### GROTHENDIECK'S DESSINS D'ENFANTS IN A WEB OF DUALITIES. II.

#### JIAN ZHOU

ABSTRACT. We show that the spectral curve for Eynard-Orantin topolgical recursions satisfied by counting Grothendieck's dessins d'enfants are related to Narayana numbers. This suggests a connection of dessins to combinatorics of Coxeter groups, noncrossing partitions, free probability theory, and cluster algebras.

#### 1. Introduction

In the first part [20] of this series of papers, we have proposed to study dualities of different topological quantum field theories using symmetries of the KP hierarchy. Such theories involve infinitely many formal variables, so it is natural to study them altogether from a point of view of infinite-dimensional geometry. One of the most developed approaches to infinite-dimensional geometry is the theory of integrable hierarchies. The Kyoto school's approach to integrable hierarchies focuses on their symmetries, described by the actions infinite-dimensional Lie groups or Lie algebras. In this approach the KP hierarchy is universal in the sense that its symmetries are described by  $\widehat{GL}(\infty)$ , into which the symmetry groups for other integrable hierarchies can be embedded.

From this point of view, different theories have equal status. Once the partition function of a particular theory is identified as a taufunction of the KP hierarchy, it can be transformed into the partition function of any other theory, which is also a tau-function of the KP hierarchy. This is very crucial to our understanding of dualities of different theories, because the constructions of two theories may be totally unrelated, but they should share exactly the same properties if they are dual to each other in our sense.

However, the democracy among different theories does not mean that some specific theories cannot play more important roles than the others. The democracy in this setting only means that before some further studies, each theory is just a candidate for some selection process after which one may or may not determine some specific theories that will play a unifying and leading role in the future studies. One of the goals of this series of papers is to show that counting of Grothendieck's dessins

of d'enfants plays such a role. In Part I of this series, we have focused on some enumeration problems. We discover that various enumeration problems are either special cases of the enumeration of dessins, or are directly related to it. We have seen that their generating series are tau-functions of the KP hierarchy. These tau-functions have very special properties. For example, they satisfy suitable versions of Virasoro constraints, and admit some cut-and-join representations. It is often possible to find explicit formulas for the bases of the elements in Sato's Grassmannian corresponding to these tau-functions, sometimes with the help of representation theory. With these bases available one can find the Kac-Schwarz operators. Furthermore, one can use the first basis vectors to write down the corresponding quantum spectral curves. To make contact with mirror symmetry, we have reformulated the differential equations involved in these examples of quantum spectral curves as differential equations of hypergeometric type.

In this Part II, we will still focus on the examples considered in Part I, but we will examine them further from the point of view of emergent geometry of KP hierarchy developed by the author in earlier works.

Emergence is an notion developed in statistical physics. Roughly it means the appearance of some structures when one deals with a system of very large degrees of freedom. We have borrowed this notion to describe some results in the study of Gromov-Witten type theories. Such a theory has a finite-dimensional phase space consisting of the primary operators. This space is called the small phase space. Intrinsic to a Gromov-Witten type theory is its coupling to 2D topological gravity. As a result, each primary operator has an infinite sequence of associated operators called its gravitational descendants. The big phase space, i.e. the space that includes all the primaries and their descendants, is then infinite-dimensional. We refer to the geometric structures that can be more easily seen from the big phase space than from the small phase space as the emergent geometry. We emphasize that such structures also exists over the small phase space, but it is more natural to think of them as the restrictions of some structures over the big phase space.

The emergent approach to the study of a Gromov-Witten type theory is then to look for some structures over the big phase space and consider their restrictions to the small phase space to get back to a finite-dimensional setting. This is in contrast to the reduction/reconstruction approach which starts with some structures on the small phase space and tries to reconstruct the whole theory based on some axioms that characterize such theories.

In earlier papers, we have shown that in some examples of Gromov-Witten type theories, the following structures emerge: a plane algebraic curve called the spectral curve, a family of deformations of this curve defined over the big phase space called its special deformation, and a conformal field theory defined over the spectral curve. More recently, it has been realized that Eynard-Orantin topological recursions can also be understood as emergent geometry. In summary, emergent geometry is a synthesis of ideas from mirror symmetry, Witten Conjecture/Kontsevich Theorem, and Eynard-Orantin topological recursions.

It is possible to study the emergent geometry of any Gromov-Witten type theory whose partition function is a tau-function of the KP hierarchy [15]. It is particularly simple when the partition function also satisfies the Virasoro constraints. The original motivation was to understand the mirror symmetry of a point, more precisely, why the n-point functions associated with the Witten-Kontsevich tau-function [12, 10] satisfied the Eynard-Orantin topological recursion on the Airy curve  $y^2 = x$ , a result proved many times by different approach in e.g. [7, 3, 13]. The proof by the author in [13] indicates that the Eynard-Orantin topological recursion in this case is equivalent to the Virasoro constraints derived by Dijkgraaf-Verlinde-Verlinde [4], and spectral curve and the Bergman kernel are determined by making sense of genus zero one-point function and the genus zero two-point function respectively. The emergent geometry in this sense of the modified partition functions of Hermitian one-matrix models with even coupling constants has been studied in [19]. The same has been done for partition functions of Hermitian one-matrix models in [21]. More recently, in Part I of this series [20] we have discovered that the enumeration of Grothendieck's dessins d'enfants lies in the center of a web of dualities of various enumerative problems, including the above three cases. One of the goals of this paper is to present the emergent geometry of Grothendick's dessins d'enfants in the same fashion as in the above three cases.

It may seem redundant to consider the Eynard-Orantin topological recursions in these cases because it is concerned with the computations of genus g, n-point functions recursively, but on the other hand we have already obtained a formula for all n-point functions, summed over all genera, in [20] based on the general results for tau-functions of the KP hierarchy developed in [15]. Our reason is that these different approaches yield different formulas that encode different information. The formulas in Part I encode information related to problems in representation theory, while the formulas in this Part encode the information related to combinatorial problems. Our second goal of this paper

is to present some unexpected connection between counting dessins and Narayana numbers. More precisely, the genus zero one-point function of Grothendieck's dessins is the generating function of the Narayana numbers. Because the latter is related to cluster algebras, root systems, Coexter groups, associahedra, etc, see e.g. [8], this suggests a connections from dessins to these mathematical objects. In particular, because Grothendieck's dessins correspond to Type A root systems, our results suggest the possibility of dessin theory of Type BCD. Note in our previous works on emergent geometry, we encounter only Catalan numbers. Because Narayana numbers refine Catalan numbers, we get another evidence that counting dessins is more fundamental than counting fat graphs (aka clean dessins).

Derivation of topological recursion from Virasoro constraints for counting Grothendieck' dessins has already been given by Kazarian and Zograf [9]. But their version is not exactly the version of Eynard and Orantin [7]. They have remarked that one can derive the latter version from their version. We will present a direct derivation of the Eynard-Orantin version in the same fashions as in [19, 21]. Our motivation is, first of all, to interpret the Eynard-Orantin recursions in these cases as emergent geometry. Secondly, as a consequence, we establish a connection between counting dessins and intersection theory on  $\overline{\mathcal{M}}_{g,n}$ .

We arrange the rest of the paper in the following way. In Section 2 we recall some earlier results on emergent geometry of some enumeration problems and the appearance of Catalan numbers in their genus zero one-point functions. We show how to compute the n-point functions in arbitrary genera for dessin tau-function using the dessin Virasoro constraints in Section 3. In particular, the genus zero one-point function in this case is shown to be the generating series of the Narayana numbers. The Eynard-Orantin topological recursions are shown to be equivalent to the dessin Virasoro constraints in Section 4. In Section 5 wee present some computations necessary for relating dessins to intersection numbers numbers using Eynard's results [6]. In he final Section 6 we present some concluding remarks.

### 2. Earlier Results on Emergent Geometry and Catalan Numbers

In this Section we recall some results on emergent geometry in the following three cases: Witten-Kontsevich tau-function, partition functions of Hermitian one-matrix models, and modified partition functions of Hermitian one-matrix models with even coupling constants. We will focus on the genus zero one-point functions and two-point functions.

We note that the genus zero one-point functions in these three cases are all related to the Catalan numbers.

### 2.1. Emergent geometry of Witten-Kontsevich tau-function.

The first example of emergent geometry of Gromov-Witten type theories appeared in [14], based on [2]. In genus zero this means the construction of the special deformation of the Airy curve defined using the genus zero one-point function associated to the Witten-Kontsevich tau-function [14]. In higher genera, this means an equivalent reformulation of the Virasoro constraints in terms of quantization of a field on the Airy curve [14], or Eynard-Orantin recursions on the Airy curve [13].

The relevant correlators are

(1) 
$$\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}.$$

The genus q free energy is defined by

(2) 
$$F_g = \sum_{n} \frac{t_{a_1} \cdots \tau_{a_n}}{n!} \langle \tau_{a_1} \cdots \tau_{a_n} \rangle_g,$$

where the summation is taken over n such that 2g - 2 + n > 0. The Virasoro constraints in terms of the free energy are given by:

(3) 
$$\frac{\partial F^{WK}}{\partial g_0} = \sum_{k=1}^{\infty} (2k+1)g_k \frac{\partial F^{WK}}{\partial g_{k-1}} + \frac{g_0^2}{2\lambda^2},$$

(4) 
$$\frac{\partial F^{WK}}{\partial g_1} = \sum_{k=0}^{\infty} (2k+1)g_k \frac{\partial F^{WK}}{\partial g_k} + \frac{1}{8},$$

(5) 
$$\frac{\partial F^{WK}}{\partial u_n} = \sum_{k=0}^{\infty} (2k+1)g_k \frac{\partial F^{WK}}{\partial g_{k+n-1}} + \frac{\lambda^2}{2} \sum_{k=0}^{n-2} \left( \frac{\partial^2 F^{WK}}{\partial g_k \partial g_{n-k-2}} + \frac{\partial F^{WK}}{\partial g_k} \frac{\partial F^{WK}}{\partial g_{n-k-2}} \right), \quad n \ge 2.$$

Usually  $\{t_n\}_{n\geq 0}$  are used to denote the coupling constants. Here we have made the following change of coordinates:

(6) 
$$t_k = (2k+1)!!g_k.$$

Correspondingly, the operator  $\tau_n$  is changed to  $\sigma_n = (2n+1)!!\tau_n$ .

Because  $\mathcal{M}_{0,1}$  and  $\mathcal{M}_{0,2}$  are not defined, the genus zero one-point function and two-point function need special care. One approach, taken in [13] and elaborated further in [15], is to introduce ghost operators  $\tau_{-n}$  and ghost variables  $t_{-n}$ . Another approach, taken in [14], is to

consider the theory in a different background, i.e., with nonzero  $g_0$ . To compute the genus zero one-point function in this background, one then needs to compute:

$$\langle \sigma_n e^{g_0 \sigma_0} \rangle_0 = \frac{g_0^{n+2}}{(n+2)!} \langle \sigma_0^{n+2} \sigma_n \rangle_0 = \frac{(2n+1)!!}{(n+2)!} g_0^{n+2}$$
$$= \frac{1}{n+2} {2n+2 \choose n+1} \cdot \frac{g_0^{n+2}}{2^{n+1}},$$

where

(7) 
$$C_n = \frac{1}{n+1} \binom{2n}{n}$$

are the *Catalan numbers*. We note the above result can be reformulated as follows:

(8) 
$$z(1 - \frac{2g_0}{z^2})^{1/2} = z - \frac{g_0}{z} - \sum_{n=0}^{\infty} \frac{\partial F_0}{\partial g_n} (g_0, 0, \dots) \cdot z^{-(2n+3)}.$$

So we define the genus zero one-point function associated to Witten-Kontsevich tau-function by:

(9) 
$$G_{0,1}^{WK}(z;g_0) = \sum_{n=0}^{\infty} \frac{\partial F_0}{\partial g_n}(g_0,0,\dots) \cdot z^{-(2n+3)} = z - \frac{g_0}{z} - z(1 - \frac{2g_0}{z^2})^{1/2}.$$

In [14], motivated by (8), we consider the following series:

(10) 
$$x = z - \sum_{n>0} (2n+1)g_n z^{2n-1} - \sum_{n>0} \frac{\partial F_0}{\partial g_n}(\mathbf{g}) \cdot z^{-2n-3}.$$

By Virasoro constraints one has

$$x^{2} = 2y \left(1 - \sum_{n \geq 1} (2n+1)g_{n}(2y)^{n-1}\right)^{2}$$

$$-2g_{0} \left(1 - \sum_{n \geq 1} (2n+1)g_{n}(2y)^{n-1}\right)$$

$$+2\sum_{n \geq 0} \sum_{k \geq n+2} (2k+1)g_{k} \cdot \frac{\partial F_{0}}{\partial g_{n}} \cdot (2y)^{k-n-2},$$

where  $y = \frac{1}{2}z^2$ . In fact, the following equation is equivalent to Virasoro constraints for  $F_0^{WK}$ :

$$(12) (x^2)_- = 0.$$

Here for a formal series  $\sum_{n\in\mathbb{Z}} a_n z^n$ ,

$$(13) \qquad (\sum_{n \in \mathbb{Z}} a_n z^n)_+ = \sum_{n > 0} a_n z^n, \quad (\sum_{n \in \mathbb{Z}} a_n z^n)_- = \sum_{n < 0} a_n z^n.$$

The genus zero two-point correlators are given by:

$$\langle \sigma_k \sigma_l e^{g_0 \sigma_0} \rangle_0 = \frac{g_0^{k+l+1}}{(k+l+1)!} \langle \sigma_k \sigma_l \sigma_0^{k+l+1} \rangle_0 = \frac{(2k+1)!!(2l+1)!!}{k!l!(k+l+1)} g_0^{k+l+1}.$$

The genus zero two-point function in this case is defined by:

$$G_{0,2}(z_1, z_2; g_0) = \sum_{n_1, n_2 = 0}^{\infty} \frac{\partial^2 F_0}{\partial g_{n_1} \partial g_{n_2}} (g_0, 0, \dots) \cdot z_1^{-(2n_1 + 3)} z_2^{-(2n_2 + 3)}$$

$$= \sum_{k, l = 0}^{\infty} \frac{(2k + 1)!!(2l + 1)!!}{k!l!(k + l + 1)} g_0^{k+l+1} \cdot z_1^{-(2k+3)} z_2^{-(2l+3)}.$$

To take the summation over k and l, we first take derivative in  $g_0$  to

$$\frac{\partial}{\partial g_0} G_{0,2}(z_1, z_2; g_0) = \sum_{k,l=0}^{\infty} \frac{(2k+1)!!(2l+1)!!}{k!l!} g_0^{k+l} \cdot z_1^{-(2k+3)} z_2^{-(2l+3)}$$

$$= \sum_{k=0}^{\infty} \frac{(2k+1)!!}{k!} g_0^k z_1^{-(2k+3)} \cdot \sum_{l=0}^{\infty} \frac{(2l+1)!!}{l!} g_0^l z_2^{-(2l+3)}$$

$$= \frac{1}{(z_1^2 - 2g_0)^{3/2} (z_2^2 - 2g_0)^{3/2}},$$

and so after integration we get:

(14) 
$$G_{0,2}(z_1, z_2; g_0) = \frac{z_1^2 + z_2^2 - 4g_0}{(z_1^2 - z_2^2)^2 (z_1^2 - 2g_0)^{1/2} (z_2^2 - 2g_0)^{1/2}} - \frac{z_1^2 + z_2^2}{(z_1^2 - z_2^2)^2 z_1 z_2}.$$

2.2. Emergent geometry of Hermitian one-matrix models. We refer to [16], [17], [18] for notations and proofs of the results in this Subsection. Roughly speaking, with the introduction of the t' Hooft coupling constant  $t = Ng_s$ , one gets the free energy functions  $F_g(\mathbf{t})$ from the fat genus expansions of Hermitian one-matrix models. The genus zero fat Virasoro constraints can be reformulated as follows: The series

$$y = \frac{1}{2} \sum_{n=0}^{\infty} \frac{t_n - \delta_{n,1}}{n!} x^n + \frac{t}{x} + \sum_{n=1}^{\infty} \frac{n!}{x^{n+1}} \frac{\partial F_0(\mathbf{t})}{\partial t_{n-1}}$$

is a solution to the equation

$$(15) (y^2)_- = 0$$

When all  $t_n$  are taken to be 0,

$$y = -\frac{x}{2} + \frac{t}{x} + \frac{t^2}{x^3} + \frac{2t^3}{x^5} + \frac{5t^4}{x^7} + \frac{14t^5}{x^9} + \cdots$$
$$= -\frac{x}{2} + \sum_{n=1}^{\infty} \frac{1}{n} {2n-2 \choose n-1} \frac{t^n}{x^{2n-1}},$$

so we again encounter Catalan numbers. Note

(16) 
$$y = \frac{-\sqrt{x^2 - 4t}}{2},$$

and so the spectral curve in terms of the field y = y(z) is given by the algebraic curve:

$$(17) x^2 - 4y^2 = 4t.$$

Define the genus zero one-point function in this case by:

(18) 
$$G_{0,1}(x) = \frac{t}{x} + \sum_{m>1} x^{-m-1} \cdot m \frac{\partial F_g}{\partial g_m} \Big|_{g_i=0},$$

then the above result shows that

(19) 
$$G_{0,1}(x) = \frac{1}{2}(x - \sqrt{x^2 - 4t}).$$

In terms of the correlators,

(20) 
$$\langle p_m \rangle_0^c(t) = \begin{cases} \frac{1}{n+1} {2n \choose n} t^{n+1} & \text{if } m = 2n, \\ 0 & \text{if } m = 2n+1. \end{cases}$$

The genus zero two-point function is computed in [21] to be:

(21) 
$$G_{0,2}(x_1, x_2) = \frac{x_1 x_2 - 4t}{2(x_1 - x_2)^2 \sqrt{(x_1^2 - 4t)(x_2^2 - 4t)}} - \frac{1}{2(x_1 - x_2)^2}.$$

It has the following expansion:

(22) 
$$G_{0,2}(x_1, x_2) = \sum_{m,n=0}^{\infty} \frac{t^{m+n+1}}{m+n+1} \frac{(2m+1)!}{(m!)^2} \cdot \frac{(2n+1)!}{(n!)^2} \frac{1}{x_1^{2m+2} x_2^{2n+2}} + \sum_{m,n=0}^{\infty} \frac{4t^{m+n+2}}{m+n+2} \frac{(2m+1)!}{(m!)^2} \cdot \frac{(2n+1)!}{(n!)^2} \frac{1}{x_1^{2m+3} x_2^{2n+3}}.$$

In terms of correlators we have

(23) 
$$\langle p_{2m+1}p_{2n+1}\rangle_0^c(t) = \frac{(2m+1)!}{(m!)^2} \cdot \frac{(2n+1)!}{(n!)^2} \cdot \frac{t^{m+n+1}}{m+n+1},$$

(24) 
$$\langle p_{2m+2}p_{2n+2}\rangle_0^c(t) = \frac{(2m+1)!}{(m!)^2} \cdot \frac{(2n+1)!}{(n!)^2} \cdot \frac{4t^{m+n+2}}{m+n+2},$$

and other genus zero two-point correlators vanish.

2.3. Emergent geometry of modified partition function. Now we recall the case of modified partition function of Hermitian onematrix model with even couplings. For details, see [19]. The Virasoro constraints in genus zero are the following sequence of differential equations:

(25) 
$$\frac{1}{2} \frac{\partial F_0}{\partial s_2} = \sum_{k>1} k s_{2k} \frac{\partial F_0}{\partial s_{2k}} + \frac{t^2}{4}$$
,

$$(26) \frac{1}{2} \frac{\partial F_0}{\partial s_{2n+2}} = \sum_{k=1}^{n-1} \frac{\partial F_0}{\partial s_{2k}} \frac{\partial F_0}{\partial s_{2n-2k}} + t \frac{\partial F_0}{\partial s_{2n}} + \sum_{k \ge 1} k s_{2k} \frac{\partial F_0}{\partial s_{2k+2n}}, \quad n \ge 1.$$

In this case, we consider:

(27) 
$$y := \frac{1}{2} \sum_{k=1}^{\infty} k(s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-1} + \frac{t}{2x} + \sum_{k=1}^{\infty} \frac{1}{x^{k+1}} \frac{\partial F_0}{\partial s_{2k}}.$$

Then by the Virasoro constraints above:

(28) 
$$(y^2)_- = \left(\frac{1}{2}t(s_2 - \frac{1}{2}) + \sum_{l \ge 1}(l+1)s_{2l+2}\frac{\partial F_0}{\partial s_{2l}}\right)x^{-1},$$

and so

$$y^{2} = \frac{1}{4} \left( \sum_{k=1}^{\infty} k(s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-1} \right)^{2}$$

$$+ \frac{t}{2} \sum_{k=1}^{\infty} k(s_{2k} - \frac{1}{2} \delta_{k,1}) x^{k-2} + \sum_{l \ge 1} \sum_{k \ge l+1} k s_{2k} \frac{\partial F_{0}}{\partial s_{2l}} x^{k-l-2}.$$

It follows that when all  $s_{2k} = 0$ , we get the spectral curve:

$$(29) y^2 = \frac{1}{16} - \frac{t}{4x}.$$

The one-point function in genus zero is:

(30) 
$$G_{0,1}(x) = \frac{1}{4} \left( 1 - \frac{2t}{x} - \sqrt{1 - \frac{4t}{x}} \right).$$

It has the following expansion:

(31) 
$$G_{0,1}(x) = \frac{1}{4} \sum_{n \ge 2} \frac{(2n-3)!!}{n!} \cdot (2t)^n x^{-n} = \frac{1}{2} \sum_{n \ge 2} \frac{(2n-2)!}{(n-1)!n!} \frac{t^n}{x^n}.$$

The two-point function in genus zero is:

(32) 
$$G_{0,2}(x_1, x_2) = \frac{1 - \frac{2t}{x_1} - \frac{2t}{x_2}}{2(x_1 - x_2)^2 \sqrt{(1 - \frac{4t}{x_1})(1 - \frac{4t}{x_2})}} - \frac{1}{2(x_1 - x_2)^2}.$$

It has the following expansion:

(33) 
$$G_{0,2}(x_1, x_2) = 2 \sum_{l=2}^{\infty} \frac{t^l}{l} \sum_{m+n=l-2} \frac{(2m+1)!}{(m!)^2} \cdot \frac{(2n+1)!}{(n!)^2} \frac{1}{x_1^{m+2} x_2^{n+2}},$$

In terms of correlators we have

(34) 
$$\langle p_{2m}p_{2n}\rangle_0^c = \frac{1}{2} \cdot \frac{(2m)!}{(m-1)!m!} \cdot \frac{(2n)!}{(n-1)!n!} \cdot \frac{t^{m+n}}{m+n}.$$

## 3. N-Point Functions of Dessin Tau-Function by Virasoro Constraints

In this Section we show how to compute the *n*-point functions associated to the dessin tau-function by the dessin Virasoro constraints.

3.1. **Dessin** *n***-point functions.** Recall the genus g, n-point functions associated to the dessin free energy  $F_{Dessins}$  [20] is defined by:

(35) 
$$G^{(n)}(\xi_1, \dots, \xi_n) = \sum_{m_1, \dots, m_n \ge 1} \prod_{j=1}^n \frac{m_j}{\xi_j^{-m_j - 1}} \frac{\partial}{\partial p_{m_j}} F_{Dessins} \Big|_{p_m = 0, m \ge 1}$$

An explicit formula for these functions based on the theory of KP hierarchy have been proved in [20]. In particular,

$$G^{(1)}(\xi) = \sum_{n\geq 1} \frac{s^n uv}{n} \sum_{i+j=n-1} \frac{(-1)^j}{i!j!} \prod_{a=1}^i (u+a)(v+a)$$
$$\cdot \prod_{b=1}^j (u-b)(v-b) \cdot \xi^{-n-1}.$$

It is actually possible to take the summation  $\sum_{i+j=n-1}$  and simplify such a formula. In this Section we will show how to recursively compute the *n*-point functions using the dessin Virasoro constraints.

The n-point correlators are defined by:

(36) 
$$\langle p_{a_1} \cdots p_{a_n} \rangle_g^c := a_1 \cdots a_n \frac{\partial^n F_{g,dessin}}{\partial p_{a_1} \cdots \partial p_{a_n}} \bigg|_{p_k = 0, k \ge 0}.$$

Define the genus q, n-point function by

(37) 
$$G_{g,n}(x_1,\ldots,x_n) = \sum_{a_1,\ldots,a_n>0} \langle p_{a_1}\cdots p_{a_n}\rangle_g^c \frac{1}{x_1^{a_1+1}\cdots x_n^{a_n+1}}.$$

3.2. Virasoro constraints for dessin tau-function. Recall the Virasoro constraints for dessins are given by [9]: For  $n \geq 0$ ,

(38) 
$$L_{n} = -\frac{n+1}{s} \frac{\partial}{\partial p_{n+1}} + (u+v)n \frac{\partial}{\partial p_{n}} + \sum_{j=1}^{\infty} p_{j}(n+j) \frac{\partial}{\partial p_{n+j}} + \sum_{i+j=n} ij \frac{\partial^{2}}{\partial p_{i} \partial p_{j}} + \delta_{n,0} uv.$$

To keep track of the genus, introduce the genus tracking parameter. This corresponds to the following change of variables:

(39) 
$$s \mapsto \lambda s, u \mapsto \lambda^{-1}u, v \mapsto \lambda^{-1}v, p_n \mapsto \lambda^{-1}p_n.$$

The dessin Virasoro operators then take the following form:

(40) 
$$L_{n} = -\frac{n+1}{s} \frac{\partial}{\partial p_{n+1}} + (u+v)n \frac{\partial}{\partial p_{n}} + \sum_{j=1}^{\infty} p_{j}(n+j) \frac{\partial}{\partial p_{n+j}} + \lambda^{2} \sum_{i+j=n} ij \frac{\partial^{2}}{\partial p_{i} \partial p_{j}} + \delta_{n,0} uv \lambda^{-2}.$$

3.3. Virasoro constraints in terms of correlators. When expressed in terms of correlators, the Virasoro constraints can be expressed as follows. First,

$$\langle p_1 \rangle_0 = suv,$$

and for  $m \geq 0$ ,

$$\frac{m+1}{s} \langle p_{m+1} \cdot p_{a_1} \cdots p_{a_n} \rangle_g$$

$$= \sum_{j=1}^{n} (a_j + m) \cdot \langle p_{a_1} \cdots p_{a_j+m} \cdots p_{a_n} \rangle_g$$

$$+ (u+v)m \langle p_m \cdot p_{a_1} \cdots p_{a_n} \rangle_g$$

$$+ \sum_{k=1}^{m-1} k(m-k) \langle p_k p_{m-k} \cdot p_{a_1} \cdots p_{a_n} \rangle_{g-1}$$

$$+ \sum_{k=1}^{m-1} \sum_{\substack{g_1+g_2=g\\I_1\coprod I_2=[n]}} k \langle p_k \cdot \prod_{i\in I_1} p_{a_i} \rangle_{g_1} \cdot (m-k) \langle p_{m-k} \cdot \prod_{i\in I_2} p_{2a_i} \rangle_{g_2},$$

where  $[n] = \{1, ..., n\}.$ 

## 3.4. Genus zero one-point function by Virasoro constraints and Narayana polynomials. From

$$\langle p_1 \rangle_0 = suv,$$

$$m \langle p_m \rangle_0^c = s \sum_{k=1}^{m-2} k \langle p_k \rangle_0 \cdot (m-1-k) \langle p_{m-1-k} \rangle_0^c$$

$$+ s(u+v) \cdot (m-1) \langle p_{m-1} \rangle_0, \quad m \ge 2,$$

we get:

$$G_{0,1}(x) = \frac{suv}{x^2} + s \sum_{n \ge 2} \frac{1}{x^{n+1}} \left( \sum_{k=1}^{n-2} k \langle p_k \rangle_0^c \cdot (n-1-k) \langle p_{n-1-k} \rangle_0 \right)$$

$$+ (u+v) \cdot (n-1) \langle p_{n-1} \rangle_0$$

$$= \frac{suv}{x^2} + \frac{s(u+v)}{x} G_{0,1}(x) + s G_{0,1}(x)^2.$$

After solving for  $G_{0,1}(x)$  we then get:

(43) 
$$G_{0,1}(x) = \frac{1}{2s} \left( 1 - \frac{s(u+v)}{x} - \sqrt{\left(1 - \frac{s(u+v)}{x}\right)^2 - \frac{4s^2uv}{x^2}} \right)$$
$$= \frac{1}{2s} \left( 1 - \frac{s(u+v)}{x} - \sqrt{1 - \frac{2s(u+v)}{x} + \frac{s^2(u-v)^2}{x^2}} \right).$$

Now recall the following identity:

(44) 
$$= \sum_{n=1}^{\infty} \frac{z^{n+1}}{n} \sum_{k=1}^{n} {n \choose k} {n \choose k-1} u^{n+1-k} v^{k}.$$

It gives the generating series of the Narayana numbers

$$(45) N_{n,k} = \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1} = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.$$

They are A001263 on [11]. The polynomials

(46) 
$$N_n(q) := \sum_{k=1}^n N_{n,k} q^k$$

is called the *n*-th Narayana polynomial. The following are the first few terms of  $G_{0,1}(x)$ :

$$G_{0,1}(x) = \frac{suv}{x^2} + \frac{s^2uv(u+v)}{x^3} + \frac{s^3uv(u^2+3uv+v^2)}{x^4} + \frac{s^4uv(u^3+6u^2v+6uv^2+v^3)}{x^5} + \frac{s^5uv(u^4+10u^3v+20u^2v^2+10uv^3+v^4)}{x^6} + \cdots$$

In terms of the correlators, we have shown that:

$$(47) \qquad \langle p_n \rangle_0 = s^n u v \sum_{k=1}^n \frac{\binom{n}{k}}{n} \frac{\binom{n}{k-1}}{n} u^{n-k} v^{k-1} = \frac{1}{n} s^n u^{n+1} N_n(\frac{v}{u}).$$

It is well-known that the Narayana numbers provide a version of q-analogues of the Catalan numbers. By taking q=1 in the Narayana polynomials, one gets the Catalan numbers:

(48) 
$$\sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} = \frac{1}{n+1} \binom{2n}{n}.$$

See e.g. [8] and [1] for references to related mathematical objects.

# 3.5. Computation of genus zero two-point function by Virasoro constraints. By equation (42),

$$\langle p_1 \cdot p_n \rangle_0 = sn \cdot \langle p_n \rangle_0,$$

and so

$$\sum_{n=1}^{\infty} n \langle p_1 \cdot p_n \rangle_0 \frac{1}{x_2^{n+1}} = \sum_{n \ge 1} sn \cdot \langle p_n \rangle_0 \frac{n}{x_2^{n+1}}$$

$$= -\frac{d}{dx_2} \left( sx_2 \sum_{n \ge 1} \langle p_n \rangle_0 \frac{n}{x_2^{n+1}} \right)$$

$$= -\frac{d}{dx_2} \left( sx_2 \cdot \frac{1}{2s} \left( 1 - \frac{s(u+v)}{x_2} - \sqrt{1 - \frac{2s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_2^2}} \right) \right)$$

$$= \frac{1 - \frac{s(u+v)}{x_2}}{2\sqrt{1 - \frac{2s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_2^2}}} - \frac{1}{2}.$$

Now recall the following combinatorial identity:

(49) 
$$\frac{1}{2} \frac{1 - (u+v)z}{\sqrt{1 - 2(u+v)z + (u-v)^2 z^2}} - \frac{1}{2}$$

$$= \sum_{n=1}^{\infty} z^{n+1} \sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} u^{n+1-k} v^k$$

$$= \sum_{n=1}^{\infty} n z^{n+1} u^{n+1} N_n(\frac{v}{u}).$$

It gives the generating series of A132812 on [11]. The first few terms are:

$$\sum_{n=1}^{\infty} n \langle p_1 \cdot p_n \rangle_0 \frac{1}{x_2^{n+1}} = \frac{uvs^2}{x_2^2} + \frac{2uv(u+v)s^3}{x_2^3} + \frac{3uv(u^2 + 3uv + v^2)s^4}{x_2^4} + \frac{4uv(u^3 + 6u^2v + 6uv^2 + v^3)s^5}{x_2^5} + \cdots$$

Next we consider the following equation:

$$(50) 2\langle p_2 \cdot p_n \rangle_0 = s(n+1) \cdot \langle p_{n+1} \rangle_0 + s(u+v) \cdot \langle p_1 \cdot p_n \rangle_0.$$

By taking generating series, one gets:

$$\sum_{n\geq 1} 2n\langle p_2 \cdot p_n \rangle_0 \frac{1}{x_2^{n+1}}$$

$$= \sum_{n\geq 1} \frac{n}{x_2^{n+1}} \left( s(n+1) \cdot \langle p_{n+1} \rangle_0 + s(u+v) \cdot \langle p_1 \cdot p_n \rangle_0 \right)$$

$$= s \sum_{n\geq 2} \frac{n(n-1)}{x_2^n} \langle p_n \rangle_0 + s(u+v) \sum_{n\geq 1} \frac{n}{x_2^{n+1}} \cdot \langle p_1 \cdot p_n \rangle_0^c$$

$$= -s \frac{d}{dx_2} \left( x_2^2 \sum_{n\geq 2} \frac{n}{x_2^{n+1}} \langle p_n \rangle_0^c \right) + s(u+v) \sum_{n\geq 1} \langle p_1 \cdot p_n \rangle_0^c \frac{1}{x_2^{n+1}}$$

$$= -s \frac{d}{dx_2} \left[ x_2^2 \cdot \frac{1}{2s} \left( 1 - \frac{s(u+v)}{x_2} - \sqrt{1 - \frac{2s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_2^2}} \right) \right]$$

$$+ s(u+v) \cdot \left( \frac{1 - \frac{s(u+v)}{x_2}}{2\sqrt{1 - \frac{2s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_2^2}}} - \frac{1}{2} \right)$$

$$= x_2 \left( \frac{1 - \frac{s(u+v)}{x_2}}{\sqrt{1 - \frac{2s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_2^2}}} - 1 \right) - \frac{\frac{2s^2uv}{x_2}}{\sqrt{1 - \frac{2s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_2^2}}}.$$

Recall the following combinatorial identity:

(51) 
$$\frac{1}{\sqrt{1-2(u+v)z+(u-v)^2z^2}} = \sum_{n=0}^{\infty} z^n \sum_{k=0}^n \binom{n}{k}^2 u^{n-k} v^k.$$

Therefore,

$$\sum_{n\geq 1} 2n \langle p_2 \cdot p_n \rangle_0^c \frac{1}{x_2^{n+1}}$$

$$= 2x_2 \sum_{n=1}^{\infty} \frac{s^{n+1}}{x_2^{n+1}} \sum_{k=1}^n \binom{n}{k} \binom{n}{k-1} u^{n+1-k} v^k$$

$$- \frac{2s^2 uv}{x_2} \sum_{n=0}^{\infty} \frac{s^n}{x_2^n} \sum_{k=0}^n \binom{n}{k}^2 u^{n-k} v^k$$

$$= 2 \sum_{n=2}^{\infty} \frac{s^{n+1}}{x_2^n} \sum_{k=1}^n \left[ \binom{n}{k} \binom{n}{k-1} - \binom{n-1}{k-1}^2 \right] u^{n+1-k} v^k,$$

with the first few terms given by:

$$\frac{2s^3uv(u+v)}{x_2^2} + \frac{2s^4uv(2u^2 + 5uv + 2v^2)}{x^3} + \cdots$$

**Proposition 3.1.** The following formula for genus zero dessin two-point function holds:

$$(52) = \frac{G_{0,2}(x_1, x_2)}{1 - \frac{s(u+v)}{x_1} - \frac{s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_1x_2}} = \frac{1 - \frac{s(u+v)}{x_1} - \frac{s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_1x_2}}{2(x_1 - x_2)^2 \sqrt{\left(1 - \frac{2s(u+v)}{x_1} + \frac{s^2(u-v)^2}{x_1^2}\right)\left(1 - \frac{2s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_2^2}\right)}} - \frac{1}{2(x_1 - x_2)^2}.$$

*Proof.* For  $m \geq 3$ ,  $n \geq 1$ ,

(53) 
$$\frac{m}{s} \langle p_m \cdot p_n \rangle_0$$

$$= (m+n-1) \cdot \langle p_{m+n-1} \rangle_0 + (u+v)(m-1) \langle p_{m-1} \cdot p_n \rangle_0$$

$$+2 \sum_{k=1}^{m-2} k \langle p_k \cdot p_n \rangle_0 \cdot (m-1-k) \langle p_{m-1-k} \rangle_0,$$

Therefore,

$$G_{0,2}(x_1, x_2) = \sum_{m,n\geq 1} \langle p_m \cdot p_n \rangle_0^c \frac{m}{x_1^{m+1}} \frac{n}{x_2^{n+1}}$$

$$= s \sum_{m,n=1}^{\infty} (m+n-1) \cdot \langle p_{m+n-1} \rangle_0 \frac{1}{x_1^{m+1}} \frac{n}{x_2^{n+1}}$$

$$+ s \sum_{m\geq 2, n\geq 1} (u+v)(m-1) \langle p_{m-1} \cdot p_n \rangle_0 \frac{1}{x_1^{m+1}} \frac{n}{x_2^{n+1}}$$

$$+ 2s \sum_{m\geq 3, n\geq 1} \sum_{k=1}^{m-2} k \langle p_k \cdot p_n \rangle_0 \cdot (m-1-k) \langle p_{m-1-k} \rangle_0 \frac{1}{x_1^{m+1}} \frac{n}{x_2^{n+1}}$$

$$= s \sum_{l\geq 1} l \cdot \langle p_l \rangle_0^c \sum_{m+n=l+1} \frac{1}{x_1^{m+1}} \frac{n}{x_2^{n+1}}$$

$$+ \frac{s(u+v)}{x_1} G_{0,2}(x_1, x_2) + 2s G_{0,1}(x_1) \cdot G_{0,2}(x_1, x_2),$$

so we have:

$$G_{0,2}(x_1, x_2) = s \left( 1 - \frac{s(u+v)}{x_1} - 2sG_{0,1}(x_1) \right)^{-1} \cdot \sum_{l \ge 1} \langle p_l \rangle_0^c \sum_{n=1}^l \frac{1}{x_1^{l+2-n}} \frac{n}{x_2^{n+1}}$$

$$= \frac{s}{\sqrt{1 - \frac{s(u+v)}{x_1} + \frac{s^2(u-v)^2}{x_1^2}}} \sum_{l \ge 1} \langle p_l \rangle_0^c \sum_{n=1}^l \frac{1}{x_1^{l+2-n}} \frac{n}{x_2^{n+1}}.$$

Furthermore,

$$\sum_{l\geq 1} \langle p_l \rangle_0^c \sum_{n=1}^l \frac{1}{x_1^{l+2-n}} \frac{n}{x_2^{n+1}}$$

$$= \frac{1}{x_1(x_1 - x_2)^2} (x_1 G_{0,1}(x_1) - x_2 G_{0,1}(x_2)) - \frac{1}{x_1(x_1 - x_2)} \frac{d}{dx_2} (x_2 G_{0,1}(x_2))$$

$$= \frac{1 - \frac{s(u+v)}{x_1} - \frac{s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_1 x_2}}{2s(x_1 - x_2)^2 \sqrt{\left(1 - \frac{2s(u+v)}{x_2} + \frac{s^2(u-v)^2}{x_2^2}\right)}}$$

$$- \frac{\sqrt{\left(1 - \frac{2s(u+v)}{x_1} + \frac{s^2(u-v)^2}{x_1^2}\right)}}{2s(x_1 - x_2)^2},$$

where in the last equality we have used (43). Therefore, we have proved (32).

## 3.6. Computation of n-point functions in arbitrary genera by Virasoro constraints. By (42) we have:

$$G_{g,n}(x_0, x_1, \dots, x_n)$$

$$= \sum_{m,a_1,\dots,a_n \ge 1} \langle p_m \cdot p_{a_1} \cdots p_{a_n} \rangle_g^c \frac{m}{x_0^{m+1}} \prod_{i=1}^n \frac{a_i}{x_i^{a_i+1}}$$

$$= s \sum_{m,a_1,\dots,a_n \ge 1} \sum_{j=1}^n \langle p_{a_1} \cdots p_{a_j+m-1} \cdots p_{2a_n} \rangle_g^c \frac{a_j + m - 1}{x_0^{m+1}} \prod_{i=1}^n \frac{a_i}{x_i^{a_i+1}}$$

$$+ s(u+v) \sum_{m \ge 2,a_1,\dots,a_n \ge 1} \langle p_{m-1} \cdot p_{a_1} \cdots p_{a_n} \rangle_0^c \frac{m-1}{x_0^{m+1}} \prod_{i=1}^n \frac{a_i}{x_i^{a_i+1}}$$

$$+ s \sum_{k=1}^{m-2} k(m-1-k) \langle p_k p_{m-1-k} \cdot p_{a_1} \cdots p_{a_n} \rangle_{g-1}^c \frac{1}{x_0^{m+1}} \prod_{i=1}^n \frac{a_i}{x_i^{a_i+1}}$$

$$+ s \sum_{k=1}^{m-2} \sum_{I_1 \coprod I_2 = [n]} k \langle p_k \cdot \prod_{i \in I_1} p_{a_i} \rangle_{g_1}^c \cdot (m-1-k) \langle p_{m-1-k} \cdot \prod_{i \in I_2} p_{a_i} \rangle_{g_2}^c$$

$$\cdot \frac{1}{x_0^{m+1}} \prod_{i=1}^n \frac{a_i}{x_i^{a_i+1}}.$$

The following operators are introduced in [19]:

$$D_{x_0,x_j}f(x_j) = \frac{x_0f(x_0) - x_jf(x_j)}{x_0(x_0 - x_j)^2} - \frac{1}{x_0(x_0 - x_j)} \frac{d}{dx_j} (x_jf(x_j))$$
$$= \frac{\partial}{\partial x_j} \left( \frac{x_0f(x_0) - x_jf(x_j)}{x_0(x_0 - x_j)} \right),$$

$$E_{x_0,u,v}f(u,v) = \lim_{u \to v} f(u,v)|_{v=x_0}.$$

With the help these operators, the dessin Virasoro constraints can be reformulated as follows.

**Proposition 3.2.** Define the renormalized operators  $\tilde{D}$  and  $\tilde{E}$  as follows:

(54) 
$$\tilde{D}_{x_0,x_j} = \frac{sD_{x_0,x_j}}{1 - 2sW_{0,1}(x_0)}, \qquad \tilde{E}_{x_0,u,v} = \frac{sE_{x_0,u,v}}{1 - 2sW_{0,1}(x_0)}.$$

Then one has:

$$G_{g,n+1}(x_0, x_1, \dots, x_n)$$

$$= \sum_{j=1}^{n} \tilde{D}_{x_0, x_j} G_{g,n}(x_1, \dots, x_n) + \tilde{E}_{x_0, u, v} G_{g-1, n+2}(u, v, x_1, \dots, x_n)$$

$$+ \sum_{\substack{g_1 + g_2 = g \\ I_1 \coprod I_2 = [n]}} {}' \tilde{E}_{x_0, u, v} \bigg( G_{g_1, |I_1| + 1}(u, x_{I_1}) \cdot G_{g_2, |I_2| + 1}(v, x_{I_2}) \bigg).$$

- 3.7. **Examples.** We now present some sample computations of  $G_{g,n}$  using (55).
- 3.7.1. Three-point function in genus zero.

$$G_{0,3}(x_0, x_1, x_2)$$

$$= \sum_{j=1}^{2} \tilde{D}_{x_0, x_j} G_{0,2}(x_1, x_2) + 2\tilde{E}_{x_0, u, v} \left( G_{0,2}(u, x_1) \cdot G_{0,2}(v, x_2) \right)$$

$$= \frac{2s^3 u v (1 - (u - v)^2 s^2 (\frac{1}{x_0 x_1} + \frac{1}{x_1 x_2} + \frac{1}{x_2 x_0}) + 2(u + v)(u - v)^2 \frac{s^3}{x_0 x_1 x_2})}{\prod_{j=0}^{2} x_j^2 (1 - 2(u + v)s/x_j + (u - v)^2 s^2/x_j^2)^{3/2}}.$$

3.7.2. One-point function in genus one. In this case (55) takes the form:

$$G_{1,1}(x_0) = \tilde{E}_{x_0,u,v}G_{0,2}(u,v) = \frac{uvs^3}{x_0^4(1-2(u+v)s/x_0+(u-v)^2/x_0^2)^{5/2}}.$$

The following are the first few terms:

$$G_{1,1}(x_0) = \frac{uvs^3}{x_0^4} + \frac{5(u+v)s^4}{x_0^5} + \frac{(15u^2 + 40uv + 15v^2)s^5}{x_0^6} + \frac{35(u+v)(u^2 + 4uv + v^2)s^6}{x_0^7} + \cdots$$

#### 4. The Emergent Geometry of Dessin Tau-Function

In this Section we show that the dessin Virasoro constraints implies that the n-point functions associated to the dessin tau-function satisfy the EO topological recursions on a suitable spectral curve. We take a different approach from that of [9].

4.1. Dessin spectral curve and its special deformation. For  $n \ge 0$ , the dessin Virasoro constraints in genus zero are:

$$(u+v)n\frac{\partial F_0}{\partial p_n} + \sum_{j=1}^{\infty} \left(p_j - \frac{\delta_{j,1}}{s}\right)(n+j)\frac{\partial F_0}{\partial p_{n+j}} + \sum_{i+j=n} ij\frac{\partial F_0}{\partial p_i}\frac{\partial F_0}{\partial p_j} + \delta_{n,0}uv = 0.$$

Consider the following series:

(56) 
$$y = \frac{1}{2} \sum_{n=1}^{\infty} (p_n - \frac{\delta_{n,1}}{s}) x^{n-1} + \frac{u+v}{2x} + \sum_{n=1}^{\infty} \frac{n}{x^{n+1}} \frac{\partial F_0(t)}{\partial p_n}.$$

One has:

$$y^{2} = \frac{1}{4} \left( \sum_{n=1}^{\infty} (p_{n} - \frac{\delta_{n,1}}{s}) x^{n-1} \right)^{2} + \frac{(u+v)^{2}}{4x^{2}} + \sum_{i,j=1}^{\infty} \frac{ij}{x^{i+j+2}} \frac{\partial F_{0}(t)}{\partial p_{i}} \frac{\partial F_{0}(t)}{\partial p_{j}} + \frac{u+v}{2} \sum_{n=1}^{\infty} (p_{n} - \frac{\delta_{n,1}}{s}) x^{n-2} + (u+v) \sum_{n=1}^{\infty} \frac{n}{x^{n+2}} \frac{\partial F_{0}(t)}{\partial p_{n}} + \sum_{j=1}^{\infty} (p_{j} - \frac{\delta_{j,1}}{s}) \sum_{k=1}^{\infty} \frac{k}{x^{k-j+2}} \frac{\partial F_{0}(t)}{\partial p_{k}}.$$

Now by Viraroso constraints,

$$y^{2} = \frac{1}{4} \left( \sum_{n=1}^{\infty} (p_{n} - \frac{\delta_{n,1}}{s}) x^{n-1} \right)^{2} + \frac{(u-v)^{2}}{4x^{2}} + \frac{u+v}{2} \sum_{n=1}^{\infty} (p_{n} - \frac{\delta_{n,1}}{s}) x^{n-2} + \sum_{k\geq 1} \sum_{j\geq k+1} (p_{j} - \frac{\delta_{j,1}}{s}) k x^{j-k-2} \frac{\partial F_{0}(t)}{\partial p_{k}}.$$

When  $p_n = 0$  for all  $p_n$ :

(57) 
$$y^2 = \frac{1}{4s^2} - \frac{u+v}{2sx} + \frac{(u-v)^2}{4x^2}.$$

We refer to this plane algebraic curve as the dessin spectral curve. We call (56) the special deformation of the dessin spectral curve.

4.2. Genus zero one-point function and the spectral curve. Let us take  $p_k = 0$  in (56) to get:

(58) 
$$y = -\frac{1}{2s} + \frac{u+v}{2x} + \sum_{k=1}^{\infty} \frac{1}{x^{k+1}} \frac{\partial F_0}{\partial p_k} \Big|_{p_n=0, n \ge 1}.$$

By the definition of correlators and  $G_{0,1}(x)$ ,

(59) 
$$y = -\frac{1}{2s} + \frac{u+v}{2x} + G_{0,1}(x).$$

By the formula (43) for  $G_{0,1}$ ,

(60) 
$$y = -\frac{1}{2s}\sqrt{1 - \frac{2(u+v)s}{x} + \frac{(u-v)^2s^2}{x^2}},$$

and so

(61) 
$$y^2 = \frac{1}{4s^2} \left( 1 - \frac{2(u+v)s}{x} + \frac{(u-v)^2 s^2}{x^2} \right).$$

Note

(62) 
$$y = 0 \Leftrightarrow x = s(u + v \pm 2\sqrt{u}) = s(\sqrt{u} \pm \sqrt{v})^2,$$

and so the above equation defines a hyperelliptic rational curve with a branched covering ramified over  $x = s(\sqrt{u} \pm \sqrt{v})^2$ . This is an example of *one-cut solution*. Following [9], introduce the following coordinate that globally parameterize the dessin spectral curve:

(63) 
$$z = \sqrt{\frac{1 - s\beta/x}{1 - s\alpha/x}}, \quad \alpha = (\sqrt{u} - \sqrt{v})^2, \quad \beta = (\sqrt{u} + \sqrt{v})^2.$$

This function in x takes the ramification point  $(x = s\beta, y = 0)$  to z = 0, and  $(x = s\alpha, y = 0)$  to  $z = \infty$ . The functions x and y are two rational functions on this curve, expressed explicitly in terms of z as follows:

(64) 
$$x(z) = \frac{s(\alpha z^2 - \beta)}{z^2 - 1}, \qquad y(z) = -\frac{(\alpha - \beta)z}{2s(\alpha z^2 - \beta)}.$$

There is a natural hyperelliptic structure on this curve: One can define an involution  $p \mapsto \sigma(p)$  by

(65) 
$$\sigma(x,y) = (x, -y).$$

On the z-coordinate, it is given by

(66) 
$$\sigma(z) = -z.$$

### 4.3. Multilinear differential forms on the dessin spectal curve.

With the introduction of the spectral curve, one can regard the genus g n-point correlation functions  $G_{g,n}(x_1,\ldots,x_n)$  as functions on it. We understand x and y as meromorphic function on the spectral curve. For a point  $p_j$  on it, we write

(67) 
$$x_j = x(p_j), y_j = y(p_j), z_j = z(p_j).$$

Then  $G_{g,n}(x_1,\ldots,x_n)$  can be written as rational functions in  $z_1,\ldots,z_n$ . One can also consider the multilinear differential forms:

(68) 
$$W_{g,n}(p_1, \dots, p_n) = \hat{G}_{g,n}(y_1, \dots, y_n) dx_1 \cdots dx_n,$$

where  $\hat{G}_{g,n}(y_1,\ldots,y_n)=G_{g,n}(y_1,\ldots,y_n)$  except for the following two exceptional cases:

(69) 
$$\hat{G}_{0,1}(y_1) = -\frac{1}{2s} + \frac{(u+v)}{2x_1} + G_{0,1}(y_1),$$

(70) 
$$\hat{G}_{0,2}(y_1, y_2) = \frac{1}{(x_1 - x_2)^2} + G_{0,2}(y_1, y_2).$$

Since  $G_{0,1}(y_1) = y_1$ , so we have:

(71) 
$$W_{0,1}(p_1) = y_1 dx_1 = \frac{(\alpha - \beta)^2 z_1^2}{(\alpha z_1^2 - \beta)(z_1^2 - 1)^2} dz_1.$$

By some straightforward computations using (52), one can get:

(72) 
$$W_{0,2}(p_1, p_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

By the results for  $G_{0,3}(x_1, x_2, x_3)$  and  $G_{1,1}(x_1)$  in §3.7, we get:

$$W_{0,3}(p_1, p_2, p_3) = G_{0,3}(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

$$W_{0,3}(p_1, p_2, p_3) = G_{0,3}(x_1, x_2, x_3) dx_1 dx_2 dx_3$$

$$= \frac{2s^3 uv(1 - (u - v)^2 s^2 \sum_{j=1}^3 \frac{1}{x_j x_{j+1}} + 2(u + v)(u - v)^2 \frac{s^3}{x_1 x_2 x_3})}{\prod_{j=1}^3 x_j^2 (-2sy_j)^3} dx_1 dx_2 dx_3$$

$$= \frac{1}{(\alpha - \beta)^2} \left(-\alpha + \frac{\beta}{z_1^2 z_2^2 z_3^2}\right) dz_1 dz_2 dz_3,$$

where  $x_4 = x_1$ , and

$$W_{1,1}(p_1) = G_{1,1}(x_1)dx_1$$

$$= \frac{uvs^3}{x_1^4(1 - 2(u+v)s/x_1 + (u-v)^2/x_1^2)^{5/2}}dx_1 = \frac{uvs^3}{x_1^4(-2sy_1)^5}dx_1$$

$$= \frac{1}{8(\alpha - \beta)^2} \left(\frac{\beta}{z^4} - \frac{2\beta + \alpha}{z^2} + (2\alpha + \beta) - \alpha z^2\right)dz.$$

4.4. Eynard-Orantin topological recursions. We use  $W_{0,2}$  as the Bergman kernel. Then

$$\int_{q=\sigma(p_2)}^{p_2} B(p_1,q) = \int_{z=-z_2}^{z_2} \frac{dz_1 dz}{(z_1-z)^2} = \frac{dz_1}{z_1-z} \bigg|_{z=-z_2}^{z_2} = \frac{2z_2 dz_1}{z_1^2-z_2^2}.$$

It follows that

(73) 
$$K(p_0, p) = \frac{2zdz_0}{2(z_0^2 - z^2) \cdot 2ydx} = \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)} \cdot \frac{dz_0}{dz}.$$

This has simple poles at z = 0 and  $z = \pm z_0$ . To understand its behavior at  $z = \infty$ , let  $z = 1/\tilde{z}$  and  $z_0 = 1/\tilde{z}_0$ . Then

(74) 
$$K(p_0, p) = \frac{(\beta \tilde{z}^2 - \alpha)(\tilde{z}^2 - 1)^2}{2(\alpha - \beta)^2 \tilde{z}(\tilde{z}_0^2 - \tilde{z}^2)} \cdot \frac{d\tilde{z}_0}{d\tilde{z}}.$$

This has a simple pole at  $\tilde{z} = 0$ .

Let us carry out the first few calculations of Eynard-Orantin recursion for the spectral curve (61) with  $\omega_{0,1} = W_{0,1}$  and  $\omega_{0,2} = W_{0,2}$  given by (71) and (72) respectively.

$$\omega_{0,3}(p_0, p_1, p_2)$$

$$= (\operatorname{Res}_{p \to p_+} + \operatorname{Res}_{p \to p_-}) \left( K(p_0, p) [W_{0,2}(p, p_1) W_{0,2}(\sigma(p), p_2) + W_{0,2}(p, p_2) W_{0,2}(\sigma(p), p_1)] \right)$$

$$= \operatorname{Res}_{z \to 0} \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)} \cdot \frac{dz_0}{dz}$$

$$\cdot \left( \frac{dz dz_1}{(z - z_1)^2} \cdot \frac{-dz dz_2}{(-z - z_2)^2} + \frac{dz dz_2}{(z - z_2)^2} \cdot \frac{-dz dz_1}{(-z - z_1)^2} \right)$$

$$+ \operatorname{Res}_{\tilde{z} \to 0} \frac{(\beta \tilde{z}^2 - \alpha)(\tilde{z}^2 - 1)^2}{2(\alpha - \beta)^2 \tilde{z}(\tilde{z}_0^2 - \tilde{z}^2)} \cdot \frac{d\tilde{z}_0}{d\tilde{z}}$$

$$\cdot \left( \frac{d\tilde{z} d\tilde{z}_1}{(\tilde{z} - \tilde{z}_1)^2} \cdot \frac{-d\tilde{z} d\tilde{z}_2}{(-\tilde{z} - \tilde{z}_2)^2} + \frac{d\tilde{z} d\tilde{z}_2}{(\tilde{z} - \tilde{z}_2)^2} \cdot \frac{-d\tilde{z} d\tilde{z}_1}{(-\tilde{z} - \tilde{z}_1)^2} \right)$$

$$= \frac{\beta}{2(\alpha - \beta)^2} \cdot \frac{dz_0}{z_0^2} \cdot \frac{dz_1}{z_1^2} \cdot \frac{dz_2}{z_2^2} + \frac{\alpha}{2(\alpha - \beta)^2} \cdot \frac{d\tilde{z}_0}{\tilde{z}_0^2} \cdot \frac{d\tilde{z}_1}{\tilde{z}_1^2} \cdot \frac{d\tilde{z}_2}{\tilde{z}_2^2}$$

$$= \left( \frac{\beta}{(\alpha - \beta)^2} \cdot \frac{1}{z_0^2 z_1^2 z_2^2} - \frac{\alpha}{(\alpha - \beta)^2} \right) dz_0 dz_1 dz_2$$

$$= W_{0,3}(p_0, p_2, p_2).$$

$$\omega_{1,1}(p_0) = (\operatorname{Res}_{p \to p_+} + \operatorname{Res}_{p \to p_-}) \left( K(p_0, p) W_{0,2}(p, \sigma(p)) \right) 
= \operatorname{Res}_{z \to 0} \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z (z_0^2 - z^2)} \cdot \frac{dz_0}{dz} \cdot \frac{-(dz)^2}{4z^2} 
+ \operatorname{Res}_{\tilde{z} \to 0} \frac{(\beta \tilde{z}^2 - \alpha)(\tilde{z}^2 - 1)^2}{2(\alpha - \beta)^2 \tilde{z} (\tilde{z}_0^2 - \tilde{z}^2)} \cdot \frac{d\tilde{z}_0}{d\tilde{z}} \cdot \frac{-(d\tilde{z})^2}{4\tilde{z}^2} 
= \frac{1}{8(\alpha - \beta)^2} \left( \frac{\beta}{z_0^4} - \frac{\alpha + 2\beta}{z_0^2} \right) dz_0 + \frac{1}{8(\alpha - \beta)^2} \left( \frac{\alpha}{\tilde{z}_0^4} - \frac{\beta + 2\alpha}{\tilde{z}_0^2} \right) d\tilde{z}_0 
= \frac{1}{8(\alpha - \beta)^2} \left( \frac{\beta}{z_0^4} - \frac{\alpha + 2\beta}{z_0^2} + (\beta + 2\alpha) - \alpha z_0^2 \right) dz_0 
= W_{1,1}(p_0).$$

**Theorem 4.1.** When 2g-2+n>0, the multi-linear differential forms  $W_{q,n}(p_1,\ldots,p_n)$  defined by (68) has the following form

(75) 
$$W_{g,n}(p_1,\ldots,p_n) = w_{g,n}(z_1,\ldots,z_n)dz_1\cdots dz_n = \tilde{w}_{g,n}(\tilde{z}_1,\ldots,\tilde{z}_n)d\tilde{z}_1\cdots d\tilde{z}_n,$$

where  $w_{g,n}(z_1, \ldots, z_n)$  is a Laurent polynomial in  $z_1, \ldots, z_n$  which has only terms of even degrees with respect to each variable, similarly for  $\tilde{w}_{g,n}(\tilde{z}_1, \ldots, \tilde{z}_n)$ . Furthermore, they satisfy the Eynard-Orantin topological recursions given by the spectral curve

(76) 
$$y^2 = \frac{1}{4s^2} - \frac{u+v}{2sx} + \frac{(u-v)^2}{4x^2}.$$

I.e., we have

$$(77) W_{g,n}(p_1,\ldots,p_n) = \omega_{g,n}(p_1,\ldots,p_n).$$

*Proof.* We have explicitly check the case of  $\omega_{0,3}$  and  $\omega_{1,1}$ . We now show that other cases can be checked by induction. First, assume the Eynard-Orantin topological recursions are satisfied. By the induction hypothesis,

$$\omega_{g,n+1}(p_0, p_1, \dots, p_n) = (\operatorname{Res}_{z \to 0} + \operatorname{Res}_{z \to \infty}) K(z_0, z)$$

$$\cdot \left[ w_{g-1,n+2}(z, -z, z_{[n]}) dz d(-z) \right]$$

$$+ \sum_{h=0}^{g} \sum_{I \subset [n]}' w_{h,|I|+1}(z, z_I) w_{g-h,n-|I|+1}(-z, z_{[n]-I}) dz d(-z) dz_1 \cdots dz_n.$$

There are three kinds of terms to consider on the right-hand side. First,

$$(\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) K(z_0, z) \cdot w_{g-1, n+2}(z, -z, z_{[n]}) dz d(-z)$$

$$= -(\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)} \cdot w_{g-1, n+2}(z, z, z_{[n]}) dz$$

$$= (\operatorname{Res}_{z\to z_0} + \operatorname{Res}_{z\to -z_0}) \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)} \cdot w_{g-1, n+2}(z, z, z_{[n]}) dz$$

$$= -\frac{(\alpha z_0^2 - \beta)(z_0^2 - 1)^2}{2(\alpha - \beta)^2 z_0^2} \cdot w_{g-1, n+2}(z_0, z_0, z_{[n]}).$$

Secondly, when none of (h, |I| + 1) and (g - h, n - |I| + 1) is (0, 2),

$$(\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) K(z_0, z) \cdot w_{h,|I|+1}(z, z_I) w_{g-h,n-|I|+1}(z, z_{[n]-I}) dz d(-z)$$

$$= -(\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)}$$

$$\cdot w_{h,|I|+1}(z, z_I) w_{g-h,n-|I|+1}(z, z_{[n]-I}) dz$$

$$= (\operatorname{Res}_{z\to z_0} + \operatorname{Res}_{z\to -z_0}) \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)}$$

$$\cdot w_{h,|I|+1}(z_0, z_I) w_{g-h,n-|I|+1}(z_0, z_{[n]-I}) dz$$

$$= -\frac{(\alpha z_0^2 - \beta)(z_0^2 - 1)^2}{2(\alpha - \beta)^2 z_0^2} \cdot w_{h,|I|+1}(z_0, z_I) w_{g-h,n-|I|+1}(z_0, z_{[n]-I}).$$

Thirdly we need to consider

$$(\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) K(z_0, z) \cdot (w_{0,2}(z, z_i) w_{g,n}(-z, z_{[n]-\{i\}}) + w_{0,2}(-z, z_i) w_{g,n}(z, z_{[n]-\{i\}})) dz d(-z)$$

$$= dz_0 \cdot (\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)} \cdot \left(\frac{1}{(z - z_i)^2} w_{g,n}(z, z_{[n]-\{i\}}) + \frac{1}{(-z - z_i)^2} \cdot w_{g,n}(z, z_{[n]-\{i\}})\right) dz.$$

Its computation can be reduced to  $dz_0$  times

$$(\operatorname{Res}_{z\to 0} - \operatorname{Res}_{z\to \infty}) \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)} \cdot \frac{2(z^2 + z_i^2)}{(z^2 - z_i^2)^2} z^{2n} dz,$$

the result is clear a Laurent series in  $z_0$  with only terms of even degrees. Let us present the result for  $n \ge 0$ , the result for  $n \ge 0$  is similar. When n=0,

$$(\operatorname{Res}_{z\to 0} - \operatorname{Res}_{z\to \infty}) \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)} \cdot \frac{2(z^2 + z_i^2)}{(z^2 - z_i^2)^2} dz$$

$$= -\frac{\beta}{(\alpha - \beta)^2 z_0^2 z_i^2} + \frac{(3z_i^2 + z_0^2 - 2)\alpha - \beta}{(\alpha - \beta)^2}.$$

For n > 0, the residue at z = 0 vanishes, and so

$$(\operatorname{Res}_{z\to 0} - \operatorname{Res}_{z\to \infty}) \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)} \cdot \frac{2(z^2 + z_i^2)}{(z^2 - z_i^2)^2} z^{2n} dz$$

$$= -\operatorname{Res}_{w\to 0} \frac{(\alpha - \beta w^2)(1 - w^2)^2}{(\alpha - \beta)^2 (z_0^2 w^2 - 1)} \cdot \frac{(1 + z_i^2 w^2)}{(1 - z_i^2 w^2)^2} w^{-2n - 3} dw$$

$$= -\frac{(\alpha - \beta w^2)(1 - w^2)^2}{(\alpha - \beta)^2 (z_0^2 w^2 - 1)} \cdot \frac{(1 + z_i^2 w^2)}{(1 - z_i^2 w^2)^2} \Big|_{w^{2n+2}}.$$

On the other hand, by (68),

$$W_{g,n+1}(y_0, \dots, y_n)$$

$$= G_{g,n+1}(y_0, y_1, \dots, y_n) dx_0 \cdots dx_n$$

$$= \sum_{j=1}^n \mathcal{D}_{y_0, y_j} G_{g,n}(y_1, \dots, y_n) \cdot dx_0 \cdots dx_n$$

$$+ \mathcal{E}_{y_0, y, y'} G_{g-1, n+2}(y, y', y_1, \dots, y_n) \cdot dx_0 \cdots dx_n$$

$$+ \sum_{\substack{g_1 + g_2 = g \\ I_1 \coprod I_2 = [n]}} {}' \mathcal{E}_{y_0, y, y'} \bigg( G_{g_1, |I_1|+1}(y, y_{I_1}) \cdot G_{g_2, |I_2|+1}(y', y_{I_2}) \bigg) \cdot dx_0 \cdots dx_n.$$

By comparing these two recursions, we see that it is easy to show that

$$(\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) K(z_0, z) \cdot \left[ \omega_{g-1, n+2}(z, -z, z_{[n]}) \right]$$

$$= \mathcal{E}_{y_0, y, y'} G_{g-1}(y, y', y_1, \dots, y_n) \cdot dx_0,$$

and when  $(h, |I| + 1) \neq (0, 2)$  and  $(g - h, n - |I| + 1) \neq (0, 2)$ ,

$$(\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) K(z_0, z) \left[ \omega_{h,|I|+1}(z, z_I) \omega_{g-h,n-|I|+1}(-z, z_{[n]-I}) \right]$$

$$= \mathcal{E}_{y_0,y,y'} \left[ G_{h,|I|+1}(y, y_I) G_{g-h,n-|I|+1}(y', y_{[n]-I}) \right] \cdot dx_0,$$

and so it suffices to show that

$$(\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) K(z_0, z) \left[ \omega_{0,2}(z, z_i) \cdot \omega_{g,n}(-z, z_1, \dots, \hat{z}_i, \dots, z_n) \right] 
+ (\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) K(z_0, z) \left[ \omega_{0,2}(-z, z_i) \cdot \omega_{g,n}(z, z_1, \dots, \hat{z}_i, \dots, z_n) \right] 
= \left( \mathcal{D}_{y_0, y_j} \left[ G_{g,n}(y_1, \dots, y_n) \right] 
+ \mathcal{E}_{y_0, y, y'} \left[ G_{0,2}(y, y_i) G_{g,n}(y', y_1, \dots, \hat{y}_i, \dots, y_n) \right] 
+ G_{g,n}(y, y_1, \dots, \hat{y}_i, \dots, y_n) G_{0,2}(y', y_i) \right] \cdot dx_0 dx_1 \cdots dx_n.$$

For the computation of the left-hand side, we have reduced to the computation with  $\omega_{g,n}(z,z_1,\ldots,\hat{z}_i,\ldots,z_n)$  replaced by  $z^{2n}dz$ , because

$$\omega_{g,n}(z, z_1, \dots, \hat{z}_i, \dots, z_n)$$

$$= G_{g,n}(y, y_1, \dots, \hat{y}_i, \dots, y_n) dx dx_1 \cdots \widehat{dx}_i \cdots dx_n,$$

this corresponds to replacing  $G_{g,n}(y, y_1, \dots, \hat{y}_i, \dots, y_N)$  by  $f_n(y)$  on the right-hand side, where  $f_n(y)$  is chosen such that:

(78) 
$$z^{2n}dz = f_n(y)dx,$$

i.e.,

(79) 
$$f_n(y) = \frac{z^{2n}}{\frac{dx}{dz}} = -\frac{(z^2 - 1)^2 z^{2n-1}}{2s(\alpha - \beta)},$$

so we need to check that:

$$(\operatorname{Res}_{z\to 0} + \operatorname{Res}_{z\to \infty}) \frac{(\alpha z^2 - \beta)(z^2 - 1)^2}{2(\alpha - \beta)^2 z(z_0^2 - z^2)} \cdot \frac{2(z^2 + z_i^2)}{(z^2 - z_i^2)^2} z^{2n} dz$$

$$= \left( \mathcal{D}_{y_0, y_i} \left[ f_n(y_i) \right] \right.$$

$$+ \left. \mathcal{E}_{y_0, y, y'} \left[ G_{0,2}(y, y_i) f_n(y') + f_n(y) \cdot G_{0,2}(y', y_i) \right] \right) \cdot \frac{dx_0}{dz_0} \frac{dx_i}{dz_i}.$$

The right-hand can be computed as follows. Note

$$\left(G_{0,2}(y_0, y_i) + \frac{1}{(x_0 - x_i)^2}\right) \frac{dx_0}{dz_0} \frac{dx_i}{dz_i} = \frac{1}{(z_0 - z_i)^2}$$

and so

$$\mathcal{E}_{y_0,y,y'} \left[ G_{0,2}(y,y_i) f_n(y') + f_n(y) \cdot G_{0,2}(y',y_i) \right] \right) \cdot \frac{dx_0}{dz_0} \frac{dx_i}{dz_i}$$

$$= -\frac{1}{2y_0} \left[ G_{0,2}(y_0,y_i) f_n(y_0) + f_n(y_0) \cdot G_{0,2}(y_0,y_i) \right] \cdot \frac{dx_0}{dz_0} \frac{dx_i}{dz_i}$$

$$= -\frac{1}{2y_0} \cdot 2 \left( \frac{1}{(z_0 - z_i)^2} \cdot \frac{1}{\frac{\partial x_0}{\partial z_0} \frac{\partial x_j}{\partial z_j}} - \frac{1}{(x_0 - x_i)^2} \right) \cdot f_n(y_0) \cdot \frac{dx_0}{dz_0} \frac{dx_i}{dz_0}$$

$$= -\frac{1}{2y_0} \cdot 2 \left( \frac{1}{(z_0 - z_i)^2} \cdot \frac{1}{\frac{\partial x_0}{\partial z_0} \frac{\partial x_j}{\partial z_j}} - \frac{1}{(x_0 - x_i)^2} \right) \cdot \frac{z_0^{2n}}{\frac{dx_0}{dz_0}} \cdot \frac{dx_0}{dz_0} \frac{dx_i}{dz_0}$$

$$= -\frac{1}{2y_0} \cdot \left( \frac{1}{(z_0 - z_i)^2} - \frac{\frac{\partial x_0}{\partial z_0} \frac{\partial x_j}{\partial z_j}}{(x - x_0)^2} \right) \frac{(z_0^2 - 1)^2}{s(\alpha - \beta)} z_0^{2n - 1}$$

$$= \frac{1}{2y_0} \cdot \frac{1}{(z_0 + z_i)^2} \cdot \frac{(z_0^2 - 1)^2}{s(\alpha - \beta)} z_0^{2n - 1}$$

$$= \frac{(z_0^2 - 1)^2 (\alpha z_0^2 - \beta)}{(\alpha - \beta)^2 (z_0 + z_i)^2} z_0^{2n - 2},$$

on the other hand,

$$\mathcal{D}_{y_0,y_i} \left[ f_n(y_i) \right] \cdot \frac{dx_0}{dz_0} \frac{dx_i}{dz_i}$$

$$= -\frac{1}{2y_0} \cdot \frac{\partial}{\partial x_i} \left( \frac{x_0 f_n(y_0) - x_i f_n(y_i)}{x_0 (x_0 - x_i)} \right) \cdot \frac{dx_0}{dz_0} \frac{dx_i}{dz_i}$$

$$= -\frac{1}{2y_0} \cdot \frac{\partial}{\partial z_i} \left( \frac{x_0 f_n(y_0) - x_i f_n(y_i)}{x_0 (x_0 - x_i)} \right) \cdot \frac{dx_0}{dz_0}$$

$$= \frac{s(\alpha z_0^2 - \beta)}{(\alpha - \beta) z_0} \cdot \frac{\partial}{\partial z_i} \left( \frac{-\frac{(\alpha z_0^2 - \beta)(z_0^2 - 1)z_0^{2n - 1}}{2(\alpha - \beta)} + \frac{(\alpha z_i^2 - \beta)(z_i^2 - 1)z_i^{2n - 1}}{2(\alpha - \beta)}}{\frac{s(\alpha z_0^2 - \beta)}{z_0^2 - 1} \left( \frac{s(\alpha z_0^2 - \beta)}{(z_0^2 - 1)} - \frac{s(\alpha z_i^2 - \beta)}{(z_i^2 - 1)} \right)} \right)$$

$$\cdot \frac{-2sz_0(\alpha - \beta)}{(z_0^2 - 1)^2}$$

$$= \frac{1}{(\alpha - \beta)^2} \cdot \frac{\partial}{\partial z_i} \left( (z_i^2 - 1) \cdot \frac{(\alpha z_0^2 - \beta)(z_0^2 - 1)z_0^{2n - 1} - (\alpha z_i^2 - \beta)(z_i^2 - 1)z_i^{2n - 1}}{z_0^2 - z_i^2} \right)$$

For n > 0,

$$\begin{split} & \frac{(\alpha - \beta w^2)(1 - w^2)^2}{(1 - z_0^2 w^2)} \cdot \frac{(1 + z_i^2 w^2)}{(1 - z_i^2 w^2)^2} \bigg|_{w^{2n+2}} \\ &= \frac{(z_0^2 - 1)^2 (\alpha z_0^2 - \beta)}{(z_0 + z_i)^2} z_0^{2n-2} \\ &+ \frac{\partial}{\partial z_i} \bigg( (z_i^2 - 1) \cdot \frac{(\alpha z_0^2 - \beta)(z_0^2 - 1)z_0^{2n-1} - (\alpha z_i^2 - \beta)(z_i^2 - 1)z_i^{2n-1}}{z_0^2 - z_i^2} \bigg). \end{split}$$

This can be proved as follows. The generating function of the left-hand side is

$$\begin{split} & \sum_{n=1}^{\infty} u^{2n+2} \cdot \frac{(\alpha - \beta w^2)(1 - w^2)^2}{(1 - z_0^2 w^2)} \cdot \frac{(1 + z_i^2 w^2)}{(1 - z_i^2 w^2)^2} \bigg|_{w^{2n+2}} \\ & = \left. \frac{(\alpha - \beta u^2)(1 - u^2)^2}{(1 - z_0^2 u^2)} \cdot \frac{(1 + z_i^2 u^2)}{(1 - z_i^2 u^2)^2} - \alpha - (3\alpha z_i^2 - 2\alpha - \beta + \alpha z_0^2)u^2, \end{split}$$

and the generating function of the right-hand side is:

$$\begin{split} & \sum_{n=1}^{\infty} u^{2n+2} \bigg( \frac{(z_0^2 - 1)^2 (\alpha z_0^2 - \beta)}{(z_0 + z_i)^2} z_0^{2n-2} \\ & + \frac{\partial}{\partial z_i} \bigg( (z_i^2 - 1) \cdot \frac{(\alpha z_0^2 - \beta)(z_0^2 - 1) z_0^{2n-1} - (\alpha z_i^2 - \beta)(z_i^2 - 1) z_i^{2n-1}}{z_0^2 - z_i^2} \bigg) \bigg) \\ & = \frac{(z_0^2 - 1)^2 (\alpha z_0^2 - \beta)}{(z_0 + z_i)^2} \cdot \frac{u^4}{1 - z_0^2 u^2} \\ & + \frac{\partial}{\partial z_i} \bigg( (z_i^2 - 1) \cdot \frac{(\alpha z_0^2 - \beta)(z_0^2 - 1) \frac{z_0 u^4}{1 - z_0^2 u^2} - (\alpha z_i^2 - \beta)(z_i^2 - 1) \frac{z_i u^4}{1 - z_i^2 u^2}}{z_0^2 - z_i^2} \bigg). \end{split}$$

It is straightforward to verify that these generating functions coincide with each other. This completes the proof.  $\Box$ 

### 5. Connection to Intersection Numbers on $\overline{\mathcal{M}}_{q,,n}$

Eynard [5, 6] related the *n*-point functions constructed from EO topological recursions to intersection theory. In this Section we carry out some computations necessary for applying his results to relating Grothendieck's dessins to intersection numbers on moduli spaces.

5.1. Local Airy coordinates. Recall the dessin spectral curve is given by (61) which we recall here:

$$y^{2} = \frac{1}{4s^{2}} \left( 1 - \frac{2(u+v)s}{x} + \frac{(u-v)^{2}s^{2}}{x^{2}} \right).$$

This defines a double covering of the complex plane ramified at two points

(80) 
$$x_{+} = s\beta = s(\sqrt{u} + \sqrt{v})^{2}, \quad x_{-} = s\alpha = s(\sqrt{u} - \sqrt{v})^{2},$$

where

(81) 
$$\alpha = (\sqrt{u} - \sqrt{v})^2, \quad \beta = (\sqrt{u} + \sqrt{v})^2.$$

Following [9], one can globally parameterize the dessin spectral curve by:

(82) 
$$z = \sqrt{\frac{1 - s\beta/x}{1 - s\alpha/x}}.$$

In this coordinate, the Bergman kernel is given by (72) which we recall here:

$$W_{0,2}(p_1, p_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}.$$

Following [5, 6], the local Airy coordinates near  $x_{\pm}$  are defined by:

(83) 
$$\xi_{\pm} = (x - x_{\pm})^{1/2}.$$

5.2. Computations of the times. Plugging  $x = x_{\pm} + \xi_{\pm}^2$  into the equation of the dessin spectral curve, one gets:

(84) 
$$y^2 = \pm \frac{\xi_{\pm}^2 (4\sqrt{uv}s \pm \xi_{\pm}^2)}{(s(\sqrt{u} \pm \sqrt{v})^2 + \xi_{\pm}^2)^2}.$$

So we get:

(85) 
$$y = \xi_{\pm} \cdot \frac{2(\sqrt{u})^{1/2} (\pm \sqrt{v})^{1/2} s^{1/2}}{s(\sqrt{u} \pm \sqrt{v})} \cdot \frac{\left(1 \pm \frac{\xi_{\pm}^2}{4\sqrt{uv}s}\right)^{1/2}}{1 + \frac{\xi_{\pm}^2}{s(\sqrt{u} \pm \sqrt{v})^2}}.$$

It is straightforward to write down the series expansions in  $\xi_{\pm}$  using

$$(1+4x)^{1/2} = 1 + 2\sum_{m=0}^{\infty} \frac{(-1)^m}{m+1} {2m \choose m} x^{m+1},$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n.$$

5.3. The expansions of the Bergman kernel in local Airy coordinates. Near  $p_1 = p_2 = p_+ = (x_+, 0)$ , we have

$$z_{j} = \sqrt{\frac{1 - s(a+b)^{2}/(s(a+b)^{2} + \xi_{j,+}^{2})}{1 - s(a-b)^{2}/(s(a+b)^{2} + \xi_{j,+}^{2})}} = \frac{\xi_{j,+}}{(4sab + \xi_{j,+}^{2})^{1/2}},$$

where  $a = \sqrt{u}$ ,  $b = \sqrt{v}$ , and so

$$B(z_{1}, z_{2}) = \frac{dz_{1}dz_{2}}{(z_{1} - z_{2})^{2}}$$

$$= \frac{d\xi_{1,+}}{(1 + \frac{\xi_{1,+}^{2}}{4abs})^{1/2}} \cdot \frac{d\xi_{2,+}}{(1 + \frac{\xi_{2,+}^{2}}{4abs})^{1/2}}$$

$$\cdot \frac{1}{(\xi_{1,+}(1 + \frac{\xi_{2,+}^{2}}{4abs})^{1/2} - \xi_{2,+}(1 + \frac{\xi_{1,+}^{2}}{4abs})^{1/2})^{2}}.$$

Its expansion can be computed using the combinatorial identity:

$$\frac{1}{(1+4tx^2)^{1/2}(1+4ty^2)^{1/2}(x(1+4ty^2)^{1/2}-y(1+4tx^2)^{1/2})^2} = \frac{1}{(x-y)^2} + \sum_{n\geq 0} (n+2)(-t)^{n+1} \sum_{k=0}^n 2\frac{\binom{n}{k}^2 \binom{2n+2}{n}}{\binom{2n+2}{2k+1}} x^{2k} y^{2n-2k}.$$

This is found and proved as follows. With the help of Maple we get:

$$\frac{1}{(1+4tx^2)^{1/2}(1+4ty^2)^{1/2}(x(1+4ty^2)^{1/2}-y(1+4tx^2)^{1/2})^2} 
= \frac{1}{(x-y)^2} - 2t + 3(2x^2 + 2y^2)t^2 - 4(5x^4 + 6x^2y^2 + 5y^4)t^3 + \cdots,$$

After consulting with [11] we find the numbers  $1, 2, 2, 5, 6, 5, \ldots$  are the integer sequence A120406 on [11], they are given by the closed formula:

(86) 
$$T(n,k) = 2 \frac{\binom{n}{k}^2 \binom{2n+2}{n}}{\binom{2n+2}{2k+1}},$$

and have the following generating series:

$$(87) \qquad = \frac{(a+b)x}{2} + \frac{(b-a)^2x^2}{8} \sum_{n\geq 0} (\sum_{0\leq k\leq n} T(n,k)a^kb^{n-k})(\frac{x}{4})^n.$$

From this we get:

$$1 - \sqrt{(1+4tx^2)(1+4ty^2)}$$

$$= -2(x^2+y^2)t + 2(x^2-y^2)^2 \sum_{n\geq 0} (-1)^n t^{n+2} \sum_{0\leq k\leq n} T(n,k) x^{2k} y^{2n-2k}.$$

After taking  $\frac{\partial}{\partial t}$  on both sides we get:

$$\sum_{n\geq 0} (n+2)(-t)^{n+1} \sum_{k=0}^{n} 2\frac{\binom{n}{k}^{2}\binom{2n+2}{n}}{\binom{2n+2}{2k+1}} x^{2k} y^{2n-2k}$$

$$= \frac{x^{2} + y^{2} + 8tx^{2}y^{2} - (x^{2} + y^{2})(1 + 4tx^{2})^{1/2}(1 + 4ty^{2})^{1/2}}{(x^{2} - y^{2})^{2}(1 + 4tx^{2})^{1/2}(1 + 4ty^{2})^{1/2}}.$$

One can then proceeds as follows:

$$\frac{1}{(1+4tx^2)^{1/2}(1+4ty^2)^{1/2}(x(1+4ty^2)^{1/2}-y(1+4tx^2)^{1/2})^2} = \frac{(x(1+4ty^2)^{1/2}+y(1+4tx^2)^{1/2})^2}{(1+4tx^2)^{1/2}(1+4ty^2)^{1/2}(x^2-y^2)^2} = \frac{x^2+y^2+8tx^2y^2+2xy(1+4tx^2)^{1/2}(1+4ty^2)^{1/2}}{(x^2-y^2)^2(1+4tx^2)^{1/2}(1+4ty^2)^{1/2}} = \frac{1}{(x-y)^2} + \sum_{n\geq 0} (n+2)(-t)^{n+1} \sum_{k=0}^{n} 2\frac{\binom{n}{k}^2\binom{2n+2}{n}}{\binom{2n+2}{2k+1}} x^{2k}y^{2n-2k}.$$

Near  $p_1 = p_2 = p_- = (x_-, 0)$ , we have

$$z_{j} = \sqrt{\frac{1 - s(a+b)^{2} / (s(a-b)^{2} + \xi_{j,-}^{2})}{1 - s(a-b)^{2} / (s(a-b)^{2} + \xi_{j,-}^{2})}} = \frac{(-4sab + \xi_{j,-}^{2})^{1/2}}{\xi_{j,-}}$$

and so

$$B(z_{1}, z_{2}) = \frac{d\xi_{1,-}}{(1 + \frac{\xi_{1,-}^{2}}{4abs})^{1/2}} \cdot \frac{d\xi_{2,-}}{(1 + \frac{\xi_{2,-}^{2}}{4abs})^{1/2}} \cdot \frac{1}{(\xi_{1,-}(1 + \frac{\xi_{2,-}^{2}}{4abs})^{1/2} - \xi_{2,-}(1 + \frac{\xi_{1,-}^{2}}{4abs})^{1/2})^{2}}.$$

Its expansion can be found in the same fashion.

Near Near  $p_1 = p_+ = (x_+, 0)$  and  $p_2 = p_- = (x_-, 0)$ , we have

$$z_1 = \frac{\xi_{1,+}}{(4sab + \xi_{1,+}^2)^{1/2}},$$
  $z_2 = \frac{(-4sab + \xi_{2,-}^2)^{1/2}}{\xi_{2,-}},$ 

In this case

$$B(z_1, z_2) = \frac{\sqrt{-1}d\xi_{1,+}d\xi_{2,-}}{4abs(1 + \frac{\xi_{1,+}^2}{4abs})^{1/2}(1 - \frac{\xi_{2,-}^2}{4abs})^{1/2}\left((1 + \frac{\xi_{1,+}^2}{4abs})^{1/2}(1 - \frac{\xi_{2,-}^2}{4abs})^{1/2} - \frac{\xi_{1,+}\xi_{2,-}}{\sqrt{-1}4abs}\right)^2}.$$

It is possible to find explicit expressions for its expansion by noting:

$$\frac{1}{(1+4tx^2)^{1/2}(1+4ty^2)^{1/2}((1+4tx^2)^{1/2}(1+4ty^2)^{1/2}+4txy)^2} 
= \frac{((1+4tx^2)^{1/2}(1+4ty^2)^{1/2}-4txy)^2}{(1+4tx^2)^{1/2}(1+4ty^2)^{1/2}(1+4tx^2+4ty^2)^2} 
= \frac{(1+4tx^2)^{1/2}(1+4ty^2)^{1/2}}{(1+4tx^2+4ty^2)^2} 
- \frac{8txy^2}{(1+4tx^2+4ty^2)^2} 
+ \frac{16t^2x^2y^2}{(1+4tx^2)^{1/2}(1+4ty^2)^{1/2}(1+4tx^2+4ty^2)^2}.$$

The Taylor expansion of the right-hand side of the second equality can be found by standard method.

### 6. Concluding Remarks

In this paper we have discussed the emergent geometry of the enumeration of Grothendieck's dessins d'enfant. This leads to Eynard-Orantin topological recursion on the spectral curve

(88) 
$$y^{2} = \frac{1}{4s^{2}} \left( 1 - \frac{2(u+v)s}{x} + \frac{(u-v)^{2}s^{2}}{x^{2}} \right).$$

This curve can be found by computing the genus zero one-point function of the dessin tau-function. Furthermore, by computing the genus zero two-point function of the dessin tau-function, the Bergman kernel is shown to be

(89) 
$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2},$$

where

(90) 
$$z_j = \sqrt{\frac{x_j - s(\sqrt{u} + \sqrt{v})^2}{x_j - s(\sqrt{u} - \sqrt{v})^2}}.$$

We have compared with the genus zero one-point functions of some related theories. It turns out that in the cases of Witten-Kontsevich tau-function (with nonzero  $t_0$ ), partition function of Hermitian onematrix models and modified partition function of Hermitian one-matrix models with even couplings (after the use of t'Hooft coupling constant), the genus zero one-point functions are all related to Catalan numbers, but for the dessin partition function, it is related to Narayana numbers of type A:

(91) 
$$N(A_n, q) = \sum_{k=0}^{n} \frac{1}{n+1} \binom{n+1}{k} \binom{n+1}{k+1} q^k.$$

More precisely,

(92) 
$$G_{0,1}(x) = \sum_{n=1}^{\infty} \frac{s^n}{x^{n+1}} \sum_{k=1}^n \frac{1}{n} \binom{n}{k} \binom{n}{k-1} u^{n+1-k} v^k = \frac{1}{2s} \left( 1 - \frac{s(u+v)}{x} - \sqrt{1 - \frac{2s(u+v)}{x} + \frac{s^2(u-v)^2}{x^2}} \right).$$

The generalized Narayana numbers of type B/C are given by:

(93) 
$$N(B_n, q) = \sum_{k=0}^{n} \binom{n}{k}^2 q^k,$$

and the generalized Narayana numbers of type D are given by:

(94) 
$$N(D_n, q) = 1 + q^n + \sum_{k=1}^{n-1} \left[ \binom{n}{k}^2 - \frac{n}{n-1} \binom{n-1}{k-1} \binom{n-1}{k} \right] q^k.$$

From the generating series

(95) 
$$\sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{n}{k}^2 y^k = \frac{1}{\sqrt{1 - 2x - 2xy + x^2 - 2x^2y + x^2y^2}}$$

we get:

$$\sum_{n=0}^{\infty} \frac{s^n}{x^{n+1}} \sum_{k=0}^{n} \binom{n}{k}^2 u^{n-k} v^k = \frac{1}{\sqrt{x^2 - 2s(u+v)x + s^2(u-v)^2}},$$

and so we conjecture that

(96) 
$$y^{2} = \frac{1}{x^{2} - 2s(u+v)x + s^{2}(u-v)^{2}}$$

is related to the spectral curve for enumeration of "Grothendieck's dessins of Type B/C". Similarly, for Type D,

$$\sum_{n=0}^{\infty} \frac{s^n}{x^{n+1}} \left( u^n + v^n + \sum_{k=1}^{n-1} \left[ \binom{n}{k}^2 - \frac{n}{n-1} \binom{n-1}{k-1} \binom{n-1}{k} \right] u^{n-k} v^k \right)$$

$$= \sum_{n=0}^{\infty} \frac{s^n}{x^{n+1}} \sum_{k=0}^{n} \binom{n}{k}^2 u^{n-k} v^k$$

$$- \sum_{n=1}^{\infty} \frac{(n+1)s^{n+1}}{x^{n+2}} \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k-1} \binom{n}{k} u^{n+1-k} v^k$$

$$= \frac{1}{\sqrt{x^2 - 2s(u+v)x + s^2(u-v)^2}}$$

$$+ s \frac{\partial}{\partial x} \sum_{n=1}^{\infty} \frac{s^n}{x^{n+1}} \sum_{k=1}^{n} \frac{1}{n} \binom{n}{k} \binom{n}{k-1} u^{n+1-k} v^k$$

$$= \frac{s(u+v)}{2x^2} + \frac{2 - \frac{s(u+v)}{x} + \frac{s^2(u-v)^2}{x^2}}{2x\sqrt{1 - \frac{2s(u+v)}{x} + \frac{s^2(u-v)^2}{x^2}}}.$$

We conjecture that it is related to the spectral curve for enumeration of "Grothendieck's dessins of Type D". One can also take u=v in the above discussions.

The Narayana numbers and Catalan numbers have a wide connection to a plethora of mathematical objects, including combinatorics of Coxeter groups, generalized noncrossing partitions, free probability, cluster algebras, etc. (See [8] and [1] for expositions.) We believe the surprising appearance of these numbers in the emergent geometry of Grothendieck's dessins and clean dessins suggest deeper connections to these objects that deserve further investigations.

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Department of Mathematical Sciences, Tsinghua University, Beijing, 100084, China

E-mail address: jianzhou@mail.tsinghua.edu.cn