ON A PROBLEM OF DE KONINCK

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ABSTRACT. Let $\sigma(n)$ and $\gamma(n)$ denote the sum of divisors and the product of distinct prime divisors of n respectively. We shall show that, if $n \neq 1, 1782$ and $\sigma(n) = (\gamma(n))^2$, then there exist odd (not necessarily distinct) primes p, p' and (not necessarily odd) distinct primes $q_i (i = 1, 2, \ldots, k)$ such that $p, p' \parallel n$, $q_i^2 \parallel n (i = 1, 2, \ldots, k)$ and $q_1 \mid \sigma(p^2), q_{i+1} \mid \sigma(q_i^2) (i = 1, 2, \ldots, k-1), p' \mid \sigma(q_k^2)$.

1. INTRODUCTION

Let $\sigma(n)$ and $\gamma(n)$ denote the sum of divisors and the product of distinct prime divisors of n, called the *radical* of n, respectively. Moreover, let $\omega(n)$ denote the number of distinct prime divisors of n. De Koninck [6] posed the problem to prove or disprove that the only solutions

$$\sigma(n) = (\gamma(n))^2 \tag{1}$$

are $n = 1, 1782$.

According to the editorial comment, it is shown that such an integer $n \neq 1, 1782$ must be even, have at least four prime factors, be neither squarefree and squarefull, be greater than 10^9 and has no prime factor raised to a power congruent to 3 (mod 4). Later, further necessary conditions to satisfy $\sigma(n) = (\gamma(n))^2$ have been shown. Broughan, De Koninck, Kátai and Luca [2] showed that, if $n > 1$, then

$$n = 2^{e_0} \prod_{i=1}^{s} p_i^{e_i}, \tag{2}$$

where p_i are distinct odd primes and e_i are positive integers satisfying (a) $p_1 \equiv 3 \pmod 8, e_1 = 1$ and the other e_i's are even, or (b) $p_1 \equiv p_2 \equiv e_1 \equiv e_2 \equiv 1 \pmod 4, \min\{e_1, e_2\} = 1$ and the other e_i's are even. Moreover, they showed that $\omega(n) \geq 5$ and n cannot be fourth power free.

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Luca [8] showed that, if a positive integer n satisfies $\omega(n) = T$ and $\sigma(n) \mid L(\gamma(n))^K$ with K, L positive integers, then

$$n < \exp\left(((K + \log L)T!)^{2^T}\right).$$

Broughan, De Koninck, Kátai and Luca [2] showed that $\omega(n) \geq 5$ and n cannot be fourth power free. Broughan, Delbourgo and Zhou [1] showed that $p_1 \geq 43$ in the case (a), $p_1 \geq 173$ in the case (b) with $\alpha_2 > \alpha_1 = 1$ and n must be divisible by the fourth power of an odd prime.

Chen and Tong [4] showed that if $n \neq 1, 1782$ satisfies (1) with (a), then n is divisible by 3 and by the fourth powers of at least two odd primes, $p_1 \geq 1571$, at most two of p_i's are greater than p_1, $e_i = 2$ for at least two i's and $e_i = 2$ for any i such that $10p_i^2 \geq p_1$. Moreover, they showed that for any n satisfying (1), at least half of the numbers among $e_i + 1$'s must be either primes or prime squares.

As usual, $p^e \mid\mid n$ denotes that $p^e \mid n$ but $p^{e+1} \nmid n$. In this paper, we shall give the following new necessary condition for an integer n to satisfy (1).

Theorem 1.1. *If an integer $n \neq 1, 1782$ of the form* (2) *satisfies* (1)*, then there exist odd (not necessarily distinct) primes p, p' and (not necessarily odd) distinct primes $q_{j_i} (i = 1, 2, \ldots, k)$ such that $p, p' \mid\mid n$, $q_i^2 \mid\mid n (i = 1, 2, \ldots, k)$ and $q_1 \mid \sigma(p^2), q_{i+1} \mid \sigma(q_i^2) (i = 1, 2, \ldots, k-1), p' \mid \sigma(q_k^2)$.*

Our idea is based on the following simple observation. Consider the special case $e_i = 1$ only for $i = 1$, $q_1 \mid \sigma(p^2)$ for two primes p and for each p, $p \mid \sigma(q_i^{e_i})$ with $e_i \geq 4$ for two primes q_i. Now we have $\sigma(q_i^{e_i})/q_i^2 > \sqrt{\sigma(q_i^{e_i})}i > p^{1/2} > q_1^{1/4}$ for each i. Hence, $((q_1 + 1)/q_1^2) \prod_i \sigma(q_i^{e_i})/q_i^2 > q_1(q_1 + 1)/q_1^2 > 1$. In order to generalized this observation, we introduce a directed multigraph related to prime power divisors of n.

In the next section, we introduce some basic terms on directed multigraphs and prove an identity on directed multigraphs. In Section 3, we introduce a certain directed multigraph related to prime power divisors of n satisfying (1) and give the key point lemma for our proofs as well as some arithmetic preliminaries.

Under our settings described in Sections 2 and 3, we shall prove the following theorem.

Theorem 1.2. *Let $n \neq 1, 1782$ be an integer of the form* (2) *satisfying* (1) *and L be the set of odd prime divisors q_i's with $e_i = 1$. Let $G(n), N = N(L), M = M(L), B = B(L)$ and $C = C(L)$ be directed multigraphs or sets defined in Section 3. Then,*

i) *If* $q_{k+1} \to q_k \to \cdots \to q_1 \to p$ *is a path from a vertex* q_{k+1} *in* B *to a vertex* p *in* L *via vertices in* M*, then* $k \leq 3$ *and* $q_i \equiv 1 \pmod 3$ *for* $1 \leq i \leq k-1$.

ii) M *contains at most two primes* $\equiv 1 \pmod 3$. *Furthermore,* $\#M \leq 6$ *if* $\#L = 1$ *and* $\#M \leq 8$ *if* $\#L = 2$.

iii) *There exists a path from* q_i *in* L *to* q_j *in* L *consisting of vertices* $q_l \in N$ *other than* q_i, q_j*, where* q_i *and* q_j *may be the same prime.*

Now Theorem 1.1 is an arithmetic translation of iii) of Theorem 1.2. In Sections 4 and 5, we prove that the directed multigraph related to prime power divisors of n defined in Section 3 cannot have some forms, which yields iii) of Theorem (1.2). Other statements of Theorem 1.2 easily follow from an elementary divisibility property of values of $\sigma(p^2)$ with p prime.

2. An identity on directed multigraphs

Before stating our result on directed multigraphs, we would like to introduce some basic terms on directed multigraphs according to [5] with some modifications. A *directed multigraph* $G = (V, A)$ consists of a set V of elements called *vertices* and a multiset A, where an element may be contained more than once, of ordered pairs of distinct elements in V called *arcs*. $V = V(G)$ and $A = A(G)$ are called the vertex set and the arc set of G respectively. For an arc (u, v) in A, which we call an arc from u to v, the former vertex u and the latter vertex v are called its *tail* and its *head* respectively. We often write $u \to v$ if $(u, v) \in A$ and $u \xrightarrow{k} v$ if $(u, v) \in A$ exactly k times.

The *subgraph* of $G = (V, A)$ spanned by a given set of vertices $S \subset V$ is the directed graph whose vertex set is S and whose arc set consists of all arcs in A whose tail and head both belong to S.

A *walk* $(a_1, a_2, \ldots, a_k)$ is a sequence of arcs $a_i = (u_i, v_i)(i = 1, 2, \ldots, k)$ such that $v_i = u_{i+1}$ for all $i = 1, 2, \ldots, k-1$. A walk $(a_1, a_2, \ldots, a_k)$ with $a_i = (u_i, u_{i+1})(i = 1, 2, \ldots, k)$ is called a *path* if $u_1, u_2, \ldots, u_k$ and v_k are all distinct and a *cycle* if $u_1, u_2, \ldots, u_k$ are all distinct and $u_1 = v_k$. A walk $(a_1, a_2, \ldots, a_k)$ with $a_i = (u_i, u_{i+1})(i = 1, 2, \ldots, k)$ is often written as $u_1 \to u_2 \to \cdots \to u_k$. A directed graph $G = (V, A)$ is acyclic if A contains no cycle.

The *out-degree* $d^+(v) = d^+_G(v)$ and the *in-degree* $d^-(v) = d^-_G(v)$ of the vertex v are the number of arcs from v and to v respectively counted with multiplicity. A vertex v is called a *sink* if $d^+(v) = 0$ and a *source* if $d^-(v) = 0$. $S(G)$ denotes the set of sources of the directed multigraph G.

Now we would like to state our identity.

Lemma 2.1. *Let G be a directed acyclic multigraph. Then, for any vertex v_0 of G with $d^-(v_0) > 0$,*

$$\sum_{\substack{P: v_k \to v_{k-1} \to \cdots \to v_0 \subset G, \\ v_n \in S(G)}} \frac{1}{\prod_{i=0}^{k-1} d_G^-(v_i)} = 1. \tag{3}$$

Proof. If G consists of only one sink v_0 and sources $u_1, u_2, \ldots, u_l$ with arcs (u_i, v_0), then (3) is clear.

For any fixed vertices $v_0, v_1, \ldots, v_{k-1}$ such that $v_{k-1} \to v_{k-2} \to \cdots \to v_0$ and any vertex $w \to v_{k-1}$ is a source in G, we have

$$\sum_{\substack{v_k \in S(G), \\ v_k \to v_{k-1} \to \cdots \to v_0 \subset G}} \frac{1}{\prod_{i=0}^{k-1} d_G^-(v_i)} = \frac{1}{\prod_{i=0}^{k-2} d_G^-(v_i)}. \tag{4}$$

Thus, setting H to be the directed multigraph obtained from G by eliminating all arcs to v_{k-1}, we have

$$\sum_{\substack{P: v_k \to v_{k-1} \to \cdots \to v_0 \subset G, \\ v_k \in S(G)}} \frac{1}{\prod_{i=0}^{k-1} d_G^-(v_i)} = \sum_{\substack{P: v_k \to v_{k-1} \to \cdots \to v_0 \subset H, \\ v_k \in S(H)}} \frac{1}{\prod_{i=0}^{k-1} d_H^-(v_i)}. \tag{5}$$

Since G is acyclic, this descent argument eventually reduces G to a directed multigraph (V, A) with $V = \{v_0, u_1, u_2, \ldots, u_l\}$ and $A = \{(u_i, v_0), i = 1, \ldots, l\}$. Now the lemma follows by induction. $\square$

3. A directed graph related to divisors of an integer

Let n be a positive integer greater than one. We define the directed multigraph $G = G(n)$ arising from n by setting its vertex set to be the set of primes dividing $n\sigma(n)$.

and each arc $p \xrightarrow{k} q$ to be of multiplicity k if $q^k \mid\mid \sigma(p^e)$ for the exponent e with $p^e \mid\mid n$. For convenience, we write $p^e \to q^f$ if $p \to q$ and $p^e, q^f \mid\mid n$ and $p^e \in S$ if $p^e \mid\mid n$ and p belongs to a set S of vertices.

For a set S of vertices $w_1, w_2, \ldots, w_k$ of G, we define their 2-*incomponent* $N(S)$ to be the subgraph of G consisting $w_1, w_2, \ldots, w_k$ themselves and the vertices w such that there exists a path $v^2 \to v_1^2 \to \cdots \to v_l^2 \to w_i$ to some vertex w_i, their 2-*boundary* $B(S)$ by the set of vertices $v \notin N(S)$ from which there exists an edge to some vertex in $N(S)$ and their 2-*closure* $C(S)$ by the subgraph whose vertex set is $N(S) \cup B(S)$ and whose arc set consists of all edges in $B(S)$ and all arcs from $N(S)$ to $B(S)$. For convenience, we simply write $N(w)$ for $N(\{w\})$ and so on. Moreover, we put $p_0 = 2$ and $M(S) = N(S) \backslash S$. We note that $C(S)$ may contain $p_0 = 2$.

Now Theorem 1.1 can be restated as in iii) of Theorem 1.2.

For a set S of prime powers, we define $h(S) = \prod_{p^e \in S} \sigma(p^e)/p^2$. Clearly, we have $h(S_0) = \sigma(n)/(\gamma(n))^2$ for the set S_0 of all prime-power divisors of n. For convenience, we write $h(p^e) = h(\{p^e\})$ for a prime power p^e and $h(n) = h(S_0)$ for the set S_0 mentioned above.

We clearly have the following lemma.

Lemma 3.1. *We have $h(m) \geq 1$ for any positive integer m with the equality just when $m = 1$. If m_1 divides m_2, then $h(m_1) \leq h(m_2)$. Furthermore, if S and T are disjoint sets of prime-power divisors of n, then $h(S \cup T) = h(S)h(T)$.*

We also use the following divisibility property of values of the polynomial $x^2 + x + 1$.

Lemma 3.2. *If m is an integer and a prime p divides $m^2 + m + 1$, then $p = 3$ or $p \equiv 1 \pmod 3$. Furthermore, 3 divides $m^2 + m + 1$ if and only if $m \equiv 1 \pmod 3$.*

Proof. The former is a special case of Theorem 94 of [7]. Indeed, if $p \neq 3$ divides $m^2 + m + 1$, then $m \not\equiv 1 \pmod p$ and $m^3 \equiv 1 \pmod p$. Hence, $m \pmod p$ has the multiplicative order 3 and therefore $p - 1$ must be divisible by 3. The latter can be easily confirmed by calculating modulo 3. □

The following lemma is the key point of our proof of Theorem 1.1.

Lemma 3.3. *Let n be an integer of the form (2) satisfying (1) and L be a set of prime power divisors of n. We define quantities κ_i for $p_i \in C = C(L)$ and λ_i for $p_i \in M = M(L)$ by*

$$\sigma(p_i^{e_i}) = \kappa_i \prod_{p_j \in N(L)} p_j^{k_{i,j}} \tag{6}$$

and

$$p_i^2 = \lambda_i \prod_{p_j \in N(L)} p_j^{k_{i,j}}. \tag{7}$$

If $N = N(L)$ is acyclic and any element of L is a sink of N, then

$$\prod_{p_i \in B} \sigma(p_i^{e_i}) = \prod_{p_i \in B} \kappa_i \prod_{p_j \in M} \lambda_j \prod_{p_i \in L} p_i^2 \tag{8}$$

and

$$h(C) > \prod_{p_i \in B} \kappa_i^{\frac{1}{2}} p_i^{\frac{e_i}{2} - 2} \prod_{p_j \in M} \frac{\sqrt{\sigma(p_j^2)}}{p_j} \prod_{p_i \in L} p_i^{e_i - 1}. \tag{9}$$

Proof. We see that

$$p_i = \lambda_i^{\frac{1}{2}} \prod_{p_i \to p_j, p_j \in N} p_j^{\frac{1}{2}} \tag{10}$$

for $p_i \in M$. Since we assume that a vertex in L must be a sink in $C = C(L)$, if $P = q_1^2 \to \cdots \to q_k^2 \to q_0$ is a path in N and a prime q in L occurs in P, then $q = q_0$. Moreover, by the assumption, N is acyclic. Hence, we iterate (10) to obtain

$$q_1 = \prod_{q_1^2 \to \cdots \to q_k^2 \to p^1, p \in L} (\lambda_{j_1}^{\frac{1}{2}} \lambda_{j_2}^{\frac{1}{4}} \cdots \lambda_{j_k}^{\frac{1}{2^k}}) q_i^{\frac{1}{2^k}} \tag{11}$$

for any $q_1 \in M$, where j_m's $(m = 1, 2, \ldots, k)$ are indices such that $p_{j_m} = q_m$.

Moreover, we see that

$$\sigma(p_i^{e_i}) = \kappa_i \prod_j p_j^{k_{i,j}} = \kappa_i \prod_{p_i \to p_j, p_j \in N} p_j \tag{12}$$

for $p_i \in B$. Combining (11) and (12), we have

$$\prod_{p_i \in B} \sigma(p_i^{e_i}) = \left(\prod_{p_i \in B} \kappa_i \right) \prod_{p_j \in M} \lambda_j^{s_j} \prod_{p_i \in L} p_j^{2s_j}, \tag{13}$$

where, observing that $d_C^-(p_i) = d_G^-(p_i) = 2$ for any $p_i \in N$ from (1),

$$s_j = \sum_{\substack{q_0 \to q_1 \to \cdots \to q_k = p_j, \\ q_0 \in B, q_1, \ldots, q_{k-1} \in N}} \frac{1}{2^k} = \sum_{\substack{q_0 \to q_1 \to \cdots \to q_k = p_j, \\ q_0 \in B, q_1, \ldots, q_{k-1} \in N}} \frac{1}{\prod_{l=1}^k d_C^-(q_l)}. \tag{14}$$

Since N is acyclic by the assumption, Lemma 2.1 gives that $s_j = 1$ for all $p_j \in N$. Thus we obtain

$$\prod_{p_i \in B} \sigma(p_i^{e_i}) = \left(\prod_{p_i \in B} \kappa_i \right) \left(\prod_{p_j \in M} \lambda_j \right) \prod_{p_i \in L} p_i^2 \tag{15}$$

and therefore

$$\begin{aligned} \prod_{p_i \in C} \frac{\sigma(p_i^{e_i})}{p_i^2} &> \prod_{p_i \in B} p_i^{\frac{e_i}{2} - 2} \sqrt{\sigma(p_i^{e_i})} \prod_{p_j \in M} \frac{\sigma(p_j^2)}{p_j^2} \prod_{p_i \in L} p_i^{e_i - 1} \\ &\geq \prod_{p_i \in B} \kappa_i^{\frac{1}{2}} p_i^{\frac{e_i}{2} - 2} \prod_{p_j \in M} \lambda_j^{\frac{1}{2}} \frac{\sigma(p_j^2)}{p_j^2} \prod_{p_i \in L} p_i^{e_i - 1}. \end{aligned} \tag{16}$$

Now the lemma immediately follows observing that $\lambda_j \geq p_j^2 / \sigma(p_j^2)$ for $p_j \in M$. $\square$

4. Acyclic cases

In this and the next section, We assume that n is an integer of the form (2) satisfying (1) and we put L to be the set of odd primes p_i with $e_i = 1$. Thus, $L = \{p_1, p_2\}$ in the case (b) with $e_1 = e_2 = 1$ and $L = \{p_1\}$ in the case (a) and the case (b) with $e_1 = 1 < e_2$. In this section, we shall show that, $N = N(L)$ must have a cycle or we must have $L = \{p_1, p_2\}$ and $p_1 \in B(p_2)$ or $p_2 \in B(p_1)$.

Lemma 4.1. *If n is divisible by 4 or 2×3^6 or n is divisible by 2 and 3 does not belong to $C = C(L)$, then $N = N(L)$ must have a cycle or we must have $L = \{p_1, p_2\}$ and $p_1 \in B(p_2)$ or $p_2 \in B(p_1)$.*

Proof. Assume that n of the form (2) is divisible by 2^2 or 2×3^6 and N is acyclic and, in the case $L = \{p_1, p_2\}$, $p_1 \notin B(p_2)$ and $p_2 \notin B(p_1)$.

We can easily see that any prime p_i in L must be a sink in N. Indeed, if $p_i \in L$ and $p_i \to p_j$ for some $p_j \in N$ not necessarily distinct from p_i, then, there exists a path from p_i to $p_j \in L$ via N, which contradicts to the assumption. Thus, we can apply Lemma 3.3 and, observing that $\kappa_i \geq 1$ for all $p_i \in B = B(L)$, we obtain

$$h(C) > \prod_{p_i \in B} p_i^{\frac{e_i}{2} - 2}. \tag{17}$$

If $4 = 2^2$ divides n, then, observing that $e_i/2 \geq 2$ for $p_i \in B$, Lemma 3.1 and (17) gives that $h(n) > h(C) > 1$.

If 2 divides n and 3 does not belong to C, then, by Lemma 3.1, we have $h(n) \geq h(C \cup \{2, 3^2\}) = h(\{2, 3^2\})h(C) > (4/3)(13/9) > 1$.

If 2×3^6 divides n and 3 belongs to C, then (17) yields that $h(C) > 3$ and $h(n) \geq (3/4)h(C) > 9/4 > 1$.

Thus, in any case, we have $h(n) > 1$ or, equivalently, $\sigma(n) > (\gamma(n))^2$, which contradicts to the assumption that n satisfies (1). $\square$

Lemma 4.2. *If $e_0 = 1$ and $3^2 \in N = N(L)$, then N must have a cycle or we must have $L = \{p_1, p_2\}$ and $p_1 \in B(p_2)$ or $p_2 \in B(p_1)$.*

Proof. Assume that $3^2 \in N$, N is acyclic and, in the case (b) with $e_1 = e_2 = 1$, $p_1 \notin B(p_2)$ and $p_2 \notin B(p_1)$. Since 3^2 belongs to N, $3^2 \to 13$ also belongs to N. If $13 \in M = M(L)$, then $3^2 \to 13^2 \to 3$, which contradicts to the assumption that N is acyclic. Thus, $13^1 \in L$. Now we may assume that $p_1 = 13$. We see that $p_2 \in L$ and $p_2 \equiv 1 \pmod 4$ since $p_1 \equiv 1 \pmod 4$. Hence, $13 \to 7^e$ divides N.

We see that $e \geq 2$ must be even since $2^3 \mid (7+1)$. If $7^2 \parallel N$, then $13 \to 7^2 \to 3^2 \to 13$, contrary to the assumption that N is acyclic. Thus, $e \geq 4$.

If $7^e \notin B(p_2)$, then, applying Lemma 3.3, we have

$$h(n) \geq h(\{2, 3^2, 13, 7^e\})h(C(p_2)) > h(C(p_2)) > 1, \tag{18}$$

which is a contradiction.

Thus, we may assume that $7^e \in B(p_2)$. If $e \geq 8$, then, Lemma 3.3 gives that

$$h(\{2, 3^2, 13\})h(C(p_2)) > 7^2 h(\{2, 3^2, 13\}) > 1, \tag{19}$$

which is a contradiction again.

Assume that $7^4 \in B(p_2)$, which immediately yields that $2801 \in N(p_2)$. If $p_2 = 2801$, then $p_2 \to 3^2 \to 13 = p_1$, contrary to the assumption that $p_2^{e_2} \notin B(p_1)$. Thus, $2801^2 \in N(p_2)$ and $2801^2 \to 37, 43, 4933$.

If $p_2 = 37$, then $p_3 = 19$ divides n. If $p_2 = 4933$, then $p_3 = 2467$ divide n. In both cases, if $p_3^2 \parallel n$, then $p_2 \to p_3^2 \to 3^2 \to 13 = p_1$, which is impossible. If $p_3^4 \mid n$, then

$$h(n) \geq h(\{2, 3^2, 13, 7^4, 2801^2, 37, 19^4\}) > 1$$

or

$$h(n) \geq h(\{2, 3^2, 13, 7^4, 2801^2, 4933, 2467^4\}) > 1.$$

Hence, $p_2 = 37$ and $p_2 = 4933$ are both impossible.

If $37^2 \in N(p_2)$, then $\sigma(37^2) = 3 \times 7 \times 67$ and therefore $67 \in N(p_2)$. Since $67 \equiv 3 \pmod 4$, we have $p_2 \neq 67$ and $37^2 \to 67^2$. But this implies that $3^3 \mid \sigma(2 \times 37^2 \times 67^2) \mid \sigma(n)$, which is a contradiction.

If $4933^2 \in N(p_2)$, then $\sigma(4933^2) = 3 \times 127 \times 193 \times 331$ and therefore $p_2 = 193$, since $p_3^2 \in N(p_2)$ with $p_3 = 127, 193$ or 331 would imply that $3^3 \mid \sigma(2 \times 4933^2 \times p_3^2)$, a contradiction. Thus $p_3 = 97$ must divide n. If $e_3 = 2$, then $3^3 \mid \sigma(2 \times 4933^2 \times 97^2) \mid \sigma(n)$, which is impossible. But, if $e_3 \geq 4$, then

$$h(n) \geq h(\{2, 3^2, 13, 7^4, 2801^2, 4933^2, 193, 97^4\}) > 1,$$

which is a contradiction again.

If $43^2 \in N(p_2)$, then $\sigma(43^2) = 3 \times 7 \times 631$ and therefore $631 \in N(p_2)$. Since $631 \equiv 3 \pmod 4$, we must have $631^2 \in N(p_2)$ and $3^3 \mid \sigma(2 \times 43^2 \times 631^2) \mid \sigma(n)$, which is impossible. Thus we see that $2801^f \notin N(p_2)$ and therefore $7^4 \notin N(p_2)$.

Now we must have $7^6 \in B(p_2)$. $\sigma(7^6) = 29 \times 4733$ must divide n. It is impossible that $p_2 = 29, 4733$ since this would imply that $p_2 \to 3^2 \to$

$13 = p_1$. If $29^2 \in N(p_2)$, then, observing that $\sigma(29^2) = 13 \times 67$ and $\sigma(67^2) = 3 \times 7^2 \times 31$, we must have $29^2 \to 67^2 \to 31^2$. However, this is impossible since $3^3 \mid \sigma(2 \times 67^2 \times 31^2)$.

If $4733^2 \in N(p_2)$, then, observing that $4733^2 + 4733 + 1 = 22406023 \equiv 3 \pmod 4$ is prime, we must have $22406023^2 \in N(p_2)$. If $22406023^2 \to p_2$, then $p_2 = 1117$ or $p_2 = 606538249$. However, neither of them is impossible since $13^3 \mid \sigma(3^2 \times 22406023^2 \times 1117)$ and $5^3 \mid \sigma(606538249)$. Hence, we must have $22406023^2 \to p_3^2 \in N(p_2)$ for some prime divisor $p_3 \neq 3$ of $\sigma(22406023^2)$. But, this is also impossible since $3^3 \mid \sigma(2 \times 22406023^2 \times p_3^2)$.

Now we conclude that 7 cannot divide N and therefore 13 cannot be in L. Hence, 3^2 cannot be in $N(L)$. This proves the lemma. □

Lemma 4.3. *If $e_0 = 1$ and $3^4 \in B = B(L)$, then $n = 1782$, $N = N(L)$ must have a cycle or we must have $L = \{p_1, p_2\}$ and $p_1 \in B(p_2)$ or $p_2 \in B(p_1)$.*

Proof. Since $3^4 \in B$, $p_1 = 11$ or $11^2 \in N$. If $p_1 = 11$, then $n = 2 \times 3^4 \times p_1 = 1782$. We note that if $n = n_0$ is a solution of (1), then $n = kn_0$ with $k > 1$ odd and $\gcd(k, n) = 1$ can never be a solution of (1). Indeed, $h(n_0) = h(kn_0) = 1$, then $h(k) = 1$. However, this is impossible since $n = 1$ is the only odd solution of (1).

Now we may assume that $11^2 \in N$. If $p_3^2 \in N$ with $p_3 = 7$ or 19, then $3^4 \to 11^2 \to p_3^2 \to 3^4$ in N, contrary to the assumption. Thus, we must have $p_1 = 19$ and $7^4 \mid N$. Since $p_1 \equiv 3 \pmod 8$, we must have $L = \{p_1\}$. Hence,

$$h(n) \geq h(\{2, 3^4, 11^2, 7^4, 19\}) > 1,$$

which is impossible again. □

5. Cyclic cases

In the previous section, we showed that, if an integer n of the form (2) satisfies (1) and L is the set of odd primes p_i with $e_i = 1$, then $N(L)$ must be cyclic or we must have $L = \{p_1, p_2\}$ and $p_1 \in B(p_2)$ or $p_2 \in B(p_1)$. In this section, we shall show that $M(L)$ must be acyclic and then complete the proof of Theorems 1.1 and 1.2. We begin by showing that $M = M(L)$ cannot contain a cycle of length ≥ 3.

Lemma 5.1. *Assume that for there exists no arc $p_i \to p_j$ from $p_i \in L$ to $p_j \in N(L)$. Then $M = M(L)$ cannot contain a cycle of length ≥ 3.*

Proof. Assume that $q_i (i = 1, 2, \ldots, l)$ is a cycle of length $l \geq 3$. We see that $q_i \equiv 1 \pmod 3$ for all i except possibly one index j, for which $q_j = 3$. We must have $l = 3$ and $q_j = 3$ for some j since otherwise we must have $q_i \equiv 1$

(mod 3) for at least three i's by Lemma 3.1 and $3^3 \mid \prod_j \sigma(q_i^2) \mid n$, which is a contradiction.

Now we see that $3^2 \to 13^2 \to 61^2 \to 3^2$ is a cycle in M and $p_1 = 97 \in L$. Hence, $97 \to 7^e$ must divide n and, observing that no more prime $p_i \equiv 1 \pmod 3$ can satisfy $p_i^2 \mid\mid N$ again, $e \geq 4$ must be even. Moreover, we must have $e_0 \geq 2$ since $3^3 \mid \sigma(2 \times 13^2 \times 61^2)$.

If $L = \{p_1\}$ and 7^6 divides n, then

$$h(n) \geq h(7^6)h(C(L)) > h(\{3^2, 13^2, 61^2, 97, 7^6\}) > 1,$$

which is a contradiction.

If $L = \{p_1, p_2\}$ and 7^{10} divides n, then, since $N(p_2)$ is acyclic, Lemma 3.3 gives

$$h(n) \geq h(\{3^2, 13^2, 61^2, 97\})h(C(p_2) \cup \{7^{10}\}) > 7^3 h(\{3^2, 13^2, 61^2, 97\}) > 1. \tag{20}$$

Now we must have $e = 4, 6$ or 8. We can never have $97 \to 7^8$ since $3^3 \mid \sigma(13^2 \times 61^2 \times 7^8)$. In both cases $e = 4$ and $e = 6$, we have a contradiction that $p^3 \mid \sigma(n) = (\gamma(n))^2$ for some prime p or $h(n) > 1$ as follows:

A. If $97 \to 7^6$, then $\sigma(7^6) = 29 \times 4733$.

A. 1. If $7^6 \to p_3^4$ with $p_3 = 29$ or 4733, then $h(\{3^2, 13^2, 61^2, 97, 7^6, p_3^4\} > 1$.

A. 2. If $p_2 = 29$ or 4733, then $3^3 \mid \sigma(13^2 \times 61^2 \times p_2)$.

A. 3. We cannot have $7^6 \to 29^2$ since $13^3 \mid \sigma(3^2 \times 61^2 \times 29^2)$.

A. 4. If $7^6 \to 4733^2$, then $4733^2 \to 22406023^2$ and $3^3 \mid \sigma(13^2 \times 61^2 \times 22406023^2)$.

B. If $97 \to 7^4$, then $7^4 \to 2801^f$ for some integer $f > 0$.

B 1. If $f \equiv 1 \pmod 4$, then $3^3 \mid \sigma(13^2 \times 61^2 \times 2801)$.

B. 2. If $f \geq 6$ and $L = \{p_1, p_2\}$, then $h(\{3^2, 13^2, 61^2, 97, 7^4, 2801^f\} \cup C(p_2)) > 2801h(\{97, 7^4\}) > 1$.

B. 3. If $f \geq 4$ and $L = \{p_1\}$, then $h(\{3^2, 13^2, 61^2, 97, 7^4, 2801^f\} \cup C(p_2)) > 2801h(\{97, 7^4\}) > 1$.

B. 4. If $f = 4, 2801^4 \notin B(p_2)$, then $h(\{3^2, 13^2, 61^2, 97, 7^4, 2801^4\} \cup C(p_2)) > h(\{97, 7^4, 2801^4\}) > 1$.

B. 5. If $f = 4, L = \{p_1, p_2\}$ and $2801^4 \in B(p_2)$, then $q \in N(p_2), q = 5, 195611, 6294091$.

B. 5. a. If $p_2 = 5$, then $3^3 \mid \sigma(13^2 \times 61^2 \times p_2)$.

B. 5. b. If $5^2 \in N(p_2)$, then $5^2 \to 31^2$ but $3^3 \mid \sigma(13^2 \times 61^2 \times 31^2)$.

B. 5. c. We cannot have $6294091^2 \in N(p_2)$ since $3^3 \mid \sigma(13^2 \times 61^2 \times 6294091^2)$.

B. 5. d. If $195611^2 \in N(p_2)$, then $\sigma(195611^2) = 211 \times 181345303$ and $195611^2 \to p_3^2$ with $p_3 = 211$ or 181345303. However, $3^3 \mid \sigma(13^2 \times 61^2 \times p_3^2)$.

B. 6. If $f = 2$, then $\sigma(2801^2) = 37 \times 43 \times 4933$.

B. 6. a. We cannot have $2801^2 \to 43^2$ since $3^3 \mid \sigma(13^2 \times 61^2 \times 43^2)$.

B. 6. b. If $2801^2 \to 43^{e_3}, e_3 \geq 6$, then $h(n) > h(\{97, 7^4, 43^{e_3}\} \cup C(p_2)) > 43h(\{97, 7^4\}) > 1$.

B. 6. c. If $2801^2 \to 43^4$ and $43^4 \notin B(p_2)$, then $h(n) > h(\{97, 7^4, 43^4\} \cup C(p_2)) > h(\{97, 7^4, 43^4\}) > 1$.

B. 6. d. If $2801^2 \to 43^4$ and $43^4 \in B(p_2)$, then $43^4 \to 3500201^2$, $\sigma(3500201^2) = 13 \times 139 \times 28411 \times 238639$. Since $q \equiv 3 \pmod 4$ for $q = 139, 28411, 238639$, $q^2 \in N(q_2)$ and $3^3 \mid \sigma(13^2 \times 61^2 \times q^2)$.

Thus we have a contradiction in any case. This yields that $3^2 \to 13^2 \to 61^2 \to 97$ is impossible. Hence, we conclude that $M = M(L)$ cannot contain a cycle of length ≥ 3, as stated in the lemma. □

Now a cycle in $M(L)$ must be of the form $p_i^2 \leftrightarrow p_j^2$. We may assume that $p_r^2 \leftrightarrow p_{r+1}^2$ for some r. In other words, we must have $p_r \mid \sigma(p_{r+1}^2)$ and $p_{r+1} \mid \sigma(p_r^2)$ for some primes $p_r, p_{r+1} \in M(L)$.

Lemma 2.6 of [4] shows that such p_r, p_{r+1} must be two consecutive terms of the binary recurrent sequence described in A101368 of OEIS. This had already been proved by Mills [9] and Chao [3]. However, this fact is not needed in our argument. We only use the fact that, if $p_{r+1} > p_r > 3$ and $p_r \leftrightarrow p_{r+1}$, then $p_r \equiv p_{r+1} \equiv 1 \pmod 3$ by Lemma 3.2.

We begin by proving that, we cannot have $p_r \leftrightarrow p_{r+1}$ if $p_{r+1} > p_r > 3$.

Lemma 5.2. *Assume that for there exists no arc $p_i \to p_j$ from $p_i \in L$ to $p_j \in N(L)$. If $M = M(L)$ contains a cycle $p_r^2 \leftrightarrow p_{r+1}^2$ of length two with $p_{r+1} > p_r$, then $(p_r, p_{r+1}) = (3, 13)$.*

Proof. We may assume that $p_r, p_{r+1} \in N(p_1)$. Hence, there exists a vertex $q \in N(p_1)$ such that $p_r \to q$ or $p_{r+1} \to q$. However, if $q \in M$, then, since $q \equiv p_{r+1} \equiv p_{r+1} \equiv 1 \pmod 3$, we must have $3^3 \mid \sigma(q^2 p_r^2 p_{r+1}^2) \mid \sigma(n)$, which is a contradiction. Thus, we must have $q \in L$.

Now we obtain a directed graph F by eliminating the arcs $p_r \leftrightarrow p_{r+1}$ and p_r or $p_{r+1} \to p_i$ with $p_i \in L$ from $C = C(L)$. Then F has two more sinks p_r, p_{r+1} as well as sinks in $C(L)$.

Proceeding as in the proof of Lemma 3.3, we have

$$\prod_{p_i \in B = B(L)} \sigma(p_i^{e_i}) = \left(\prod_{p_i \in B} \kappa_i\right) \prod_{p_j \in M, j \neq r, r+1} \lambda_j^{s_j} \prod_{p_i \in L \cup \{p_r, p_{r+1}\}} p_j^{2s_j}, \tag{21}$$

where

$$s_j = \sum_{\substack{q_0 \to q_1 \to \cdots \to q_k = p_j, \\ q_0 \in B, q_1, \ldots, q_{k-1} \in N}} \frac{1}{2^k}. \tag{22}$$

Let f_i be the exponent $p_i^{f_i} \mid\mid \sigma(p_r^2 p_{r+1}^2)$ for $p_i \in L$. We observe that $d_F^-(p_i) = 2 - f_i$ for $p_i \in L$, $d_F^-(p_r) = d_F^-(p_{r+1}) = 1$ and $d_F^-(p_j) = 2$ for any other vertices p_j in N. Hence,

$$s_j = t_j \sum_{\substack{q_0 \to q_1 \to \cdots \to q_k = p_j, \\ q_0 \in B, q_1, \ldots, q_{k-1} \in N}} \frac{1}{\prod_{l=1}^k d_F^-(q_l)}, \tag{23}$$

where $t_j = (2 - f_i)/2$ for $p_j \in L$, $1/2$ for $j = r, r+1$ and $t_j = 1$ for any other j such that $p_j \in N$. By Lemma 2.1, we have $s_j = t_j$ for any j such that $p_j \in N$ and, as in Lemma 3.3,

$$h(C) > \prod_{p_i \in B} \kappa_i^{\frac{1}{2}} p_i^{\frac{e_i}{2} - 2} \prod_{p_j \in M, j \neq r, r+1} \frac{\sqrt{\sigma(p_j^2)}}{p_j} \frac{p_r^{\frac{1}{2}} p_{r+1}^{\frac{1}{2}}}{p_1^{\frac{f_1}{2}} p_2^{\frac{f_2}{2}}} > \sqrt{\frac{p_r p_{r+1}}{p_1^{f_1} p_2^{f_2}}}. \tag{24}$$

If $p_r > 3$, then we have $p_r \equiv p_{r+1} \equiv 1 \pmod 3$ and $p_1^{f_1} p_2^{f_2} \leq \sigma(p_r^2 p_{r+1}^2)/(9 p_r p_{r+1})$. Hence, we must have

$$h(n) > h(C) > \frac{3 p_r p_{r+1}}{\sigma(p_r^2 p_{r+1}^2)} > 1, \tag{25}$$

which is a contradiction. Thus, we must have $(p_r, p_{r+1}) = (3, 13)$. □

Now the only remaining case is $3^2 \leftrightarrow 13^2 \to 61$.

Lemma 5.3. *Assume that for there exists no arc $p_i \to p_j$ from $p_i \in L$ to $p_j \in N = N(L)$. Then, $3^2 \leftrightarrow 13^2 \to 61$ is impossible.*

Proof. We immediately have $L = \{61\}$ or $L = \{61, p_2\}$ with $p_2 \equiv 1 \pmod 4$ and $61 \to 31^{e_3}$.

If $e_3 \geq 8$, then Lemma 3.3 gives

$$h(n) \geq h(\{2, 3^2, 13^2, 61\}) h(C(p_2) \cup \{31^{e_3}\}) > 31^2 h(\{2, 61\}) > 1.$$

If $e_3 \geq 4$ and $L = \{p_1\}$, then Lemma 3.3 gives

$$h(n) \geq h(\{2, 3^2, 13^2, 61, 31^{e_3}\}) \geq h(\{2, 61, 31^4\}) > 1.$$

If $e_3 \geq 4$, $L = \{p_1, p_2\}$ and $31^{e_3} \notin B(p_2)$, then Lemma 3.3 gives

$$h(n) \geq h(\{2, 3^2, 13^2, 61, 31^{e_3}\})h(C(p_2)) > h(\{2, 61, 31^4\}) > 1.$$

Thus, in these three cases, we are led to $h(n) > 1$, which is a contradiction. Hence, we must have (I) $L = \{p_1, p_2\}$, $e_3 \in \{4, 6\}$ and $31^{e_3} \in B(p_2)$ or (II) $e_3 = 2$. In both cases (I) and (II), we have a contradiction that $p^3 \mid \sigma(n) = (\gamma(n))^2$ for some prime p or $h(n) > 1$ as follows:

I. A. If $31^6 \in B(p_2)$, then $p_2 = 917087137$ or $917087137^2 \in N(p_2)$.

I. A. 1. In the case $p_2 = 917087137$, we observe that $p_4^{e_4} \to p_2$ for a prime $p_4 \neq 31$.

I. A. 1. a. If $e_4 = 2$, then $p_4 \geq 20612597323$ and, since $3^2, 13^2, 61^1, 31^6, p_2^1 \notin C(p_4)$ (we observe that $p_2^1 \in C(p_4)$ implies that $N(p_2)$ must contain a cycle $p_2 \to \cdots \to p_4^2 \to p_2$), Lemma 3.3 yields that

$$h(n) > h(\{2, 61, 31^6, p_2\})h(C(p_4)) > p_4 h(\{2, 61, 31^6, p_2\}) > 1.$$

I. A. 1. b. If $e_4 > 2$, then $h(C(p_2)) > 31p_4 > 31^2$ by Lemma 3.3 and therefore

$$h(n) > h(\{2, 61\})h(C(p_2)) > 31^2 h(\{2, 61\}) > 1.$$

I. A. 2. If $917087137^2 \in N(p_2)$, then, since any prime factor of $\sigma(p_4^2)$ is $\equiv 3 \pmod 4$, we must have $p_4^2 \to p_5^2$ with $p_5 = 43, 4447, 38647$ or 38533987, which is impossible since $3^3 \mid \sigma(13^2 p_4^2 p_5^2)$.

I. B. If $31^4 \in B(p_2)$, then one of $5, 5^2, 11^2, 17351^2$ must belong to $N(p_2)$.

I. B. 1. If $p_2 = 5$, then $h(n) > h(\{2, 61, 31^4, 5\}) > 1$, a contradiction.

I. B. 2. We cannot have $5^2 \in N(p_2)$ since $\sigma(5^2) = 31 \in B(p_2)$.

I. B. 3. If $11^2 \in N(p_2)$, then $7^2 \in N(p_2)$ or $19^2 \in N(p_2)$. Since $\sigma(7^2) = 3 \times 19$, we have $19^2 \in N(p_2)$ in any case. Now we must have $19^2 \to 127^2 \in N(p_2)$. Thus, $3^3 \mid \sigma(13^2 \times 19^2 \times 127^2) \mid \sigma(n)$, a contradiction.

I. B. 4. If $17351^2 \in N(p_2)$, then $1063^2 \in N(p_2)$ or $21787^2 \in N(p_2)$.

I. B. 4. a. If $1063^2 \in N(p_2)$, then we must have $1063^2 \to 377011^2 \in N(p_2)$ and $3^3 \mid \sigma(13^2 \times 1063^2 \times 377011^2)$, which is a contradiction.

I. B. 4. b. If $21787^2 \in N(p_2)$, then $p_2 = 5104249$ or $5104249^2 \in N(p_2)$. Neither of them is possible since $5^3 \mid (5104249+1)$ and $3^3 \mid \sigma(13^2 \times 21787^2 \times 5104249^2)$.

II. If $61 \to 31^2$, then we must have $31^2 \to 331^{e_3}$ for some e_3. Since $331 \equiv 3 \pmod 4$, $p_2 \neq 331$ and e_3 must be even.

II. 1. $e_3 = 2$ is impossible since $3^3 \mid \sigma(13^2 \times 31^2 \times 331^2)$.

II. 2. If $e_3 \geq 6$ and $31^{e_3} \in B(p_2)$, then Lemma 3.3 gives

$$h(n) \geq h(\{2, 61\})h(C(p_2) \cup \{331^{e_3}\}) > 331h(\{2, 61\}) > 1.$$

II. 3. If $e_3 \geq 4$ and $L = \{p_1\}$, then Lemma 3.3 gives

$$h(n) \geq h(\{2, 61, 331^{e_3}\}) \geq h(\{2, 61, 331^4\}) > 1.$$

II. 4. If $e_3 \geq 4$, $L = \{p_1, p_2\}$ and $331^{e_3} \notin B(p_2)$, then Lemma 3.3 gives

$$h(n) \geq h(\{2, 61, 331^{e_3}\})h(C(p_2)) > h(\{2, 61, 331^4\}) > 1.$$

II. 5. If $e_3 = 4$, $L = \{p_1, p_2\}$ and $331^4 \in B(p_2)$, then $p_2 = 5, 37861, 62601$ or $331^4 \to p_4^2 \in N(p_2)$ with $p_4 = 37861$ or 63601 (we see that since $\sigma(5^2) = 31$, we cannot have $5^2 \in N(p_2)$).

II. 5. a. $331^4 \to p_4^2 \in N(p_2)$ is impossible since $3^3 \mid \sigma(13^2 \times 31^2 p_4^2)$.

II. 5. b. If $p_2 = 5, 37861$ or 63601, then, observing that $p_5^{e_5} \to 37861$ with $e_5 \geq 10$, Lemma 3.3 gives,

$$h(n) \geq h(\{2, 61, 331^4, p_5^{e_5}, p_2\}) > 37861^{4/5}h(\{2, 61, 331^4, p_2\}) > 1,$$

which is a contradiction.

Thus we have a contradiction in any case. This shows that $3^2 \leftrightarrow 13^2 \to 61$ is impossible, as desired. □

Now we can easily prove Theorem 1.1. Let n be an integer of the form (2) satisfying (1) and L be the set of odd primes p_i such that $p_i \mid\mid n$. If there exists no path between two vertices in L, then, by Lemmas 4.1, 4.2 and 4.3, $N(L)$ must have a cycle but, by Lemmas 5.1, 5.2 and 5.3, $M(L)$ cannot have a cycle. Hence, $G(n)$ must have a path between two vertices in L or a cycle in $N(L)$ containing a vertex in L. This proves iii) of Theorem 1.2 and therefore Theorem 1.1.

The remaining statements of Theorem 1.2 can be easily deduced from Lemma 3.2. Let g_1 and g_2 be the number of primes $\equiv 1 \pmod 3$ and $\not\equiv 1 \pmod 3$ in M respectively. i) and the former statement of ii) immediately follow from Lemma 3.2 and the fact that $3^3 \nmid (\gamma(n))^2 = \sigma(n)$. Thus, $g_1 \leq 2$. If p_i is a prime $\not\equiv 1 \pmod 3$ in M, then $p_i^2 \to p_j^2$ for some prime $p_j \equiv 1 \pmod 3$ in M or $p_i^2 \to p_l$ for some prime $p_l \in L$. Hence, we obtain $g_2 + g_1 \leq 2(g_1 + \#L)$ and $g_2 \leq g_1 + 2\#L \leq 2(1 + \#L)$. Now the latter statement of ii) follows. This completes the proof of our theorems.

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