The Fibonacci Sequence and Schreier-Zeckendorf Sets

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Abstract

A finite subset of the natural numbers is *weak-Schreier* if $\min S \geq |S|$, *strong-Schreier* if $\min S > |S|$, and *maximal* if $\min S = |S|$. Let M_n be the number of weak-Schreier sets with n being the largest element and $(F_n)_{n\geq -1}$ denote the Fibonacci sequence. A finite set is said to be Zeckendorf if it does not contain two consecutive natural numbers. Let E_n be the number of Zeckendorf subsets of $\{1, 2, \ldots, n\}$. It is well-known that $E_n = F_{n+2}$. In this paper, we first show four other ways to generate the Fibonacci sequence from counting Schreier sets. For example, let C_n be the number of weak-Schreier subsets of $\{1, 2, \ldots, n\}$. Then $C_n = F_{n+2}$. To understand why $C_n = E_n$, we provide a bijective mapping to prove the equality directly. Next, we prove linear recurrence relations among the number of Schreier-Zeckendorf sets. Lastly, we discover the Fibonacci sequence by counting the number of subsets of $\{1, 2, \ldots, n\}$ such that two consecutive elements in increasing order always differ by an odd number.

1 Background and main results

Let the Fibonacci sequence be $F_{-1} = 1$, $F_0 = 0$, and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 1$. We only concern ourselves with finite subsets of natural numbers greater than 0 and use $\mathbb{N}$ for the set $\{1, 2, 3, \ldots\}$. We define a set to be

- *weak-Schreier* if $\min S \geq |S|$,
- *strong-Schreier* if $\min S > |S|$ and
- *maximal* if $\min S = |S|$,

where $|S|$ is the cardinality of set S. Schreier sets are named after Schreier who defined them to solve a problem in Banach space theory in 1930 [10]. These sets were also independently discovered in combinatorics and are connected to Ramsey-type theorems for subsets of $\mathbb{N}$.

For each $n \in \mathbb{N}$, let M_n be the number of weak-Schreier sets with n being the largest element. In notation,

$$M_n = |\{S \subseteq \mathbb{N} : \min S \geq |S| \text{ and } \max S = n\}|.$$

The first few values of M_n are $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$; indeed, it is known that $M_n = F_n$ for all n [12]. However, the author is unable to locate the first person to prove this result. If we look at either strong-Schreier sets or maximal sets instead, we can also generate the Fibonacci sequence. Let

- A_n be the number of strong-Schreier sets S with $\max S = n$,
- B_n be the number of maximal sets S with $\max S = n$,
- C_n be the number of weak-Schreier subsets of $\{1, 2, \ldots, n\}$ (including the empty set),
- D_n be the number of strong-Schreier subsets of $\{1, 2, \ldots, n\}$ (including the empty set).

For our sequence $(C_n)_{n\geq 1}$ and $(D_n)_{n\geq 1}$, we relax the condition about the maximum of our sets. Clearly, for each $n \in \mathbb{N}$, $M_n = A_n + B_n$, $C_n = \sum_{k=1}^n M_k + 1$ and $D_n = \sum_{k=1}^n A_n + 1$.

Theorem 1. *For each $n \in \mathbb{N}$, we have $A_n = F_{n-1}$, $B_n = F_{n-2}$, $C_n = F_{n+2}$ and $D_n = F_{n+1}$*

The Fibonacci representation of natural numbers was first studied by Ostrowski [9] and Lekkerkerker [8]. In 1972, Zeckendorf proved that every positive integer can be uniquely written as a sum of non-consecutive Fibonacci numbers [11]. Since then, many papers have generalized this result and explored properties of the Zeckendorf decomposition: see [1, 2, 3, 4, 5, 6, 8]. We instead focus on the important requirement for uniqueness of the Zeckendorf decomposition; that is, our set contains no two consecutive Fibonacci numbers. We give the same definition for natural numbers.

Definition 2. A finite set of natural numbers is Zeckendorf if the set does not contain two consecutive natural numbers.

Let E_n be the number of subsets of $\{1, 2, \ldots, n\}$ that satisfy the Zeckendorf condition. It is well-known that $E_n = F_{n+2}$.

Two different ways of counting subsets of $\{1, 2, \ldots, n\}$ give the same number; that is, $C_n = E_n$. To understand the connection, we construct a bijective mapping to show that $C_n = E_n$ directly. Our proof is independent of the fact that $C_n = E_n = F_{n+2}$ and thus, provides insight into the seemingly mysterious equality.

Theorem 3. *For each $n \in \mathbb{N}$, $C_n = E_n$.*

Next, a natural question is about sequences formed by the number of sets that satisfy both the Schreier and the Zeckendorf conditions. In particular, we say that a set satisfies the k-Zeckendorf condition if two arbitrary numbers in the set are at least k apart. We discover linear recurrence relations among the number of sets satisfying both the Schreier and the k-Zeckendorf conditions.

For each $n \in \mathbb{N}$, let $H_{k,n}$ be the number of subsets of $\{1, 2, \ldots, n\}$ that

(1) satisfy the k-Zeckendorf condition;

(2) contain n; and

(3) are weak-Schreier.

Theorem 4. *Fix $k \in \mathbb{N}_{\geq 2}$. We have*

$$H_{k,n} = \begin{cases} 1, & \text{if } 1 \leq n \leq k+1; \\ H_{k,n-1} + H_{k,n-(k+1)}, & \text{if } n > k+1. \end{cases}$$

Using the exact same argument as in the proof of Theorem 4, we can also deduce the following theorems regarding strong and maximal Schreier sets. For each $n \in \mathbb{N}$, let $I_{k,n}$ be the number of subsets of $\{1, 2, \ldots, n\}$ that (1) satisfy the k-Zeckendorf condition, (2) contain n, and (3) are strong-Schreier.

Theorem 5. *Fix $k \in \mathbb{N}_{\geq 2}$. We have*

$$I_{k,n} = \begin{cases} 0, & \text{if } n = 1; \\ 1, & \text{if } 2 \leq n \leq k+2; \\ I_{k,n-1} + I_{k,n-(k+1)}, & \text{if } n > k+2. \end{cases}$$

For each $n \in \mathbb{N}$, let $J_{k,n}$ be the number of subsets of $\{1, 2, \ldots, n\}$ that

(1) satisfy the k-Zeckendorf condition;

(2) contain n; and

(3) are maximal.

Theorem 6. *Fix $k \in \mathbb{N}_{\geq 2}$. We have*

$$J_{k,n} = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } 2 \leq n \leq k+1; \\ 1, & \text{if } k+1 < n \leq 2k+2; \\ J_{k,n-1} + J_{k,n-(k+1)}, & \text{if } n > 2k+2. \end{cases}$$

We give the following definition that is useful for the statement of our last result.

Definition 7. Let $A = \{a_1, a_2, \ldots, a_k\}$ $(a_1 < a_2 < \cdots < a_k)$ for some $k \in \mathbb{N}_{\geq 2}$. The difference set of A is $\{a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}\}$. The empty set and a set with exactly one element do not have a difference set.

We end with the following small result.

Theorem 8. *Fix $n \in \mathbb{N}$. The number of subsets of $\{1, 2, \ldots, n\}$*

1. *that contain n and whose difference sets contain only odd numbers is F_{n+1},*

2. *whose difference sets contain only odd numbers (the empty set and sets with exactly one element vacuously satisfy this requirement) is $F_{n+3} - 1$.*

2 Proof of Theorem 1

Proof of Theorem 1. We first prove item (1). Simple computation gives $A_1 = 0 = F_0$, $A_2 = 1 = F_1$, $A_3 = 1 = F_2$, $A_4 = 2 = F_3$, and $A_5 = 3 = F_4$. It suffices to prove that $A_n + A_{n+1} = A_{n+2}$ for $n \geq 4$. Fix $n \geq 4$ and let us find a formula for A_n. The minimum number k in our sets can take values from 1 to n. For each value of k, there are $n-k-1$ numbers strictly between k and n. Because our sets are strong-Schreier, they contain at most $k-3$ numbers out of these $n-k-1$ numbers. Hence, our formula for A_n is

$$A_n \ = \ \sum_{k=1}^{n-1}\sum_{j=0}^{k-3}\binom{n-k-1}{j}+1.$$

Note that the number 1 in our formula accounts for the set $\{n\}$. It remains to show that $A_n + A_{n+1} = A_{n+2}$ or equivalently, $A_{n+2} - A_{n+1} = A_n$ for $n \geq 4$. We have

$$\begin{aligned} A_{n+2}-A_{n+1} \ &= \ \sum_{k=1}^{n+1}\sum_{j=0}^{k-3}\binom{n-k+1}{j} - \sum_{k=1}^{n}\sum_{j=0}^{k-3}\binom{n-k}{j} \\ &= \ \sum_{k=1}^{n}\sum_{j=0}^{k-3}\left(\binom{n-k+1}{j} - \binom{n-k}{j}\right) + \sum_{j=0}^{n-2}\binom{0}{j} \\ &= \ \sum_{k=1}^{n}\sum_{j=1}^{k-3}\binom{n-k}{j-1} + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} A_{n+2}-A_{n+1}-A_n \ &= \ \sum_{k=1}^{n}\sum_{j=1}^{k-3}\binom{n-k}{j-1} - \sum_{k=1}^{n-1}\sum_{j=0}^{k-3}\binom{n-k-1}{j} \\ &= \ \sum_{k=4}^{n}\sum_{j=1}^{k-3}\binom{n-k}{j-1} - \sum_{k=3}^{n-1}\sum_{j=0}^{k-3}\binom{n-k-1}{j} \ = \ 0. \end{aligned}$$

The last equality is because for each $4 \leq t \leq n$, we have $\sum_{j=1}^{t-3}\binom{n-t}{j-1} = \sum_{j=0}^{(t-1)-3}\binom{n-(t-1)-1}{j}$. Hence, $A_{n+2} = A_{n+1} + A_n$ and we are done.

Next, we prove item (2), which follows immediately from item (1). We know that

$$B_n \ = \ M_n - A_n \ = \ F_n - F_{n-1} \ = \ F_{n-2}.$$

We prove item (3). Fix $n \geq 1$. We have

$$C_n = \sum_{k=1}^{n} M_k + 1 \ = \ \sum_{k=1}^{n} F_k + 1 \ = \ (F_{n+2}-1)+1 \ = \ F_{n+2},$$

as desired. The number 1 accounts for the empty set. The fact that $\sum_{k=1}^{n} F_k = F_{n+2} - 1$ is due to Lucas [7, p. 4].

Similarly, we prove item (4). Fix $n \geq 1$. We have

$$D_n = \sum_{k=1}^{n} A_k + 1 \ = \ \sum_{k=1}^{n} F_{k-1} + 1 \ = \ (F_{n+1} - 1) + 1 \ = \ F_{n+1}.$$

We complete our proof of Theorem 1. ◻

Let L_n^w be the number of weak-Schreier sets as subsets of $\{1, 2, \ldots, n\}$ with an even maximum.

Corollary 9. *For each $n \in \mathbb{N}$,*

$$L_n^w \ = \ \begin{cases} F_n, & \text{if } n \text{ is odd;} \\ F_{n+1}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. We have

$$L_n^w \ = \ \sum_{\substack{1 \leq k \leq n \\ 2 | k}} M_k + 1 \ = \ \sum_{\substack{1 \leq k \leq n \\ 2 | k}} F_k + 1.$$

The number 1 accounts for the empty set.

If n is even,

$$L_n^w \ = \ \sum_{\substack{1 \leq k \leq n \\ 2 | k}} F_k + 1 \ = \ (F_{n+1} - 1) + 1 \ = \ F_{n+1}.$$

If n is odd,

$$L_n^w \ = \ \sum_{\substack{1 \leq k \leq n \\ 2 | k}} F_k + 1 \ = \ (F_n - 1) + 1 = F_n.$$

◻

Let L_n^s be the number of strong-Schreier sets as subsets of $\{1, 2, \ldots, n\}$ with an odd maximum.

Corollary 10. *For each $n \in \mathbb{N}$,*

$$L_n^s \ = \ \begin{cases} F_n, & \text{if } n \text{ is odd;} \\ F_{n-1}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. We have

$$L_n^s \ = \ \sum_{\substack{1 \leq k \leq n \\ 2 \nmid k}} A_k + 1 \ = \ \sum_{1 \leq k \leq n, 2 \nmid k} F_{k-1} + 1.$$

If n is even,

$$L_n^s = \sum_{\substack{1\le k\le n\\ 2\nmid k}} F_{k-1} + 1 = (F_{n-1} - 1) + 1 = F_{n-1}.$$

If n is odd,

$$L_n^s = \sum_{\substack{1\le k\le n\\ 2\nmid k}} F_{k-1} + 1 = (F_n - 1) + 1 = F_n.$$

□

3 Proof of Theorem 3 — Explanation of the mysterious identity

Recall that C_n is the number of weak-Schreier sets as subsets of $\{1, 2, \ldots, n\}$, while E_n is the number of subsets of $\{1, 2, \ldots, n\}$ that do not contain two consecutive numbers. At the first glance, C_n and E_n are little related, so it is surprising to see that $C_n = E_n$ for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, let X_n denote the set of weak-Schreier sets as subsets of $\{1, 2, \ldots, n\}$ and let Y_n denote the set of subsets of $\{1, 2, \ldots, n\}$ that do not contain two consecutive numbers. In this section, we construct a bijective function $f : X_n \to Y_n$ to prove that $|X_n| = |Y_n|$.

Proof of Theorem 3. Fix $n \in \mathbb{N}$. Let $A = \{a_1, a_2, \ldots, a_{k-1}, a_k\}$ $(a_1 < a_2 < \cdots < a_k)$ be a weak-Schreier subset of $\{1, 2, \ldots, n\}$. Our mapping f acts on A as follows

$$f(A) = f(\{a_1, a_2, \ldots, a_{k-1}, a_k\}) = \{a_1 - (k-1), a_2 - (k-2), \ldots, a_{k-1} - 1, a_k\}.$$

Define $f(\emptyset) = \emptyset$. To show that f is well-defined, we show that $\{a_1 - (k-1), a_2 - (k-2), \ldots, a_{k-1} - 1, a_k\}$ is in Y_n. Because A is weak-Schreier, $k \le a_1 < a_2 < \cdots < a_k$. Hence,

$$1 \le a_1 - (k-1) < a_2 - (k-2) < \cdots < a_{k-1} - 1 < a_k \le n.$$

Let $t_i = a_i - (k - i)$ for $1 \le i \le k$. If $k = 1$, then $\{t_1\}$ is clearly in Y_n. If $k \ge 2$, then for each $2 \le i \le k$, we have

$$t_i - t_{i-1} = (a_i - (k-i)) - (a_{i-1} - (k - (i-1))) = (a_i - a_{i-1}) + 1 \ge 2.$$

Therefore, $\{t_1, t_2, \ldots, t_k\} \in Y_n$. So, f is well-defined.

Next, we prove that f is injective. Suppose that $f(A) = f(B)$. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_k\}$, where $a_1 < a_2 < \cdots < a_k$ and $b_1 < b_2 < \cdots < b_k$. Because

$$\begin{aligned} a_1 - (k-1) &< a_2 - (k-2) < \cdots < a_{k-1} - 1 < a_k, \\ b_1 - (k-1) &< b_2 - (k-2) < \cdots < b_{k-1} - 1 < b_k, \end{aligned}$$

we know that $f(A) = f(B)$ implies $a_i - (k - i) = b_i - (k - i)$ for all $1 \le i \le k$. Hence, $a_i = b_i$, which shows that $A = B$. Therefore, f is injective.

Finally, we prove that f is surjective. Let $C = \{c_1, c_2, \ldots, c_k\} \in Y_n$ be chosen, where $c_1 < c_2 < \cdots < c_k$. We claim that

$$D = \{c_1 + (k-1), c_2 + (k-2), \ldots, c_{k-1} + 1, c_k\}$$

satisfies $f(D) = C$ and $D \in X_n$. Because C do not contain two consecutive numbers, we know that

$$k \le c_1 + (k-1) < c_2 + (k-2) < \cdots < c_{k-1} + 1 < c_k \le n.$$

Hence, $D \in X_n$.

We have shown that f is both well-defined and bijective. Therefore, $|X| = |Y|$ or $C_n = E_n$, as desired. □

Remark 11. We would like to discuss the motivation for the bijection f used in the proof of Theorem 3. Let A be a Schreier set. The map f serves to increase the gap between adjacent elements of A by 1, thus fulfilling the Zeckendorf condition that adjacent elements differ by at least 2. Furthermore, the weak-Schreier condition that $\min A \ge |A|$ ensures that the resulting set is in $\{1, 2, \ldots, n\}$.

4 Proof of Theorem 4

Before we prove Theorem 4, we need a simple proposition.

Proposition 12. *For $n, k \in \mathbb{Z}$, the following claims hold.*

1. *If $\lfloor \frac{n-2}{k+1} \rfloor = \lfloor \frac{n-k-2}{k+1} \rfloor$, then $\lfloor \frac{n-1}{k+1} \rfloor = \lfloor \frac{n-2}{k+1} \rfloor + 1$.*
2. *If $\lfloor \frac{n-2}{k+1} \rfloor > \lfloor \frac{n-k-2}{k+1} \rfloor$, then $\lfloor \frac{n-1}{k+1} \rfloor < \lfloor \frac{n-2}{k+1} \rfloor + 1$.*
3. *If $\lfloor \frac{n-k-2}{k+1} \rfloor = \lfloor \frac{n-2}{k+1} \rfloor$, then $\frac{n-k-2}{k+1} = \lfloor \frac{n-2}{k+1} \rfloor$.*

Proof. We prove claim (1). We have

$$\left\lfloor \frac{n-2}{k+1} \right\rfloor = \left\lfloor \frac{n-k-2}{k+1} \right\rfloor = \left\lfloor \frac{n-1}{k+1} - 1 \right\rfloor = \left\lfloor \frac{n-1}{k+1} \right\rfloor - 1.$$

Therefore,

$$\left\lfloor \frac{n-1}{k+1} \right\rfloor = \left\lfloor \frac{n-2}{k+1} \right\rfloor + 1.$$

Next, we prove claim (2). We have

$$\left\lfloor \frac{n-2}{k+1} \right\rfloor > \left\lfloor \frac{n-k-2}{k+1} \right\rfloor = \left\lfloor \frac{n-1}{k+1} - 1 \right\rfloor = \left\lfloor \frac{n-1}{k+1} \right\rfloor - 1.$$

Therefore,

$$\left\lfloor \frac{n-1}{k+1} \right\rfloor \;<\; \left\lfloor \frac{n-2}{k+1} \right\rfloor + 1.$$

Lastly, we prove claim (3). Write $n-k-2 = (k+1)p + q$ for some $0 \leq q \leq k$. Then

$$\frac{n-2}{k+1} \;=\; \frac{(k+1)p+q+k}{k+1} \;=\; p + \frac{q+k}{k+1} \;=\; p+1+\frac{q-1}{k+1}.$$

If $q \geq 1$, then $\left\lfloor \frac{n-2}{k+1} \right\rfloor = p+1 > p = \left\lfloor \frac{n-k-2}{k+1} \right\rfloor$, a contradiction. So, $q = 0$, implying that $\frac{n-k-2}{k+1} = \left\lfloor \frac{n-2}{k+1} \right\rfloor$. □

The following lemma is from [6, Lemma 2.1] by Kologlŭ et al.

Lemma 13. *The number of solutions to $y_1 + \cdots + y_p = n$ with $y_i \geq c_i$ (each c_i a non-negative integer) is $\binom{n-(c_1+\cdots+c_p)+p-1}{p-1}$.*

Proof of Theorem 4. Fix $k \geq 2$. We now find a formula for $H_{k,n}$ for all $n \in \mathbb{N}$. Fix $1 \leq \ell \leq n-1$. Suppose that the set $\{a_1, \ldots, a_\ell, n\}$ satisfies all of our requirements. (For $\ell = 0$, we have the set $\{n\}$.) In particular,

1. $a_1 \geq \ell + 1$,
2. $d_i = a_{i+1} - a_i \geq k$ and $d_\ell = n - a_\ell \geq k$.

Note that

$$a_1 + \sum_{i=1}^{\ell} d_i \;=\; n. \tag{1}$$

By Lemma 13, the number of sets satisfying Equation (1) is

$$\binom{n-(\ell+1+k\ell)+(\ell+1)-1}{(\ell+1)-1} \;=\; \binom{n-k\ell-1}{\ell}.$$

Therefore, the number of sets containing n that are k-Zeckendorf and weak-Schreier is

$$H_{k,n} \;=\; \sum_{\ell=1}^{\left\lfloor \frac{n-1}{k+1} \right\rfloor} \binom{n-k\ell-1}{\ell} + 1.$$

The number 1 accounts for the set $\{n\}$ and we only let ℓ run up to $\left\lfloor \frac{n-1}{k+1} \right\rfloor$ to make sure that $n - k\ell - 1 \geq \ell$. It can be easily verified that $H_{k,n} = 1$ for $1 \leq n \leq k+1$ because $\left\lfloor \frac{n-1}{k+1} \right\rfloor = 0$

for $1 \leq n \leq k+1$. It suffices to show that for $n \geq k+2$, $H_{k,n} = H_{k,n-1} + H_{k,n-(k+1)}$. Equivalently,

$$\sum_{\ell=1}^{\left\lfloor \frac{n-1}{k+1} \right\rfloor} \binom{n-k\ell-1}{\ell} = \sum_{\ell=1}^{\left\lfloor \frac{n-2}{k+1} \right\rfloor} \binom{n-k\ell-2}{\ell} + \sum_{\ell=1}^{\left\lfloor \frac{n-(k+1)-1}{k+1} \right\rfloor} \binom{n-k\ell-1-(k+1)}{\ell} + 1. \quad (2)$$

Equivalently, noting that the $+1$ term cancels with the $l = 1$ term in the left hand side summation

$$\sum_{\ell=2}^{\left\lfloor \frac{n-2}{k+1} \right\rfloor} \left(\binom{n-k\ell-1}{\ell} - \binom{n-k\ell-2}{\ell} \right) + \sum_{\left\lfloor \frac{n-2}{k+1} \right\rfloor+1}^{\left\lfloor \frac{n-1}{k+1} \right\rfloor} \binom{n-k\ell-1}{\ell} \quad (3)$$

$$= \sum_{\ell=1}^{\left\lfloor \frac{n-k-2}{k+1} \right\rfloor} \binom{n-k(\ell+1)-2}{\ell}.$$

We can simplify Equation (3) further by applying the binomial coefficient recurrence

$$\sum_{\ell=2}^{\left\lfloor \frac{n-2}{k+1} \right\rfloor} \binom{n-k\ell-2}{\ell-1} + \sum_{\left\lfloor \frac{n-2}{k+1} \right\rfloor+1}^{\left\lfloor \frac{n-1}{k+1} \right\rfloor} \binom{n-k\ell-1}{\ell} = \sum_{\ell=1}^{\left\lfloor \frac{n-k-2}{k+1} \right\rfloor} \binom{n-k(\ell+1)-2}{\ell}.$$

Reindexing ℓ in the first summation, we have

$$\sum_{\ell=1}^{\left\lfloor \frac{n-2}{k+1} \right\rfloor-1} \binom{n-k(\ell+1)-2}{\ell} + \sum_{\left\lfloor \frac{n-2}{k+1} \right\rfloor+1}^{\left\lfloor \frac{n-1}{k+1} \right\rfloor} \binom{n-k\ell-1}{\ell} = \sum_{\ell=1}^{\left\lfloor \frac{n-k-2}{k+1} \right\rfloor} \binom{n-k(\ell+1)-2}{\ell}.$$

Subtract the first summation from both sides to have

$$\sum_{\left\lfloor \frac{n-2}{k+1} \right\rfloor+1}^{\left\lfloor \frac{n-1}{k+1} \right\rfloor} \binom{n-k\ell-1}{\ell} = \sum_{\ell=\left\lfloor \frac{n-2}{k+1} \right\rfloor}^{\left\lfloor \frac{n-k-2}{k+1} \right\rfloor} \binom{n-k(\ell+1)-2}{\ell}. \quad (4)$$

We now prove that Equation (4) is correct, which implies that Equation (2) is correct.

Case 1: $\left\lfloor \frac{n-k-2}{k+1} \right\rfloor < \left\lfloor \frac{n-2}{k+1} \right\rfloor$. Then $\left\lfloor \frac{n-2}{k+1} \right\rfloor + 1 > \left\lfloor \frac{n-1}{k+1} \right\rfloor$ by Proposition 12. Therefore, two sides of Equation (4) are identically 0.

Case 2: $\left\lfloor \frac{n-k-2}{k+1} \right\rfloor = \left\lfloor \frac{n-2}{k+1} \right\rfloor$. Then $\left\lfloor \frac{n-2}{k+1} \right\rfloor + 1 = \left\lfloor \frac{n-1}{k+1} \right\rfloor$ and $\frac{n-k-2}{k+1} = \left\lfloor \frac{n-2}{k+1} \right\rfloor$ by Proposition 12. Therefore, the left side of Equation (4) is

$$\binom{n-k(\left\lfloor \frac{n-2}{k+1} \right\rfloor+1)-1}{\left\lfloor \frac{n-2}{k+1} \right\rfloor+1} = 1$$

because $\frac{n-k-2}{k+1} = \left\lfloor \frac{n-2}{k+1} \right\rfloor$. Similarly, the right side is also equal to 1.

In both cases, Equation (4) is correct. This completes our proof. □

5 Proof of Theorem 8—A new way to generate the Fibonacci sequence

Proof of Theorem 8. First, we prove item (1). Let P_n be the number of subsets of $\{1, 2, \ldots, n\}$ that contain n and whose difference sets contain only odd numbers.

Base cases: For $n = 1$, we have $\{1\}$ to be the only subset of $\{1\}$ that satisfies our requirement. So, $P_1 = 1 = F_2$. For $n = 2$, we have $\{2\}$ and $\{1, 2\}$ to be the only two subsets of $\{1, 2\}$ that satisfy our requirement. So, $P_2 = 2 = F_3$.

Inductive hypothesis: Suppose that there exists $k \geq 2$ such that for all $n \leq k$, $P_n = F_{n+1}$. We show that $P_{k+1} = F_{k+2}$. Let O_n denote the set of subsets of $\{1, 2, \ldots, n\}$ that satisfy our requirement. Observe that unioning a set in O_{n-1-2i} (for $i \geq 0$) with n produces a set in O_n and any set in O_n is of the form of a set in O_{n-1-2i} plus the element n. Therefore,

$$P_{k+1} \ = \ |O_{k+1}| \ = \ \sum_{\substack{1 \leq i \leq k \\ 2 \nmid i}} |O_{k+1-i}| + 1 \ = \ P_k + \sum_{\substack{3 \leq i \leq k \\ 2 \nmid i}} |O_{k+1-i}| + 1.$$

The number 1 accounts for the set $\{n\}$. If k is odd,

$$\sum_{\substack{3 \leq i \leq k \\ 2 \nmid i}} |O_{k+1-i}| \ = \ |O_1| + |O_3| + \cdots + |O_{k-2}|$$

$$= \ |F_2| + |F_4| + \cdots + |F_{k-1}| \ = \ F_k - 1 \ = \ P_{k-1} - 1.$$

If k is even,

$$\sum_{\substack{3 \leq i \leq k \\ 2 \nmid i}} |O_{k+1-i}| \ = \ |O_2| + |O_4| + \cdots + |O_{k-2}|$$

$$= \ |F_3| + |F_5| + \cdots + |F_{k-1}| \ = \ F_k - 1 \ = \ P_{k-1} - 1.$$

In both cases, we have $\sum_{\substack{3 \leq i \leq k \\ 2 \nmid i}} |O_{k+1-i}| = P_{k-1} - 1$. Therefore, $P_{k+1} = P_k + P_{k-1} = F_{k+1} + F_k = F_{k+2}$, as desired.

Next, we prove item (2). Let Q_n be the number of subsets of $\{1, 2, \ldots, n\}$ whose difference sets contain only odd numbers is Q_n (the empty set and sets with exactly one element vacuously satisfy this requirement). Note that by definition of P_n and Q_n, we have

$$Q_n \ = \ 1 + \sum_{k=1}^{n} |P_k| \ = \ 1 + \sum_{k=1}^{n} F_{k+1} \ = \ \sum_{k=1}^{n+1} F_k \ = \ F_{n+3} - 1,$$

as desired. (The +1 before the first summation accounts for the empty set.) □

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References

[1] E. Burger, D. Clyde, C. Colbert, G. Shin, and Z. Wang, Canonical diophantine representations of natural numbers with respect to quadratic "bases", *J. Number Theory* **133** (2013), 1372–1388.

[2] O. Beckwith, A. Bower, L. Gaudet, R. Insoft, S. Li, S. Miller, and P. Tosteson, The average gap distribution for generalized Zeckendorf decompositions, *Fibonacci Quart.* **51** (2013), 13–27.

[3] A. Best, P. Dynes, X. Edelsbrunner, B. McDonald, S. Miller, C. Turnage-Butterbaugh, and M. Weinstein, Benford behavior of Zeckendorf decompositions, *Fibonacci Quart.* **52** (2014), 35–46.

[4] P. Demontigny, T. Do, A. Kulkarni, S. Miller, D. Moon, and U. Varma, Generalizing Zeckendorf's theorem to f-decompositions, *J. Number Theory* **141** (2014), 136–158.

[5] T. J. Keller, Generalizations of Zeckendorf's theorem, *Fibonacci Quart.* **10** (1972), 95–102.

[6] G. S. Kopp, M. Kologlu, S. Miller, and Y. Wang, On the number of summands in Zeckendorf decompositions, *Fibonacci Quart.* **49** (2011), 116–130.

[7] E. Lucas. *Theorie Des Nombres*, Gauthier-Villars, 1891.

[8] C. G. Lekkerkerker, Voorstelling van natuurlyke getallen door een som van getallen van Fibonacci, *Simon Stevin* **29** (1951–1952), 190–195.

[9] A. Ostrowski, *Bemerkungen zur Theorie der diophantischen Approximationen*, Hambg. Abh. **1** (1922), 77–98.

[10] J. Schreier, Ein Gegenbeispiel zur Theorie der schwachen Konvergentz, *Studia Math.* **2** (1962), 58–62.

[11] E. Zeckendorf, Representation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. Roy. Sci. Liege* **41** (1972), 179–182.

[12] Unknown author, Jozef Schreier, Schreier sets and the Fibonacci sequence, 2012. Available at https://outofthenormmaths.wordpress.com/2012/05/13/jozef-schreier-schreier-sets-and-the-fibonacci-sequence/.

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