log(2n_2)+0.14, 10.5\}\right)^2.$$

By comparing the above inequality with (26), we get

$$2\ell_3 \log\alpha - \log(240 n_2) < 270\left(\max\{\log(2n_2)+0.14, 10.5\}\right)^2.$$

If $n_2 \le 15785$, then $\log(2n_2)+0.14 < 10.5$. Thus, the last inequality above gives

$$2\ell_3 \log\alpha < 270 \times 10.5^2 + \log(240 \times 15785),$$

giving $\ell_3 < 16000$ in this case. Otherwise, $n_2 > 15785$, and we get

$$2\ell_3 \log\alpha < 270(1+\log n_2)^2 + \log(240 n_2) < 280(1+\log n_2)^2,$$

which gives

$$\ell_3 < 160(1+\log n_2)^2.$$

We record what we have proved

Lemma 5. *If $\ell_3 > 100$, then either*

(i) $n_2 \le 15785$ *and* $\ell_3 < 16000$ *or*
(ii) $n_2 > 15785$, *in which case* $\ell_3 < 160(1+\log n_2)^2$.

Now suppose that some ℓ is fixed in (6), or at least we have some good upper bounds on it. We rewrite (6) using (1) and (12) as

$$\frac{1}{2}(\delta^n + \eta^n) = \frac{P_\ell}{2\sqrt{2}}(\alpha^m - \beta^m),$$

so

$$\delta^n\left(\frac{\sqrt{2}}{P_\ell}\right)\alpha^{-m} - 1 = -\frac{\sqrt{2}}{P_\ell}\eta^n\alpha^{-m} - (\beta\alpha^{-1})^m.$$

Since $\ell \ge 1$, $\beta = -\alpha^{-1}$, $\eta = \varepsilon\delta^{-1}$ and $\delta^n > \alpha^{\ell+m-3}$, we get

$$\begin{aligned}\left|\delta^n\left(\frac{\sqrt{2}}{P_\ell}\right)\alpha^{-m} - 1\right| &\le \frac{\sqrt{2}}{P_\ell\delta^n\alpha^m} + \frac{1}{\alpha^{2m}} \le \frac{\sqrt{2}\alpha^4}{\alpha^{2(\ell+m)}} + \frac{1}{\alpha^{2m}}\\ &\le \frac{\sqrt{2}\alpha^4+1}{\alpha^{2m}} < \frac{50}{\alpha^{2m}},\end{aligned}$$

where we have used the fact that $m \ge \ell \ge 1$ and $\sqrt{2}\alpha^4 + 1 < 50$. Hence,

$$|\Lambda_1| := \left|\delta^n\left(\frac{\sqrt{2}}{P_\ell}\right)\alpha^{-m} - 1\right| < \frac{50}{\alpha^{2m}}. \tag{27}$$

We assume that $m_3 > 100$. In particular, $\frac{50}{\alpha^{2m}} < \frac{1}{2}$ for $m \in \{m_1, m_2\}$, so we get by the previous argument that

$$|\Gamma_1| := \left|n\log\delta + \log(\sqrt{2}/P_\ell) - m\log\alpha\right| < \frac{100}{\alpha^{2m}}. \tag{28}$$

We are now set to apply Theorem 2 on the left-hand side of (27) with the data

$$t := 3, \quad \gamma_1 := \delta, \quad \gamma_2 := \sqrt{2}/P_\ell, \quad \gamma_3 := \alpha, \quad b_1 := n, \quad b_2 := 1, \quad b_3 := -m.$$

First, we need to check that $\Lambda_1 := \delta^n(\sqrt{2}/P_\ell)\alpha^{-m} - 1 \ne 0$. If not, then $\delta^n = \alpha^m P_\ell/\sqrt{2}$. The left-hand side belongs to the field $\mathbb{Q}(\sqrt{d})$ but not rational while the right-hand side belongs to the field $\mathbb{Q}(\sqrt{2})$. This is not possible unless $d = 2$. In this last case, δ is a unit in $\mathbb{Q}(\sqrt{2})$ while $P_\ell/\sqrt{2}$ is not a unit in $\mathbb{Q}(\sqrt{2})$ since the norm of this last element is $P_\ell^2/2 \neq \pm 1$. So $\Lambda_1 \neq 0$. Thus, we can Theorem 2. We have the field $\mathbb{K} := \mathbb{Q}(\sqrt{d}, \sqrt{2})$ which has degree $D \leq 4$. We also have

$$h(\gamma_1) = \frac{1}{2}\log\delta, \quad h(\gamma_2) = \max\left\{\frac{1}{2}\log 2, \log P_\ell\right\} \quad \text{and} \quad h(\gamma_3) = \frac{1}{2}\log\alpha.$$

Since $P_\ell \leq \alpha^{\ell-1} < 2^{2\ell}$, we can take

$$A_1 := 2\log\delta, \quad A_2 := 8\ell\log 2 \quad \text{and} \quad A_3 := 2\log\alpha.$$

Then, by Theorem 2 we get

$$\begin{aligned}\log|\Lambda_1| &> -1.4\times 30^6\times 3^{4.5}\times 4^2(1+\log 4)(1+\log m)(2\log\delta)(8\ell\log 2)(2\log\alpha)\\ &> -7.58\times 10^{13}\ell\log\delta(1+\log m).\end{aligned}$$

Comparing the above inequality with (27), we get

$$2m\log\alpha - \log 50 < 7.58\times 10^{13}\ell\log\delta(1+\log m),$$

which implies that

$$m < 4.30\times 10^{13}\ell\log\delta(1+\log m). \tag{29}$$

We record what we have proved.

Lemma 6. *If $x_n = P_\ell P_m$ with $m \geq \ell \geq 1$, then we have*

$$m < 4.30\times 10^{13}\ell\log\delta(1+\log m).$$

Note that we did not use the assumption that $\ell_3 > 100$ of that $m_3 > 100$ for Lemma 6 since we worked with the inequality (27) not with the inequality (28). We now assume that $m_3 > 100$ and in particular (28) holds for $(n,\ell,m) = (n_i,\ell_i,m_i)$ for both $i = 1,2$. By the previous procedure, we also eliminate the term involving $\log\delta$ as follows

$$\begin{aligned}\left|n_2\log(\sqrt{2}/P_{\ell_1}) - n_1\log(\sqrt{2}/P_{\ell_2}) - (n_2m_1 - n_1m_2)\log\alpha\right| &< \frac{100n_2}{\alpha^{2m_1}} + \frac{100n_1}{\alpha^{2m_2}}\\ &< \frac{200n_2}{\alpha^{2m_3}}.\end{aligned} \tag{30}$$

We assume that $\alpha^{2m_3} > 400n_2$. If we put

$$\Gamma_2 := n_2\log(\sqrt{2}/P_{\ell_1}) - n_1\log(\sqrt{2}/P_{\ell_2}) - (n_2m_1 - n_1m_2)\log\alpha,$$

we have that $|\Gamma_2| < 1/2$. We then get that

$$|\Lambda_2| := |e^{\Gamma_2} - 1| < 2|\Gamma_2| < \frac{400n_2}{\alpha^{2m_3}}. \tag{31}$$

We apply Theorem 2 to

$$\Lambda_2 := (\sqrt{2}/P_{\ell_1})^{n_2}(\sqrt{2}/P_{\ell_2})^{-n_1}\alpha^{-(n_2m_1-n_1m_2)} - 1.$$

First, we need to check that $\Lambda_2 \neq 0$. Well, if it were, then it would follow that

$$\frac{P_{\ell_2}^{n_1}}{P_{\ell_1}^{n_2}} = 2^{(n_1-n_2)/2}\alpha^{n_2m_1-n_1m_2}. \tag{32}$$

By squaring the above relation, we get that $\alpha^{2(n_2m_1-n_1m_2)} \in \mathbb{Q}$, so $n_2m_1 = m_2n_1$. Thus, $P_{\ell_2}^{n_1}/P_{\ell_1}^{n_2} = 2^{(n_1-n_2)/2}$. If $n_1 = n_2$, then together with $n_2m_1 = n_2m_2$ we get $m_1 = m_2$ and now from $x_{n_i} = P_{\ell_i}P_{m_i}$, we get that $P_{\ell_1} = P_{\ell_2}$, so $\ell_1 = \ell_2$. This is impossible. If $\ell_4 \geq 2$ then the Carmichael Primitive Divisor Theorem for Pell numbers says that if $\ell_3 \neq \ell_4$ (so $\ell_1 \neq \ell_2$), then P_{ℓ_4} has a multiple of a prime ≥ 2 which does not divide P_{ℓ_3}. This is not possible in our case. So, still under the assumption that $\ell_4 \geq 2$, we get that $\ell_1 = \ell_2$ so $P_{\ell_1}^{n_1-n_2} = 2^{(n-n_1)/2}$, giving that $P_\ell = \sqrt{2}$, a contradiction. Thus, $\ell_4 \leq 2$. Also the previous argument shows that $\ell_1 \neq \ell_2$. We now list all the Pell numbers with indices at most 2. The only ones which is a multiple of 2 is $P_2 = 2$. So $2 \in \{\ell_1, \ell_2\}$. It follows that the other index has to be 1 since the only indices $k < 2$ such that P_k is a power of 2. Since $n_1 < n_2$, the exponent $(n_1 - n_2)/2$ of 2 is negative, so it follows that $\ell_1 = 2$ and $\ell_2 = 1$. So we get the equation $2^{-n_2} = 2^{(n_1-n_2)/2}$, which does not yield positive integer solutuions in n_1, n_2. So $\Lambda_2 \neq 0$. Thus, we can now apply Theorem 2 with the data

$$t := 3, \quad \gamma_1 := \sqrt{5}/P_{\ell_1}, \quad \gamma_2 := \sqrt{5}/P_{\ell_1}, \quad \gamma_3 := \alpha, \quad b_1 = n_2,$$
$$b_2 := -n_1, \quad b_3 := -(n_2m_1 - n_1m_2).$$

We have $\mathbb{K} = Q(\sqrt{2})$ which has degree $D = 2$. Also, using (16), we can take $B := 2m_4^2$. We can also take $A_1 := 4\ell_1 \log 2$, $A_2 := 4\ell_2 \log 2$ and $A_3 := \log \alpha$. Theorem 2 gives that

$$\begin{aligned} \log|\Lambda_2| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1 + \log 2)(1 + \log(2m_4^2))(4l_1 \log 2)(4l_2 \log 2) \log \alpha, \\ &> -6.57 \times 10^{12}\ell_1\ell_2(1 + \log(2m_4^2)). \end{aligned}$$

By comparing this with the inequality (31), we get

$$2m_3 \log \alpha - \log(400n_2) < 6.57 \times 10^{12}\ell_1\ell_2(1 + \log(2m_4^2)).$$

Since $n_2 < 2m_4$ and $m_4 > 100$, we get that $\log(48n_2) < 1 + \log(2m_4^2)$. Thus,

$$m_3 < 6.6 \times 10^{12}\ell_1\ell_2(1 + \log(2m_4^2)). \tag{33}$$

All this was done under the assumption that $\alpha^{2m_3} > 400n_2$. But if that inequality fails, then

$$m_3 < c_1 \log(400n_2) < 12(1 + \log(2m_4^2)),$$

which is much better than (33). Thus, (33) holds in all cases. Next, we record what we have proved.

Lemma 7. *Assume that $m_3 > 100$, then we have*

$$m_3 < 6.6 \times 10^{12}\ell_1\ell_2(1 + \log(2m_4^2)).$$

We now start finding effective bounds for our variables.

Case 1. $\ell_4 \leq 100$.

Then $\ell_1 < 100$ and $\ell_2 < 100$. By Lemma 7, we get that

$$m_3 < 6.6 \times 10^{16}(1 + \log(2m_4^2)).$$

By Lemma 4, we get

$$\log \delta < 2m_3 \log \alpha < 6.6 \times 10^{16}(1 + \log(2m_4^2)).$$

By the inequality (15), we have that

$$\begin{array}{rcl} m_4 & \leq & m_4+\ell_4-1 \\ & < & n_2 c_1 \log\delta + 2 \\ & < & \dfrac{1}{\log\alpha}(5.36\times 10^{13}(1+\log(2m_4)))(6.6\times 10^{16}(1+\log(2m_4^2))) \\ & < & 4\times 10^{30}\log(1+\log(2m_4))(1+\log(2m_4^2)). \end{array}$$

With the help of *Mathematica*, we get that $m_4 < 5.3\times 10^{34}$. Thus, using (16), we get

$$\max\{n_2, m_4\} < 1.1\times 10^{35}.$$

We record what we have proved.

Lemma 8. *If* $\ell_4 := \max\{\ell_1,\ell_2\} \leq 100$, *then*

$$\max\{n_2, m_4\} < 1.1\times 10^{35}.$$

From now on, we assume that $\ell_4 > 100$. Note that either $\ell_3 \leq 100$ or $\ell_3 > 100$ case in which by Lemma 5 and the inequality 16, we have $\ell_3 \leq 160(1+\log(2m_4))^2$ provided that $m_4 > 10000$, which we now assume.

We let $i \in \{1,2\}$ be such that $\ell_i = \ell_3$ and j be such that $\{i,j\} = \{1,2\}$. We assume that $m_3 > 100$. We work with (28) for i and (21) for j and noting the conditions $m_i > 100$ and $\ell_j = \ell_4 > 100$ are fullfilled. That is,

$$\begin{array}{rcl} \left|n_i\log\delta + \log(\sqrt{2}/P_{\ell_i}) - m_i\log\alpha\right| & < & \dfrac{100}{\alpha^{2m_i}}, \\ |n_j\log\delta - 2\log 2 - (\ell_j+m_j)\log\alpha| & < & \dfrac{120}{\alpha^{2\ell_j}}. \end{array}$$

By a similar procedure as before, we eliminate the term involving $\log\delta$. We multiply the first inequality by n_j, the second inequality by n_i, subtract the resulting inequalities and apply the triangle inequalty to get

$$\begin{array}{rcl} \left|n_j\log(\sqrt{2}/P_{\ell_i}) - 2n_i\log 2 - (n_jm_i - n_im_j + n_i\ell_j)\log\alpha\right| & < & \dfrac{100n_j}{\alpha^{2m_i}} + \dfrac{120n_i}{\alpha^{2l_j}} \\ & < & \dfrac{220n_2}{\alpha^{2\min\{m_i,\ell_j\}}}. \end{array} \quad (34)$$

Assume that $\alpha^{2\min\{m_i,\ell_j\}} > 440n_2$. We put

$$\Gamma_3 := n_j\log(\sqrt{2}/P_{\ell_i}) - 2n_i\log 2 - (n_jm_i - n_im_j + n_i\ell_j)\log\alpha.$$

We can write $\Lambda_3 := (\sqrt{2}/P_{\ell_i})^{n_j}2^{-2n_i}\alpha^{-(n_jm_i+n_im_j-n_i\ell_j)} - 1$. Under the above assumption and using (34), we get that

$$|\Lambda_3| = |e^{\Gamma_3} - 1| < 2|\Gamma_3| < \frac{440n_2}{\alpha^{2\min\{m_i,\ell_j\}}}. \quad (35)$$

We are now set to apply Theorem 2 on Λ_3. First, we need to check that $\Lambda_3 \neq 0$. Well, if it were, then we would get that

$$P_{\ell_i}^{n_j} = 2^{-2n_i+n_j/2}\alpha^{n_jm_i-n_im_j+n_i\ell_j}. \quad (36)$$

By similar arguments as before and the Carmichael Primitive Divisor Theorem for Pell numbers, we get a contradiction on (36). Thus, $\Lambda_3 \neq 0$. So we can apply Theorem 2 with the data

$$t := 3, \quad \gamma_1 := \sqrt{2}/P_{\ell_i}, \quad \gamma_2 := 2 \quad \gamma_3 := \alpha \quad b_1 := n_j,$$
$$b_2 := -2n_i, \quad b_3 := -(n_j m_i - n_i m_j + n_i \ell_j).$$

From the previous calculations, we know that $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ which has degree $D = 2$ and $A_1 := 4\ell_i \log 2$, $A_2 := 2\log 2$ and $A_3 := 2\log\alpha$. We also take $B := 2m_4^2$. By Theorem 2, we get that

$$\begin{aligned} \log|\Lambda_3| &> -1.4 \times 30^6 \times 3^{4.5} \times 2^2(1+\log 2)(1+\log(2m_4^2))(4\ell_i \log 2)(2\log 2)\log\alpha, \\ &> -3.30 \times 10^{12}\ell_i(1+\log(2m_4^2)). \end{aligned}$$

Comparing the above inequality with (35), we get

$$2\min\{m_i, \ell_j\}\log\alpha - \log(440n_2) < 3.30 \times 10^{12} l_i(1+\log(2m_4)).$$

Since $m_4 > 100$, we get using (16) that $n_2 < 2m_4$. Hence,

$$\min\{m_i, l_j\} < \frac{c_1}{2} 3.30 \times 10^{12} \times 160(1+\log(2m_4))^2(1+\log(2m_4^2)) + \frac{c_1}{2}\log(880m_4^2),$$

which implies that

$$\min\{m_i, \ell_j\} < 3 \times 10^{15}(1+\log(2m_4))^2(1+\log(2m_4^2)). \tag{37}$$

All this was under the assumptions that $m_4 > 10000$, and that $\alpha^{2\min\{m_i,\ell_j\}} > 440n_2$. But, still under the condition that $m_4 > 10000$, if $\alpha^{2\min\{m_i,\ell_j\}} < 440n_2$, then we get an inequality for $\min\{m_i, \ell_j\}$ which is even much better than (37). So, (37) holds provided that $m_4 > 10000$. Suppose say that $\min\{m_i, \ell_j\} = \ell_j$. Then we get that

$$\ell_3 < 160(1+\log(2m_4))^2, \quad \ell_4 < 3 \times 10^{15}(1+\log(2m_4))^2(1+\log(2m_4^2)).$$

By Lemma 7, since $m_3 > 100$, we get

$$\begin{aligned} m_3 &< (6.6 \times 10^{12})(160(1+\log(2m_4))^2)(1+\log(2m_4^2)) \\ &\quad \times 3 \times 10^{15}(1+\log(2m_4))^2(1+\log(2m_4^2)) \\ &< 3.2 \times 10^{30}(1+\log(2m_4^2))^6. \end{aligned}$$

Together with Lemma 4, we get

$$\log\delta < 3.2 \times 10^{30}(1+\log(2m_4^2))^6,$$

which together with Lemma 6 gives

$$\begin{aligned} m_4 < 4.30 \times 10^{13}&(3 \times 10^{15}(1+\log(2m_4))^2(1+\log(2m_4^2))) \\ &\times (3.2 \times 10^{30}(1+\log(2m_4^2))^6)(1+\log m_4), \end{aligned}$$

which implies that

$$m_4 < 4.1 \times 10^{59}(1+\log(2m_4^2))^{10}. \tag{38}$$

With the help of *Mathematica* we get that $m_4 < 3.8 \times 10^{85}$. This was proved under the assumption that $m_4 > 10000$, but the situation $m_4 \leq 10000$ already provides a better bound than $m_4 < 3.8 \times 10^{85}$. Hence,

$$\max\{n_2, m_1, m_2\} < 3.8 \times 10^{85}. \tag{39}$$

This was when $\ell_j = \min\{m_i, \ell_j\}$. Now we assume that $m_i = \min\{m_i, \ell_j\}$. Then we get

$$m_i < 3 \times 10^{15}(1 + \log(2m_4^2))^3.$$

By Lemma 4, we get that

$$\log \delta < 3 \times 10^{15}(1 + \log(2m_4^2))^3.$$

Now by Lemma 7 together with Lemma 4 to bound l_4 give

$$\begin{aligned} m_4 &< 4.30 \times 10^{13}(5.36 \times 10^{13}(3 \times 10^{15}(1 + \log(2m_4^2))^3)(1 + \log(2m)))^2 \\ &\quad \times (1 + \log(2m_4^2)))(3 \times 10^{15}(1 + \log(2m_4^2))^3)(1 + \log m_4), \\ &< 2 \times 10^{58}(1 + \log(2m_4^2))^{10}. \end{aligned}$$

This gives, $m_4 < 1.6 \times 10^{84}$ which is a better bound than 3.8×10^{85}. We record what we have proved.

Lemma 9. *If* $\ell_4 := \max\{\ell_1, \ell_2\} > 100$ *and* $m_3 := \min\{m_1, m_2\} > 100$*, then*

$$\max\{n_2, m_1, m_2\} < 3.8 \times 10^{85}.$$

It now remains the case when $\ell_4 > 100$ and $m_3 \leq 100$. But then, by Lemma 4, we get $\log \delta < 100$ and now Lemma 4 together with Lemma 7 give

$$m_4 < 2 \times 10^{31}(1 + \log(2m_4^2))^3,$$

which implies that $m_4 < 10^{38}$ and further $\max\{n_1, m_1, m_2,\} < 10^{40}$. We record what we have proved.

Lemma 10. *If* $\ell_4 > 100$ *and* $m_3 \leq 100$*, then*

$$\max\{n_1, m_1, m_2,\} < 10^{40}.$$

5. The final computations

We return to (26) and we set $s := n_2 - n_1$ and $r := n_2(\ell_1 + m_1) - n_1(\ell_2 + m_2)$ and divide both sides by $s \log \alpha$ to get

$$\left| \frac{\log 4}{\log \alpha} - \frac{r}{s} \right| < \frac{240 n_2}{\alpha^{2\ell_3} s \log \alpha}. \tag{40}$$

We assume that ℓ_3 is so large that the right-hand side of the inequality in (40) is smaller than $1/(2s^2)$. This certainly holds if

$$\alpha^{2\ell_3} > 480 n_2^2 / \log \alpha. \tag{41}$$

Since $n_2 < 3.8 \times 10^{85}$, it follows that the last inequality (41) holds provided that $\ell_3 \geq 227$, which we now assume. In this case r/s is a convergent of the continued fraction of $\tau := \log 4 / \log \alpha$ and $s < 3.8 \times 10^{85}$. We are now set to apply Lemma 1.

We write $\tau := [a_0; a_1, a_2, a_3, \ldots] = [1, 1, 1, 2, 1, 13, 2, 1, 5, 4, 1, 3, 1, 8, 1, 10, 1, 1, 2, 3, \ldots]$ for the continued fraction of τ and p_k/q_k for the $k-$th convergent. We get that $r/s = p_j/q_j$ for some $j \le 170$. Furthermore, putting $a(M) := \max\{a_j : j = 0, 1, \ldots, 170\}$, we get $a(M) := 1469$. By Lemma 1, we get

$$\frac{1}{1471s^2} = \frac{1}{(a(M)+2)s^2} \le \left|\tau - \frac{r}{s}\right| < \frac{240n_2}{\alpha^{2l_3} s \log \alpha},$$

giving

$$\alpha^{2\ell_3} < \frac{1471 \times 240 n_2^2}{\log \alpha} < \frac{1471 \times 240 \times (3.8 \times 10^{85})^2}{\log \alpha},$$

leading to $\ell_3 \le 230$. We record what we have just proved.

Lemma 11. *We have* $\ell_3 \le 230$.

If $\ell_1 = \ell_3$, then we have $i = 1$ and $j = 2$, otherwise $\ell_2 = \ell_3$ implying that we have $i = 2$ and $j = 1$. In both cases, the next step is the application of Lemma 3 (LLL algorithm) for (34), where $n_i < 3.8 \times 10^{85}$ and $|n_j m_i - n_i m_j + n_i \ell_j| < 10^{90}$. For each $\ell_j \in [1, 230]$ and

$$\Gamma_3 := n_j \log(\sqrt{2}/P_{\ell_i}) - 2n_i \log 2 - (n_j m_i - n_i m_j + n_i \ell_j) \log \alpha,$$

we apply the LLL algorithm on Γ_3 with the data

$$t := 3, \quad \tau_1 := \log(\sqrt{2}/P_{\ell_i}), \quad \tau_2 := \log 4, \quad \tau_3 := \log \alpha$$
$$x_1 := n_j, \quad x_2 := n_i, \quad x_3 := n_j m_i - n_i m_j + n_i \ell_j.$$

Further, we set $X := 10^{90}$ as an upper bound to $|x_i| < 2n_2$ for $i = 1, 2$, and $C := (5X)^5$. A computer search in *Mathematica* allows us to conculde, together with the inequality 34, that

$$2 \times 10^{-220} < \min_{1 \le \min\{m_i, l_j\} \le 230} |\Gamma_3| < \frac{220 n_2}{\alpha^{2\min\{m_i, l_j\}}}. \tag{42}$$

Thus, $\min\{m_i, \ell_j\} \le 401$.

We assume first that $i = 1, j = 2$. Thus, $\min\{m_1, \ell_2\} \le 401$ can be split into two branches. If $m_1 \le 401$, then $\ell_1 + m_1 \le 631$, and by (15) we obtain $n_1 < 556$. For $\ell_2 \le 401$ we run the LLL algorithm on (30) with $2 \le \ell_1 \le 230$ and $\ell_1 \le \ell_2 \le 401$ for each $n_i < 3.8 \times 10^{85}$ and further $|n_2 m_1 - n_1 m_2| < 10^{90}$. This results in the upper bound $m_3 \le 412$. This in turn splits into either $m_1 \le 412$ or $m_2 \le 412$. Suppose that $m_1 \le 412$, together with $\ell_1 \le 230$ and (15), it yields $n_1 \le 565$. For $m_2 \le 412$ and that $\ell_2 \le 401$, and then (15) gives $n_2 \le 716$. Clearly, now $n_1 \le 715$. The symmetric case $i = 2, j = 1$ with $\min\{m_2, \ell_1\} \le 401$ is anologous. We record the results of the computation in the table below.

	ℓ_1	m_1	n_1	ℓ_2	m_2	n_2
1.	230	401	556			
2.	230	412	565	401		
3.	230		715	401	412	716
4.			552	230	401	556
5.	401	412	716	230		
6.	401		556	230	412	565

By similar arguments given in Kafle et al. [8] by applying Lemma 1, Lemma 2 and Lemma 3 on the appropraite linear forms in logarithms, we can further reduce these bounds to

$$\ell_1 \leq 200, \quad m_1 \leq 200, \quad \ell_2 \leq 120, \quad m_2 \leq 120, \quad n_2 \leq 150. \tag{43}$$

The final verification of our results was carried out according to the bounds in (43) to check all the possibilities. With the help of a computer search in *Mathematica* we found no values of d that lead to at least two positive integer solutions to (6). This completes the proof of Theorem 1. □

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