BIASED PERMUTATIVE EQUIVARIANT CATEGORIES

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Abstract. For a finite group G, we introduce the complete suboperad $\mathcal{Q}_G$ of the categorical G-Barratt–Eccles operad $\mathcal{P}_G$. We prove that $\mathcal{P}_G$ is not finitely generated, but $\mathcal{Q}_G$ is finitely generated and is a genuine E_∞ G-operad (i.e., it is N_∞ and includes all norms). For G cyclic of order 2 or 3, we determine presentations of the object operad of $\mathcal{Q}_G$ and conclude with a discussion of algebras over $\mathcal{Q}_G$, which we call biased permutative equivariant categories.

Introduction

The classifying space functor from categories to topological spaces allows the construction of spaces with desired structure from categories with similar, but usually easier to handle, structure. This is especially true for symmetric monoidal categories (categories with a binary operation that is unital, associative, and commutative up to coherent natural isomorphisms), which give rise to infinite loop spaces. This was proven independently by Segal [Seg74] and May [May72, May74], the latter using the theory of operads.

The particular operad of interest in [May74] is the categorical Barratt-Eccles operad $\mathcal{P}$. Its algebras are *unbiased* permutative categories. On one hand, a (biased) permutative category is a symmetric monoidal category that is strictly associative and unital. Its structure is specified by a finite amount of information: the unit object (0-ary operation), the monoidal product (2-ary operation), and the symmetry (2-ary morphism). This structure is subject to a finite number of axioms. On the other hand, an unbiased permutative category, defined as an algebra over $\mathcal{P}$, is given by a collection of n-ary operations for all $n \geq 0$, that are compatible with each other in a way encoded by the operad. One can easily check that one obtains a biased permutative category from an unbiased one by restricting the structure. A harder result, that relies on the coherence theorem for symmetric monoidal categories [ML63], is that every biased permutative category gives rise to an unbiased one, thus giving a one-to-one correspondence between the two kinds of structure.

One perspective on this correspondence is that the operad given by the objects of $\mathcal{P}$, thought of as an operad in **Set**, is finitely presented. More precisely, this operad is generated by a 0-ary operation (encoding the unit) and by a 2-ary operation (encoding the monoidal product), and all other operations can be obtained from these two using the symmetric group actions and the operad composition. As such, this is all the structure one needs to specify to give an algebra over $P = \operatorname{Ob}\mathcal{P}$. The coherence theorem may then be used to understand $\mathcal{P}$-algebras.

The operad $\mathcal{P}$ is constructed such that its classifying space is an E_∞ operad in spaces, and thus, the classifying space of a permutative category is an E_∞ space, and hence, an infinite loop space upon group completion. In [GM17], Guillou and May construct an equivariant analogue of the categorical Barratt-Eccles operad for a finite group G. This operad, $\mathcal{P}_G$, has the property that its classifying space is a genuine E_∞ G-operad, and thus, its algebras give rise to genuine equivariant infinite loop spaces. Because of this, Guillou and May define permutative G-categories as algebras over $\mathcal{P}_G$.

Following [GMMO18], one may ask if there is a biased definition of permutative G-categories, as there is for permutative categories. One of the main results of this paper, Theorem 2.15 is that in the strictest sense, the answer is no for nontrivial groups G. Indeed, we prove that the object part of $\mathcal{P}_G$ is *not* finitely generated, meaning that one needs to specify infinitely many operations to give an algebra over it.

Using the work of Rubin [Rub17, Rub18], we construct suboperads $\mathcal{Q}_G$ of $\mathcal{P}_G$ that are still E_∞, yet are finitely generated. The key insight from Rubin, which is inspired by the work on N_∞ operads of Blumberg and Hill [BH15], is that the full suboperad generated by a collection of norms will be E_∞, as long as one includes all the norms for orbits as generators.

Finally, in Theorems 3.5 and 3.7 we give explicit presentations for the operads $\mathcal{Q}_G$ in the cases where $G = C_2$ and $G = C_3$. Although the statements of the proofs look very similar, the proofs that the relations given are sufficient are strikingly different. We use these results together with Rubin's coherence theorem for normed symmetric monoidal categories [Rub18] to give a biased definition of $\mathcal{Q}_G$-algebras.

Organization. In Section 1, we recall necessary preliminary notions regarding permutations, operads in general, and the categorical G-Barratt–Eccles operad $\mathcal{P}_G$ and its operad of objects P_G. In Section 2, we prove that P_G is not finitely generated for nontrivial G (Theorem 2.15). In Section 3, we introduce the finitely generated E_∞ G-operads $\mathcal{Q}_G$ and determine presentations of the operad of objects when $G = C_2$ or C_3. Finally, in Section 4, we define the notion of a biased permutative G-category for $G = C_2$ or C_3 and prove that these are in one-to-one correspondence with $\mathcal{Q}_G$-algebras.

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1. Preliminaries

1.1. **Permutations.** Let Σ_n be the symmetric group on n letters. Throughout the paper we denote elements in Σ_n using cycle notation. For $\sigma \in \Sigma_n$, let M_σ denote the permutation matrix representing σ, that is,

$$M_\sigma = \begin{pmatrix} e_{\sigma(1)} & \cdots & e_{\sigma(n)} \end{pmatrix}.$$

For $\sigma \in \Sigma_n$, $k_1, ..., k_n \geq 0$, let $k = k_1 + \cdots + k_n$, and think of $\{k_1, \ldots, k_n\}$ as a partition of $\{1, \ldots, k\}$ into n (possibly empty) blocks. We define the *block permutation* $\sigma\langle k_1, \ldots, k_n\rangle$ to be the permutation in Σ_k that permutes the k blocks according to σ. For example, if $\sigma = (1\ 2\ 3) \in \Sigma_3$, then $M_{\sigma\langle k_1,k_2,k_3\rangle}$ is the block matrix

$$\begin{pmatrix} 0 & 0 & I_{k_3} \\ I_{k_1} & 0 & 0 \\ 0 & I_{k_2} & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix.

Let $\tau_j \in \Sigma_{k_j}$ for $j = 1, \dots, n$. Define the *block sum* $\tau_1 \oplus ... \oplus \tau_n \in \Sigma_k$ to be the permutation that permutes via τ_j within the j-th block. Using permutation matrices, we have

$$M_{\tau_1 \oplus \cdots \oplus \tau_n} = \begin{pmatrix} M_{\tau_1} & 0 & \cdots & 0 \\ 0 & M_{\tau_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{\tau_n} \end{pmatrix}.$$

We may combine these two constructions to form

$$\sigma\langle \tau_1, \dots, \tau_n \rangle = \sigma\langle k_1, \dots, k_n \rangle \cdot (\tau_1 \oplus \cdots \oplus \tau_n)$$

where $\sigma \in \Sigma_n$ and $\tau_i \in \Sigma_{k_i}$ for $i = 1, \dots, n$. For instance,

$$M_{(1\ 2\ 3)\langle \tau_1, \tau_2, \tau_3 \rangle} = \begin{pmatrix} 0 & 0 & M_{\tau_3} \\ M_{\tau_1} & 0 & 0 \\ 0 & M_{\tau_2} & 0 \end{pmatrix}.$$

Finally, we take this opportunity to define two special classes of permutations which we will need to reference in our subsequent work.

Definition 1.1. A permutation $\sigma \in \Sigma_n$ is *simple* if, for any $k, k' \in \{1, \dots, n\}$ and $0 < m < n - 1$, σ does not map $\{k, k+1, \dots, k+m\}$ to $\{k', k'+1, \dots, k'+m\}$. That is, σ does not map any nontrivial proper interval to another nontrivial proper interval. We call a permutation *nonsimple* if it is not simple.

Example 1.2. The permutation $(2\ 3) \in \Sigma_3$ is nonsimple since it takes $\{2, 3\}$ to itself. The permutation $(1\ 2\ 4\ 3) \in \Sigma_4$ is simple. Asymptotically, the fraction of simple permutations in Σ_n is $\frac{1}{e^2}(1 - \frac{4}{n})$ [AAK03].

Remark 1.3. It is perhaps easiest to recognize a nonsimple permutation via its permutation matrix, which necessarily has a block decomposition with one block of size strictly between 1 and n.

We also need the notion of a separable permutation.

Definition 1.4. The *skew sum* of permutations σ and τ is $(1\ 2)\langle \sigma, \tau \rangle$. A permutation is *separable* if it can be obtained from the trivial permutation 1_{Σ_1} by a finite number of block and skew sums.

Example 1.5. The permutation with matrix

$$\begin{pmatrix} & & & 1 & & & & \\ & & 1 & & & & & \\ & & & & 1 & & & \\ 1 & & & & & & & \\ & 1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & 1 \\ & & & & & 1 & & \end{pmatrix}$$

(where the 0's are omitted) is separable.

1.2. **Operads.** The purpose of an operad is to encode families of operations. Since we will study them intensively throughout the rest of the paper, we provide a brief introduction to operads and their algebras here and set notation.

Definition 1.6. Let $\mathcal{V}$ be a cartesian category. An *operad* $\mathcal{O}$ in $\mathcal{V}$ consists of a sequence $\{\mathcal{O}(n)\}_{n\geq 0}$ of objects in $\mathcal{V}$ such that $\mathcal{O}(n)$ has a right Σ_n-action, together with morphisms

$$\gamma\colon \mathcal{O}(n)\times\mathcal{O}(k_1)\times\cdots\times\mathcal{O}(k_n)\longrightarrow\mathcal{O}(k_1+\cdots+k_n)$$

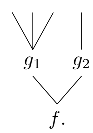
and

$$\mathbb{1}\colon * \longrightarrow \mathcal{O}(1),$$

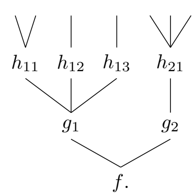
satisfying associativity, unitality and equivariance axioms. See [May72] or [Yau16] for a complete description.

In this paper we will concentrate on operads in **Set**, **Cat**, G**Set** and G**Cat**, where G is a finite group. Note that in these cases we can think of $\mathbb{1}$ as a (G-fixed) element, respectively object, in the (G-)set, respectively (G-)category, $\mathcal{O}(1)$. If $f\in\mathcal{O}(n)$ we say f has *arity* n and write $|f|=n$. In the case of G**Set** and G**Cat**, we often think of $\mathcal{O}(n)$ as a left $G\times\Sigma_n$-object via $(g,\sigma)\cdot f = g\cdot f\cdot\sigma^{-1}$.

Elements of $\mathcal{O}(n)$ should be thought of as operations with n inputs and 1 output, so as such, they will be depicted as trees, with γ depicted as grafting. For example, if $f\in\mathcal{O}(2)$, $g_1\in\mathcal{O}(3)$, and $g_2\in\mathcal{O}(1)$, we depict $\gamma(f;g_1,g_2)\in\mathcal{O}(4)$ as



Associativity of γ can then be interpreted as saying that the grafting of three levels can be done in any order, yielding the same result. Thus the tree



has a unique interpretation as

$$\gamma(f;\gamma(g_1;h_{11},h_{12},h_{13}),\gamma(g_2;h_{21})) = \gamma(\gamma(f;g_1,g_2);h_{11},h_{12},h_{13},h_{21}).$$

For $m\geq 1$, $n\geq 0$ and $1\leq i\leq m$, we define the ith *partial composition*

$$\circ_i\colon \mathcal{O}(m)\times\mathcal{O}(n)\longrightarrow\mathcal{O}(m+n-1)$$

as the composite

$$\mathcal{O}(m)\times\mathcal{O}(n)\to\mathcal{O}(m)\times\mathcal{O}(1)^{i-1}\times\mathcal{O}(n)\times\mathcal{O}(1)^{m-i}\xrightarrow{\gamma}\mathcal{O}(m+n-1),$$

where the first arrow is induced by the map $\mathbb{1}\colon *\to\mathcal{O}(1)$. In terms of elements, the ith partial composition is given by

$$f\circ_i g = \gamma(f;\mathbb{1},\ldots,\mathbb{1},g,\mathbb{1},\ldots,\mathbb{1}),$$

where g is in the ith position of the m-tuple. It should be thought of as grafting g onto the ith leaf of f and prolonging the rest of the leaves appropriately.

Definition 1.7. An *$\mathcal{O}$-algebra* is given by a pair (X, μ), where X is an object of $\mathcal{V}$, and μ is a collection of morphisms

$$\mu_n \colon \mathcal{O}(n) \times X^n \longrightarrow X$$

in $\mathcal{V}$ satisfying equivariance conditions and compatibility with γ and $\mathbb{1}$.

Definition 1.8. Let $\mathcal{N}$ and $\mathcal{O}$ be operads in $\mathcal{V}$. A *map of operads* $f \colon \mathcal{N} \to \mathcal{O}$ consists of a Σ_n-equivariant morphism $f_n \colon \mathcal{N}(n) \to \mathcal{O}(n)$ for all $n \geq 0$, such that they respect the unit and the operadic composition.

When $\mathcal{V} = G\mathbf{Cat}$, we say f is an *equivalence* if the map of fixed points

$$f_n^\Gamma \colon \mathcal{N}(n)^\Gamma \to \mathcal{O}(n)^\Gamma$$

is a weak equivalence on passage to classifying spaces for all $\Gamma \subseteq G \times \Sigma_n$.

We note that if $F \colon \mathcal{V} \to \mathcal{W}$ is a product-preserving functor and $\mathcal{O}$ is an operad in $\mathcal{V}$, then $F(\mathcal{O})$ will form an operad in $\mathcal{W}$ with all the structure induced from that of $\mathcal{O}$ (see [Yau16, Theorem 11.5.1] for a more general version of this result). Most of the operads used in this paper will be constructed this way from the following example.

Example 1.9. The *associativity operad* in **Set** is given by the sequence $\mathbf{Assoc}(n) = \Sigma_n$, with (right) Σ_n-action given by right multiplication. The composition

$$\gamma \colon \Sigma_n \times \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \longrightarrow \Sigma_{k_1 + \cdots + k_n}$$

is given by

$$\gamma(\sigma; \tau_1, \ldots, \tau_k) = \sigma\langle \tau_1, \ldots, \tau_n \rangle.$$

The identity $\mathbb{1}$ is given by $1_{\Sigma_1} \in \Sigma_1$. Algebras over P are (unbiased) associative and unital monoids.

1.3. **The categorical Barratt-Eccles operad and its equivariant analogue.** The categorical Barratt-Eccles operad plays an important role in the theory of infinite loop spaces. To construct it, we first recall the chaotic category functor $\widetilde{(-)} \colon \mathbf{Set} \to \mathbf{Cat}$.

Definition 1.10. Given a set X, we denote by $\widetilde{X}$ the category with objects given by X and a unique morphism between any two objects. It is called the *chaotic category on X*.

The construction above extends to a functor $\widetilde{(-)} \colon \mathbf{Set} \to \mathbf{Cat}$ that is right adjoint to object functor $\mathbf{Cat} \to \mathbf{Set}$. As a right adjoint, $\widetilde{(-)}$ preserves products and hence, sends operads in **Set** to operads in **Cat**.

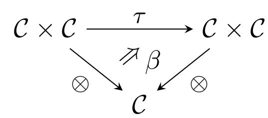
Definition 1.11. The *categorical Barratt-Eccles operad* $\mathcal{P}$ is the operad in **Cat** defined as $\widetilde{\mathbf{Assoc}}$. In particular, $\mathcal{P}(n) = \widetilde{\Sigma_n}$.

We recall the definition of a permutative category.

Definition 1.12. A *permutative category* consists of

- a category $\mathcal{C}$;
- an object $e \in \mathcal{C}$;
- a functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;

- a natural isomorphism



  called the *symmetry*, whose components are given by morphisms $\beta_{a,b} \colon a \otimes b \to b \otimes a$ in $\mathcal{C}$.

The data above are subject to the following axioms

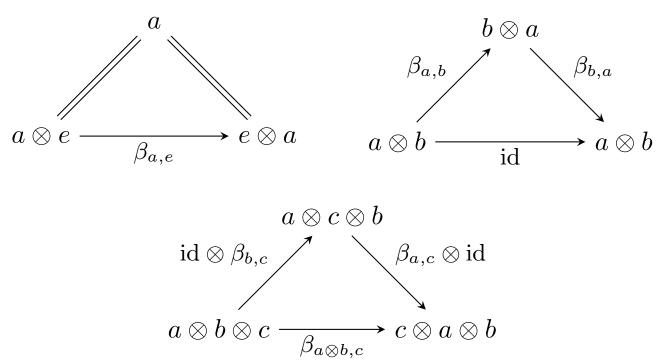
(i) e is a strict two-sided unit for $\otimes$ that is, for all $a \in \mathcal{C}$,

$$e \otimes a = a = a \otimes e;$$

(ii) $\otimes$ is strictly associative: for all $a, b, c \in \mathcal{C}$,

$$a \otimes (b \otimes c) = (a \otimes b) \otimes c;$$

(iii) for all $a, b, c \in \mathcal{C}$, the following diagrams commute



As noted in the introduction, algebras over $\mathcal{P}$ are in one-to-one correspondence with permutative categories [May74] with e and $\otimes$ induced by 1_{Σ_0} and 1_{Σ_2}, respectively, and β induced by the unique morphism in $\mathcal{P}(2)$ from 1_{Σ_2} to $(1\ 2)$.

For a finite group G, we also define the categorical G-equivariant Barratt-Eccles operad. We use the functor $\mathbf{Set}(G, -) \colon \mathbf{Set} \to G\mathbf{Set}$ that takes a set X to the set $\mathbf{Set}(G, X)$ of all functions from G to X with left G-action given as follows. For $g \in G$ and $f \in \mathbf{Set}(G, X)$, the function $g \cdot f$ sends $h \in G$ to $f(hg)$. This is a product-preserving functor, and as such we can use it to transfer operads from $\mathbf{Set}$ to $G\mathbf{Set}$.

Definition 1.13. The operad P_G in $G\mathbf{Set}$ is defined as $\mathbf{Set}(G, \mathbf{Assoc})$. In particular, an element in $P_G(n)$ is a function (not necessarily a group homomorphism) $f \colon G \to \Sigma_n$. The *categorical G-equivariant Barratt-Eccles operad* $\mathcal{P}_G$ is the operad in $G\mathbf{Cat}$ defined as $\widetilde{P_G}$. Algebras over $\mathcal{P}_G$ are called *permutative G-categories.*

Remark 1.14. A standard calculation shows that the operad $\mathcal{P}_G$ can be alternatively defined as $\mathbf{Cat}(\widetilde{G}, \widetilde{\mathbf{Assoc}})$. As noted in [Rub17, Example 3.8], the operad $\mathcal{P}_G$ is isomorphic but not equal to the one defined in [GM17], the main difference being that the G-actions are slightly different. There the authors prove that upon geometric realization, one obtains an E_∞ G-operad in $G\mathbf{Top}$.

1.4. **Presentations for operads in GSet.** We conclude this section by recalling how presentations of operads work. The basic characters are free operads (in G-sets) and quotients. In [Rub17, §4], Rubin presents a model for the free G-operad on a Σ-free symmetric sequence of G-sets in terms of labeled trees. Our starting point is a sequence of sets, from which we build the free Σ-free symmetric sequence of G-sets and then apply Rubin's construction; in fact, this simplifies the construction somewhat since the intermediate step comes equipped with a canonical choice of Σ_n-orbit representatives. (In Rubin's notation, $R_n = G \times S_n \times \{1_{\Sigma_n}\}$.)

Definition 1.15. Let $S = \{S_n\}$ be a sequence of sets and take $G \times S \times \Sigma = \{G \times S_n \times \Sigma_n\}$. Then the *free operad* in G-sets on S, denoted $\mathbb{F}S$, consists of Rubin's construction applied to $G \times S \times \Sigma$.

In the end, elements of $\mathbb{F}S(n)$ are isomorphism classes of finite rooted planar trees with n leaves and k-ary nodes labeled by elements of $G \times S_k$; the entire $(G \times S)$-labeled tree is then further labeled by an element of Σ_n. The Σ-action is the obvious one, operadic composition is given by grafting trees, and the G-action simply multiplies the G-label of each node. (We do not need the recursive definition of [Rub17, §4.3.1] because of the special form of $G \times S \times \Sigma$.) For further details and instructive pictures, we send the reader to [Rub17, §4].

We now move on to quotients of free operads. Our primary reference is [Rub18, §4.3].

Definition 1.16. Let $\mathcal{O}$ be an operad in G-sets. A *congruence relation* on $\mathcal{O}$ is a graded equivalence relation $\sim\, = (\sim_n)_{n\geq 0}$ which respects the $G \times \Sigma$-action and operadic composition. If $R = (R_n)_{n\geq 0}$ is a graded binary relation on $\mathcal{O}$, then the smallest congruence relation containing $\mathcal{O}$, denoted $\langle R\rangle$, is the *congruence relation generated by R.*

Remark 1.17. Congruence relations are closed under intersection, and $R = (\mathcal{O}(n))_{n\geq 0}$ constitutes a congruence relation, so we may construct $\langle R\rangle$ by taking the intersection of all congruence relations containing R.

We can form quotients of operads in G**Set** by congruence relations satisfying the expected universal property; see [Rub18, Proposition 4.7]. To be specific, given an operad $\mathcal{O}$ and a congruence relation $\sim$, there is a G-operad $\mathcal{O}/\sim$ and operad map $\mathcal{O} \to \mathcal{O}/\sim$ such that any other operad map $\mathcal{O} \to \mathcal{O}'$ which respects $\sim$ factors uniquely through $\mathcal{O}/\sim$.

Finally, we note the following proposition which will be important when we pass from operads in G-sets to operads in G-categories via the chaotic functor.

Proposition 1.18 ([Rub18, Proposition 4.12])**.** *Suppose that $\mathcal{O}$ is an operad in G-sets, R is a binary relation on $\mathcal{O}$, and $f : \widetilde{\mathcal{O}} \to \mathcal{N}$ is a map of operads in G-categories. Then f factors through the quotient map $\widetilde{\mathcal{O}} \to \widetilde{\mathcal{O}}/\langle R\rangle = \widetilde{\mathcal{O}/\langle R\rangle}$ if and only if f respects R on objects and $f(x \to y) = \mathrm{id} : f(x) \to f(y)$ whenever xRy. In such a case, the induced map $\widetilde{\mathcal{O}}/\langle R\rangle \to \mathcal{N}$ is unique.*

2. The G-Barratt-Eccles operad is not finitely generated

In this section we prove that P_G is not finitely generated for $|G| > 1$. To do so, we introduce the notion of the suboperad generated by a sequence of subsets.

Recall that $\mathcal{N} \subseteq \mathcal{O}$ is a suboperad if $\mathcal{N}(n) \subseteq \mathcal{O}(n)$ is a Σ_n-subset for all n, $\mathbb{1} \in \mathcal{N}(1)$, and $\mathcal{N}$ is closed under the operadic composition for $\mathcal{O}$.

Definition 2.1. For an operad $\mathcal{O}$ in G**Set** and a sequence of subsets $S = \{S_i : S_i \subseteq \mathcal{O}(i)\}$, the suboperad generated by S, denoted $\langle S\rangle$ is the smallest suboperad of $\mathcal{O}$ such that $S_i \subseteq \langle S\rangle(i)$ for all i. The operad $\mathcal{O}$ is called *finitely generated* if there exists such an S with $\left|\bigcup_{i\in\mathbb{N}} S_i\right| < \infty$ and $\langle S\rangle = \mathcal{O}$.

Remark 2.2. The definition permits S_i to be empty, and it is necessary that $S_i = \varnothing$ for sufficiently large i in order for S to witness finite generation of $\mathcal{O}$.

Remark 2.3. Note that taking the elements of S as abstract symbols, one can construct a surjective map of operads $\mathbb{F}S \to \langle S\rangle$, and thus, one can construct the latter as a quotient of the former.

The reader may check that we may explicitly model $\langle S\rangle$ in the following fashion.

Proposition 2.4. *If $\mathcal{O}$ is an operad in G-sets and $S = \{S_i : S_i \subseteq \mathcal{O}(i)\}$, then $\langle S\rangle(k)$ is the set of Σ_k-actions on operadic compositions of G-actions on elements of S,* i.e.,

$$\langle S\rangle(k) = \left\{ ((g_0 s_0) \circ_{i_1} (g_1 s_1) \circ_{i_2} \cdots \circ_{i_m} (g_m s_m)) \cdot \sigma \;\middle|\; \begin{array}{c} m \in \mathbb{N}, g_i \in G, s_i \in S_{k_i}, \\ \sum k_i = k + m, \sigma \in \Sigma_k \end{array} \right\}.$$

We now introduce two further notions of generation that will be important in our proof that P_G is not finitely generated.

Definition 2.5. An element $f \in P_G(n)$ is *γ-generated from below* if there exist $s, h_1, \ldots, h_{|s|} \in \bigcup_{i=0}^{n-1} P_G(i)$ such that

$$f = \gamma(s; h_1, \ldots, h_{|s|}).$$

An element $f' \in P_G(n)$ is *generated from below* if it is of the form $f \cdot \sigma$ for f γ-generated from below and $\sigma \in \Sigma_n$.

Remark 2.6. The G-equivariance axiom on γ guarantees that the set of elements of arity n that are generated from below is closed under the G-action.

We now consider how the notions of γ-generation from below and generation from below interact with a special class of elements of P_G, the primitive ones:

Definition 2.7. Call $f \in P_G(n)$ *primitive* if $f(1_G) = 1_{\Sigma_n}$. If f is not primitive, we call it *nonprimitive*.

Remark 2.8. For each $f \in P_G(n)$, the permutation $f(1_G)^{-1}$ is the unique $\sigma \in \Sigma_n$ such that $f \cdot \sigma$ is primitive.

Lemma 2.9. *Suppose f is γ-generated from below with $f = \gamma(s; h_1, \ldots, h_{|s|})$, $|s|; |h_1|, \ldots, |h_{|s|}| < n$. Then f is primitive if and only if it can be written as $f = \gamma(s'; h'_1, \ldots, h'_{|s'|})$, where $s', h'_1, \ldots, h_{s'}$ are all primitive elements of arity greater than 0 and less than n.*

Remark 2.10. This might seem obvious, but becomes less so once one considers the (unique) 0-ary operation e. If we view $s(1_G)$ as a permutation matrix, the following proof essentially justifies ignoring the instances of e in the set $\{h_i\}_i$ by changing s appropriately.

Proof of Lemma 2.9. The reverse direction is trivial.

For the forward direction, suppose $f = \gamma(s; h_1, h_2, \dots, h_{|s|})$. Let $h'_1, h'_2, \dots$ be the terms of $h_1, \dots, h_{|s|}$ not equal to e in ascending order, and let

$$s' = \gamma(s; t_1, \dots, t_{|s|}) \quad \text{where} \quad t_j = \begin{cases} \mathbb{1} & \text{if } h_j \neq e \\ e & \text{if } h_j = e. \end{cases}$$

Thus by associativity and unitality of γ, we have that $f = \gamma(s'; h'_1, h'_2, ..h'_{|s'|})$ and hence,

$$1_{\Sigma_{|f|}} = f(1_G) = s'(1_G)\langle h'_1(1_G), \dots, h'_{|s'|}(1_G)\rangle.$$

Note that if $h_i(1_G)$ is not the identity for some i, then the expression on the right hand side cannot be the identity. Since all of the h_i are of arity at least 1, $s'(1_G)$ must also be the identity, as desired. □

Lemma 2.11. *If $f \in P_G(n)$ is primitive and generated from below (but not necessarily γ-generated from below), then f can be written in the form $f = \gamma(s; h_1, \dots, h_{|s|})$ where $s, h_1, \dots, h_{|s|}$ are all primitive and of arity less than n.*

Proof. Suppose some primitive $f \in P_G(n)$ is generated from below, meaning

$$f = \gamma(s; h_1, h_2, ..h_{|s|}) \cdot \sigma$$

for some $\sigma \in \Sigma_n$ and $|s|; |h_1|, \dots, |h_{|s|}| < n$. Then let $\rho = s(1_G)$, $s' = s \cdot \rho^{-1}$, $\tau_i = h_i(1_G)$, and

$$h'_i = h_i \cdot \tau_i^{-1}$$

for $i = 1, \dots, |s|$. Note that $s'_i; h'_1, \dots, h'_{|s|}$ are all necessarily primitive. It follows from the Σ-equivariance axioms of an operad that

$$f = \gamma(s'; h'_{\rho^{-1}(1)}, \dots, h'_{\rho^{-1}(|s|)}) \cdot (\rho\langle \tau_1, \dots, \tau_{|s|}\rangle\sigma).$$

Let $\sigma' = \rho\langle \tau_1, \dots, \tau_{|s|}\rangle\sigma$ and $f' = \gamma(s'; h'_{\rho^{-1}(1)}, \dots, h'_{\rho^{-1}(|s|)})$, meaning f' is primitive, since all the arguments are primitive. Then $f = f' \cdot \sigma'$. Since f is also primitive, we know that $\sigma' = 1_{\Sigma_{|f|}}$ and thus $f = f'$, which is the desired form. □

Recall the notion of a *nonsimple permutation* from Definition 1.1.

Lemma 2.12. *Suppose $f \in P_G(n)$ is primitive and γ-generated from below. Then $f(g)$ is nonsimple for all $g \in G$.*

Proof. Assume $f = \gamma(s; h_1, \dots, h_{|s|})$ with the arity of all arguments less than n. By Lemma 2.9, we may suppose $|h_i| \geq 1$ for all i. Moreover, since we assumed f is generated from below, $1 < |h_k| < n$ for some k. Fix an arbitrary $g \in G$. Then

$$f(g) = s(g)\langle h_1(g), \dots, h_{|s|}(g)\rangle.$$

Thus, by definition, $f(g)$ permutes the intervals of length $|h_1|, \dots, |h_{|s|}|$ according to $s(g)$, making it nonsimple. □

Proposition 2.13. *If an element $f \in P_G(n)$ is generated from below, then $f(g)f(1_G)^{-1}$ is a nonsimple permutation for all $g \in G$.*

Proof. Suppose $f \in P_G(n)$ is generated from below, and let $f' = f \cdot f(1_G)^{-1}$. Then f' is primitive and generated from below, so by Lemma 2.11 it is γ-generated from below. Thus, by Lemma 2.12, $f'(g) = f(g)f(1_G)^{-1}$ is nonsimple for all $g \in G$. □

Corollary 2.14. *Let G be a nontrivial finite group. For $n > 3$, there exists at least one element $f \in P_G(n)$ such that f is not generated from below.*

Proof. Note that at any arity $n > 3$, there exists at least one simple permutation. Therefore, at any arity $n > 3$, there exists a primitive element $f \in P_G$ and $g \in G \smallsetminus \{1_G\}$ such that $f(g)$ is simple, and so f is not generated from below. □

Theorem 2.15. *The operad P_G is not finitely generated for $|G| > 1$.*

Proof. In any candidate finite generating set S, there is an element of highest arity. Call this highest arity m. Let $m' = m + 4$ (to ensure $m' > 3$). Then $\langle S \rangle \subseteq \langle \bigcup_{i=0}^{m'-1} P_G(i) \rangle$. There exists an element in $P_G(m')$ that is not generated from below, and therefore not generated by S. □

3. Finitely generated E_∞ G-operads

We have seen that the object operad P_G of the G-equivariant Barratt-Eccles operad is not finitely generated for nontrivial G. This makes the task of explicitly describing $\mathcal{P}_G$-algebras with a finite amount of data seem intractable. Fortunately, P_G admits finitely generated suboperads $Q_G \subseteq P_G$ such that $\mathcal{Q}_G := \widetilde{Q_G} \simeq \mathcal{P}_G$. In this section, we introduce the operads Q_G for arbitrary G, and then give explicit presentations of Q_G for G a cyclic group of order 2 or 3.

3.1. **The operads $\mathcal{Q}_G$.** Fix a finite group G. Morally speaking, the suboperad Q_G is generated by the operations $e \in P_G(0)$, $\otimes \in P_G(2)$ (the constant function at 1_{Σ_2}), and norms for all G-orbits. In order to make the last notion precise, we make the following three definitions.

Definition 3.1 (cf. [Rub17, Definition 2.5]). Given a finite ordered G-set T, write $\otimes_T : G \to \Sigma_{|T|}$ for the permutation representation of T. Considered as an element of $P_G(|T|)$, we call $\otimes_T$ an *external norm for T*.

Note that e is the norm for $\varnothing$ and $\otimes$ is the norm for any 2-element set with trivial G-action.

Definition 3.2. For a finite group G, let $\mathfrak{O}$ be a set of ordered transitive G-sets. Call $\mathfrak{O}$ a *complete set of ordered G-orbits* if it contains exactly one non-trivial transitive G-set of each isomorphism class (forgetting ordering).

Clearly, we may produce a complete set of ordered G-orbits by arbitrarily ordering each G/H as H ranges through a set of representatives of conjugacy classes of subgroups of G.

Definition 3.3. Given a complete set of ordered G-orbits $\mathfrak{O}$, let

$$Q_{G,\mathfrak{O}} := \langle \otimes_T \mid T \in \mathfrak{O} \cup \{\varnothing, \{0,1\}\} \rangle \subseteq P_G$$

where $\{0,1\}$ has trivial G-action and $0 < 1$. We define $\mathcal{Q}_{G,\mathfrak{O}}$ as the chaotic operad on $Q_{G,\mathfrak{O}}$, and we note that it is a full suboperad of $\mathcal{P}_G$. We call $\mathcal{Q}_{G,\mathfrak{O}}$ the *complete suboperad of $\mathcal{P}_G$ relative to $\mathfrak{O}$*. If the choice of $\mathfrak{O}$ is understood from context, then we will write $\mathcal{Q}_G$ for $\mathcal{Q}_{G,\mathfrak{O}}$ and call it a *complete suboperad of $\mathcal{P}_G$*.

Since any complete set of ordered G-orbits is finite, the operads $Q_{G,\mathfrak{O}}$ are finitely generated. They also have the following remarkable property.

Theorem 3.4. *For any finite group G and complete set of ordered G-orbits $\mathfrak{O}$, $\mathcal{Q}_{G,\mathfrak{O}}$ is an E_∞ G-operad and the inclusion $\mathcal{Q}_{G,\mathfrak{O}} \hookrightarrow \mathcal{P}_G$ is an equivalence of G-operads.*

Proof. The operad $Q_{G,\mathfrak{O}}$ is Σ-free since it is a suboperad of P_G, and it is a quotient of $\mathbb{F}(\{\mathfrak{O} \cup \{\varnothing, \{0,1\}\})$. As noted in [Rub18, Example 6.5], the latter is an E_∞ G-operad, i.e., it is an N_∞ operad that contains all norms. Thus, the same is true for $Q_{G,\mathfrak{O}}$ and the result follows. □

3.2. **Presentation for Q_{C_2}.** In this subsection, we specialize to $G = C_2$, which we consider to have generator g. Note that the operad Q_{C_2} has three generators:

(1) $e \in P_{C_2}(0)$,
(2) $\otimes \in P_{C_2}(0)$, which is the function constant at the identity permutation, and
(3) $\boxtimes \in P_{C_2}(2)$, which sends 1_{C_2} to the identity permutation 1_{Σ_2} and g to the permutation $(1\ 2)$.

We thus have a map of operads $\phi : \mathbb{F}\{e, \otimes, \boxtimes\} \to Q_{C_2}$ determined by sending each of the generators to its namesake. The following theorem gives a presentation for Q_{C_2}.

Theorem 3.5. *The operad Q_{C_2} is isomorphic to $\mathbb{F}\{e, \otimes, \boxtimes\}/\langle R\rangle$, where R consists of the following:*

(1) Strict unit:

$$\gamma(\boxtimes; \mathbb{1}, e) = \gamma(\boxtimes; e, \mathbb{1}) = \gamma(\otimes; \mathbb{1}, e) = \gamma(\otimes; e, \mathbb{1}) = \mathbb{1};$$

(2) Strict associativity: for any primitive $\diamondsuit \in Q_{C_2}(2)$,

$$\gamma(\diamondsuit; \diamondsuit, \mathbb{1}) = \gamma(\diamondsuit; \mathbb{1}, \diamondsuit);$$

(3) Group action: $g \cdot \boxtimes = \boxtimes \cdot (1\ 2)$, $g \cdot \otimes = \otimes$, and $g \cdot e = e$.

The reader can check that all these relations are indeed satisfied in P_{C_2}, and hence in Q_{C_2}. The hard work is to show that these generate all the relations.

We are now ready to prove the main theorem of this subsection.

Proof of Theorem 3.5. Let $\mathcal{O} = \mathbb{F}\{e, \otimes, \boxtimes\}/\langle R\rangle$. By Remark 2.3, the map ϕ induces a level-wise surjective map $\phi : \mathcal{O} \to Q_{C_2}$. Call an element of $\mathcal{O}(n)$ *primitive* when it has a representative for which all nodes are labeled by 1_G and the tree is labeled by 1_{Σ_n}; call an element $f \in Q_{C_2}(n)$ *primitive* when $f(1_G) = 1_{\Sigma_n}$. By the equivariance axiom and relation (3), it suffices to prove that ϕ induces a bijection $\operatorname{Prim}\mathcal{O}(n) \to \operatorname{Prim} Q_{C_2}(n)$ for all n.

Let $\mathcal{T}_n \subset \mathbb{F}\{e, \otimes, \boxtimes\}(n)$ be the set of planar rooted trees with nodes labeled by e, $\otimes$ and $\boxtimes$, i.e., without using the G and Σ actions. Let $\mathcal{CT}_n \subseteq \mathcal{T}_n$ denote the elements of $\mathbb{F}\{e, \otimes, \boxtimes\}$ derived from trees with only $\otimes$ and $\boxtimes$ nodes and such that no instance of $\otimes$ is grafted directly to the right branch of another instance of $\otimes$, and similarly, no instance of $\boxtimes$ is grafted directly to the right branch of another instance of $\boxtimes$; we call elements of $\mathcal{CT}_n$ *canonical trees*. We claim that all primitive elements of $\mathcal{O}(n)$ are represented uniquely by a canonical tree $t \in \mathcal{CT}_n$.

To prove this claim, we use an inductive argument on the number of *violations*, $v(t)$. For $t \in \mathcal{T}_n$, define $v(t)$ to be the number of instances of $\diamondsuit$ being the right branch of another instance of $\diamondsuit$, where $\diamondsuit = \otimes$ or $\boxtimes$. Then $t \in \mathcal{CT}_n$ if and only if $v(t) = 0$. If $f \in \operatorname{Prim}\mathcal{O}(n)$ is represented by some $t \in \mathcal{T}_n$ with $v(t) > 0$, then we may use the associativity relation to replace t with t' where $v(t') < v(t)$ and t' still represents f. This proves the claim by induction on $v(t)$.

We now know that ϕ induces a surjection from $\mathcal{CT}_n$ to $\operatorname{Prim} Q_{C_2}(n)$. Furthermore, for $t \in \mathcal{CT}_n$, the value of $\phi(t)$ on the generator of C_2 is the separable permutation produced by interpreting the leaves as 1_{Σ_1}, $\otimes$ as block sum, and $\boxtimes$ as skew sum. By the proof of [SS91, Theorem 1], separable permutations are in fact in bijection with canonical trees, and we conclude that the restriction of ϕ onto $\mathcal{CT}_n$ is a bijection onto the primitive elements of $Q_{C_2}(n)$. It follows that $\phi : \operatorname{Prim}\mathcal{O}(n) \cong \operatorname{Prim} Q_{C_2}(n)$, as desired. □

As an aside, we note that it is possible to enumerate $\operatorname{Prim} Q_{C_2}(n)$ in terms of the large Schröder numbers.

Definition 3.6. The *large Schröder numbers* are the integers S_i with $S_0 = 1$, $S_1 = 2$, and S_n for $n \geq 2$ given by the recurrence relation

$$S_n = 3S_{n-1} + \sum_{k=1}^{n-2} S_k S_{n-k-1}.$$

The first several terms in the sequence are

$$1,\ 2,\ 6,\ 22,\ 90,\ 394,\ 1,806,\ 8,558,\ 41,586,\ 206,098,\ 1,037,718,\ \ldots.$$

By [SS91, Theorem 1] and the bijection in our proof of Theorem 3.5, we know that

$$|\operatorname{Prim} Q_{C_2}(n)| = S_{n-1}.$$

We initially discovered the connection between Q_{C_2} and separable permutations via computer experimentation and reference to Sloane's OEIS [Slo19].

3.3. **Presentation for Q_{C_3}.** Now we consider $G = C_3$, and we will denote one of the generators by g. As it is the case for C_2, the operad Q_{C_3} has three generators:

(1) $e \in P_{C_3}(0)$,
(2) $\otimes \in P_{C_3}(2)$, which is the function constant at the identity permutation, and
(3) $\boxtimes \in P_{C_3}(3)$, which sends 1_{C_3} to the identity permutation 1_{Σ_3}, g to the permutation $(1\ 2\ 3)$, and g^2 to $(1\ 3\ 2)$.

We thus have a map of operads $\phi\colon \mathbb{F}\{e, \otimes, \boxtimes\} \to Q_{C_3}$ determined by sending each of the generators to its namesakes. The main goal of this section is to prove the following theorem.

Theorem 3.7. *There is an isomorphism of operads $\mathbb{F}\{e, \otimes, \boxtimes\}/\langle R\rangle \cong Q_{C_3}$, where R consists of the following:*

(1) Reduction to identity: there is only one element in $P_{C_3}(1)$, hence

$$\gamma(\boxtimes; \mathbb{1}, e, e) = \gamma(\boxtimes; e, \mathbb{1}, e) = \gamma(\boxtimes; e, e, \mathbb{1}) = \gamma(\otimes; \mathbb{1}, e) = \gamma(\otimes; e, \mathbb{1}) = \mathbb{1};$$

(2) Strict associativity: primitive elements in $Q_{C_3}(2)$ follow strict associativity, i.e., for any $\diamondsuit \in Q_{C_3}(2)$,

$$\gamma(\diamondsuit; \diamondsuit, \mathbb{1}) = \gamma(\diamondsuit; \mathbb{1}, \diamondsuit);$$

(3) Group action: $g \cdot \boxtimes = \boxtimes \cdot (1\ 2\ 3)$, $g \cdot \otimes = \otimes$, $g \cdot e = e$.

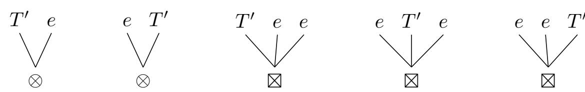
The reader can check that all these relations are indeed satisfied in P_{C_3}, and hence in Q_{C_3}. The hard work is to show that these generate all the relations.

As in the C_2 case, consider the set $\mathcal{T}_n \subset \mathbb{F}\{e, \otimes, \boxtimes\}$ given by those planar rooted trees with nodes labeled by e, $\otimes$ and $\boxtimes$, i.e., without using the G and Σ actions.

Definition 3.8. Let f be a primitive element in $Q_{C_3}(n)$. A *tree representation* of f is a planar rooted tree $T \in \mathcal{T}_n$ such that $\phi(T) = f$. We call a tree representation *reduced* if at any node the number of branches that are not marked by e is at least 2.

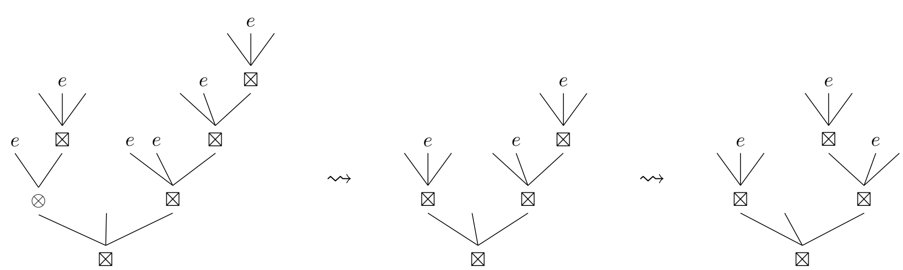
A *canonical tree representation* of f is a reduced tree representation of f, such that for any primitive binary node $\diamondsuit$, the node grafted on its right input, if there is any, is different than $\diamondsuit$. Let $\mathcal{CT}_n$ denote the subset of $\mathcal{T}_n$ given by canonical trees.

Remark 3.9. Given a tree representation T of $f \in Q_{C_3}(n)$, we can get a reduced tree representation T_r by replacing every instance of any of



by T' itself. This corresponds to instances of the first relation. Then we can also get a canonical tree representation T_c of T from T_r by rotating nodes to the left when possible, i.e., replace every instance of

where $\diamondsuit \in Q_{C_3}(2)$ is primitive, and T_1, T_2, T_3 are trees. For example, the process can be described visually as follows:



As in the proof of Theorem 3.5, we have a surjective map

$$\mathcal{CT}_n \to \operatorname{Prim} Q_{C_3}(n).$$

Moreover, we can think of $\mathcal{CT}_n$ as a choice of representatives for the set of equivalence classes of $\mathcal{T}_n$ modulo the reduction to identity and strict associativity relations of Theorem 3.7. Thus, if we can prove this map is injective we will prove that there are no other relations amongst primitive elements in Q_{C_3}. We will show this by examining the matrices given by these trees when evaluated at g and g^2.

Definition 3.10. Let $T \in \mathcal{CT}_n$. An *uncovered node* of T is a node such that no node of arity greater than 2 is grafted upon it, i.e., all its leaves are marked by e or unmarked.

Definition 3.11. Let A, B be $n \times n$ permutation matrices and C, D be $k \times k$ permutations matrix for some $k \leq n$. We say C and D occur *j-column-simultaneously* *in A and B, respectively* if C and D appear as blocks within A and B, starting on the j-th column.

Lastly, we will call the collection of $\boxtimes$ and the four canonical binary trees, which are $\otimes$ and the three possible graftings of e on $\boxtimes$, *the essential nodes*. Their corresponding functions in Q_{C_3} have outputs as in Table 3.12. Since their corresponding functions are distinct, we will not distinguish between these nodes and their associated function. The following lemma describes how the uncovered nodes in a canonical tree T are detected by the outputs of the corresponding function $\phi(T)$.

essential primitive nodes	output of g	output of g^2
$\otimes$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\boxtimes$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
$\gamma(\boxtimes; e, \mathbb{1}, \mathbb{1})$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\gamma(\boxtimes; \mathbb{1}, e, \mathbb{1})$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\gamma(\boxtimes; \mathbb{1}, \mathbb{1}, e)$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

TABLE 3.12. Outputs of essential primitive elements for C_3

Proposition 3.13 (Uncovered Nodes)**.** *Let $T \in \mathcal{CT}_n$, $f = \phi(T)$ be the primitive element in $Q_{C_3}(n)$ it represents, t an essential node of arity a, and $j = 1, \ldots, n-a+1$. Then the following are equivalent:*

(1) The permutation matrices for $f(g)$ and $f(g^2)$ contain a j-column-simultaneous instance of $t(g)$ and $t(g^2)$, respectively, but not a $(j-1)$-column simultaneous instance.

(2) There exists $T' \in \mathcal{CT}_{n-a+1}$ such that $T = T' \circ_j t$.

Proof. Proving that (2) implies (1) for all five nodes is trivial, since this follows from the definition of the operadic composition γ and the fact that we are dealing with canonical trees. The real heart of the matter is the converse, which is a five-in-one proof, one for each of the five essential primitive nodes. The structure of each is similar: induction on n for all canonical trees at level n.

Let us begin with $t = \boxtimes$. The base case is to consider $\mathcal{CT}_3$, which contains 29 distinct elements. Thus $\boxtimes$ is the only element of $\mathcal{CT}_3$ that contains a column-simultaneous $\boxtimes$-pattern.

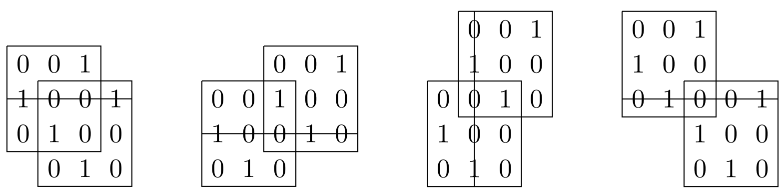
Now suppose that the statement is true in $\mathcal{CT}_m$ for all $m \leq n$, and consider a canonical tree in $\mathcal{CT}_{n+1}$ representing some function f. Assume that we have a j-simultaneous $\boxtimes$-pattern. There must be an uncovered node, t' of arity a, attached to the k-th position of some canonical tree of lesser arity.

If $k = j$ and $t' = \boxtimes$, we are done. If not, we remove t' and call the new associated function f'. The matrices for $f'(g)$ and $f'(g^2)$ are obtained from the matrices of $f(g)$ and $f(g^2)$, respectively, by removing columns k through $k + a - 1$

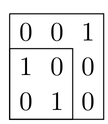
and the corresponding a rows in which the k-column-simultaneous pattern appear, say row i through $i+a-1$, and then put the k-th column and i-th row back, with their intersection entry being 1 and all other entries 0. This 1 corresponds to the unmarked leaf at the k-th entry of the new canonical tree.

Now we want to show that not removing the entire j-column-simultaneous $\boxtimes$-pattern implies we have not removed any of the pattern. If this is the case, we have removed a node and arrived at a smaller arity canonical tree where we can use our induction hypothesis to say that the column-simultaneous $\boxtimes$-pattern indeed corresponds to an uncovered node, and that is not changed when we restore the removed node t'. We check for the impossibility of partial intersection for each of the five essential primitive nodes.

We begin with $\boxtimes$ itself. Here it suffices to show that there is simply no way for $\boxtimes(g)$ to partially-intersect another $\boxtimes(g)$ without destroying the permutation matrix. Note that if the intersection contains a column or row with only 0s, then the pattern will contain two 1s in the same column or row, which means it can't be a subpattern of a permutation matrix. One can check that this is the case with the four possible intersections

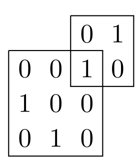


Next is to check for the intersection of the $\boxtimes$-pattern with a primitive essential node t' such that $t'(g) = 1_{\Sigma_2}$. There are six possible ways to intersect for $t'(g)$ and the $\boxtimes(g)$ pattern to intersect, with only one of them being feasible, the rest having the same issue as above, that the intersection contains a row or column comprised of 0s:

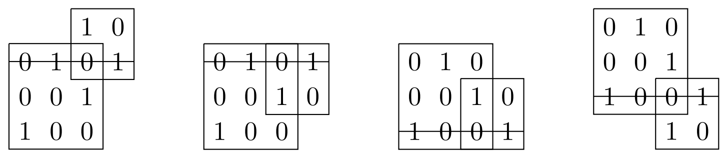


This case is a lost cause since we assume there is a column-simultaneous $\boxtimes(g)$ and $\boxtimes(g^2)$ pattern at that point. Regardless of $t'(g^2)$, it is an uncovered binary node, so we would need the first two columns of $\boxtimes(g^2)$ to form a 2×2 permutation matrix, but this is not the case, since $\boxtimes(g^2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

Next, we assume $t'(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that this is not a submatrix in $\boxtimes(g)$, so in light of that, there are five possible overlaps, only one of which is feasible:



Again, this case runs into problems when considering g^2. No matter what $t'(g^2)$ is, we would break the permutation matrix for a similar intersection to occur on $\boxtimes(g^2)$:



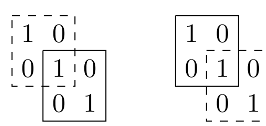
This covers the case for $\boxtimes$. We can do a similar argument for binary nodes. Let us begin with $t = \otimes$. The base case is for $\mathcal{CT}_2$, where the distinction of canonical trees and column-simultaneous $\otimes$-patterns is simple to check.

Like before, assuming the statement for $\mathcal{CT}_m$ for $m \leq n$, we take a canonical tree in $\mathcal{CT}_{n+1}$, assume there is a j-column-simultaneous $\otimes$-pattern, and remove a top node t' grafted in the k-th position. Assuming that the node t' does not entirely intersect the $\otimes$-pattern, we must show that the node cannot intersect at all.

We need not consider the case of the removed node being a $\boxtimes$, since that was covered by the earlier case to not have a feasible intersection with $\otimes$. So instead, assume the removed node t' satisfies $t'(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. It is not hard to check that there are no feasible intersections.

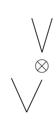
This argument generalizes to show the impossibility of any binary node intersecting with any binary node distinct from itself. So it suffices hereon to only discuss the case of self-intersection, which is taken care of by the assumption of canonical trees.

The two cases to consider are



where the uncovered node t' is denoted by the dashed line and the original j-column-simultaneous instanced is denoted by the solid line. Note that the first case cannot happen because of our assumption that there is no $(j-1)$-column simultaneous $\otimes$-pattern.

For the second case, note that after removing the uncovered node t', which was grafted at $k = j+1$, we are left with $T' \in \mathcal{CT}_n$ which contains a j-column simultaneous instance of t. By the inductive hypothesis we have that $T' = T'' \circ_j t$, and thus $T = (T'' \circ_j t) \circ_{j+1} t'$, meaning that T contains



as a subtree, implying that T is not canonical. Thus, this case cannot happen either. This argument generalizes for the remaining binary nodes. $\square$

Remark 3.14. The proof of Proposition 3.13 also tells us that there is no relation between $\boxtimes$ and itself, and there is no relation between any two different essential primitive nodes. Also, any time we see a column-simultaneous pattern, of size larger than 3, consisting of either the identity matrix or the antidiagonal, the left most 2×2 block corresponds to the only uncovered node in the canonical tree corresponding to the pattern.

We are now ready to prove the main theorem of this subsection.

Proof of Theorem 3.7. By the equivariance axiom and relation (3), it suffices to show that if two canonical trees represent the same primitive function in Q_{C_3} then they are the same tree. We proceed by induction on the arity of the trees. If $n = 2$, there are only 4 elements in $\mathcal{CT}_2$, namely the binary essential primitive nodes, and they have different outputs at g and g^2. We have already shown in the proof of Proposition 3.13 that elements in $\mathcal{CT}_3$ are representing different functions. Suppose this is true for all $m \leq n$, and consider the case $n + 1$.

Let $T_1,\ T_2 \in \mathcal{CT}_{n+1}$ such that they represent the same primitive function $f \in Q_{C_3}(n+1)$. Let t be the left-most uncovered node in T_1. It corresponds to a j-column-simultaneous t-pattern in $f(g)$ and $f(g^2)$ satisfying condition (1) of Proposition 3.13. Thus, we have that $T_1 = T_1' \circ_j t$ and $T_2 = T_2' \circ_j t$ for some canonical trees of lesser arity. Since the T_1 and T_2 represent the same function, the same is true for T_1' and T_2', since we are removing the same uncovered node. The inductive hypothesis tells us that $T_1' = T_2'$, and hence $T_1 = T_2$. ☐

4. Biased permutative equivariant categories for cyclic groups of order two and three

Now that we have explicit descriptions for the generators and the relations on Q_G for $G = C_2$ and C_3, we turn to their categorical analogues and via Theorem 4.3 give explicit biased descriptions of their algebras.

We start with the definition of biased permutative equivariant categories, separating the cases of C_2 and C_3. For both, we denote by g a chosen generator for the group.

Definition 4.1. A *biased permutative C_2-category* consists of

- a C_2-category $\mathcal{C}$;
- a C_2-fixed object $e \in \mathcal{C}$;
- a C_2-equivariant functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;
- a nonequivariant functor $\boxtimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;
- a C_2-natural isomorphism

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \xrightarrow{\ \tau\ } & \mathcal{C} \times \mathcal{C} \\ & {\scriptstyle\otimes}\searrow \ \nearrow\!\!\!\!\nearrow \beta \ \swarrow{\scriptstyle\otimes} & \\ & \mathcal{C}; & \end{array}$$

- a nonequivariant natural isomorphism $\upsilon\colon \boxtimes \Rightarrow \otimes$ called the *untwistor*, with components given by morphisms $\upsilon_{a,b}\colon a \boxtimes b \to a \otimes b$;

subject to the following axioms:

(i) $(\mathcal{C}, \otimes, e, \beta)$ is a permutative category;
(ii) e is a strict two-sided unit for $\boxtimes$, that is, for all $a \in \mathcal{C}$,

$$e \boxtimes a = a = a \boxtimes e \qquad \text{and} \qquad \upsilon_{e,a} = \mathrm{id}_a = \upsilon_{a,e};$$

(iii) $\boxtimes$ is strictly associative: for all $a, b, c \in \mathcal{C}$,

$$a \boxtimes (b \boxtimes c) = (a \boxtimes b) \boxtimes c$$

and the following diagram commutes

$$\begin{array}{ccc}
a \boxtimes (b \boxtimes c) & = & (a \boxtimes b) \boxtimes c \\
\downarrow{\scriptstyle \upsilon_{a,b\boxtimes c}} & & \downarrow{\scriptstyle \upsilon_{a\boxtimes b,c}} \\
a \otimes (b \boxtimes c) & & (a \boxtimes b) \otimes c \\
\downarrow{\scriptstyle \mathrm{id}_a \otimes \upsilon_{b,c}} & & \downarrow{\scriptstyle \upsilon_{a,b} \otimes \mathrm{id}_c} \\
a \otimes (b \otimes c) & = & (a \otimes b) \otimes c;
\end{array}$$

(iv) for all $a, b \in \mathcal{C}$

$$g \cdot (a \boxtimes b) = (g \cdot b) \boxtimes (g \cdot a)$$

and similarly for morphisms in $\mathcal{C}$;

(v) for all $a, b \in \mathcal{C}$, the following diagram commutes

$$\begin{array}{ccc}
g \cdot (a \boxtimes b) & = & (g \cdot b) \boxtimes (g \cdot a) \\
 & & \downarrow{\scriptstyle \upsilon_{gb,ga}} \\
\downarrow{\scriptstyle g \cdot \upsilon_{a,b}} & & (g \cdot b) \otimes (g \cdot a) \\
 & & \downarrow{\scriptstyle \beta_{gb,ga}} \\
g \cdot (a \otimes b) & = & (g \cdot a) \otimes (g \cdot b).
\end{array}$$

Definition 4.2. A *biased permutative C_3-category* consists of

- a C_3-category $\mathcal{C}$;
- a C_3-fixed object $e \in \mathcal{C}$;
- a C_3-equivariant functor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;
- a functor $\boxtimes \colon \mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;
- a C_3-natural isomorphism

$$\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} & \xrightarrow{\tau} & \mathcal{C} \times \mathcal{C} \\
 {\scriptstyle \otimes}\searrow & \nearrow\!\!\!\!\nearrow \beta & \swarrow{\scriptstyle \otimes} \\
 & \mathcal{C}; &
\end{array}$$

- a nonequivariant natural isomorphism υ called the *untwistor*, whose components are given by morphisms $\upsilon_{a,b,c} \colon \boxtimes(a, b, c) \to (a \otimes b) \otimes c$.

To list the axioms, we note there are four (a priori distinct) binary operations: $\otimes$ and the three obtained from inserting e in any of the positions of $\boxtimes$. There is also an associated instance of the untwistor for each of them. For example, if $a \diamondsuit b$ (temporarily) denotes $\boxtimes(e, a, b)$, we have

$$\upsilon^{\diamondsuit}_{a,b} = \upsilon_{e,a,b} \colon a \diamondsuit b = \boxtimes(e, a, b) \longrightarrow (e \otimes a) \otimes b = a \otimes b.$$

The data above are subject to the following axioms

(i) $(\mathcal{C}, \otimes, e, \beta)$ forms a permutative category;

(ii) e is a strict two-sided unit for all binary operations, that is, for all $a \in \mathcal{C}$ and all binary operations $\diamondsuit$,

$$e\diamondsuit a = a = a\diamondsuit e \qquad \text{and} \qquad \upsilon_{e,e,a} = \upsilon_{e,a,e} = \upsilon_{a,e,e} = \mathrm{id}_a;$$

(iii) all the binary operations are strictly associative: for all $a, b, c \in \mathcal{C}$ and all binary operations $\diamondsuit$,

$$a\diamondsuit(b\diamondsuit c) = (a\diamondsuit b)\diamondsuit c$$

and the following diagram commutes

$$\begin{array}{ccc}
a\diamondsuit(b\diamondsuit c) & = & (a\diamondsuit b)\diamondsuit c \\
\downarrow{\scriptstyle \upsilon^{\diamondsuit}_{a,b\diamondsuit c}} & & \downarrow{\scriptstyle \upsilon^{\diamondsuit}_{a\diamondsuit b,c}} \\
a \otimes (b\diamondsuit c) & & (a\diamondsuit b) \otimes c \\
\downarrow{\scriptstyle \mathrm{id}_a \otimes \upsilon^{\diamondsuit}_{b,c}} & & \downarrow{\scriptstyle \upsilon^{\diamondsuit}_{a,b} \otimes \mathrm{id}_c} \\
a \otimes (b \otimes c) & = & (a \otimes b) \otimes c;
\end{array}$$

(iv) for all $a, b, c \in \mathcal{C}$

$$g \cdot \boxtimes(a, b, c) = \boxtimes(g \cdot c, g \cdot a, g \cdot b)$$

and similarly for morphisms in $\mathcal{C}$;

(v) for all $a, b \in \mathcal{C}$, the following diagram commutes

$$\begin{array}{ccc}
g \cdot \boxtimes(a, b, c) & = & \boxtimes(g \cdot c, g \cdot a, g \cdot b) \\
 & & \downarrow{\scriptstyle \upsilon_{gc,ga,gb}} \\
\downarrow{\scriptstyle g \cdot \upsilon_{a,b,c}} & & (g \cdot c) \otimes (g \cdot a) \otimes (g \cdot b) \\
 & & \downarrow{\scriptstyle \beta_{gc,ga\otimes gb}} \\
g \cdot (a \otimes b \otimes c) & = & (g \cdot a) \otimes (g \cdot b) \otimes (g \cdot c).
\end{array}$$

Note that in both definitions, $(\mathcal{C}, \otimes, e, \beta)$ forms a naive permutative G-category, that is, a permutative category in which all the pieces are appropriately G-equivariant.

Recall from Definition 3.3 that $\mathcal{Q}_G$ is defined as the chaotic operad on Q_G, and thus, by the results of the previous section, we can describe $\mathcal{Q}_G$ as $\mathbb{F}\{\widetilde{e, \otimes, \boxtimes}\}/\langle R\rangle$, where R is given by Theorem 3.5 for $G = C_2$ and by Theorem 3.7 for $G = C_3$.

Theorem 4.3. *For $G = C_2$ and C_3, there is a one-to-one correspondence between biased permutative G-categories and algebras over $\mathcal{Q}_G$.*

Proof. This follows from [Rub18, Theorem 2.10] and Proposition 1.18. More precisely, consider the operad

$$\mathcal{O} = \mathbb{F}\{\widetilde{e, \otimes, \boxtimes}\}/\langle g \cdot e = e, g \cdot \otimes = \otimes, g \cdot \boxtimes = \boxtimes \cdot \sigma\rangle,$$

where $\sigma = (1\ 2)$ if $G = C_2$ and $(1\ 2\ 3)$ if $G = C_3$. Note that $\mathcal{O}$ is the operad $\mathcal{SM}_{\mathcal{N}}$ of [Rub18, Definition 2.17] in the case where $\mathcal{N}$ contains the single G-set given by G itself with action by left multiplication (and a choice of ordering). Thus, [Rub18, Theorem 2.10] implies that $\mathcal{O}$-algebras correspond precisely to $\mathcal{N}$-normed symmetric monoidal categories (cf. [Rub18, Definition 2.3] for details). Note that this definition is very similar to ours, with the exception that there is an underlying symmetric monoidal structure (not necessarily strictly associative and unital), and that axiom (v) is required to hold for all elements of the group.

The relations from Theorems 3.5 and 3.7 and Proposition 1.18 imply that the underlying symmetric monoidal structure in our algebras will be strictly associative and unital, with the associator α and the unit constraints λ and ρ equal to the identity. Similarly, these results give the relations between the different instances of υ. Requiring axiom (v) for a group generator ensures it holds for all elements of the group. This gives the desired result. □

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