

# ON THE DIVISIBILITY OF BINOMIAL COEFFICIENTS

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ABSTRACT. In Pacific J. Math. 292 (2018), 223–238, Shareshian and Woodroffe asked if for every positive integer  $n$  there exist primes  $p$  and  $q$  such that, for all integers  $k$  with  $1 \leq k \leq n - 1$ , the binomial coefficient  $\binom{n}{k}$  is divisible by at least one of  $p$  or  $q$ . We give conditions under which a number  $n$  has this property and discuss a variant of this problem involving more than two primes. We prove that every positive integer  $n$  has infinitely many multiples with this property.

## 1. INTRODUCTION

Binomial coefficients display interesting divisibility properties. Conditions under which a prime power  $p^a$  divides a binomial coefficient  $\binom{n}{k}$  are given by Kummer's Theorem [10] and also by a generalized form of Lucas' Theorem [5, 12].

Still, there are problems involving divisibility of binomial coefficients that remain unsolved. In this article we investigate the following question, which was asked by Shareshian and Woodroffe in [15].

**Question 1.1.** Is it true that for every positive integer  $n$  there exist primes  $p$  and  $q$  such that, for all integers  $k$  with  $1 \leq k \leq n - 1$ , the binomial coefficient  $\binom{n}{k}$  is divisible by  $p$  or  $q$ ?

As in [15], we say that  $n$  satisfies *Condition 1* if such primes  $p$  and  $q$  exist for  $n$ . In this article we discuss sufficient conditions under which an integer  $n$  satisfies Condition 1. In Sections 2 and 3 we prove a variation of the Sieve Lemma from [15] and use it to show that  $n$  satisfies Condition 1 if certain inequalities hold. In Section 5 we infer that every positive integer has infinitely many multiples for which Condition 1 is satisfied.

The collection of numbers for which Condition 1 is not known to hold has asymptotic density 0 assuming the truth of Cramér's conjecture (as first shown in [15]) and includes most *primorials*  $p_1 p_2 \cdots p_i$ , where  $p_1, \dots, p_i$  are the first  $i$  primes, namely those primorials such that  $(p_1 p_2 \cdots p_i) - 1$  is not a prime.

In addition, we introduce the following variant of Condition 1:

**Definition 1.2.** A positive integer  $n$  satisfies the  *$N$ -variation* of Condition 1 if there exist  $N$  different primes  $p_1, \dots, p_N$  such that if  $1 \leq k \leq n - 1$  then  $\binom{n}{k}$  is divisible by at least one of  $p_1, \dots, p_N$ .

For example, it follows from Kummer's Theorem or from Lucas' Theorem that a positive integer  $n$  satisfies the 1-variation of Condition 1 if and only if  $n$  is a prime power, and every integer  $n$  satisfies the  $m$ -variation of Condition 1 if  $n = p_1^{a_1} \cdots p_m^{a_m}$  where  $p_1, \dots, p_m$  are distinct primes. In Section 4 we discuss upper bounds on  $N$  so that a given  $n$  satisfies the  $N$ -variation of Condition 1.

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## 2. AN EXTENDED SIEVE LEMMA

Our results in this section will be based on Lucas' Theorem:

**Theorem 2.1** (Lucas [12]). *Let  $p$  be a prime and let*

$$\begin{aligned} n &= n_r p^r + n_{r-1} p^{r-1} + \cdots + n_1 p + n_0 \\ k &= k_r p^r + k_{r-1} p^{r-1} + \cdots + k_1 p + k_0 \end{aligned}$$

*be base  $p$  expansions of two positive integers, where  $0 \leq n_i < p$  and  $0 \leq k_i < p$  for all  $i$ , and  $n_r \neq 0$ . Then*

$$\binom{n}{k} \equiv \prod_{i=0}^r \binom{n_i}{k_i} \pmod{p}.$$

By convention, a binomial coefficient  $\binom{n_i}{k_i}$  is zero if  $n_i < k_i$ . Hence, if any of the digits of the base  $p$  expansion of  $n$  is 0 whereas the corresponding digit in the base  $p$  expansion of  $k$  is nonzero, then  $\binom{n}{k}$  is divisible by  $p$ . As a particular case, if a prime power  $p^a$  with  $a > 0$  divides  $n$  and does not divide  $k$ , then  $\binom{n}{k}$  is divisible by  $p$ .

Observe that, if  $n$  satisfies Condition 1 with two primes  $p$  and  $q$ , then at least one of these primes has to be a divisor of  $n$ , because otherwise  $\binom{n}{1}$  would not be divisible by any of them. The next two results are elementary consequences of Lucas' Theorem.

**Proposition 2.2.** *If  $n = p^a + 1$  with  $p$  a prime and  $a > 0$ , then  $n$  satisfies Condition 1 with  $p$  and any prime dividing  $n$ .*

*Proof.* If  $n - 1$  is a prime power then the two summands in the left-hand term of the equality

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

are divisible by  $p$  by Lucas' Theorem if  $2 \leq k \leq n - 2$ , and hence  $\binom{n}{k}$  is also divisible by  $p$ . When  $k = 1$  or  $k = n - 1$ , we have that  $\binom{n}{k} = n$ , so any prime factor of  $n$  divides  $\binom{n}{k}$ .  $\square$

**Proposition 2.3.** *If a positive integer  $n$  is equal to the product of two prime powers  $p_1^a$  and  $p_2^b$  with  $a > 0$ ,  $b > 0$ , and  $p_1 \neq p_2$ , then  $n$  satisfies Condition 1 with  $p_1$  and  $p_2$ .*

*Proof.* The base  $p_1$  expansion of  $n$  ends with  $a$  zeroes and the base  $p_2$  expansion of  $n$  ends with  $b$  zeroes. Because a positive integer  $k$  smaller than  $n$  cannot be divisible by both  $p_1^a$  and  $p_2^b$ , it is not possible that  $k$  ends with  $a$  zeroes in base  $p_1$  and  $b$  zeroes in base  $p_2$ . Consequently, we can apply Lucas' Theorem modulo  $p_1$  if  $p_1^a$  does not divide  $k$  or modulo  $p_2$  if  $p_2^b$  does not divide  $k$ .  $\square$

Proposition 2.3 generalizes as follows.

**Proposition 2.4.** *If  $p_1, \dots, p_m$  are distinct primes and  $n = p_1^{a_1} \cdots p_m^{a_m}$  with  $a_i > 0$  for all  $i$ , then  $n$  satisfies the  $m$ -variation of Condition 1 with  $p_1, \dots, p_m$ .*

*Proof.* If  $1 \leq k \leq n - 1$ , then the base  $p_i$  expansion of  $k$  ends with less zeroes than the base  $p_i$  expansion of  $n$  for at least one prime factor  $p_i$  of  $n$ .  $\square$

The following result extends [15, Lemma 4.3]. It is the starting point of our discussion of Question 1.1 in the next sections.

**Theorem 2.5.** *Let  $n$  be a positive integer and suppose that  $p^a$  divides  $n$  where  $p$  is a prime and  $a > 0$ . Suppose that there is a prime  $q$  with  $n/(d+1) < q < n/d$ , where  $d \geq 1$ . Then  $\binom{n}{k}$  is divisible by  $p$  or  $q$  except possibly when  $k$  is a multiple of  $p^a$  belonging to one of the intervals  $[cq, cq + \beta]$  with  $\beta = n - dq$  and  $0 \leq c < (d+1)/2$ .*

*Proof.* By symmetry, we only need to consider those values of  $k$  with  $k \leq n/2$ . Moreover, we may restrict our study further to those values of  $k$  that are multiples of  $p^a$ , since otherwise  $\binom{n}{k}$  is divisible by  $p$ .

Since  $q < n/d$ , the number  $\beta = n - dq$  is positive. If  $k \leq \beta$  then  $k$  is in the interval  $[0, \beta]$ , which is the case  $c = 0$  in the statement of the theorem.

The assumption that  $n/(d+1) < q$  is equivalent to assuming the inequality  $n - dq < q$ , which implies that the last digit in the base  $q$  expansion of  $n$  is equal to  $\beta$ . Hence, if  $\beta < k < q$  then we may infer from Lucas' Theorem that  $\binom{n}{k}$  is divisible by  $q$ .

The remaining range of values of  $k$  to be considered is  $q \leq k \leq n/2$ . In this case we look at the last digit of the base  $q$  expansion of  $k$ . If this last digit is bigger than  $\beta$ , then  $\binom{n}{k}$  is again divisible by  $q$ . Thus the undecided cases are those in which the residue of  $k \bmod q$  is smaller than or equal to  $\beta$ . This happens when  $cq \leq k \leq cq + \beta$  for some positive integer  $c$ , and if  $cq \leq k \leq n/2$  then  $c \leq n/(2q) < (d+1)/2$ .  $\square$

By the Bertrand–Chebyshev Theorem [2], for every integer  $n > 2$  there exists a prime  $q$  such that  $n/2 < q < n$ . This yields the following particular instance of Theorem 2.5, which is also a special case of [15, Lemma 4.3].

**Corollary 2.6.** *For a positive integer  $n$ , suppose that  $p^a$  divides  $n$  where  $p$  is a prime and  $a > 0$ . If  $q$  is a prime such that  $n/2 < q < n$  and  $n - q < p^a$ , then  $n$  satisfies Condition 1 with  $p$  and  $q$ .*

*Proof.* Pick  $d = 1$  in Theorem 2.5.  $\square$

Note that, under the assumptions of Corollary 2.6, the equality  $n - q = p^a$  cannot hold, since  $p$  divides  $n$  and  $p \neq q$  because  $q$  does not divide  $n$ . Hence there remains to study the case when  $n - q > p^a$  and  $q$  is the largest prime smaller than  $n$  while  $p^a$  is the largest prime power dividing  $n$ . In other words, Condition 1 holds for  $n$  whenever there is a prime between  $n - p^a$  and  $n$ .

The sequence of integers  $n$  for which there is no prime between  $n - p^a$  and  $n$  can be found in The On-Line Encyclopedia of Integer Sequences (OEIS [3]) with the reference A290203. Its first terms are the following:

$$(2.1) \quad 126, 210, 330, 630, 1144, 1360, 2520, 2574, 2992, 3432, 3960, 4199 \dots$$

*Banderier's conjecture* [1] claims that if  $p_n\#$  denotes the  $n$ -th primorial, that is,

$$p_n\# = p_1 p_2 \cdots p_n$$

where  $p_1, \dots, p_n$  are the first  $n$  primes, and  $q$  is the largest prime below  $p_n\#$ , then either  $p_n\# - q = 1$  or  $p_n\# - q$  is a prime.

**Proposition 2.7.** *If Banderier's conjecture is true, then the sequence (2.1) contains all primorials  $p_n\#$  such that  $p_n\# - 1$  is not a prime.*

*Proof.* If  $p_n\# - 1$  is not a prime, then  $p_n\# - q$  is a prime according to Banderier's conjecture. Since  $p_n\# - q$  does not divide  $p_n\#$ , we infer that  $p_n\# - q$  is bigger than  $p_n$ , which is the largest prime power dividing  $p_n\#$ .  $\square$

The first primorials  $p_n\#$  such that  $p_n\# - 1$  is not a prime are

$$p_4\# = 210, \quad p_7\# = 510510, \quad p_8\# = 9699690, \quad p_9\# = 223092870.$$

Inspecting this list could be a strategy to seek for a counterexample for Question 1.1. The complementary list of primorials can be found in OEIS with reference A057704.

For any fixed value of  $d$ , the number  $\beta$  in Theorem 2.5 is smallest when  $q$  is as close as possible to  $n/d$ . For this reason, we focus our attention on the largest prime  $q_d$  below  $n/d$  for various values of  $d$ . This motivates the next definition.

**Definition 2.8.** For positive integers  $n$  and  $1 \leq d < n/2$ , let  $q_d$  be the largest prime smaller than  $n/d$  and let  $\beta_d = n - dq_d$ . For each integer  $c$  with  $0 \leq c < (d + 1)/2$ , we call  $[cq_d, cq_d + \beta_d]$  a *dangerous interval*.

By Theorem 2.5, if we attempt to prove that Condition 1 holds with  $p$  and  $q_d$  assuming that  $q_d > n/(d + 1)$ —that is, assuming that the dangerous intervals are disjoint—we only need to care about values of  $k$  that lie in a dangerous interval and are multiples of the largest power of  $p$  dividing  $n$ .

In the case  $d = 1$ , the only dangerous interval below  $n/2$  is  $[0, n - q_1]$ . When  $d = 2$ , we have that  $[0, n - 2q_2]$  and  $[q_2, n - q_2]$  are dangerous intervals. Since  $n - q_2 > n/2$ , the second interval may be replaced by  $[q_2, n/2]$  to carry our study further, as we do in the next section.

**Example 2.9.** The largest prime below  $n = p_7\# = 510510$  is  $q_1 = 510481$  and the largest prime dividing  $n$  is  $p = 17$ . Here  $n - q_1 = 29$  and therefore  $\binom{n}{k}$  is divisible by 17 or 510481 for all  $k$  except for  $k = 17$ .

On the other hand, the largest prime below  $n/2 = 255255$  is  $q_2 = 255253$ . Thus  $\beta_2 = n - 2q_2 = 4$  and therefore  $[0, 4]$  and  $[255253, 255257]$  are dangerous intervals. The second interval contains a multiple of 17, namely  $n/2$ . However, since

$$510510 = 6 \cdot 17^4 + 1 \cdot 17^3 + 15 \cdot 17^2 + 8 \cdot 17$$

$$255255 = 3 \cdot 17^4 + 0 \cdot 17^3 + 16 \cdot 17^2 + 4 \cdot 17,$$

we infer from Lucas' Theorem that  $\binom{510510}{255255}$  is divisible by 17. Consequently,  $\binom{n}{k}$  is divisible by 17 or 255253 for all  $k$ .

### 3. USING THE NEAREST PRIME BELOW $n/2$

Nagura showed in [13] that, if  $m \geq 25$ , then there is a prime between  $m$  and  $(1 + 1/5)m$ . Therefore, there is a prime  $q$  such that  $5n/6 < q < n$  when  $n \geq 30$ . This implies that, if  $n \geq 30$  and the largest prime-power divisor  $p^a$  of  $n$  satisfies  $p^a \geq n/6$ , then there is a prime  $q$  between  $n - p^a$  and  $n$  and hence Condition 1 holds for  $n$  with  $p$  and  $q$ .

The following result is sharper.

**Proposition 3.1.** *If  $n \geq 2010882$  and the largest prime-power divisor  $p^a$  of  $n$  satisfies  $p^a \geq n/16598$ , then  $n$  satisfies Condition 1 with  $p$  and the nearest prime  $q$  below  $n$ .*

*Proof.* Schoenfeld proved in [14] that for  $m \geq 2010760$  there is a prime between  $m$  and  $(1 + 1/16597)m$ . Hence, if  $n \geq 2010882$  and the largest prime-power divisor  $p^a$  of  $n$  satisfies  $p^a \geq n/16598$  then there is a prime between  $n - p^a$  and  $n$ , and therefore Condition 1 holds for  $n$  by Corollary 2.6.  $\square$

The following are consequences of Nagura's and Schoenfeld's bounds.

**Lemma 3.2.** *Let  $q_d$  be the largest prime below  $n/d$  for positive integers  $n$  and  $d$ .*

- (a) *If  $n \geq 120$  and  $d < 5$ , then  $n/(d+1) < q_d$ .*
- (b) *If  $n \geq 3.34 \cdot 10^{10}$  and  $d < 16597$ , then  $n/(d+1) < q_d$ .*

*Proof.* By Nagura's bound [13], if  $n/d \geq 30$ , then  $5n/6d < q_d < n/d$ . Therefore,  $n - dq_d < n/6$ . If  $d < 5$ , then  $6d < 5(d+1)$  and hence

$$n < \frac{5n(d+1)}{6d} < q_d(d+1),$$

as claimed. The proof of part (b) is analogous using Schoenfeld's bound [14].  $\square$

In order to apply Theorem 2.5 with  $d = 2$  for a given  $n$ , we need that there is a prime  $q$  such that  $n/3 < q < n/2$ . If  $q_2$  denotes the nearest prime below  $n/2$ , then the inequality  $n/3 < q_2$  holds if  $n \geq 120$  by Lemma 3.2. Since by (2.1) we have that  $n - q_1 < p^a$  if  $n < 126$ , we may assume that  $n/3 < q_2$  without any loss of generality.

Note that the inequality  $n/3 < q$  is equivalent to  $n - 2q < q$ , so the intervals  $[0, n - 2q]$  and  $[q, n - q]$  are disjoint.

**Theorem 3.3.** *For an odd positive integer  $n$  and a prime power  $p^a$  dividing  $n$ , suppose that there is a prime  $q$  with  $n/3 < q < n/2$  and  $n - 2q < p^a$ . Then  $n$  satisfies Condition 1 with  $p$  and  $q$ .*

*Proof.* By Theorem 2.5, in order to infer that  $\binom{n}{k}$  is divisible by  $p$  or  $q$ , the only cases that we need to discuss are those values of  $k$  that are multiples of  $p^a$  with  $k \in [0, n - 2q]$  or  $k \in [q, n - q]$ . By assumption, there are no multiples of  $p^a$  in  $[0, n - 2q]$ . Since  $n - q > n/2$ , we may focus on the interval  $[q, n/2]$ . Since  $n$  is odd,  $n/2$  is not an integer; hence we are only left to prove that there is no multiple  $k$  of  $p^a$  with  $q \leq k < n/2$ . We will prove this by contradiction.

Thus suppose that  $q \leq \lambda p^a < n/2$  for some integer  $\lambda$ . The assumption that  $n - 2q < p^a$  implies that  $n - p^a < 2q$  and hence

$$n/2 - p^a/2 < q \leq \lambda p^a.$$

Consequently,  $\lambda p^a < n/2 < (\lambda + 1/2)p^a$ . If we now write  $n = mp^a$ , we obtain that  $2\lambda < m < 2\lambda + 1$ , which is impossible for an integer  $m$ .  $\square$

The rest of this section is devoted to the case when  $n$  is even.

**Lemma 3.4.** *Suppose that  $n$  is even and there is a prime  $q$  with  $q < n/2$  and  $n - 2q < p^a$ , where  $p^a$  is the largest power of  $p$  dividing  $n$ . If there is a multiple  $k$  of  $p^a$  in the interval  $[q, n/2]$ , then  $p$  is odd and  $k = n/2$ .*

*Proof.* Suppose first that  $p$  is odd. Then the integer  $n/2$  is a multiple of  $p^a$ , so we may write  $n/2 = \lambda p^a$  for some integer  $\lambda$ . If there is another multiple of  $p^a$  in the interval  $[q, n/2]$ , then  $q \leq (\lambda - 1)p^a < n/2$ , and this implies that

$$n/2 - p^a = \lambda p^a - p^a = (\lambda - 1)p^a \geq q.$$

Hence  $n - 2q \geq 2p^a$ , which is incompatible with our assumption that  $n - 2q < p^a$ .

In the case  $p = 2$  (so that  $2^a$  is the largest power of 2 dividing  $n$ ), we have that  $n/2$  is divisible by  $2^{a-1}$ , and we may write  $n/2 = \lambda 2^{a-1}$  with  $\lambda$  odd. If there is a multiple

of  $2^a$  in the interval  $[q, n/2)$ , then  $q \leq \mu 2^a < n/2$ , so  $\mu < \lambda/2$  and  $\mu \leq (\lambda - 1)/2$  because  $\lambda$  is odd. Therefore

$$n/2 - 2^{a-1} = (\lambda - 1)2^{a-1} \geq \mu 2^a \geq q.$$

Hence, as above,  $n - 2q \geq 2^a$ , which contradicts that  $n - 2q < 2^a$ .  $\square$

**Theorem 3.5.** *For an even positive integer  $n$ , suppose that there is a prime  $q$  with  $n/3 < q < n/2$  and  $n - 2q < p^a$ , where  $p^a$  is the largest power of  $p$  dividing  $n$ .*

- (a) *If  $p = 2$ , then  $n$  satisfies Condition 1 with 2 and  $q$ .*
- (b) *If  $p \neq 2$ , then  $n$  satisfies Condition 1 with  $p$  and  $q$  if and only if  $\binom{n}{n/2}$  is divisible by  $p$ .*

*Proof.* By Theorem 2.5 and Lemma 3.4, the only case left is  $k = n/2$  for  $p$  odd. Consequently, if  $\binom{n}{n/2}$  is divisible by  $p$ , then  $n$  satisfies Condition 1 with  $p$  and  $q$ . Moreover,  $\binom{n}{n/2}$  is not divisible by  $q$ , since the base  $q$  expansions of  $n$  and  $n/2$  are, respectively,  $2 \cdot q + (n - 2q)$  and  $1 \cdot q + (n/2 - q)$ . Hence the assumption that  $\binom{n}{n/2}$  be divisible by  $p$  is necessary.  $\square$

Our last remarks in this section correspond to the case when  $n$  is even, and they are only relevant if  $p \neq 2$ , by Theorem 3.5. Sufficient conditions are given to infer that a prime  $p$  divides  $\binom{n}{n/2}$ . The greatest integer less than or equal to a real number  $x$  is denoted by  $\lfloor x \rfloor$ , and we write  $v_p(n) = a$  if  $p^a$  is the maximum power of  $p$  such that  $p^a$  divides  $n$ .

Recall from [11] that

$$(3.1) \quad v_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor = \frac{n - s_p(n)}{p - 1},$$

where  $s_p(n)$  denotes the sum of all the digits in the base  $p$  expansion of  $n$ .

**Proposition 3.6.** *Suppose that  $n$  is even. A prime  $p$  divides  $\binom{n}{n/2}$  if and only if at least one of the numbers  $\lfloor n/p^r \rfloor$  with  $r \geq 1$  is odd.*

*Proof.* By comparing  $v_p(n!)$  and  $v_p((n/2)!)$  we see that, for each  $r$ ,

$$\left\lfloor \frac{n}{p^r} \right\rfloor = 2 \left\lfloor \frac{n/2}{p^r} \right\rfloor$$

if  $\lfloor n/p^r \rfloor$  is even. If  $\lfloor n/p^r \rfloor$  is even for all  $r$ , we conclude that  $v_p(n!) = 2v_p((n/2)!)$ , and hence  $p$  does not divide  $\binom{n}{n/2}$ . However, if  $\lfloor n/p^r \rfloor$  is odd, then

$$\left\lfloor \frac{n}{p^r} \right\rfloor = 2 \left\lfloor \frac{n/2}{p^r} \right\rfloor + 1$$

and consequently  $v_p(n!)$  is greater than  $2v_p((n/2)!)$ .  $\square$

**Corollary 3.7.** *If  $n$  is even and  $(n - s_p(n))/(p - 1)$  is odd, then  $p$  divides  $\binom{n}{n/2}$ .*

*Proof.* This follows from Proposition 3.6 and Legendre's formula (3.1).  $\square$

**Corollary 3.8.** *Suppose that  $n$  is even.*

- (a) *If any of the digits in the base  $p$  expansion of  $n/2$  is larger than  $\lfloor p/2 \rfloor$ , then  $p$  divides  $\binom{n}{n/2}$ .*

(b) *If one of the digits in the base  $p$  expansion of  $n$  is odd, then  $p$  divides  $\binom{n}{n/2}$ .*

*Proof.* If a digit of  $n/2$  in base  $p$  is larger than  $\lfloor p/2 \rfloor$ , then when we add  $n/2$  to itself in base  $p$  to obtain  $n$  there is at least one carry. Similarly, if  $n$  has an odd digit in base  $p$ , then there is a carry when adding  $n/2$  and  $n/2$  in base  $p$ . Hence, by Kummer's Theorem [10] with  $k = n/2$ , if there is at least one carry when adding  $n/2$  to itself in base  $p$ , then  $p$  divides  $\binom{n}{n/2}$ .  $\square$

**Corollary 3.9.** *Let  $n$  be an even positive integer. Suppose that there is a prime  $q$  such that  $n/3 < q < n/2$  and  $n - 2q < p^a$ , where  $p^a$  denotes the largest power of  $p$  dividing  $n$ . If  $p^{\lfloor \log n / \log p \rfloor} > n/2$ , then  $p$  divides  $\binom{n}{n/2}$  and therefore  $n$  satisfies Condition 1 with  $p$  and  $q$ .*

*Proof.* The largest value of  $r$  such that  $p^r < n < p^{r+1}$  is  $\lfloor \log n / \log p \rfloor$ . Therefore, in Proposition 3.6, the exponent  $r$  is bounded by  $\lfloor \log n / \log p \rfloor$ . Also note that  $r \geq a$ , where  $a$  is the largest exponent of  $p$  such that  $p^a$  divides  $n$ . If  $p^{\lfloor \log n / \log p \rfloor} > n/2$ , then  $\lfloor n/p^r \rfloor = 1$ . Because this is odd,  $p$  divides  $\binom{n}{n/2}$  by Proposition 3.6.  $\square$

In those cases when the inequalities  $n - q_1 < p^a$  and  $n - 2q_2 < p^a$  both fail for the largest prime power  $p^a$  dividing  $n$ , a possible strategy is to analyze the inequality  $n - dq_d < p^a$  for bigger values of  $d$ , where  $q_d$  is the largest prime below  $n/d$ .

Up to 1,000,000 there are 88 integers that do not satisfy  $n - 2q_2 < p^a$ , where  $p^a$  is the largest prime power dividing  $n$ . The On-Line Encyclopedia of Integer Sequences has published these numbers [4] with the reference A290290. Among these, there are 25 that do not satisfy the inequality  $n - 3q_3 < p^a$ ; there are 7 that do not satisfy the inequality  $n - 4q_4 < p^a$  either; there are 5 for which the inequality  $n - 5q_5 < p^a$  also fails, and there is only one integer for which the inequality  $n - 6q_6 < p^a$  still fails (namely,  $n = 875160$ ). However, the value of  $n - dq_d$  need not decrease as  $d$  grows, and the number of dangerous intervals that one needs to inspect when  $n - dq_d < p^a$  increases linearly with  $d$ . Therefore this strategy is not conclusive, although it often works in practice.

**Example 3.10.** The largest prime power dividing  $n = p_{14}\# = 13082761331670030$  is  $p = 43$ . In this case,  $n - q_1 = 89$  and  $n - 2q_2 = 268$ . Thus, Condition 1 fails for  $p$  and  $q_1$  and it also fails for  $p$  and  $q_2$ . Nevertheless,  $n - 3q_3 = 27$  works, as the dangerous interval  $[q_3, n - 2q_3]$  contains one multiple of 43, namely  $n/3$ , and  $\binom{n}{n/3}$  is divisible by 43. Therefore Condition 1 holds for  $p = 43$  and  $q_3 = 4360920443890001$ .

**Example 3.11.** For  $n = 210$ , the inequality  $n - q_1 < 7$  fails while  $n - 2q_2 < 7$  is true. However,  $\binom{210}{105}$  is not divisible by 7. Hence we look for greater values of  $d$  and find that  $n - 5q_5 < 7$  with  $q_5 = 41$ . Now  $42 \in [41, 46]$  and  $84 \in [82, 87]$ , yet  $\binom{210}{42}$  and  $\binom{210}{84}$  are both divisible by 7. Hence Condition 1 is satisfied with  $p = 7$  and  $q_5 = 41$ .

**Example 3.12.** For  $n = 875160$ , the inequality  $n - dq_d < 17$  is satisfied with  $d = 11$  but not with any smaller value of  $d$ . There are 6 dangerous intervals of length  $n - 11q_{11} = 11$ . Each of these intervals (except the first) contains one multiple of 17, and in each case the corresponding binomial coefficient  $\binom{n}{k}$  happens to be divisible by 17. Therefore Condition 1 is satisfied with  $p = 17$  and  $q_{11} = 79559$ .

4. ON THE  $N$ -VARIATION OF CONDITION 1

Recall from Definition 1.2 that  $n$  satisfies the  $N$ -variation of Condition 1 if there are  $N$  primes  $p_1, \dots, p_N$  such that if  $1 \leq k \leq n-1$  then  $\binom{n}{k}$  is divisible by at least one of  $p_1, \dots, p_N$ .

**Theorem 4.1.** *If an even positive integer  $n$  satisfies  $n - 2q < p^a$  for a prime  $q$  with  $n/3 < q < n/2$ , where  $p^a$  is the largest power of  $p$  dividing  $n$  and  $p \neq 2$ , then  $n$  satisfies the 3-variation of Condition 1 with  $p, q$  and any prime that divides  $\binom{n}{n/2}$ .*

*Proof.* According to part (b) of Theorem 3.5, the only binomial coefficient  $\binom{n}{k}$  with  $1 \leq k \leq n-1$  that might fail to be divisible by  $p$  or  $q$  is  $\binom{n}{n/2}$ . Hence it suffices to add an extra prime with this purpose.  $\square$

**Proposition 4.2.** *For a positive integer  $n$ , let  $q_1$  be the largest prime smaller than  $n$ , let  $p_1^{a_1}$  be the largest prime-power divisor of  $n$  and let  $p_2^{a_2}$  be the second largest prime-power divisor of  $n$ . If  $p_1^{a_1} p_2^{a_2} > n - q_1$ , then  $n$  satisfies the 3-variation of Condition 1 with  $p_1, p_2$  and  $q_1$ .*

*Proof.* By Lucas' Theorem, for any  $k$  such that  $1 \leq k < p_1^{a_1}$ , the binomial coefficient  $\binom{n}{k}$  is divisible by  $p_1$ , and for any  $k$  such that  $n - q_1 < k \leq n/2$  the binomial coefficient  $\binom{n}{k}$  is divisible by  $q_1$ . Thus we need to add a prime that divides at least the binomial coefficients  $\binom{n}{k}$  with  $p_1^{a_1} \leq k \leq n - q_1$  in which  $k$  is a multiple of  $p_1^{a_1}$ . For this, we pick  $p_2$  and therefore we only need to consider those values of  $k$  that are, in addition, multiples of  $p_2^{a_2}$ . The least  $k$  that is a multiple of both prime powers is  $p_1^{a_1} p_2^{a_2}$ . Therefore, if  $p_1^{a_1} p_2^{a_2} > n - q_1$ , then all values of  $k$  lying in the interval  $p_1^{a_1} \leq k \leq n - q_1$  are such that  $\binom{n}{k}$  is divisible by  $p_1$  or  $p_2$ .  $\square$

In the statement of Proposition 4.2, the condition that  $p_1^{a_1} p_2^{a_2} > n - q_1$  holds by Nagura's bound [13] if we impose instead that  $p_1^{a_1} p_2^{a_2} > n/6$ .

For each  $n$ , we are interested in the minimum number  $N$  of primes such that  $n$  satisfies the  $N$ -variation of Condition 1. We next discuss upper bounds for  $N$ .

**Proposition 4.3.** *For positive integers  $n$  and  $d$ , suppose that there is a prime  $q$  such that  $n/(d+1) < q < n/d$  and a prime-power divisor  $p^a$  of  $n$  such that  $n - dq < p^a$ . Then  $n$  satisfies the  $N$ -variation of Condition 1 with  $N = 2 + \lfloor d/2 \rfloor$ .*

*Proof.* By Theorem 2.5, the binomial coefficients  $\binom{n}{k}$  are divisible by  $q$  except possibly if  $k$  lies in a dangerous interval. In the dangerous intervals we only need to consider those integers that are multiples of  $p^a$ , since otherwise  $\binom{n}{k}$  is divisible by  $p$ . Since we are assuming that  $n - dq < p^a$ , we know that in each dangerous interval there is at most one multiple of  $p^a$ . This means that the worst case is the one in which there is a multiple of  $p^a$  in every dangerous interval  $[cq, cq + \beta]$  with  $1 \leq c \leq \lfloor d/2 \rfloor$ . Hence we pick one extra prime for each such interval.  $\square$

**Corollary 4.4.** *If  $1 < d < 5$  and  $p^a > q_d + \beta_d$  where  $p^a$  divides  $n$  and  $q_d$  is the largest prime below  $n/d$ , and  $\beta_d = n - dq_d$ , then  $n$  satisfies Condition 1 with  $p$  and  $q_d$ .*

*Proof.* By Lemma 3.2, we may assume that  $n/(d+1) < q_d$ . If  $1 < d < 5$ , then  $\lfloor d/2 \rfloor$  equals 1 or 2. If  $\lfloor d/2 \rfloor = 1$ , then the assumption that  $p^a > q_d + \beta_d$  implies that no multiple of  $p^a$  falls into any dangerous interval until  $n/2$ . If  $\lfloor d/2 \rfloor = 2$ , then we need to check that  $2p^a > 2q_d + \beta_d$  in order to ensure that  $2p^a$  does not fall into the third

dangerous interval. The minimum value of  $p^a$  such that our assumption  $p^a > q_d + \beta_d$  holds is  $q_d + \beta_d + 1$ . The next multiple of  $q_d + \beta_d + 1$  is  $2q_d + 2\beta_d + 2$ , which is greater than  $2q_d + \beta_d$  and therefore  $2p^a$  does not fall into the third dangerous interval.  $\square$

In order to refine the conclusion of Proposition 4.3, we consider the Diophantine equation

$$(4.1) \quad p^a x - q_d y = \delta,$$

for  $0 \leq \delta \leq \beta_d = n - dq_d$ , where  $p^a$  is a prime-power divisor of a given number  $n$  and  $q_d$  is the largest prime below  $n/d$  with  $d \geq 1$ . We keep assuming, as above, that  $q_d > n/(d+1)$ . We will also assume that  $p \neq q_d$ , which guarantees that (4.1) has infinitely many solutions for each value of  $\delta$ . Specifically, if  $(x_1, y_1)$  is a particular solution for some value of  $\delta$ , then the general solution for this  $\delta$  is

$$x = x_1 + r q_d, \quad y = y_1 + r p^a,$$

where  $r$  is any integer. In the next theorem we denote by  $N(\delta)$  the number of solutions  $(x, y)$  of (4.1) with  $x > 0$  and  $0 \leq y \leq \lfloor d/2 \rfloor$  for each value of  $\delta$  with  $0 \leq \delta \leq \beta_d$ . Thus  $N(\delta) = 0$  precisely when (4.1) has no solution  $(x, y)$  subject to these conditions.

**Theorem 4.5.** *For positive integers  $n$  and  $d$ , suppose that the largest prime  $q_d$  below  $n/d$  satisfies  $q_d > n/(d+1)$ , and let  $\beta_d = n - dq_d$ . Let  $p^a$  be a prime power dividing  $n$  with  $p \neq q_d$ . Then  $n$  satisfies the  $N$ -variation of Condition 1 with*

$$N = 2 + \sum_{\delta=0}^{\beta_d} N(\delta),$$

where  $N(\delta)$  is the number of solutions  $(x, y)$  of  $p^a x - q_d y = \delta$  with  $x > 0$  and  $0 \leq y \leq \lfloor d/2 \rfloor$  for each value of  $\delta$  with  $0 \leq \delta \leq \beta_d$ .

*Proof.* The number  $N(\delta)$  counts how many times a multiple of  $p^a$  falls into a dangerous interval  $[cq_d, cq_d + \beta_d]$  at a distance  $\delta$  from the origin of that interval. Thus we pick an extra prime for each such case, and add two to the sum in order to account for  $p$  and  $q_d$ .  $\square$

**Example 4.6.** The largest prime-power divisor of  $n = 96135$  is  $p = 29$ . For  $d = 4$  we find that  $q_4 = 24029$  and  $\beta_4 = 19$ . Since  $24029 \equiv 17 \pmod{29}$ , the only solution  $(x, y)$  of the Diophantine equation  $29x - 24029y = \delta$  with  $x > 0$  and  $0 \leq y \leq 2$  is  $(829, 1)$  for  $\delta = 12$ . Thus,  $N(12) = 1$  and  $N = 3$  for  $d = 4$ . In other words, the only occurrence of a multiple of 29 in a dangerous interval for  $d = 4$  is  $24041 \in [24029, 24048]$ . This example shows that the bound  $2 + \lfloor d/2 \rfloor$  given in Proposition 4.3 can be lowered.

The number  $N$  given by Theorem 4.5 is not a sharp bound. For those multiples  $p^a x$  of  $p^a$  falling into a dangerous interval  $[cq_d, cq_d + \beta_d]$ , it often happens that the corresponding binomial coefficient  $\binom{n}{p^a x}$  is divisible by  $p$ , as in Example 4.6 or in other examples given in the previous sections. It could also be divisible by  $q_d$  if  $d \geq q_d$ . When  $d < q_d$ , we have that  $n$  satisfies Condition 1 with  $p$  and  $q_d$  if and only if the binomial coefficient  $\binom{n}{p^a x}$  is divisible by  $p$  for every solution  $(x, y)$  of (4.1) with  $x > 0$  and  $0 \leq y \leq \lfloor d/2 \rfloor$ , since  $n = dq_d + \beta_d$  and  $p^a x = yq_d + \delta$  with  $\delta \leq \beta_d < q_d$  and  $y \leq \lfloor d/2 \rfloor < d$ , so  $\binom{n}{p^a x}$  is not divisible by  $q_d$  by Lucas' Theorem. Note also, for practical purposes, that  $\binom{n}{p^a x} \equiv \binom{n/p^a}{x} \pmod{p}$ .

## 5. EVERY NUMBER HAS MULTIPLES FOR WHICH CONDITION 1 HOLDS

We next prove that every positive integer  $n$  has infinitely many multiples for which Condition 1 holds. We are indebted to R. Woodroffe for simplifying and improving our earlier statement of this result, which was based on prime gap conjectures.

It follows from the Prime Number Theorem [7] that, given any real number  $\varepsilon > 0$ , there is a prime between  $m$  and  $m(1 + \varepsilon)$  for sufficiently large  $m$ . This fact can be used to prove the following:

**Theorem 5.1.** *For every positive integer  $n$  and every prime  $p$ , the number  $np^k$  satisfies Condition 1 with  $p$  and another prime, for all sufficiently large values of  $k$ .*

*Proof.* For any prime  $p$  and any  $k > 0$ , let  $m = np^k - p^k = p^k(n - 1)$ . Then

$$np^k = m + p^k = m \left( 1 + \frac{1}{n-1} \right).$$

Therefore, by the Prime Number Theorem, there is a prime between  $m$  and  $np^k$  for all sufficiently large values of  $k$ . Choose the largest prime  $q$  with this property. Thus,

$$np^k - p^k < q < np^k,$$

so  $np^k - q < p^k$ , from which it follows, according to Corollary 2.6, that  $np^k$  satisfies Condition 1 with  $p$  and  $q$ .  $\square$

**Theorem 5.2.** *For every positive integer  $n$  there is a number  $M$  such that if  $p$  is any prime with  $p > M$  then  $np$  satisfies Condition 1 with  $p$  and another prime.*

*Proof.* Given  $n$ , let  $\varepsilon = 1/(n - 1)$ . Choose  $m_0$  such that there is a prime between  $m$  and  $m(1 + \varepsilon)$  for all  $m \geq m_0$ , and let  $M = \varepsilon m_0$ . If  $p$  is any prime such that  $p > M$ , then for  $m = p(n - 1)$  we have

$$np = m + p = m \left( 1 + \frac{p}{m} \right) = m \left( 1 + \frac{1}{n-1} \right) = m(1 + \varepsilon).$$

Therefore, by our choice of  $m_0$ , there is a prime between  $m$  and  $np$ . If  $q$  is the largest prime with this property, then  $np - p < q < np$ , and consequently  $np$  satisfies Condition 1 with  $p$  and  $q$ .  $\square$

Prime gap conjectures provide information relevant to our problem. For example, if  $p_i$  denotes the  $i$ -th prime, then Cramér's conjecture [6] claims that there exist constants  $M$  and  $N$  such that if  $p_i \geq N$  then

$$p_{i+1} - p_i \leq M(\log p_i)^2.$$

**Proposition 5.3.** *Let  $m$  be the number of distinct prime factors of  $n$ . If Cramér's conjecture is true and  $n$  grows sufficiently large keeping  $m$  fixed, then  $n$  satisfies Condition 1.*

*Proof.* If  $n$  has  $m$  distinct prime factors, then  $\sqrt[m]{n} \leq p^a$ , where  $p^a$  is the largest prime-power divisor of  $n$ . Let  $M$  and  $N$  be the constants given by Cramér's conjecture. Pick  $n_0$  such that if  $n \geq n_0$  then  $M(\log n)^2 < \sqrt[m]{n}$ . For every  $n$  such that  $n \geq n_0$  and  $N \leq p_i < n \leq p_{i+1}$  (where  $p_i$  denotes the  $i$ -th prime), we have

$$n - p_i \leq p_{i+1} - p_i \leq M(\log p_i)^2 < M(\log n)^2 < \sqrt[m]{n} \leq p^a,$$

from which it follows that  $n$  satisfies Condition 1 with  $p$  and  $p_i$ .  $\square$

We note that the argument used in the proof of Proposition 5.3 yields an alternative proof of the fact that Condition 1 holds for a set of integers of asymptotic density 1 if Cramér's conjecture holds, a result first found in [15, § 5]:

**Theorem 5.4** ([15]). *If Cramér's conjecture is true, then the set of numbers in the sequence (2.1) has asymptotic density zero.*

*Proof.* Suppose that Cramér's conjecture holds with constants  $M$  and  $N$ , and denote by  $\omega(n)$  the number of distinct prime divisors of  $n$ . Thus  $n^{1/\omega(n)} \leq p^a$ , where  $p^a$  is the largest prime-power divisor of  $n$ . According to [8, § 3.2], for every  $\varepsilon > 0$  the inequality

$$(5.1) \quad \omega(n) < (1 + \varepsilon) \log \log n$$

holds for all  $n$  except those of a set of asymptotic density zero. Since

$$\lim_{n \rightarrow \infty} \frac{n^{1/\log \log n}}{(\log n)^k} = \infty$$

for all  $k$ , there is an  $n_0$  such that  $n^{1/\omega(n)} > M(\log n)^2$  if  $n \geq n_0$ . Now, if  $n$  is bigger than  $n_0$  and satisfies  $N \leq p_i < n \leq p_{i+1}$ , and moreover  $n$  is not in the set of integers for which (5.1) fails, then

$$n - p_i \leq p_{i+1} - p_i \leq M(\log p_i)^2 < M(\log n)^2 < n^{1/\omega(n)} \leq p^a.$$

Therefore,  $n$  satisfies Condition 1 with  $p$  and  $p_i$ . □

## 6. MULTINOMIALS

We also consider a generalization of Condition 1 to multinomials. We say that a positive integer  $n$  satisfies *Condition 1 for multinomials of order  $m$*  if there are primes  $p$  and  $q$  such that the multinomial coefficient

$$\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \cdots k_m!}$$

is divisible by either  $p$  or  $q$  whenever  $k_1 + \cdots + k_m = n$  with  $1 \leq k_i \leq n - 1$  for all  $i$ .

**Proposition 6.1.** *If  $n$  satisfies Condition 1 with two primes  $p$  and  $q$ , then  $n$  satisfies Condition 1 for multinomials of any order  $m \leq n$  with  $p$  and  $q$ .*

*Proof.* This follows from the equality

$$\binom{n}{k_1, k_2, \dots, k_m} = \binom{n}{k_1} \binom{n - k_1}{k_2} \binom{n - k_1 - k_2}{k_3} \cdots \binom{k_m}{k_m},$$

and the fact that  $\binom{n}{k_1}$  is divisible by  $p$  or  $q$  by assumption. □

Therefore, if Condition 1 is proven for binomial coefficients, then it automatically holds for multinomial coefficients.

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