New identities for some symmetric polynomials and their applications

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Abstract

We give new identities for some symmetric polynomials: elementary, complete homogeneous, power symmetric polynomials, which are relationships between symmetric polynomials in $2r$ variables z_1, z_1^{-1}, …, z_r, z_r^{-1} and r variables $z_1+z_1^{-1}$, …, $z_r+z_r^{-1}$. As applications of these identities, we obtain some formulas for a higher order analogue of Fibonacci and Lucas numbers.

1 Introduction

Throughout the paper, we denote the set of non negative integers by $\mathbb{Z}_{\geq 0}$, the field of real numbers by $\mathbb{R}$, the field of complex numbers by $\mathbb{C}$ and $\mathbb{C}^\times := \mathbb{C}\setminus\{0\}$. Let $z_1,\dots,z_r$ be r independent variables and $\mathbf{z} := (z_1,\dots,z_r)$. For each non-negative integer n, the nth elementary symmetric polynomial $e_n^{(r)}$, complete homogeneous polynomials $h_n^{(r)}$ and power symmetric polynomial $p_n^{(r)}$ are defined by

$$e_n^{(r)} = e_n^{(r)}(\mathbf{z}) := \begin{cases} \sum_{1\leq j_1<\cdots<j_n\leq r} z_{j_1}\cdots z_{j_n} & (1\leq n\leq r) \\ 1 & (n=0) \\ 0 & (n>r) \end{cases}, \tag{1.1}$$

$$h_n^{(r)} = h_n^{(r)}(\mathbf{z}) := \sum_{m_1+\cdots+m_r=n} z_1^{m_1}\cdots z_r^{m_r}, \tag{1.2}$$

$$p_n^{(r)} = p_n^{(r)}(\mathbf{z}) := \sum_{j=1}^{r} z_j^n \tag{1.3}$$

respectively. For $z=(z_1,\dots,z_r)\in {\mathbb{C}^\times}^r$, we put

$$(\mathbf{z},\mathbf{z}^{-1}) := (z_1,\dots,z_r,z_1^{-1},\dots,z_r^{-1}) \in \mathbb{C}^{2r},$$
$$(\mathbf{z}+\mathbf{z}^{-1}) := (z_1+z_1^{-1},\dots,z_r+z_r^{-1}) \in \mathbb{C}^{r}.$$

Our main results are new identities for these three types of symmetric polynomials $f = e, h, p$, which are relationships between $f_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1})$ and $f_n^{(r)}(\mathbf{z}+\mathbf{z}^{-1})$. More precisely, we determine the following expansion coefficients $a_{n,k}^{(f)}$ and $b_{n,k}^{(f)}$,

$$f_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) = \sum_{k=0}^{n} a_{n,k}^{(f)} f_k^{(r)}(\mathbf{z} + \mathbf{z}^{-1}), \tag{1.4}$$

$$f_n^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) = \sum_{k=0}^{n} b_{n,k}^{(f)} f_k^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}). \tag{1.5}$$

The explicit expression of $a_{n,k}^{(f)}$ and $b_{n,k}^{(p)}$ may be well known. Actually we determine $a_{n,k}^{(f)}$ and $b_{n,k}^{(p)}$ immediately by only using the binomial formula. However we have not found appropriate references about explicit formulas of $a_{n,k}^{(f)}$ and their interesting applications which are to derive some new formulas of Fibonacci and Lucas numbers etc. The proofs of $b_{n,k}^{(e)}$ and $b_{n,k}^{(h)}$ are more difficult than $a_{n,k}^{(e)}$ and $a_{n,k}^{(h)}$. In particular we need a hypergeometric identity in proofs of the explicit formula of $b_{n,k}^{(e)}$ or $b_{n,k}^{(h)}$. Therefore we give their proofs and applications together in this article. For examples, by some specializations of our results we obtain the following type formulas (see Corollary 16):

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} = \begin{cases} 1 & (n \equiv 0, 1 \,(\mathrm{mod}\, 6)) \\ -1 & (n \equiv 3, 4 \,(\mathrm{mod}\, 6)) \\ 0 & (\text{others}) \end{cases},$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k+1}{k} F_{n-2k+1} = \begin{cases} 1 & (n \equiv 0, 1 \,(\mathrm{mod}\, 10)) \\ -1 & (n \equiv 5, 6 \,(\mathrm{mod}\, 10)) \\ 0 & (\text{others}) \end{cases},$$

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{2} \left((-1)^{\lfloor \frac{m-2k}{3} \rfloor} - (-1)^{\lfloor \frac{m-2k-2}{3} \rfloor} \right) \left(\binom{m}{k} - \binom{m}{k-1} \right) = 1,$$

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{2} \left((-1)^{\lfloor \frac{m-2k}{5} \rfloor} - (-1)^{\lfloor \frac{m-2k-2}{5} \rfloor} \right) \left(\binom{m+1}{k} - \binom{m+1}{k-1} \right) = F_{m+1},$$

where n and m are non negative integers, $\lfloor x \rfloor$ is the greatest integer not exceeding $x \in \mathbb{R}$, F_{m+1} is the classical Fibonacci numbers.

The content of this article is as follows. In Section 2, we refer some basic formulas for symmetric polynomials and Gauss hypergeometric function from [AAR] and [M]. Section 3 is the main part of this article. In this section, we determine $a_{n,k}^{(f)}$ and $b_{n,k}^{(f)}$ explicitly. We also consider principle specializations $z_i = q^i$ of our formulas. In Section 4, we introduce a natural generalization of Fibonacci or Lucas numbers and give their fundamental formulas by applying our main results. Further, we consider some specializations of our results and drive some interesting binomial sum formulas and others.

2 Preliminaries

2.1 Symmetric polynomials

Refer to Macdonald [M] for the details in this subsection. We fix a positive integer r, and denote the partition set of length r by

$$\lambda \in \mathcal{P}_r := \{\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{Z}_{\geq 0}^r \mid \nu_1 \geq \ldots \geq \nu_r\}$$

and the symmetric group of degree r by $\mathfrak{S}_r$. For some special partitions, we use the following notations

$$\begin{aligned} (n) &:= (n, 0, \ldots, 0) \in \mathcal{P}_r, \\ (1^n) &:= (1, \ldots, 1, 0, \ldots, 0) \in \mathcal{P}_r, \quad (n = 1, \ldots, r). \end{aligned}$$

The symmetric group $\mathfrak{S}_r$ acts on $\mathbf{z} = (z_1, \ldots, z_r) \in \mathbb{C}^r$ by

$$\sigma.\mathbf{z} := (z_{\sigma(1)}, \ldots, z_{\sigma(r)}).$$

For any partition λ, we define Schur polynomial $s_\lambda(\mathbf{z})$ and monomial symmetry polynomial $m_\lambda(\mathbf{z})$ by

$$s_\lambda(\mathbf{z}) := \frac{\det\left(z_i^{\lambda_j+r-j}\right)_{i,j=1,\ldots r}}{\det\left(z_i^{r-j}\right)_{i,j=1,\ldots r}} = \frac{\det\left(z_i^{\lambda_j+r-j}\right)_{i,j=1,\ldots r}}{\prod_{1\leq i<j\leq r}(z_i - z_j)}, \tag{2.1}$$

$$m_\lambda(\mathbf{z}) := \sum_{\nu \in \mathfrak{S}_r.\lambda} z_1^{\nu_1} \cdots z_r^{\nu_r}, \tag{2.2}$$

where det is the usual determinant and

$$\mathfrak{S}_r.\lambda := \{\sigma.\lambda := (\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(r)}) \mid \sigma \in \mathfrak{S}_r\}.$$

We remark that Schur polynomial extends to Schur function $s_\lambda(\mathbf{z})$ for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r$ by (2.1).

It is well known that

$$e_n^{(r)}(\mathbf{z}) = s_{(1^n)}(\mathbf{z}) = m_{(1^n)}(\mathbf{z}), \tag{2.3}$$

$$p_n^{(r)}(\mathbf{z}) = m_{(n)}(\mathbf{z}), \tag{2.4}$$

$$h_n^{(r)}(\mathbf{z}) = s_{(n)}(\mathbf{z}), \tag{2.5}$$

which is Schur $s_{(1^n)}$ or monomial $m_{(1^n)}$ with one column are $e_n^{(r)}$, and monomial $m_{(n)}$ with one row is $p_n^{(r)}$, and Schur $s_{(n)}$ with one row is $h_n^{(r)}$ ([M] Chapter I Section 3 (3.9)). From (2.5), we extend the complete homogeneous polynomials to $h_n^{(r)}(\mathbf{z})$ ($n \in \mathbb{Z}$): namely

$$h_{-n}^{(r)}(\mathbf{z}) := s_{(-n)}(\mathbf{z}) \quad (n \geq 0). \tag{2.6}$$

By this extension (2.6) and the definition of Schur function, for any r we have

$$h_{-n}^{(r)}(\mathbf{z}) = 0 \quad (n = 1, 2, \ldots, r-1). \tag{2.7}$$

We list up some required formulas for symmetric polynomials in [M].

Lemma 1. *(1) Generating functions*

$$\prod_{j=1}^{r}(1 + z_j y) = \sum_{n=0}^{r} e_n^{(r)}(\mathbf{z}) y^n, \tag{2.8}$$

$$\prod_{j=1}^{r} \frac{1}{1 - z_j y} = \sum_{n=0}^{\infty} h_n^{(r)}(\mathbf{z}) y^n, \tag{2.9}$$

$$\sum_{j=1}^{r} \frac{1}{1 - z_j y} = \sum_{n=0}^{\infty} p_n^{(r)}(\mathbf{z}) y^n. \tag{2.10}$$

(2) q-binomial formula

$$\prod_{j=0}^{n-1}(1 + q^j y) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{2}} y^k, \tag{2.11}$$

$$\prod_{j=0}^{n-1} \frac{1}{1 - q^j y} = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q y^k, \tag{2.12}$$

where $\binom{n}{k}_q$ *denotes the q-binomial coefficient*

$$\binom{n}{k}_q := \frac{(1-q^n)\cdots(1-q^{n-k+1})}{(1-q)\cdots(1-q^k)}.$$

(3) Wronski relation and Newton's formula

$$\sum_{j=0}^{\min(n,r)} (-1)^{j-1} e_j^{(r)}(\mathbf{z}) h_{n-j}^{(r)}(\mathbf{z}) = 0, \tag{2.13}$$

$$\sum_{j=0}^{\min(n,r)} (-1)^{n-j} p_{n-j+1}^{(r)}(\mathbf{z}) e_j^{(r)}(\mathbf{z}) = (n+1) e_{n+1}^{(r)}(\mathbf{z}). \tag{2.14}$$

Actually, (2.8) is [M] p19 (2.2) and (2.9) is [M] p21 (2.5) exactly. For (2.11) and (2.12), see [M] p26 Examples 3. Similarly, (2.13) and (2.14) are [M] p21 (2.6′) and p23 (2.11′) respectively.

2.2 Gauss hypergeometric functions

Let a, b, c, z be complex numbers such that c is not non negative integers, and $(a)_m$ be the raising factorial defined by

$$(a)_m := \begin{cases} a(a+1)\cdots(a+m-1) & (m \neq 0) \\ 1 & (m = 0) \end{cases}.$$

We recall Gauss hypergeometric function

$${}_2F_1(z) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) := \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{m!(c)_m} z^m \quad (|z| < 1),$$

and for any complex numbers α and x we put

$$\psi(\alpha; x) := \sum_{k=0}^{\infty} \frac{(\alpha)_{2k}}{k!(\alpha+1)_k} x^k \quad (|4x| < 1). \tag{2.15}$$

Since

$$\psi(\alpha; x) = {}_2F_1\left(\begin{matrix} \frac{\alpha}{2}, \frac{\alpha+1}{2} \\ \alpha+1 \end{matrix}; 4x\right),$$

$\psi(\alpha; x)$ is analytically continued to $4x \in \mathbb{C} \setminus \{0, 1\}$ by analytic continuation of ${}_2F_1(z)$.

Lemma 2. (1) *Another expression*

$$\psi(\alpha; x) = \sum_{k=0}^{\infty} c_{\alpha+2k-1,k} x^k, \tag{2.16}$$

where

$$c_{a,k} := \binom{a}{k} - \binom{a}{k-1}, \quad \binom{a}{k} := \begin{cases} \frac{a(a-1)\cdots(a-k+1)}{k!} & (k \neq 0) \\ 1 & (k = 0) \end{cases}.$$

(2) *Closed form*

$$\psi(\alpha; x) = \left(\frac{1 - \sqrt{1-4x}}{2x}\right)^{\alpha}. \tag{2.17}$$

(3) *Index law*

$$\psi(\alpha; x)\psi(\beta; x) = \psi(\alpha + \beta; x). \tag{2.18}$$

(4) *Quadratic formula*

$$x(x\psi(1; x^2))^2 - (x\psi(1; x^2)) + x = 0. \tag{2.19}$$

Proof. (1) By the definition of $\psi(\alpha; x)$ and $c_{a,k}$, we have

$$\begin{aligned}
\psi(\alpha; x) &= \sum_{k=0}^{\infty} \frac{(\alpha)_{2k}}{k!(\alpha+1)_k} x^k \\
&= 1 + \sum_{k=1}^{\infty} (\alpha + k - k) \frac{(\alpha+k+1)_{k-1}}{k!} x^k \\
&= 1 + \sum_{k=1}^{\infty} \left(\frac{(\alpha+k)_k}{k!} - \frac{(\alpha+k+1)_{k-1}}{(k-1)!} \right) x^k \\
&= 1 + \sum_{k=1}^{\infty} \left(\binom{\alpha+2k-1}{k} - \binom{\alpha+2k-1}{k-1} \right) x^k \\
&= \sum_{k=0}^{\infty} c_{\alpha+2k-1,k} x^k.
\end{aligned}$$

(2) We remark a hypergeometric transformation [AAR] (3.1.10)

$${}_2F_1\left(\begin{matrix} \frac{\alpha}{2}, \frac{\alpha+1}{2} \\ \alpha-\beta+1 \end{matrix}; x \right) = \left(2\frac{1-\sqrt{1-x}}{x} \right)^{\alpha} {}_2F_1\left(\begin{matrix} \alpha, \beta \\ \alpha-\beta+1 \end{matrix}; \frac{1-\sqrt{1-x}}{1+\sqrt{1-x}} \right).$$

Thus,

$$\psi(\alpha; x) = {}_2F_1\left(\begin{matrix} \frac{\alpha}{2}, \frac{\alpha+1}{2} \\ \alpha+1 \end{matrix}; 4x \right) = \left(\frac{1-\sqrt{1-4x}}{2x} \right)^{\alpha}.$$

The formulas (2.18) and (2.19) follow from (2.17) immediately. □

Remark 3. We mention some properties of $c_{n,k}$. For non negative integers n and k, we make the table of $c_{n,k} > 0$. This table is determined exactly by initial conditions

$$\begin{aligned}
c_{n,0} =& 1 \quad (n \geq 0), \\
c_{n,k} :=& 0 \quad \left(n > 0, \left\lfloor \frac{n}{2} \right\rfloor < k \right), \\
c_{2n,n+1} =& \frac{1}{n+2} \binom{2n+2}{n+1} \quad (n \geq 0), \\
c_{2n,n} =& \frac{1}{n+1} \binom{2n}{n} : \text{Catalan numbers} \quad (n \geq 0)
\end{aligned}$$

and a recursion formula

$$c_{n,k} = c_{n-1,k-1} + c_{n-1,k}, \tag{2.20}$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding $x \in \mathbb{R}$. The sequence $c_{n,k}$ is a kind of Clebsch-Gordan coefficients for sl_2. Actually, from the above initial conditions and recursion of $c_{n,k}$, we have

$$\left(\frac{\sin(2\theta)}{\sin\theta} \right)^n = (2\cos\theta)^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,k} \frac{\sin((n-2k+1)\theta)}{\sin\theta}, \tag{2.21}$$

$n \setminus k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1										
2	1	1									
3	1	2									
4	1	3	2								
5	1	4	5								
6	1	5	9	5							
7	1	6	14	14							
8	1	7	20	28	14						
9	1	8	27	48	42						
10	1	9	35	75	90	42					
11	1	10	44	110	165	132					
12	1	11	54	154	275	297	132				
13	1	12	65	208	429	572	429				
14	1	13	77	273	637	1001	1001	429			
15	1	14	90	350	910	1638	2002	1430			
16	1	15	104	440	1260	2548	3640	3432	1430		
17	1	16	119	544	1700	3808	6188	7072	4862		
18	1	17	135	663	2244	5508	9996	13260	11934	4862	
19	1	18	152	798	2907	7752	15504	23256	25194	16796	
20	1	19	170	950	3705	10659	23256	38760	48450	41990	16796
21	1	20	189	1120	4655	14364	33915	62016	87210	90440	58786

Table 1: $c_{n,k}$

that is the classical Clebsch-Gordan rule for sl_2 exactly.

Further $c_{n,k}$ is a typical example of Kostka numbers $K_{\lambda\mu}$ (see [S] Chapter 2 Section 2.11 and Chapter 4 Section 4.9). We remark Young's rule

$$s_{(\mu_1)}(\mathbf{z}) \cdots s_{(\mu_n)}(\mathbf{z}) = \sum_{\lambda} K_{\lambda\mu} s_\lambda(\mathbf{z}) \quad (\mu_1 \geq \cdots \geq \mu_n \geq 1), \tag{2.22}$$

and

$$s_{(\lambda_1,\lambda_2)}\left(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}\right) = \frac{e^{(\lambda_1-\lambda_2+1)\sqrt{-1}\theta} - e^{-(\lambda_1-\lambda_2+1)\sqrt{-1}\theta}}{e^{\sqrt{-1}\theta} - e^{-\sqrt{-1}\theta}} = \frac{\sin\left((\lambda_1-\lambda_2+1)\theta\right)}{\sin\theta}.$$

By putting $r = 2$, $\mu_1 = \cdots = \mu_n = 1$, $z_1 = e^{\sqrt{-1}\theta}$, $z_2 = e^{-\sqrt{-1}\theta}$ in (2.22), we have

$$\left(\frac{\sin(2\theta)}{\sin\theta}\right)^n = s_{(1,0)}\left(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}\right)^n$$
$$= \sum_{\lambda} K_{\lambda(1^n)} s_\lambda\left(e^{\sqrt{-1}\theta}, e^{-\sqrt{-1}\theta}\right)$$
$$= \sum_{\lambda} K_{\lambda(1^n)} \frac{\sin((\lambda_1 - \lambda_2 + 1)\theta)}{\sin\theta}. \tag{2.23}$$

Finally, by comparing (2.21) and (2.23), we have

$$K_{(n-k,k),(1^n)} = c_{n,k}.$$

In Section 4 we show that a higher order analogue of Fibonacci numbers appear in the Table 1, that is one of our main results (4.18).

3 Main results

From (2.8), (2.9) and

$$(1 \pm z_j y)(1 \pm z_j^{-1} y) = 1 \pm (z_j + z_j^{-1})y + y^2 = (1 + y^2)\left(1 \pm (z_j + z_j^{-1})\frac{y}{1 + y^2}\right),$$

we obtain the following key lemma.

Lemma 4. *If*

$$|z_j y|, |z_j^{-1} y|, \left|(z_j + z_j^{-1})\frac{y}{1 + y^2}\right| < 1 \quad (j = 1, \dots, r), \tag{3.1}$$

then

$$\sum_{n=0}^{2r} e_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) y^n = \prod_{j=1}^{r} (1 + z_j y)(1 + z_j^{-1} y)$$
$$= (1 + y^2)^r \sum_{m=0}^{r} e_m^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) \left(\frac{y}{1 + y^2}\right)^m, \tag{3.2}$$
$$\sum_{n=0}^{\infty} h_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) y^n = \prod_{j=1}^{r} \frac{1}{(1 - z_j y)(1 - z_j^{-1} y)}$$
$$= (1 + y^2)^{-r} \sum_{m=0}^{\infty} h_m^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) \left(\frac{y}{1 + y^2}\right)^m. \tag{3.3}$$

First, we prove the formulas of type (1.4).

Theorem 5. (1) *For $n = 0, 1, \ldots, 2r$, we have*

$$e_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) = \sum_{k=\max\left\{\left\lfloor \frac{n-r}{2} \right\rfloor, 0\right\}}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{r-n+2k}{k} e_{n-2k}^{(r)}(\mathbf{z} + \mathbf{z}^{-1}). \tag{3.4}$$

(2) *For any non negative integer n,*

$$h_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k+r-1}{k} h_{n-2k}^{(r)}(\mathbf{z} + \mathbf{z}^{-1}). \tag{3.5}$$

(3) *For any positive integer n,*

$$p_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) = 2 \sum_{k=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{2k-n-1}{k} p_{n-2k}^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) - \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{2k-n}{k} p_{n-2k}^{(r)}(\mathbf{z} + \mathbf{z}^{-1}). \tag{3.6}$$

Proof. (1) From (3.2) and binomial theorem, we have

$$\begin{aligned}
\sum_{n=0}^{2r} e_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) y^n &= \sum_{m=0}^{r} e_m^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) y^m (1+y^2)^{r-m} \\
&= \sum_{m=0}^{r} \sum_{k=0}^{r-m} \binom{r-m}{k} e_m^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) y^{m+2k} \\
&= \sum_{n=0}^{2r} \sum_{k=\max\left\{\left\lfloor \frac{n-r}{2} \right\rfloor, 0\right\}}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{r-n+2k}{k} e_{n-2k}^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) y^n.
\end{aligned}$$

(2) Similarly, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} h_n^{(2r)}(\mathbf{z}, \mathbf{z}^{-1}) y^n &= \sum_{m=0}^{\infty} h_m^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) y^m (1+y^2)^{-m-r} \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(m+r)_k}{k!} h_m^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) y^{m+2k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(n-2k+r)_k}{k!} h_{n-2k}^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) y^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-k+r-1}{k} h_{n-2k}^{(r)}(\mathbf{z} + \mathbf{z}^{-1}) y^n.
\end{aligned}$$

(3) It is enough to show that the case of $r = 1$ which is

$$z^n + z^{-n} = 2\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{2k-n-1}{k}(z+z^{-1})^{n-2k} - \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{2k-n}{k}(z+z^{-1})^{n-2k}.$$

From (3.2) and simple calculus,

$$\begin{aligned}\sum_{n=0}^{\infty}(z^n+z^{-n})y^n &= \frac{1}{1-zy}+\frac{1}{1-z^{-1}y}\\ &= \frac{2-(z+z^{-1})y}{1-(z+z^{-1})y+y^2}\\ &= \frac{1}{1+y^2}\frac{2-(z+z^{-1})y}{1-(z+z^{-1})\frac{y}{1+y^2}}.\end{aligned}$$

If $|zy| < 1$, $|z^{-1}y| < 1$ and $|y| < 1$, then

$$\begin{aligned}\sum_{n=0}^{\infty}(z^n+z^{-n})y^n &= \sum_{m=0}^{\infty}\left(\frac{2}{y}-(z+z^{-1})\right)(z+z^{-1})^m\left(\frac{y}{1+y^2}\right)^{m+1}\\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left(\frac{2}{y}-(z+z^{-1})\right)(z+z^{-1})^m y^{m+1}\frac{(m+1)_k}{k!}(-1)^k y^{2k}\\ &= \sum_{n=0}^{\infty}2\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{2k-n-1}{k}(z+z^{-1})^{n-2k}y^n\\ &\quad -\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{2k-n}{k}(z+z^{-1})^{n-2k+1}y^{n+1}.\end{aligned}$$

□

To prove (1.5), we prepare the following formula which follows from Lemma 2 immediately.

Lemma 6. *If*

$$y := \frac{1-\sqrt{1-4x^2}}{2x},$$

then

$$x = \frac{y}{1+y^2}$$

and for any non negative integer N

$$y^N = x^N\psi\left(N;x^2\right) = x^N\sum_{k=0}^{\infty}c_{N+2k-1,k}x^{2k}. \tag{3.7}$$

Proof. By quadratic formula,

$$y = x\frac{1-\sqrt{1-4x^2}}{2x^2} = x\psi\left(1; x^2\right). \tag{3.8}$$

Thus, we have

$$y^N = x^N \psi\left(1; x^2\right)^N = x^N \psi\left(N; x^2\right) = x^N \sum_{k=0}^{\infty} c_{N+2k-1,k} x^{2k}.$$

Here the second and third equalities follow from (2.18) and (2.16) respectively. □

Next, we obtain the formulas of type (1.5).

Theorem 7. *For any non negative integer m, we have the following identities.*
(1)

$$\sum_{k=\max\left(\left\lfloor \frac{m}{2}\right\rfloor - r, 0\right)}^{\left\lfloor \frac{m}{2}\right\rfloor} c_{m-r-1,k} e^{(2r)}_{m-2k}(\mathbf{z}, \mathbf{z}^{-1}) = \begin{cases} e^{(r)}_m(\mathbf{z}+\mathbf{z}^{-1}) & (m = 0, 1, \ldots, r) \\ 0 & (\text{others}) \end{cases}. \tag{3.9}$$

(2)

$$\sum_{k=0}^{\left\lfloor \frac{m}{2}\right\rfloor} c_{m+r-1,k} h^{(2r)}_{m-2k}(\mathbf{z}, \mathbf{z}^{-1}) = h^{(r)}_m(\mathbf{z}+\mathbf{z}^{-1}). \tag{3.10}$$

(3)

$$\frac{1}{2}\sum_{k=0}^{m} \binom{m}{k} p^{(2r)}_{|m-2k|}(\mathbf{z}, \mathbf{z}^{-1}) = p^{(r)}_m(\mathbf{z}+\mathbf{z}^{-1}). \tag{3.11}$$

Proof. (1) From (3.2),

$$\sum_{n=0}^{2r} e^{(2r)}_n(\mathbf{z}, \mathbf{z}^{-1}) y^{n-r} = x^{-r} \sum_{m=0}^{r} e^{(r)}_m(\mathbf{z}+\mathbf{z}^{-1}) x^m, \tag{3.12}$$

where

$$x = \frac{y}{1+y^2}.$$

Hence, from (3.7) we have

$$\begin{aligned} \sum_{n=0}^{2r} e^{(2r)}_n(\mathbf{z}, \mathbf{z}^{-1}) y^{n-r} &= \sum_{n=0}^{2r} e^{(2r)}_n(\mathbf{z}, \mathbf{z}^{-1}) x^{n-r} \sum_{k=0}^{\infty} c_{n-r+2k-1,k} x^{2k} \\ &= x^{-r} \sum_{m=0}^{\infty} \sum_{k=\max\left(\left\lfloor \frac{m}{2}\right\rfloor - r, 0\right)}^{\left\lfloor \frac{m}{2}\right\rfloor} e^{(2r)}_{m-2k}(\mathbf{z}, \mathbf{z}^{-1}) c_{m-r-1,k} x^m. \end{aligned} \tag{3.13}$$

By comparing coefficients of (3.12) and (3.14), we obtain the conclusion.
(2) Similarly, from (3.3)

$$\sum_{n=0}^{\infty} h_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})y^{n+r} = x^r \sum_{m=0}^{\infty} h_m^{(r)}(\mathbf{z}+\mathbf{z}^{-1})x^m \tag{3.14}$$

and (3.7) we have

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})y^{n+r} &= \sum_{n=0}^{\infty} h_n^{(2r)}(\mathbf{z},\mathbf{z}^{-1})x^{n+r}\sum_{k=0}^{\infty} c_{n+r+2k-1,k}x^{2k} \\ &= x^r \sum_{m=0}^{\infty}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} h_{m-2k}^{(2r)}(\mathbf{z},\mathbf{z}^{-1})c_{m+r-1,k}x^m.\end{aligned}$$

(3) We prove this formula without generating function and other Lemmas. Actually, by applying the usual binomial formula we have

$$\begin{aligned}p_m^{(r)}(\mathbf{z}+\mathbf{z}^{-1}) &= \sum_{i=1}^{r}(z_i+z_i^{-1})^m \\ &= \sum_{i=1}^{r}\sum_{k=0}^{m}\binom{m}{k} z_i^{(2k-m)} \\ &= \frac{1}{2}\sum_{i=1}^{r}\sum_{k=0}^{m}\left(\binom{m}{k} z_i^{2k-m} + \binom{m}{m-k} z_i^{m-2k}\right) \\ &= \frac{1}{2}\sum_{k=0}^{m}\binom{m}{k}\sum_{i=1}^{r}(z_i^{2k-n}+z_i^{n-2k}) \\ &= \frac{1}{2}\sum_{k=0}^{m}\binom{m}{k} p_{|m-2k|}^{(2r)}(\mathbf{z},\mathbf{z}^{-1}).\end{aligned}$$

□

Finally, we consider principal specialization of Theorem 5 and Theorem 7. Let

$$\boldsymbol{q}^{\pm\delta} := (q^{\pm(r-1)},\dots,q^{\pm 1}).$$

In this special case, we evaluate $e_n^{(2r)}(\boldsymbol{q}^{+\delta},\boldsymbol{q}^{-\delta})$, $h_n^{(2r)}(\boldsymbol{q}^{+\delta},\boldsymbol{q}^{-\delta})$ and $p_n^{(2r)}(\boldsymbol{q}^{+\delta},\boldsymbol{q}^{-\delta})$ explicitly.

Proposition 8. *For any non negative integer n, we have the following identities.*
(1)

$$\sum_{k=0}^{\min(n,2r+1)} (-1)^{n-k}\binom{2r+1}{k}_q q^{\frac{k(k-2r-1)}{2}} = \begin{cases} e_n^{(2r)}(\boldsymbol{q}^{+\delta},\boldsymbol{q}^{-\delta}) & (n=0,1,\dots,2r) \\ 0 & (\text{others})\end{cases}. \tag{3.15}$$

(2)

$$h_n^{(2r)}(\boldsymbol{q}^{+\delta}, \boldsymbol{q}^{-\delta}) = q^{-nr}\left(\binom{2r+n}{n}_q - \binom{2r+n-1}{n-1}_q q^r\right). \tag{3.16}$$

(3)

$$p_n^{(2r)}(\boldsymbol{q}^{+\delta}, \boldsymbol{q}^{-\delta}) = -1 + q^{-rn}\frac{1-q^{(2r+1)n}}{1-q^n}. \tag{3.17}$$

Proof. (1) From generating function of elementary symmetric polynomials (2.8),

$$\sum_{n=0}^{2r} e_n^{(2r)}(\boldsymbol{q}^{+\delta}, \boldsymbol{q}^{-\delta})y^n = \prod_{j=1}^{r}(1+q^jy)(1+q^{-j}y) = \frac{1}{1+y}\prod_{j=0}^{2r}(1+q^jq^{-r}y).$$

By q-binomial formula (2.11), we have

$$\begin{aligned}\sum_{n=0}^{2r} e_n^{(2r)}(\boldsymbol{q}^{+\delta}, \boldsymbol{q}^{-\delta})y^n &= \sum_{i=0}^{\infty}(-1)^iy^i\sum_{k=0}^{2r+1}\binom{2r+1}{k}_q q^{\frac{k(k-1)}{2}}q^{-kr}y^k \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\min(n,2r+1)}(-1)^{n-k}\binom{2r+1}{k}_q q^{\frac{k(k-2r-1)}{2}}y^n.\end{aligned}$$

(2) Similarly, we have

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^{(2r)}(\boldsymbol{q}^{+\delta}, \boldsymbol{q}^{-\delta})y^n &= \prod_{j=1}^{r}\frac{1}{(1-q^jy)(1-q^{-j}y)} \\ &= (1-y)\prod_{j=0}^{2r}\frac{1}{1-q^jq^{-r}y} \\ &= (1-y)\sum_{k=0}^{\infty}\binom{2r+k}{k}_q q^{-kr}y^k \\ &= \sum_{n=0}^{\infty}\left(\binom{2r+n}{n}_q - \binom{2r+n-1}{n-1}_q q^r\right)q^{-nr}y^n.\end{aligned}$$

(3) By the definition of power sum and geometric series, we have

$$p_n^{(2r)}(\boldsymbol{q}^{+\delta}, \boldsymbol{q}^{-\delta}) = -1 + \sum_{k=-r}^{r} q^{kn} = -1 + \frac{q^{-rn} - q^{(r+1)n}}{1-q^n}.$$

We remark this formula holds on the limit $q \to 1$. □

Corollary 9. (1) *For $n=0,1,\dots,2r$,*

$$\sum_{k=\max\{\lfloor\frac{n-r}{2}\rfloor,0\}}^{\lfloor\frac{n}{2}\rfloor}\binom{r-n+2k}{k}e_{n-2k}^{(r)}(\boldsymbol{q}^{+\delta}+\boldsymbol{q}^{-\delta})=\sum_{k=0}^{n}(-1)^k\binom{2r+1}{k}_q q^{\frac{k(k-2r-1)}{2}}. \tag{3.18}$$

For any non negative integer m,

$$\begin{aligned}&\sum_{k=\max\{\lfloor\frac{m-r}{2}\rfloor,0\}}^{\lfloor\frac{m}{2}\rfloor}\sum_{l=0}^{\min(m-2k,2r+1)}(-1)^{m-l}\binom{2r+1}{l}_q q^{\frac{l(l-2r-1)}{2}}c_{m-r-1,k}\\&=\begin{cases}e_m^{(r)}(\boldsymbol{q}^{+\delta}+\boldsymbol{q}^{-\delta}) & (m=0,1,\dots,r)\\ 0 & (\text{others})\end{cases}.\end{aligned} \tag{3.19}$$

(2) *For any non negative integers n and m, we have*

$$\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n-k+r-1}{k}h_{n-2k}^{(r)}(\boldsymbol{q}^{+\delta}+\boldsymbol{q}^{-\delta})=q^{-nr}\left(\binom{2r+n}{n}_q-\binom{2r+n-1}{n-1}_q q^r\right), \tag{3.20}$$

$$\begin{aligned}&\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^m q^{-(m-2k)r}\left(\binom{2r+m-2k}{m-2k}_q-\binom{2r+m-2k-1}{m-2k-1}_q q^r\right)c_{m+r-1,k}\\&=h_m^{(r)}(\boldsymbol{q}^{+\delta}+\boldsymbol{q}^{-\delta}).\end{aligned} \tag{3.21}$$

(3) *For any positive integer n,*

$$\begin{aligned}&2\sum_{k=0}^{\lfloor\frac{n+1}{2}\rfloor}\binom{2k-n-1}{k}p_{n-2k}^{(r)}(\boldsymbol{q}^{+\delta}+\boldsymbol{q}^{-\delta})-\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{2l-n}{k}p_{n-2k}^{(r)}(\boldsymbol{q}^{+\delta}+\boldsymbol{q}^{-\delta})\\&=-1+\frac{q^{-rn}-q^{(r+1)n}}{1-q^n}.\end{aligned} \tag{3.22}$$

For any non negative integer m,

$$-2^{m-1}+\frac{1}{2}\sum_{k=0}^{m}\binom{m}{k}\frac{1-q^{(2r+1)|m-2k|}}{1-q^{|m-2k|}}q^{-|m-2k|r}=p_m^{(r)}(\boldsymbol{q}^{+\delta}+\boldsymbol{q}^{-\delta}). \tag{3.23}$$

4 Applications

In this section we investigate more specializations of Theorem 5 and Theorem 7, that is including new identities for the classical Fibonacci and Lucas numbers. Let

$$\boldsymbol{\zeta}^{\pm\delta}:=\left(e^{\pm\frac{2\pi\sqrt{-1}r}{2r+1}},\dots,e^{\pm\frac{2\pi\sqrt{-1}}{2r+1}}\right).$$

We put

$$F_{n+1}^{(r)} := h_n^{(r)}\left(-\boldsymbol{\zeta}^{+\delta}-\boldsymbol{\zeta}^{-\delta}\right) = h_n^{(r)}\left(-\cos\left(\frac{2\pi}{2r+1}\right),\dots,-\cos\left(\frac{2\pi r}{2r+1}\right)\right), \tag{4.1}$$

$$L_n^{(r)} := p_n^{(r)}\left(-\boldsymbol{\zeta}^{+\delta}-\boldsymbol{\zeta}^{-\delta}\right) = p_n^{(r)}\left(-\cos\left(\frac{2\pi}{2r+1}\right),\dots,-\cos\left(\frac{2\pi r}{2r+1}\right)\right), \tag{4.2}$$

that is a higher order analogue of the classical Fibonacci and Lucas numbers. Actually, the case of $r=1$ is trivial sequences

$$F_{n+1}^{(1)} = L_n^{(1)} = 1, \quad (n \geq 0) \tag{4.3}$$

and the case of $r=2$ is the classical Fibonacci numbers F_{n+1} and Lucas numbers L_n

$$F_{n+1}^{(2)} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right) =: F_{n+1}, \tag{4.4}$$

$$L_n^{(2)} = \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} =: L_n. \tag{4.5}$$

Further, under this specialization, $e_n^{(2r)}(-\boldsymbol{\zeta}^{+\delta},-\boldsymbol{\zeta}^{-\delta})$, $h_n^{(2r)}(-\boldsymbol{\zeta}^{+\delta},-\boldsymbol{\zeta}^{-\delta})$, $p_n^{(2r)}(-\boldsymbol{\zeta}^{+\delta},-\boldsymbol{\zeta}^{-\delta})$ and $e_m^{(r)}(-\boldsymbol{\zeta}^{+\delta}-\boldsymbol{\zeta}^{-\delta})$ have the following simple expressions.

Proposition 10. (1) *We have*

$$e_n^{(2r)}(-\boldsymbol{\zeta}^{+\delta},-\boldsymbol{\zeta}^{-\delta}) = 1 \quad (n=0,1,\dots,2r), \tag{4.6}$$

$$\sum_{k=\max\left\{\left\lfloor\frac{m-r}{2}\right\rfloor,0\right\}}^{\left\lfloor\frac{m}{2}\right\rfloor} c_{m-r-1,k} = \begin{cases} e_m^{(r)}(-\boldsymbol{\zeta}^{+\delta}-\boldsymbol{\zeta}^{-\delta}) & (m=0,1,\dots,r) \\ 0 & (\text{others}) \end{cases}. \tag{4.7}$$

In particular, for $m=0,1,\dots,r$, we have

$$e_m^{(r)}(-\boldsymbol{\zeta}^{+\delta}-\boldsymbol{\zeta}^{-\delta}) = \sum_{k=0}^{\left\lfloor\frac{m}{2}\right\rfloor} c_{m-r-1,k} = \binom{m-r-1}{\left\lfloor\frac{m}{2}\right\rfloor} = (-1)^{\left\lfloor\frac{m}{2}\right\rfloor}\binom{r-\left\lfloor\frac{m+1}{2}\right\rfloor}{\left\lfloor\frac{m}{2}\right\rfloor}. \tag{4.8}$$

(2) *For any non negative integer n,*

$$\begin{aligned} h_n^{(2r)}(-\boldsymbol{\zeta}^{+\delta},-\boldsymbol{\zeta}^{-\delta}) &= \frac{1}{2}\left((-1)^{\left\lfloor\frac{n}{2r+1}\right\rfloor} - (-1)^{\left\lfloor\frac{n-2}{2r+1}\right\rfloor}\right) \\ &= \begin{cases} 1 & (n \equiv 0,1 \pmod{2(2r+1)}) \\ -1 & (n \equiv 2r+1, 2r+2 \pmod{2(2r+1)}) \\ 0 & (\text{others}) \end{cases}. \end{aligned} \tag{4.9}$$

(3) *For any non negative integer* n,

$$p_n^{(2r)}(-\boldsymbol{\zeta}^{+\delta}, -\boldsymbol{\zeta}^{-\delta}) = (-1)^n(-1 + (2r+1)\delta_{2r+1|n}) = \begin{cases} (-1)^{n-1} & (2r+1 \nmid n) \\ (-1)^n 2r & (2r+1 \mid n) \end{cases}, \tag{4.10}$$

where

$$\delta_{2r+1|n} := \begin{cases} 0 & (2r+1 \nmid n) \\ 1 & (2r+1 \mid n) \end{cases}.$$

Proof. (1) From (2.8), we have

$$\begin{aligned} \sum_{n=0}^{2r} e_n^{(2r)}(-\boldsymbol{\zeta}^{+\delta}, -\boldsymbol{\zeta}^{-\delta}) y^n &= \prod_{j=1}^{r} \left(1 - e^{2\pi\sqrt{-1}\frac{j}{2r+1}} y\right)\left(1 - e^{-2\pi\sqrt{-1}\frac{j}{2r+1}} y\right) \\ &= \frac{1 - y^{2r+1}}{1-y} \\ &= \sum_{n=0}^{2r} y^n. \end{aligned}$$

(2) From (2.9), we have

$$\begin{aligned} \sum_{n=0}^{\infty} h_n^{(2r)}(-\boldsymbol{\zeta}^{+\delta}, -\boldsymbol{\zeta}^{-\delta}) y^n &= \prod_{j=1}^{r} \frac{1}{\left(1 + e^{2\pi\sqrt{-1}\frac{j}{2r+1}} y\right)\left(1 + e^{-2\pi\sqrt{-1}\frac{j}{2r+1}} y\right)} \\ &= \frac{1+y}{1+y^{2r+1}} \\ &= \sum_{k=0}^{\infty} (-1)^k (y^{(2r+1)k} + y^{(2r+1)k+1}) \\ &= \sum_{n=0}^{\infty} \frac{1}{2}\left((-1)^{\lfloor \frac{n}{2r+1} \rfloor} - (-1)^{\lfloor \frac{n-2}{2r+1} \rfloor}\right) y^n. \end{aligned}$$

(3) By the definition of $p_n^{(2r)}$,

$$p_n^{(2r)}(-\boldsymbol{\zeta}^{+\delta}, -\boldsymbol{\zeta}^{-\delta}) = (-1)^n \left(-1 + \sum_{k=-r}^{r} \zeta_{2r+1}^{kn}\right) = (-1)^n(-1 + (2r+1)\delta_{2r+1|n}).$$

□

Remark 11. For (4.8), we give another proof without using (3.9). Let $x := z + z^{-1}$. First, we remark

$$\sum_{k=-r}^{r} z^k = \prod_{j=1}^{r} \left(x - 2\cos\left(\frac{2\pi j}{2r+1}\right)\right) = \sum_{m=0}^{r} e_m^{(r)}(-\boldsymbol{\zeta}^{+\delta} - \boldsymbol{\zeta}^{-\delta}) x^{r-m}.$$

On the other hand, if $|u| < |z| < |u|^{-1}$, then

$$\begin{aligned}
\sum_{r=0}^{\infty} u^r \sum_{k=-r}^{r} z^k &= \sum_{r=0}^{\infty} u^r \left(\frac{z^{-r}}{1-z} - \frac{z^{r+1}}{1-z} \right) \\
&= \frac{1}{1-z} \left(\frac{1}{1-z^{-1}u} - \frac{z}{1-zu} \right) \\
&= \frac{1+u}{(1-z^{-1}u)(1-zu)} \\
&= \frac{1+u}{1+u^2} \frac{1}{1-x\frac{u}{1+u^2}} \\
&= (1+u) \sum_{N=0}^{\infty} x^N u^N (1+u^2)^{-N-1} \\
&= (1+u) \sum_{N=0}^{\infty} x^N \sum_{k=0}^{\infty} \binom{-N-1}{k} u^{N+2k} \\
&= \sum_{r=0}^{\infty} u^r \left(\sum_{m=0}^{\lfloor \frac{r}{2} \rfloor} (-1)^m \binom{r-m}{m} x^{r-2m} + \sum_{m=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^m \binom{r-1-m}{m} x^{r-1-2m} \right).
\end{aligned}$$

Hence we obtain the conclusion (4.8).

From the Wronski, Newton relations and (4.8), we have the recursion formulas of $F_{n+1}^{(r)}$ and $L_n^{(r)}$.

Proposition 12. *For any non negative integer n,*

$$\sum_{j=0}^{\min(n,r)} (-1)^{\lfloor \frac{j-1}{2} \rfloor} \binom{r - \lfloor \frac{j+1}{2} \rfloor}{\lfloor \frac{j}{2} \rfloor} F_{n-j+1}^{(r)} = 0, \tag{4.11}$$

$$\begin{aligned}
&\sum_{j=0}^{\min(n,r)} (-1)^{n-\lfloor \frac{j+1}{2} \rfloor} \binom{r - \lfloor \frac{j+1}{2} \rfloor}{\lfloor \frac{j}{2} \rfloor} L_{n-j+1}^{(r)} \\
&= \begin{cases} (-1)^{\lfloor \frac{n+1}{2} \rfloor} (n+1) \binom{r-\lfloor \frac{n}{2} \rfloor - 1}{\lfloor \frac{n+1}{2} \rfloor} & (n = 0, 1, \ldots, r-1) \\ 0 & (n \geq r) \end{cases}.
\end{aligned} \tag{4.12}$$

We remark that a generalized Fibonacci numbers $\{F_{n+1}^{(r)}\}_{n \geq 0}$ are determined by the vanishing property (2.7)

$$F_0^{(r)} = \cdots = F_{-(r-2)}^{(r)} = 0 \tag{4.13}$$

and the initial condition $F_1^{(r)} = h_0^{(r)} = 1$, and the recursion formula

$$F_n^{(r)} = \sum_{j=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^j \binom{r-1-j}{j} F_{n-1-2j}^{(r)} + \sum_{j=0}^{\lfloor \frac{r-2}{2} \rfloor} (-1)^j \binom{r-1-j}{j+1} F_{n-2-2j}^{(r)}. \tag{4.14}$$

Example 13. $\underline{r=1}$ $\quad F_1^{(1)}=1,\ F_{n+1}^{(1)}=F_n^{(1)}.$

$$1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\dots$$

$\underline{r=2\text{ (Fibonacci numbers)}}$ $\quad F_0^{(2)}=0,\ F_1^{(2)}=1,\ F_{n+2}^{(2)}=F_{n+1}^{(2)}+F_n^{(2)}.$

$$1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,2584,4181,6765,10946,\dots$$

$\underline{r=3\text{ (OEIS A006053)}}$ $\quad F_{-1}^{(3)}=F_0^{(3)}=0,\ F_1^{(3)}=1,\ F_{n+3}^{(3)}=F_{n+2}^{(3)}+2F_{n+1}^{(3)}-F_n^{(3)}.$

$$1,1,3,4,9,14,28,47,89,155,286,507,924,1652,2993,5373,9707,17460,31501,56714,\dots$$

$\underline{r=4\text{ (OEIS A188021)}}$ $\quad F_{-2}^{(4)}=F_{-1}^{(4)}=F_0^{(4)}=0,\ F_1^{(4)}=1,$

$$F_{n+4}^{(4)}=F_{n+3}^{(4)}+3F_{n+2}^{(4)}-2F_{n+1}^{(4)}-F_n^{(4)}.$$

$$1,1,4,5,14,20,48,75,165,274,571,988,1988,3536,6953,12597,24396,44745,85786,\dots$$

$\underline{r=5\text{ (OEIS A231181)}}$ $\quad F_{-3}^{(5)}=F_{-2}^{(5)}=F_{-1}^{(5)}=F_0^{(5)}=0,\ F_1^{(5)}=1,$

$$F_{n+5}^{(5)}=F_{n+4}^{(5)}+4F_{n+3}^{(5)}-3F_{n+2}^{(5)}-3F_{n+1}^{(5)}+F_n^{(5)}.$$

$$1,1,5,6,20,27,75,110,275,429,1001,1637,3639,6172,13243,23104,48280,86090,\dots$$

$\underline{r=6}$ $\quad F_{-4}^{(6)}=F_{-3}^{(6)}=F_{-2}^{(6)}F_{-1}^{(6)}=F_0^{(6)}=0,\ F_1^{(6)}=1,$

$$F_{n+6}^{(6)}=F_{n+5}^{(6)}+5F_{n+4}^{(6)}-4F_{n+3}^{(6)}-6F_{n+2}^{(6)}+3F_{n+1}^{(6)}+F_n^{(6)}.$$

$$1,1,6,7,27,35,110,154,429,637,1638,2548,6188,9995,23255,38741,87190,149017,\dots$$

On the other hand, calculation of a generalized Lucas numbers $\{L_n^{(r)}\}_{n\geq 0}$ is more complicated. First, from the initial condition $L_0^{(r)}=p_0^{(r)}=r$ and (4.12), we calculate $L_1^{(r)},\ \dots,\ L_{r-1}^{(r)}$. Next, by using the recursion

$$L_n^{(r)}=\sum_{j=0}^{\lfloor\frac{r-1}{2}\rfloor}(-1)^j\binom{r-1-j}{j}L_{n-1-2j}^{(r)}+\sum_{j=0}^{\lfloor\frac{r-2}{2}\rfloor}(-1)^j\binom{r-1-j}{j+1}L_{n-2-2j}^{(r)} \tag{4.15}$$

and values of $L_1^{(r)},\ \dots,\ L_{r-1}^{(r)}$, we obtain the following examples.

Example 14. $\underline{r=1}$ $\quad L_0^{(1)}=1,\ L_{n+1}^{(1)}=L_n^{(1)}.$

$$1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\dots$$

$\underline{r=2\text{ (Lucas numbers)}}$ $\quad L_0^{(2)}=2,\ L_1^{(2)}=1,\ L_{n+2}^{(2)}=L_{n+1}^{(2)}+L_n^{(2)}.$

$$2,1,3,4,7,11,18,29,47,76,123,199,322,521,843,1364,2207,3571,5778,9349,\dots$$

$\underline{r = 3 \text{ (OEIS A096975)}}$ $\quad L_0^{(3)} = 3,\ L_1^{(3)} = 1,\ L_2^{(3)} = 5,\ L_{n+3}^{(3)} = L_{n+2}^{(3)} + 2L_{n+1}^{(3)} - L_n^{(3)}.$

$$3, 1, 5, 4, 13, 16, 38, 57, 117, 193, 370, 639, 1186, 2094, 3827, 6829, 12389, 22220, 40169, \dots$$

$\underline{r = 4 \text{ (OEIS A094649)}}$ $\quad L_0^{(4)} = 4,\ L_1^{(4)} = 1,\ L_2^{(4)} = 7,\ L_3^{(4)} = 4,$

$$L_{n+4}^{(4)} = L_{n+3}^{(4)} + 3L_{n+2}^{(4)} - 2L_{n+1}^{(4)} - L_n^{(4)}.$$

$$4, 1, 7, 4, 19, 16, 58, 64, 187, 247, 622, 925, 2110, 3394, 7252, 12289, 25147, 44116, 87727, \dots$$

$\underline{r = 5 \text{ (OEIS A189234)}}$ $\quad L_0^{(5)} = 5,\ L_1^{(5)} = 1,\ L_2^{(5)} = 9,\ L_3^{(5)} = 4,\ L_4^{(5)} = 25,$

$$L_{n+5}^{(5)} = L_{n+4}^{(5)} + 4L_{n+3}^{(5)} - 3L_{n+2}^{(5)} - 3L_{n+1}^{(5)} + L_n^{(5)}.$$

$$5, 1, 9, 4, 25, 16, 78, 64, 257, 256, 874, 1013, 3034, 3953, 10684, 15229, 38017, 58056, \dots$$

$\underline{r = 6}$ $\quad L_0^{(6)} = 6,\ L_1^{(6)} = 1,\ L_2^{(6)} = 11,\ L_3^{(6)} = 4,\ L_4^{(6)} = 31,\ L_5^{(6)} = 16,$

$$L_{n+6}^{(6)} = L_{n+5}^{(6)} + 5L_{n+4}^{(6)} - 4L_{n+3}^{(6)} - 6L_{n+2}^{(6)} + 3L_{n+1}^{(6)} + L_n^{(6)}.$$

$$6, 1, 11, 4, 31, 16, 98, 64, 327, 256, 1126, 1024, 3958, 4083, 14116, 16189, 50887, 63768, \dots$$

By comparing the table of $c_{n,k}$ and the above examples of $\{F_{n+1}^{(r)}\}_{n\geq 0}$, we find some relations between special Kostka numbers $c_{n,k}$ and our generalized Fibonacci numbers $F_{n+1}^{(r)}$. Actually, from our main results Theorem 5, Theorem 7 and Proposition 10, we obtain the following formulas.

Theorem 15. (1) *For $n = 0, 1, \dots, 2r$,*

$$\sum_{k=\max\{\lfloor \frac{n-r}{2} \rfloor, 0\}}^{\lfloor \frac{n}{2} \rfloor} \binom{r-n+2k}{k} \binom{n-2k-r-1}{\lfloor \frac{n}{2} \rfloor - k} = 1. \tag{4.16}$$

(2) *For any non negative integers n and m, we have*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k+r-1}{k} F_{n-2k+1}^{(r)} = \begin{cases} 1 & (n \equiv 0, 1 \pmod{2(2r+1)}) \\ -1 & (n \equiv 2r+1, 2r+2 \pmod{2(2r+1)}) \\ 0 & (\text{others}) \end{cases}, \tag{4.17}$$

$$F_{m+1}^{(r)} = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{2} \left((-1)^{\lfloor \frac{m-2k}{2r+1} \rfloor} - (-1)^{\lfloor \frac{m-2k-2}{2r+1} \rfloor} \right) c_{m+r-1,k}. \tag{4.18}$$

(3) *For any positive integer n,*

$$2 \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{2k-n-1}{k} L_{n-2k}^{(r)} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2k-n}{k} L_{n-2k}^{(r)} = \begin{cases} (-1)^{n-1} & (2r+1 \nmid n) \\ (-1)^n 2r & (2r+1 \mid n) \end{cases}. \tag{4.19}$$

For any non negative integer m,

$$L_m^{(r)} = (-1)^{m+1}2^{m-1} + (-1)^m \frac{2r+1}{2} \sum_{k=0}^{m} \binom{m}{k} \delta_{2r+1|m-2k}. \tag{4.20}$$

In particular, we have

$$L_{2m}^{(r)} = -2^{2m-1} + \frac{2r+1}{2} \sum_{k=-\lfloor \frac{m}{2r+1} \rfloor}^{\lfloor \frac{m}{2r+1} \rfloor} \binom{2m}{m-(2r+1)k}, \tag{4.21}$$

$$L_{2m+1}^{(r)} = 4^m - \frac{2r+1}{2} \sum_{k=-\lfloor \frac{m+r+1}{2r+1} \rfloor}^{\lfloor \frac{m-r}{2r+1} \rfloor} \binom{2m+1}{m-(2r+1)k-r}. \tag{4.22}$$

From specializations of Theorem 15, we obtain the following intersting binomial sum formulas and others.

Corollary 16. *For any non negative integers n and m, we have*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} = \begin{cases} 1 & (n \equiv 0,1 \pmod 6) \\ -1 & (n \equiv 2r+1, 2r+2 \pmod 6) \\ 0 & (\text{others}) \end{cases}, \tag{4.23}$$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k+1}{k} F_{n-2k+1} = \begin{cases} 1 & (n \equiv 0,1 \pmod{10}) \\ -1 & (n \equiv 2r+1, 2r+2 \pmod{10}) \\ 0 & (\text{others}) \end{cases} \tag{4.24}$$

and

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{2} \left((-1)^{\lfloor \frac{m-2k}{3} \rfloor} - (-1)^{\lfloor \frac{m-2k-2}{3} \rfloor} \right) c_{m,k} = 1, \tag{4.25}$$

$$\sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{2} \left((-1)^{\lfloor \frac{m-2k}{5} \rfloor} - (-1)^{\lfloor \frac{m-2k-2}{5} \rfloor} \right) c_{m+1,k} = F_{m+1}. \tag{4.26}$$

Corollary 17. (1) *For any positive integer n, we have*

$$2\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{2k-n-1}{k} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2k-n}{k} = \begin{cases} (-1)^{n-1} & (3 \nmid n) \\ (-1)^n 2 & (3 \mid n) \end{cases}, \tag{4.27}$$

$$2\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{2k-n-1}{k} L_{n-2k} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2k-n}{k} L_{n-2k} = \begin{cases} (-1)^{n-1} & (5 \nmid n) \\ (-1)^n 4 & (5 \mid n) \end{cases}. \tag{4.28}$$

For any non negative integer m, we have

$$\frac{3}{2}\sum_{k=-\lfloor\frac{m}{3}\rfloor}^{\lfloor\frac{m}{3}\rfloor}\binom{2m}{m-3k}=2^{2m-1}+1, \tag{4.29}$$

$$\frac{3}{2}\sum_{k=-\lfloor\frac{m+2}{3}\rfloor}^{\lfloor\frac{m-1}{3}\rfloor}\binom{2m+1}{m-3k-1}=3\sum_{k=0}^{\lfloor\frac{m-1}{3}\rfloor}\binom{2m+1}{m-3k-1}=4^m-1, \tag{4.30}$$

$$L_{2m}=-2^{2m-1}+\frac{5}{2}\sum_{k=-\lfloor\frac{m}{5}\rfloor}^{\lfloor\frac{m}{5}\rfloor}\binom{2m}{m-5k}, \tag{4.31}$$

$$L^{(r)}_{2m+1}=4^m-\frac{5}{2}\sum_{k=-\lfloor\frac{m+3}{5}\rfloor}^{\lfloor\frac{m-2}{5}\rfloor}\binom{2m+1}{m-5k-2}. \tag{4.32}$$

(2) *If $m<2r+1$, then*

$$L^{(r)}_{2m}=-2^{2m-1}+\frac{2r+1}{2}\binom{2m}{m}. \tag{4.33}$$

(3) *If $m<r$, then*

$$L^{(r)}_{2m+1}=4^m. \tag{4.34}$$

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