# Bijections between directed animals, multisets and Grand-Dyck paths

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#### Abstract

An *n*-multiset of  $[k] = \{1, 2, ..., k\}$  consists of a set of *n* elements from [k] where each element can be repeated. We present the bivariate generating function for *n*-multisets of [k] with no consecutive elements. For n = k, these multisets have the same enumeration as directed animals in the square lattice. Then we give constructive bijections between directed animals, multisets with no consecutive elements and Grand-Dyck paths avoiding the pattern DUD, and we show how classical and novel statistics are transported by these bijections.

Keywords: Multisets; directed animals; Grand-Dyck paths; Motzkin, Catalan.

### **1** Preliminaries

An *n*-multiset on  $[k] = \{1, 2, ..., k\}$  consists of a set of *n* elements from [k] where we permit each element to be repeated [6, 10]. Throughout this paper, an *n*-multiset  $\pi$  will be represented by the unique sequence  $\pi_1 \pi_2 ... \pi_n$  of its elements ordered in non-decreasing order, *e.g.* the multiset  $\{1, 1, 2, 2, 3, 3\}$  will be written 112233. For  $n, k \ge 1$ , let  $\mathcal{M}_{n,k}$ be the set of *n*-multisets of [k]. We set  $\mathcal{M}_n = \mathcal{M}_{n,n}$  and  $\mathcal{M} = \bigcup_{n\ge 1} \mathcal{M}_n$ . For instance, we have  $\mathcal{M}_{3,2} = \{111, 112, 122, 222\}$  and  $\mathcal{M}_3 = \mathcal{M}_{3,2} \cup \{113, 123, 133, 223, 233, 333\}$ .

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The graphical representation of a multiset  $\pi \in \mathcal{M}_{n,k}$  is the set of points in the plane at coordinates  $(i, \pi_i)$  for  $i \in [n]$ . Whenever none of the points  $(i, \pi_i)$  lie below the diagonal  $y = x, i.e., i \leq \pi_i$  for all  $i \in [n], \pi$  will be called *superdiagonal*. Let  $\mathcal{M}_{n,k}^s$  (resp.  $\mathcal{M}_n^s$ ) be the set of superdiagonal *n*-multisets of [k] (resp. of [n]) and  $\mathcal{M}^s = \bigcup_{n>1} \mathcal{M}_n^s$ .

From a multiset  $\pi \in \mathcal{M}_{n,k}$ , we consider the path of length n + k on its graphical representation with up and right moves along the edges of the squares that goes from the lower-left corner (0,0) to the upper-right corner (n,k) and leaving all the points  $(i, \pi_i), i \in [n]$ , to the right and remaining always as close to the line x = n as possible (see the left part of Figure 1 for an example). Then, the number of n-multisets of [k]is the number of possibilities to choose n right moves among k + n - 1 moves (the first is necessary an up move), that is the binomial coefficient  $\binom{k+n-1}{n}$  (see for instance [10]). Reading this path from left to right, we construct a lattice path of length n + kfrom (0,0) to (n+k, n-k) by replacing any up-move with up step U = (1,1) and any right-move with down step D = (1, -1). Clearly, this path starts with U and consists of n up steps and k down steps. As a byproduct whenever n = k, this construction induces a bijection  $\Phi$  between  $\mathcal{M}_n$  and the set  $\mathcal{GD}_n$  of Grand-Dyck paths of semilength n starting with an up-step, that is the set of paths from (0,0) to (2n,0) starting with U and consisting of U and D steps. Moreover, the image by  $\Phi$  of  $\mathcal{M}_n^s$  is the set  $\mathcal{D}_n$ of Dyck paths of semilength n, *i.e.* the subset of paths in  $\mathcal{GD}_n$  that do not cross the x-axis. See Figure 1 for two examples of this construction.

**Theorem 1.** The map  $\Phi$  is a bijection from  $\mathcal{M}_n$  to  $\mathcal{GD}_n$ , and the image of  $\mathcal{M}_n^s$  is  $\mathcal{D}_n$ .

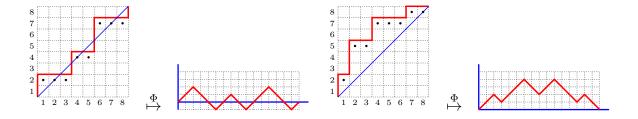


Figure 1: Illustration of the bijection  $\Phi$  between multisets and lattice paths

Now, let us define the set  $\mathcal{M}_{n,k}^{\star}$  of *n*-multisets of [k] with no consecutive integers, *i.e.*, multisets  $\pi$  such that  $\pi_{i+1} \neq \pi_i + 1$  for all  $i \in [n-1]$ . Then we set  $\mathcal{M}_n^{\star} = \mathcal{M}_{n,n}^{\star}$ ,  $\mathcal{M}_n^{s,\star} = \mathcal{M}_n^{\star} \cap \mathcal{M}_n^s$ ,  $\mathcal{M}^{\star} = \bigcup_{n \geq 1} \mathcal{M}_n^{\star}$  and  $\mathcal{M}^{s,\star} = \bigcup_{n \geq 1} \mathcal{M}_n^{s,\star}$ . On the other hand, if  $\mathcal{P}$ is a set of lattice paths consisting of U and D steps, then we denote by  $\mathcal{P}^{\star}$  the subset of  $\mathcal{P}$  consisting of paths that do not contain any occurrence of the pattern DUD.

Considering these notations, it is straightforward to obtain the following theorem.

**Theorem 2.** The map  $\Phi$  induces a bijection from  $\mathcal{M}_n^{\star}$  to  $\mathcal{GD}_n^{\star}$ , and from  $\mathcal{M}_n^{s,\star}$  to  $\mathcal{D}_n^{\star}$ .

It is well known (see for instance [7, 8, 11]) that the cardinality of  $\mathcal{D}_n^{\star}$  is given by the general term of Motzkin sequence A001006 in [9]. Then, using Theorem 2 this also is for the set  $\mathcal{M}_n^{s,\star}$ . Now, using combinatorial arguments we prove that the cardinality of  $\mathcal{M}_n^{\star}$  is given by the general term of the sequence A005773 in [9] which also counts directed animals with a given area on the square lattice.

Let  $f(z, u) = \sum_{n,k\geq 1} f_{n,k} z^n u^k$  be the bivariate generating function for the set  $\mathcal{M}_{n,k}^*$ , *i.e.*, the coefficient  $f_{n,k}$  is the number of *n*-multisets of [k] with no consecutive integers. We can build such a multiset by considering each integer from  $\{1, \ldots, k\}$  in turn and marking how many times it occurs in the multiset. First, we choose the (possibly empty) initial sequence of integers not in the multiset, which corresponds to the term  $\operatorname{Seq}[u] = \frac{1}{1-u}$ . Next, we choose the first integer in the multiset occurring one or more times, which corresponds to the term  $u \cdot \operatorname{Seq}_1[z]$  where  $\operatorname{Seq}_1[z] = \frac{z}{1-z}$ . We choose subsequent integers in the multiset, occurring one or more times, preceded each time by a non-empty sequence of integers not in the multiset (to guarantee no consecutive integers), which corresponds to the term  $\operatorname{Seq}\left[\operatorname{Seq}_1[u] \cdot u \cdot \operatorname{Seq}_1[z]\right]$ . Finally, we choose the (possibly empty) final sequence of integers not in the multiset, which corresponds to the term  $\operatorname{Seq}[u]$ . Thus, we obtain

$$\begin{split} f(z,u) &= \operatorname{Seq}[u] \cdot u \cdot \operatorname{Seq}_1[z] \cdot \operatorname{Seq}\left[\operatorname{Seq}_1[u] \cdot u \cdot \operatorname{Seq}_1[z]\right] \cdot \operatorname{Seq}[u] \\ \text{where } \operatorname{Seq}[z] &= \frac{1}{1-z} \text{ and } \operatorname{Seq}_1[z] = \frac{z}{1-z}. \\ \text{So, we deduce} \end{split}$$

$$f(z,u) = \frac{uz}{(1-u)(1-z-u+uz-u^2z)}$$

and we refer to Table 1 for small values of  $f_{n,k}$ .

As a byproduct of the bijection  $\Phi$ , the set of lattice paths of length n + k starting at (0,0), ending at (n+k, n-k) consisting of n up steps and k down steps and avoiding the pattern DUD has a bivariate generating function given by f(zu, z/u).

In order to obtain the generating function for the set  $\mathcal{M}_n^* = \mathcal{M}_{n,n}^*$ , we require the diagonal of  $f: \Delta(f)(z) = [u^0]f(z/u, u)$ . Extracting the constant term of a Laurent series is a standard procedure (Stanley Vol 2, Section 6.3), and in this case it yields the same algebraic generating function that counts directed animals (when a term for the empty multiset is added):

$$\Delta(f)(z) = \frac{3 z - \sqrt{1 - 2 z - 3 z^2} - 1}{6 z - 2}.$$

The first terms of the Taylor expansion are  $1 + z + 2z^2 + 5z^3 + 13z^4 + 35z^5 + 96z^6 + 267z^7 + 750z^8 + 2123z^9$  which correspond to the sequence A005773 in [9].

$k \backslash n$	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	4	5	6	7	8	9	10	11
4	4	7	10	13	16	19	22	25	28
5	5	11	18	26	35	45	56	68	81
	6	16	30	48	70	96	126	160	198

Table 1: Coefficients  $f_{n,k}$  of f(z, u) for  $1 \le n \le 9$  and  $1 \le k \le 6$ .

## 2 From multisets to directed animals via Grand-Dyck paths

A directed animal A of area n (or equivalently with n nodes) in the square (resp. triangular) lattice is a subset of n points in the lattice containing (0,0) and where any point in A can be reached from (0,0) with up-moves (0,1) and right-moves (1,0) (resp. and diagonal moves (1,1)) by staying always in A. See the left part of Figure 2 for an example of directed animal in the triangular lattice, and we refer to [3, 4, 5] for several combinatorial studies on these objects. Let  $\mathcal{Q}_n$  (resp.  $\mathcal{T}_n$ ) be the set of directed animals with n nodes in the square (resp. triangular) lattice, then its cardinality is given by the nth term of the sequence A005773 in [9] (resp. by the binomial coefficient  $\binom{2n-1}{n}$ ). We set  $\mathcal{Q} = \bigcup_{n\geq 1} \mathcal{Q}_n$ ,  $\mathcal{T} = \bigcup_{n\geq 1} \mathcal{T}_n$  and obviously we have  $\mathcal{Q} \subset \mathcal{T}$ .

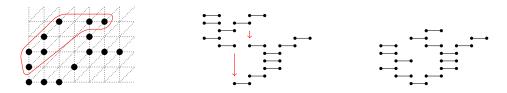


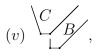
Figure 2: A directed animal and its associated heap

In the literature [3, 12, 13], directed animals are often viewed as heaps obtained by dropping vertically dimers such that each dimer (except the first) touches the one below by at least one of its extremities. Indeed, from  $A \in \mathcal{T}$ , we apply a counterclockwise rotation of 45 degree of its graphical representation and we replace each point of A with a dimer of width  $\sqrt{2}/2$ . See Figure 2 for an example of such a representation. Notice that directed animals in  $\mathcal{Q}$  correspond to heaps of dimers where no dimer has another dimer directly above it (such a heap will be called *strict*). Let  $\mathcal{T}^s$  (resp.  $\mathcal{Q}^s$ ) be the set of all subdiagonal directed animals in  $\mathcal{T}$  (resp.  $\mathcal{Q}$ ), *i.e.*, directed animals where all its points (i, j) satisfy  $j \leq i$ .

Without losing accuracy, the sets  $\mathcal{T}$  and  $\mathcal{Q}$  will also be used to designate respectively the set of heaps of dimers and the set of strict heaps of dimers. Then, any heap  $A \in \mathcal{T}^s$ has a unique factorization of one of the four following forms (see [3]):

$$(i) \hfill (iii) \hfill (iii) \hfill (iv) \hfill (iv$$

where  $B, C \in \mathcal{T}^s$ . Moreover, any heap  $A \in \mathcal{T} \setminus \mathcal{T}^s$  has a unique factorization:



where  $B \in \mathcal{T}^s$  and  $C \in \mathcal{T}$ .

The factorization of  $A \in \mathcal{Q}$  (resp.  $A \in \mathcal{Q}^s$ ) is obtained after omitting the case (*iii*). Translating these factorizations using functional equations involving the generating functions T(z) and  $T^s(z)$  for  $\mathcal{T}$  and  $\mathcal{T}^s$  (resp. Q(z) and  $Q^s(z)$  for  $\mathcal{Q}$  and  $\mathcal{Q}^s$ ), we obtain

$$T^{s}(z) = \frac{1 - 2z - \sqrt{1 - 4z}}{2z}, \quad T(z) = \frac{1 - 4z - \sqrt{1 - 4z}}{8z - 2},$$
$$1 - z - \sqrt{1 - 2z - 3z^{2}}, \quad 1 - 3z - \sqrt{1 - 2z - 3z}$$

$$Q^{s}(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^{2}}}{2z}$$
, and  $Q(z) = \frac{1 - 3z - \sqrt{1 - 2z - 3z^{2}}}{6z - 2}$ .

The coefficients of  $z^n$  in the Taylor expansion of  $Q^s(z)$  (resp. Q(z),  $T^s(z)$  and T(z)) generate a shift of the Motzkin sequence A001006 (resp. A005773, the Catalan sequence A000108 and A001700) in [9]).

Now, we construct a bijection from  $\mathcal{M}_n$  to the set  $\mathcal{T}_n$  of directed animals in the triangular lattice which transports  $\mathcal{M}_n^*$  into  $\mathcal{Q}_n$ . We proceed in two steps. Firstly, we define a bijection from  $\mathcal{T}_n^s$  to  $\mathcal{M}_n^s$  for  $n \geq 1$ , and secondly we extend it from  $\mathcal{T}_n$  to  $\mathcal{M}_n$ . For the first step, and according to the above bijection  $\Phi$  from  $\mathcal{M}_n^s$  to  $\mathcal{D}_n$ , it suffices to define a one-to-one correspondence  $\Psi$  between  $\mathcal{T}_n^s$  and  $\mathcal{D}_n$ . Let A be a directed animal in  $\mathcal{T}^s$ , we define  $\Psi(A)$  with respect to its four possible factorizations:

- if A satisfies (i) then  $\Psi(A) = UD$ ,
- if A satisfies (ii) then  $\Psi(A) = U\Psi(B)D$ ,
- if A satisfies (*iii*) then  $\Psi(A) = \Psi(B)UD$ ,
- if A satisfies (iv) then  $\Psi(A) = \Psi(C)U\Psi(B)D$ .

Due to the recursive definition, the image by  $\Psi$  of a directed animal in  $\mathcal{T}_n^s$  is a Dyck path of semilength n, and the image of an element of  $\mathcal{Q}_n^s$  is a Dyck path with no pattern DUD, *i.e.* in  $\mathcal{D}_n^{\star}$ .

**Theorem 3.** For  $n \ge 1$ , the map  $\Phi^{-1} \cdot \Psi$  is a bijection from  $\mathcal{T}_n^s$  to  $\mathcal{M}_n^s$ , and the image of  $\mathcal{Q}_n^s$  is  $\mathcal{M}_n^{s,\star}$ .

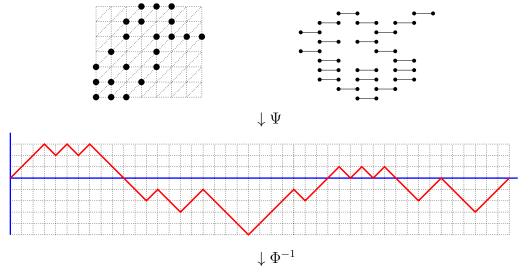
Proof. Since  $\Phi^{-1}$  is a bijection from  $\mathcal{D}_n$  to  $\mathcal{M}_n^s$ , it suffices to prove that  $\Psi$  is a bijection from  $\mathcal{T}_n^s$  to  $\mathcal{D}_n$ . As these two last sets are both enumerated by the Catalan numbers, it suffices to prove the injectivity of  $\Psi$ . We proceed by induction on n. The case n = 1holds trivially. We assume that  $\Psi$  is injective for  $k \leq n$ , and we prove the result for n + 1. By definition, the image by  $\Psi$  of animals satisfying (i) and (ii) are Dyck paths with only one return on the x-axis, *i.e.* with only one down step D that touches the x-axis. Animals satisfying (*iii*) are sent by  $\Psi$  to Dyck paths ending with DUD and with at least two return on the x-axis. Animals satisfying (*iv*) are sent to Dyck paths with at least two down steps at the end, and with at least two returns. Then, for  $A, A' \in \mathcal{T}_{n+1}^s$ ,  $\Psi(A) = \Psi(A')$  implies that A and A' belong to the same case (*i*), (*ii*), (*iii*) or (*iv*). The recurrence hypothesis induces A = A' which completes the induction. Moreover, in the case where  $A \in \mathcal{Q}_n^s$ , it does not satisfy (*iii*) and this implies that  $\Psi(A)$  is a Dyck path avoiding DUD. Finally, a cardinality argument proves that  $\Psi(\mathcal{Q}_n^s) = \mathcal{M}_n^{s,\star}$ .

Now we extend the map  $\Psi$  from  $\mathcal{T}_n$  to  $\mathcal{M}_n$  as follows. Let A be a directed animal in  $\mathcal{T}_n \setminus \mathcal{T}_n^s$ , then A can be factorized as (v) with  $B \in \mathcal{T}^s$  and  $C \in \mathcal{T}$ . In the subcase where  $C \in \mathcal{T} \setminus \mathcal{T}^s$ , C satisfies the case (v), and let  $D \in \mathcal{T}^s$ ,  $E \in \mathcal{T}$  be the two parts of its factorization. According to these two cases, we set:

$$\Psi(A) = \begin{cases} \Psi(B)\Psi(C)^r & \text{if } C \in \mathcal{T}^s, \\ \Psi(B)\Psi(D)^r\Psi(E) & \text{otherwise,} \end{cases}$$

where  $P^r$  is obtained from P by reading the Dyck path P from right to left (for instance, if P = UUDUUDDD then  $P^r = DDDUUDUU$ ). Less formally,  $\Psi$  maps successive components from  $\mathcal{T}^s$  to Dyck paths alternalely above and below the *x*-axis. See Figure 3 for an illustration of the map  $\Psi$ . **Theorem 4.** For  $n \ge 1$ , the map  $\Phi^{-1} \cdot \Psi$  is a bijection from  $\mathcal{T}_n$  to  $\mathcal{M}_n$ , and the image of  $\mathcal{Q}_n$  is  $\mathcal{M}_n^{\star}$ .

Proof. Let us prove that  $\Psi$  is a bijection from  $\mathcal{T}_n$  to  $\mathcal{M}_n$ . As these two sets have the same cardinality, it suffices to prove the injectivity of  $\Psi$ . Using Theorem 3, it remains to prove that directed animals  $A \in \mathcal{T}_n \setminus \mathcal{T}_n^s$  are sent bijectively by  $\Psi$  to Grand-Dyck paths in  $\mathcal{GD}_n \setminus \mathcal{D}_n$ . Due to the definition of  $\Psi$  whenever  $A \in \mathcal{T}_n \setminus \mathcal{T}_n^s$ , we have either  $\Psi(A) = \Psi(B)\Psi(C)^r$  or  $\Psi(A) = \Psi(B)\Psi(D)^r\Psi(E)$  with  $B, C, D \in \mathcal{T}^s$  and  $E \in \mathcal{T}$ . Then, the path  $\Psi(A)$  starts with an up-step (the first step of the non-empty Dyck path  $\Psi(B)$ ), and since the first step of  $\Psi(C)^r$  (resp.  $\Psi(D)^r$ ) is a down-step,  $\Psi(A)$  crosses the x-axis which ensures that  $\Psi(B)\Psi(C)^r$  (resp.  $\Psi(B)\Psi(D)^r$ ) belongs to  $\mathcal{GD}_n \setminus \mathcal{D}_n$ . We complete the proof with a simple induction on n. Whenever  $A \in \mathcal{Q}$ , Theorem 3 ensures that  $\Psi(B), \Psi(C)$  and  $\Psi(D)$  avoid the pattern DUD. By symmetry, the paths  $\Psi(C)^r$  and  $\Psi(D)$  avoid DUD which implies that the Grand-Dyck paths  $\Psi(B)\Psi(C)^r$ and  $\Psi(B)\Psi(D)^r$  do not contain DUD. By induction,  $\Psi(A)$  belongs to  $\mathcal{M}_n^*$ .



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Figure 3: Bijection  $\Phi^{-1}\Psi$  between directed animals and multisets via Grand-Dyck paths.

Now we define some statistics and parameters on  $\mathcal{T}_n$ ,  $\mathcal{M}_n$  and  $\mathcal{GD}_n$ , and we show how the bijections  $\Phi$ ,  $\Psi$  and  $\Phi^{-1} \cdot \Psi$  establish correspondences between them. Table 2 summarizes these correspondences. For a directed animal  $A \in \mathcal{T}_n$ , we set:

- Area(A) = number of points in A,
- $\mathbf{Lw}(A) = \text{left width}, i.e. \max\{i \ge 0 \text{ such that the line } y = x + i \text{ meets } A\},\$
- $\mathbf{Rw}(A) = \text{right width}, i.e. \max\{i \ge 1 \text{ such that the line } y = x i + 1 \text{ meets } A\},\$
- $\mathbf{Width}(A) = \mathbf{Lw}(A) + \mathbf{Rw}(A) = \text{width},$
- $\mathbf{Diag}(A) =$ number of  $\overset{\checkmark}{\bullet} \times$  in A, where  $\times$  means a site without point in A,
- Nbp(A, i) = number of points of A on the line y = x i + 1,

For a multiset  $\pi \in \mathcal{M}_n$ , we define  $\delta(\pi_i) = 0$  if  $\pi_i < i$  and 1 otherwise, and we set:

- Length $(\pi) = n$ ,
- $\mathbf{Cross}(\pi) = \operatorname{card}\{i \in [n-1], \delta(\pi_i) \neq \delta(\pi_{i+1})\},\$
- $\operatorname{Adj}(\pi) = \operatorname{number} \text{ of adjacencies, } i.e., \operatorname{card}\{i \in [n-1], \operatorname{such that } \pi_{i+1} = \pi_i + 1\},\$
- $\operatorname{Gap}(\pi, i) = |\pi_i i| c_i$  where  $c_i = \operatorname{card}\{j \le i 1, \delta(\pi_j) \ne \delta(\pi_{j+1})\},\$
- $\operatorname{Gap}(\pi) = \max_{i \in [n]} \operatorname{Gap}(\pi, i).$

For a Grand-Dyck path  $P \in \mathcal{GD}_n$ , the height  $\mathbf{h}(a, b)$  of a point  $(a, b) \in P$  is the ordinate b, and  $\mathbf{h}(P) = \max{\{\mathbf{h}(a, b), (a, b) \in P\}}$ . Here we consider a new height function defined by  $\mathbf{Height}(a, b) = |b| - c_a$  where  $c_a$  is the number of the x-axis crossings before the line x = a, and we set:

- Semilength(P) = number of up-steps U,
- $\mathbf{Cross}(P) =$ number of crossings of the *x*-axis,
- $\operatorname{Height}(P) = \max_{(a,b) \in P} \operatorname{Height}(a,b),$
- $\mathbf{Dud}(P) =$ number of pattern DUD,
- Nbu(P, i) = number of U having endpoint (a, b) satisfying Height(a, b) = i + 1.

**Theorem 5.** The bijections  $\Phi$  and  $\Psi$  induce correspondences between statistics as summarized in Table 2.

$A \in \mathcal{T}_n$	$P = \Psi(A) \in \mathcal{GD}_n$	$\pi = \Phi^{-1}(P) \in \mathcal{M}_n$
$\mathbf{Area}(A)$	$\mathbf{Semilength}(P)$	$\mathbf{Length}(\pi)$
$\mathbf{Lw}(A)$	$\mathbf{Cross}(P)$	$\mathbf{Cross}(\pi)$
$\mathbf{Rw}(A)$	$\mathbf{Height}(P)$	$\mathbf{Gap}(\pi)$
$\mathbf{Width}(A)$	$\mathbf{Cross}(P) + \mathbf{Height}(P)$	$\mathbf{Cross}(\pi) + \mathbf{Gap}(\pi)$
$\mathbf{Diag}(A)$	$\mathbf{Dud}(P)$	$\mathbf{Adj}(\pi)$
$\mathbf{Nbp}(A, i)$	$\mathbf{Nbu}(P,i)$	$\mathbf{Gap}(\pi, i)$

Table 2: Statistic correspondences by the bijections  $\Psi$  and  $\Phi$ .

*Proof.* The statistic correspondences induced by  $\Phi$  are easy to check. So, we only prove the correspondences generated by  $\Psi$  from directed animals to Grand-Dyck paths.

When  $A \in \mathcal{T}^s$ , we have  $\mathbf{Lw}(A) = \mathbf{Cross}(\Psi(A)) = 0$ . When  $A \in \mathcal{T} \setminus \mathcal{T}^s$ , A satisfies (v) with  $B \in \mathcal{T}^s$  and  $C \in \mathcal{T}$ . Then  $\mathbf{Lw}(A) = 1 + \mathbf{Lw}(C)$ . We assume the recurrence hypothesis  $\mathbf{Lw}(C) = \mathbf{Cross}(\Psi(C))$ , which implies  $\mathbf{Lw}(A) = 1 + \mathbf{Cross}(\Psi(C))$ . Using the recursive definition of  $\Psi$ , we have  $\mathbf{Cross}(\Psi(A)) = 1 + \mathbf{Cross}(\Psi(C))$  which gives by induction  $\mathbf{Lw}(A) = \mathbf{Cross}(\Psi(A))$ .

When  $A \in \mathcal{T}^s$ , it satisfies (i), (ii), (iii) or (iv), and the recursive definition of  $\Psi$ implies that  $\mathbf{Rw}(A) = \mathbf{h}(\Psi(A)) = \mathbf{Height}(\Psi(A))$ . Otherwise, if A is factorized as (v) with  $B \in \mathcal{T}^s$  and  $C \in \mathcal{T}^s$ , then

$$\mathbf{Rw}(A) = \max\{\mathbf{Rw}(B), \mathbf{Rw}(C) - 1\}$$
  
= max{ $\mathbf{h}(\Psi(B)), \mathbf{h}(\Psi(C)) - 1$ }  
= max{ $\mathbf{Height}(a, b), (a, b) \in \Psi(B)\Psi(C)^r$ },

which is equal to  $\operatorname{Height}(\Psi(A))$ . If A is factorized as (v) with  $B \in \mathcal{T}^s$  and  $C \in \mathcal{T} \setminus \mathcal{T}^s$ , then C can be factorized as (v) with  $D \in \mathcal{T}^s$  and  $E \in \mathcal{T}$ , and using an induction we have:

$$\mathbf{Rw}(A) = \max\{\mathbf{h}(\Psi(B)), \mathbf{h}(\Psi(D)) - 1, \mathbf{Height}(\Psi(E)) - 2\} \\ = \max\{\mathbf{Height}(a, b), (a, b) \in \Psi(B)\Psi(D)^r\Psi(E)\},\$$

which gives exactly  $\mathbf{Height}(\Psi(A))$ .

When  $A \in \mathcal{T}^s$ , it satisfies (i), (ii), (iii) or (iv), and the recursive definition of  $\Psi$  implies that  $\mathbf{Nbp}(A, i) = \mathbf{Nbu}(\Psi(A), i)$ . Otherwise, if A is factorized as (v) with

 $B \in \mathcal{T}^s$  and  $C \in \mathcal{T}^s$ , then  $\mathbf{Nbp}(A, i) = \mathbf{Nbp}(B, i) + \mathbf{Nbp}(C, i + 1)$ , and using the recurrence hypothesis it is equal to

$$\mathbf{Nbu}(\Psi(B), i) + \mathbf{Nbu}(\Psi(C), i+1) = \mathbf{Nbu}(\Psi(B)\Psi(C)^r, i) = \mathbf{Nbu}(\Psi(A), i).$$

Whenever A is factorized as (v) with  $D \in \mathcal{T}^s$  and  $E \in \mathcal{T}$ , a similar argument completes the proof.

Due to the symmetry  $\sigma$  about the diagonal y = x, the two statistics  $\mathbf{Lw}(\cdot) + 1$  and  $\mathbf{Rw}(\cdot)$  have the same distribution on directed animals in  $\mathcal{T}$  and  $\mathcal{Q}$ . Using Theorem 5 and Table 2, this induces that  $\mathbf{Cross}(\cdot) + 1$  and  $\mathbf{Height}(\cdot)$  also have the same distribution in  $\mathcal{GD}$  and  $\mathcal{GD}^*$ .

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