

# Quantum Theory as Symmetry Broken by Vitality

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I summarize a research program that aims to reconstruct quantum theory from a fundamental physical principle that, while a quantum system has no intrinsic hidden variables, it can be understood using a *reference measurement*. This program reduces the physical question of why the quantum formalism is empirically successful to the mathematical question of why complete sets of equiangular lines appear to exist in complex vector spaces when they do not exist in real ones. My primary goal is to clarify motivations, rather than to present a closed book of numbered theorems, and consequently the discussion is more in the manner of a colloquium than a PRL.

## I. INTRODUCTION

I have been looking up science shows I saw as a kid and rewatching them, partly to see how well their content has aged, and partly as “relaxation tapes” to help me decompress. On occasion, a bit will jump out at me and become particularly relevant to my current interests. The two examples I have in mind right now arose from the Public Broadcasting System of ages past, so what follows was made possible by the generosity of Viewers Like You. First, there’s David L. Goodstein in *The Mechanical Universe and Beyond* (1986):

The science of thermodynamics is based on four fundamental postulates, or axioms, which are called the Four Laws of Thermodynamics. Of these four laws, the second law was discovered first, and the first law was discovered second, and the third to be discovered was called the zeroth law, and the fourth law is called the third law. Now, all of that makes perfect sense because thermodynamics is the most implacably logical of all the sciences [1].

The second example is Timothy Ferris in *The Creation of the Universe* (1985):

Perfect symmetry may be beautiful, but it’s also sterile.

In this essay, I will attempt to explain why particular statements ought to be true, not necessarily proving them yet in all detail. My goal is to lay out my motivations and heuristics, rather than to make a closed book of definitions and lemmas and corollaries. My approach will be grounded in previous QBist and QBist-adjacent writing on the reconstruction of quantum theory. Chiefly, this means our paper “Introducing the Qplex” [2], and a line of thinking that had been confined to an appendix therein. My conceit is that there are Laws of Quantum Mechanics, analogous to the Laws of Thermodynamics and to Einstein’s postulates for special relativity.

0. Two states of expectation are equivalent for an arbitrary measurement when they are equivalent for the reference measurement.
1. Certainty is achievable, and it defines the boundary of state space. In particular, the states most closely associated with the reference measurement each imply certainty for some experiment.

2. Yet certainty is *not* about hidden variables: *Unperformed experiments have no results*.
3. As in classical probability, states in the interior of the state space can be reversibly mapped to one another.

Asher Peres’ slogan, “Unperformed experiments have no results” [3], is typically said to summarize the fact that “no-go theorems” rule out hidden-variable completions of quantum mechanics. It is convenient to have a positive expression of what is normally stated negatively, and so we will entertain a counterpart slogan, *quantum physics has vitality* [4].

The *physical* assumption underlying all of this development is the possibility of expressing this vitality using a *reference measurement*. We have taken this route before, but compared to what I would like to do, our prior work jumped somewhat into the middle, and it did not fully embrace the theme that I intend to explore now. If there is truly one fundamental mystery and all the rest of the quantum formalism is mathematical niceties, then perhaps those should be expressed as mathematicians do — that is, by talk of symmetries, and the properties left invariant by transformations. Perfect symmetry is sterile; all the life lies in the fundamental axiom. Therefore, all the later steps in the derivation of the quantum follow the route of maximal symmetry.

A conceptual tool that I will use to narrow the mathematical possibilities is van Fraassen’s *reflection principle*, which gives meaning to convex combinations of probabilities and provides a reason to believe in linearity [5–8].

## II. PRELIMINARIES AND DEFINITIONS

When I read someone else’s math,  
I always hope the author  
will have included a reason  
and not just a proof.  
— Eugenia Cheng [9]

Our goal is the finite-dimensional quantum theory familiar from quantum information and computation. That is, we wish to explain why applying the theory means associating a physical system with a complex Hilbert space, why positive semidefinite operators on that Hilbert space are so important for multiple purposes, why the textbook rule for calculating probabilities is the correct one, and so on. Efforts to put quantum theory on a more principled foundation are almost as old as quantum physics itself. However, many of these research endeavors antedate the mature understanding of what the most enigmatic features of quantum physics truly are. Even the modern renaissance of quantum reconstructions, beginning around the turn of this century, has largely been preoccupied with making the theory seem as “benignly humdrum” as possible [10]. Rather than starting with a remarkable phenomenon — say, the violation of a Bell inequality — and building the subject up from there, the reconstructors’ ethos has mostly been to draw up lists of postulates that, individually, sound thoroughly pedestrian. From such a list, the quantum formalism is rederived. And then, given that formalism, the remarkable features can be exhibited as they had been before. The undeniable strangeness of quantum phenomena is not written in any one axiom, but somehow interleaved between them, sometimes in the tacit conditions accepted without demur but not given bold text and bullet points. The bolded axioms themselves are often

satisfied by fundamentally classical theories, like the Spekkens toy model [11]! Thus, while the mathematics may be sound — the density of errata is probably no worse than average for the *Physical Review* family — the result is vaguely dispiriting all the same. Reconstructing the quantum was supposed to free us from antiquated debates over “interpretations”, but how can it do that if the lists of new operational postulates themselves multiply like the “interpretations” have?

Accordingly, while we will lean upon the algebraic achievements of these reconstruction efforts, we will arrive at the point where that algebra can be invoked in our own way.

The present work is firmly in the personalist Bayesian tradition. Apart from some changes in emphasis, the QBist view on how to interpret probability is closely kin to that espoused by Diaconis and Skyrms [12], and it has been developed over several previous publications [8, 13–15]. Elsewhere, I have built up the theory pedagogically to the point where one can do nonequilibrium statistical physics with it [16]. Well before I acquired a serious interest in quantum foundations, I had a nudge in this general direction, thanks to taking statistical mechanics from Mehran Kardar. To quote his *Statistical Physics of Particles* [17], “All assignments of probability in statistical mechanics are subjectively based.” (For all its talk of “ensembles”, statistical mechanics really gives lessons in how to adopt particular priors [18]. Indeed, as the concepts are applied in practice, the term “ensemble” becomes less and less an appeal to relative frequencies, and more a purely conventional meat-noise.) If anyone is aghast at the spectacle of a youth turning to QBism, well, they can blame the Massachusetts Institute of Technology.

Topology is the study of properties invariant under continuous transformations, and differential geometry studies quantities that are invariant under coordinate changes. Terry Tao suggests that we should think of probability theory analogously, as studying those concepts and operations that are preserved by extending sample spaces [19]. This is not dissimilar to the Dutch-book view, where one imposes the rule that two propositions should be ascribed the same probability if they are equivalent by Boolean grammar. Moreover, this motivates the idea that we should be able to apply our theory to systems of arbitrary dimensionality, once we have properly made precise what “dimensionality” means.

Let  $d$  be an integer greater than 1, and consider the vector space  $\mathbb{C}^d$ . A *SIC* is a set of  $d^2$  unit vectors  $\{|\pi_j\rangle\}$  in this space which enjoy the property that

$$|\langle\pi_j|\pi_k\rangle|^2 = \frac{d\delta_{jk} + 1}{d + 1}. \quad (1)$$

The acronym SIC (pronounced “seek”) stands for “Symmetric Informationally Complete” and refers to the fact that such a set represents a measurement that can be performed upon a quantum system of Hilbert-space dimension  $d$  [20–23]. Because the operators  $\Pi_j = |\pi_j\rangle\langle\pi_j|$  span the space of Hermitian operators on  $\mathbb{C}^d$ , any quantum state  $\rho$  that one might ascribe to the system can be expressed in terms of its inner products with those operators, which up to normalization are just the probabilities for the outcomes of the SIC measurement:

$$p(j) = \frac{1}{d}\text{tr}(\rho\Pi_j). \quad (2)$$

Prior work has established the virtues of SICs as reference measurements [14, 24–26]. They are also of interest for purely mathematical reasons [27–32]. SICs are the largest possible sets of equiangular lines; that is, one cannot have more than  $d^2$  unit vectors in  $\mathbb{C}^d$  such that  $|\langle\pi_j|\pi_k\rangle|$  is constant for all  $j \neq k$ . As of this writing, SICs are known numerically for all  $d$

up to 181, and in irregular cases up to  $d = 2208$ , while exact solutions have been found for all  $d$  up to 21 and irregularly up to  $d = 1299$  [33].

Now, we have defined enough terminology to know what we mean when we say, for example, “Bengtsson, Blanchfield and Cabello [34] proved a quantum vitality theorem using a SIC in dimension 3.”

A *qplex* is a subset of the probability simplex of normalized, entrywise nonnegative vectors in  $\mathbb{R}^{d^2}$ . Any two points within a qplex are said to be *consistent* in that their Euclidean inner product satisfies the inequalities

$$\frac{1}{d(d+1)} \leq \langle p_1, p_2 \rangle \leq \frac{2}{d(d+1)}. \quad (3)$$

Moreover, a qplex is a *maximal* consistent set: If one tries to introduce even a single additional point, inconsistencies arise. A *Hilbert qplex* is one whose symmetry group is isomorphic to the projective extended unitary group  $\text{PEU}(d)$ . Hilbert qplexes are images of quantum state spaces under the mappings defined by SICs; the above inequalities are the image of the statement that  $\text{tr} \rho_1 \rho_2$  lies in the unit interval for any two quantum states  $\rho_1$  and  $\rho_2$ . A primary goal of this essay is to replace the assumption of projective unitary group symmetry with a postulate that is less specific but just as powerful.

It follows from the maximality property that a qplex is necessarily convex and closed. Thanks to Huangjun Zhu, we know many more things about qplexes. For instance, we know that any qplex is a *self-polar* set. Let  $H$  be the hyperplane in  $\mathbb{R}^{d^2}$  consisting of vectors whose elements sum to unity, i.e., the hyperplane of probabilities and quasiprobabilities. The polar of a point in  $H$  is the set of all points in  $H$  whose inner product with the given point is greater than the lower bound in the fundamental inequalities. The polar of a set of points is the set of all points which are in the polars of all the given points. (This terminology is adapted from the study of polytopes.) The operation of taking the polar reverses inclusion, so the polar of a set that lies within the probability simplex contains the polar of the probability simplex, which is another simplex whose vertices are the probability distributions

$$e_j(i) = \frac{1}{d+1} \delta_{ij} + \frac{1}{d(d+1)}. \quad (4)$$

Note that the constant term is just the lower bound. We refer to these vectors as the *basis distributions*. When considered together, they form a matrix whose inverse is

$$\Phi = (d+1)I - \frac{1}{d}J, \quad (5)$$

using  $J$  to stand for the all-ones matrix (the Hadamard identity).

Another important fact about qplexes is the maximal size of a *mutually maximally distant* (MMD) set. Let  $\{p_j : j = 1, \dots, m\}$  be a set of points in a qplex, such that  $\langle p_j, p_j \rangle$  equals the upper bound and, when  $j \neq k$ ,  $\langle p_j, p_k \rangle$  equals the lower bound. Such a set can only be so big. In fact,  $m \leq d$ .

A *generalized qplex* is defined similarly as a maximal consistent set of probability vectors in  $\mathbb{R}^N$ , where consistency is with respect to the inequalities

$$L \leq \langle p_1, p_2 \rangle \leq U. \quad (6)$$

Polarity works much as before in this more general setting. The basis distributions are the vectors

$$e_k(j) = (1 - NL)\delta_{jk} + L, \quad (7)$$

again found by taking all entries save one to be the lower bound. The matrix  $\Phi$  that we defined in Eq. (5) generalizes to

$$\Phi = \frac{1}{1 - NL}(I - LJ). \quad (8)$$

It will be helpful later to note that

$$\langle p_1, \Phi p_2 \rangle = \frac{1}{1 - NL} \langle p_1, p_2 - LJp_2 \rangle = \frac{1}{1 - NL} (\langle p_1, p_2 \rangle - L). \quad (9)$$

One intriguing property enjoyed by generalized qplexes is that the number of zeros in any probability vector cannot exceed  $N - 1/U$ .

Marcus Appleby made a new branch of mathematics available to us by defining what I will call a *qplectic cone theory* [35]. First, we need the notion of a *convex theory*, which is a triple  $(V, \mathcal{C}, \mathcal{I})$ , where  $V$  is a real  $d^2$ -dimensional vector space whose inner product we write  $(\cdot, \cdot)$ . The cone  $\mathcal{C} \subset V$  is self-dual with respect to this inner product, and  $\mathcal{I}$  is an element in the interior of  $\mathcal{C}$  that we'll call the *order unit*. We use the order unit to define the hyperplane of normalized elements:

$$\mathcal{H} = \{v \in V : (\mathcal{I}, v) = 1\}. \quad (10)$$

The intersection of this hyperplane with the cone  $\mathcal{C}$  is the *state space*:

$$\Omega = \mathcal{H} \cap \mathcal{C}. \quad (11)$$

A qplectic cone theory is a convex theory  $(V, \mathcal{C}, \mathcal{I}, \{e_j\})$  where the state space  $\Omega$  contains a regular simplex  $\{e_j : j = 1, \dots, d^2\}$  such that

$$(e_j, e_k) = \frac{d\delta_{jk} + 1}{d + 1}. \quad (12)$$

As a minor technical point, the inner product and order unit in a qplectic cone theory are scaled such that the supremum of  $|p|^2$  over all  $p \in \Omega$  is unity.

The elements  $\{e_j\}$  can be shown to form a basis, with a dual basis

$$\bar{e}_j = \frac{d + 1}{d} e_j - c, \quad (13)$$

where we have defined the center point

$$c = \frac{\mathcal{I}}{(\mathcal{I}, \mathcal{I})} = \frac{1}{d} \mathcal{I}. \quad (14)$$

The dual basis is a resolution of the order unit:

$$\sum_k \bar{e}_k = \mathcal{I}; \quad (15)$$

as is the original basis, up to scaling:

$$\sum_k e_k = d\mathcal{I}. \quad (16)$$

The key result is the following. Let  $(V, \mathcal{C}, \mathcal{I}, \{e_j\})$  be a qplectic cone theory. Define  $f : V \rightarrow \mathbb{R}^{d^2}$  to be the affine bijection

$$f(p) = (p_1, \dots, p_{d^2})^T, \quad p_j = \frac{d(\bar{e}_j, p) + 1}{d(d+1)}. \quad (17)$$

Then  $f(\Omega)$  is a qplex. Conversely, given an arbitrary qplex, we can realize it as the image under the appropriate affine bijection of the state space  $\Omega$  of some qplectic cone. This correspondence carries over to measurements, which in a qplectic cone theory are resolutions of the order unit  $\mathcal{I}$  into elements of the cone  $\mathcal{C}$ .

John DeBrotta has worked out the details for generalized qplexes, where the dimension and the bounds are not fixed to the values familiar from quantum theory. In this broader context, our convex theory will again be a tuple  $(V, \mathcal{C}, \mathcal{I}, \{e_i\})$ , but with

$$(e_i, e_j) = 1 + NL(\delta_{ij} - 1). \quad (18)$$

The barycenter  $c$  is given by

$$c = \frac{1}{N} \sum_k e_k = \frac{\mathcal{I}}{(\mathcal{I}, \mathcal{I})}. \quad (19)$$

The scaling of the  $\mathcal{I}$  is

$$(\mathcal{I}, \mathcal{I}) = \frac{1}{1 + L - NL}. \quad (20)$$

The dual basis to  $\{e_i\}$  is given by the points

$$\bar{e}_i = \frac{1}{NL} \left( e_i - \frac{1 - NL}{1 + L - NL} c \right). \quad (21)$$

It follows that the dual basis is again a resolution of the order unit. As before, we take a generalized qplex to be a set  $\mathcal{P} \subset \Delta$  such that for all  $p, p' \in \mathcal{P}$ , we have

$$L \leq \langle p, p' \rangle \leq U, \quad (22)$$

and  $U$  is given by the length of the basis distributions:

$$U = 1 + L(N - 1)(NL - 2). \quad (23)$$

The vertices of the basis simplex have coordinates

$$e_k(j) = (1 - NL)\delta_{jk} + L. \quad (24)$$

A bijection is established by  $f : V \rightarrow \mathbb{R}^N$ , where

$$f(v) = (p_1, \dots, p_N)^T, \quad p_i = (1 - NL)(\bar{e}_i, v) + L. \quad (25)$$

For any  $v \in \Omega$ ,

$$v = \sum_j \frac{p_j - L}{1 - NL} e_j. \quad (26)$$

Convex cones can be classified once we postulate a sufficient degree of symmetry. It is known that convex cones which are *homogeneous* are all isomorphic to algebraic structures of a genre that, in turn, has been classified exhaustively. Homogeneous cones are those for

which any point in the interior of the cone can be mapped to any other by some isomorphism of the cone. The classification task is achieved by the Koecher–Vinberg theorem, which establishes that there exists a one-to-one correspondence between these highly symmetric cones and “formally real Jordan algebras” [36, 37]. The Koecher–Vinberg theorem is one of the two mathematical results upon which this approach to reconstructing quantum theory relies; the other, more empirical in character, is that SICs appear to be a quintessentially complex-vector-space phenomenon.

### III. MANAGING EXPECTATIONS IN A REALITY TOO RICH FOR TURING MACHINES

In earlier work, I have drawn a connection between SICs and thermodynamically significant quantities [38]. Here, I want instead to make a much higher-level argument, following the analogy that Chris Fuchs and I articulated in our “Hero’s Handbook” paper [39]:

We can illustrate the trouble with quantum mechanics by comparing it with other areas of physics in which we have collectively honed our understanding to a high degree of sophistication. Two examples that come to mind are the science of thermodynamics and the special theory of relativity. An old joke has it that the three laws of thermodynamics are “You can’t win”, “You can’t break even”, and “You can’t get out of the game.” To these, we ought to prepend the zeroth law, which we could state as, “At least the scoring is fair.” But consider the premise of the joke, which is really rather remarkable: There *are* laws of thermodynamics — a concise list of deep physical principles that underlie and nourish the entire subject. Likewise for special relativity: Inertial observers Alice and Bob can come to agree on the laws of physics, but no experiment they could ever do can establish that one is “really moving” and the other “really standing still” — not even measuring the speed of light. We invest a little mathematics, and then close and careful consideration of these basic principles yields all the details of the formal apparatus, with its nasty square roots or intermingling partial derivatives.

This level of understanding brings many advantages. Having the deep principles set out in explicit form points out how to *test* a theory in the most direct manner. Moreover, it greatly aids us when we *teach* the theory. We do not have to slog through all the confusions that bedeviled the physicists who first developed the subject, to say nothing of the extra confusions created by the fact that “historical” pedagogy is almost inevitably a caricature. In addition, a principled understanding helps us *apply* a theory. As we work our way into a detailed calculation, we can cross-check against the basic postulates. Does our calculation imply that signals travel faster than light? Does our seventeenth equation imply that entropy is flowing the wrong way? We must have made an error! And, when we found our theory upon its deep principles, we have a guide for *extending* our theory, because we know what properties must obtain in order for a new, more general theory to reduce to our old one in a special case.

To our great distress, we must admit that in the matter of quantum mechanics, the physics profession lacks this level of understanding.

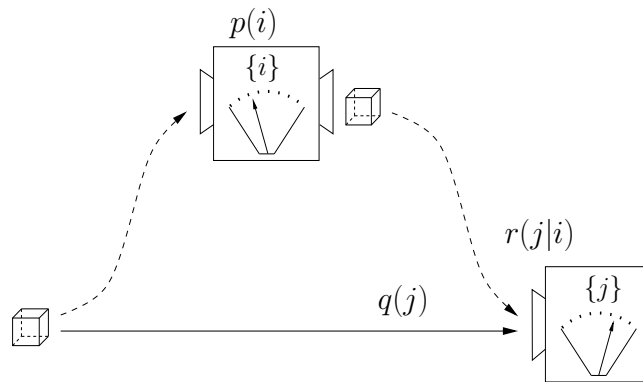


FIG. 1: A choice between two experiments. In one scenario (solid line), a system is fed directly into a measuring apparatus. In the other (dashed line), the system is sent through the reference measurement first. Probability theory does not itself enforce a relation between an agent’s probabilities for these two scenarios. Different conditions, different probabilities! The classical intuition that the reference measurement just reads off the system’s intrinsic degrees of freedom leads to using the Law of Total Probability to relate expectations between the two scenarios. Quantum theory, on the other hand, provides its own relation. Our goal is to identify exactly what physical principle implies the quantum relation. (After [25].)

We can make this analogy more exact [40]. Both in special relativity and in thermodynamics, one of the postulates has the character of a guarantee: Inertial observers Alice and Bob can come to agree on the laws of physics; energy is conserved. Then comes a postulate that serves as a dramatic foil to the first, verging on contradicting it. No matter what they do, Alice and Bob cannot agree on a standard of rest, even if they go so far as measuring the speed of light. And in thermodynamics, energy is conserved, but *useful* energy diminishes in all but the most ideal processes. In special relativity, we derive a statement of unattainability: Massive bodies cannot attain light speed. Meanwhile, over in thermodynamics, we assume a form of unattainability in the Third Law.

What about the Zeroth Law? While the ordinary statement of it disguises this a little, the way we use it in practice indicates that it is a statement that holds moment-to-moment. At each instant of a “quasi-static” evolution, a system can be taken to be in equilibrium with a fiducial heat bath at the appropriate temperature. This is analogous to the clock postulate, the rule which Einstein invoked but did not grant a number. Just as the moment-to-moment statement of the Zeroth Law implies that we can use temperature as a reference scale even during a nontrivial time evolution, the clock postulate lets us use *velocity* as a reference scale, analyzing accelerated motion using a series of momentarily co-moving inertial frames.

Thus, we have a pedagogical schema that applies to both theories:

0. Equivalence relation, implying a scale of reference
1. Reassuring guarantee
2. Dramatic foil with metaphysical weight
3. Unattainability, of an asymptotic flavor

What, then, about quantum mechanics?

The view developed in the “Hero’s Handbook” review [39] is that systems can have an arbitrarily rich supply of physical properties, but these attributes do not, either singly or



in combination, compel the outcomes of measurements, or even the probabilities that an agent should ascribe to them. The system attribute to which *the basic quantum formalism* is sensitive is the “creative capacity” that manifests as Hilbert-space dimension, or the quantity in a more ambitious theory that reduces to it.

The “zereth law of quantum theory” proposed above tells us that we can think in terms of a reference measurement. In any situation, we can calculate the probabilities we need in terms of the probabilities  $\{p(i)\}$  for the possible outcomes of the reference measurement, and the conditional probabilities  $\{r(j|i)\}$  for the result of our other experiment given an outcome of the reference measurement:

$$q(j) = \mu(p, r), \tag{27}$$

where  $p$  is a vector and  $r$  is a matrix. The number of outcomes necessary for a reference measurement,  $N$ , is regarded as an intrinsic property of a physical phenomenon. It is how the quantity that we evocatively described as “creative capacity” enters the theory. Our goal now is to characterize the function  $\mu$ . Note that classically — or perhaps better put, according to a sentiment of classicality more primitively rooted than classical mechanics — an ideal “reference measurement” would simply read off the system’s intrinsic physical degrees of freedom, and we would have

$$q(j) = \sum_i p(i)r(j|i). \tag{28}$$

This formula is the *Law of Total Probability*, and it expresses the intuition that classical uncertainty is ignorance of a system’s “physical condition” (to use Einstein’s terminology [41]) or “ontic state” (in a more modern turn of phrase). We will show that the function  $\mu$  cannot be of this form, and that the correct quantum form of  $\mu$  follows from an expression of quantum vitality.

To vary our vocabulary, we will sometimes speak of the “preparation” of a system rather than the “state” ascribed to it. This is with the understanding that, for example, picking up a rock from the beach is a “preparation”. We make no claim of expense or exactitude; a preparation has the same status as a prior, with all the personalism inherent in that. A preparation is a prior with different scansion. Likewise, we call two preparations “distinguishable” if they imply sharply discrepant predictions for some experiment.

Other approaches have been criticized for saying that the laws of physics “state by fiat that a particular type of behavior is rationally compulsory for rational agents” [42]. We are aiming for something more subtle and robust. In our approach, we acknowledge that we bring to the table certain choices — about how to represent the reasoning process, what mathematical entities can stand for intensities of belief, etc. — and we then manage our expectations *in light of* the natural world and its character.

The most arbitrary-looking mathematical decision occurs where the rubber meets the road, i.e., where the character of natural phenomena must find a representation within our conventions. For our purposes, this means the way we choose to encode the nonexistence of intrinsic hidden variables. In a classical theory, a reference measurement would at the simplest just read off the intrinsic “physical condition” of the system being measured. All other measurements could be understood as coarse-grainings of this reference measurement, which we can represent probabilistically as a filtering by a stochastic process. Even if the reference measurement were not feasible in practice, we could tie together our probabilities for those experiments that *are* practical, introducing a reference experiment of which they

are all post-processings, without fear of inconsistency. One consequence of this is that if two preparations are distinguishable with respect to some measurement, they must also be so by the reference measurement. Post-processing by a stochastic coarse-graining cannot eliminate an overlap between probability distributions.

We posit that *rejecting* this statement of how distinguishability behaves captures a deep truth about quantum physics. Two orthogonal quantum states are perfectly distinguishable with respect to some experiment, yet in terms of the reference measurement, they are *inevitably overlapping* probability distributions. The idea that any two valid probability distributions for the reference measurement must overlap, and that the minimal overlap in fact corresponds to distinguishability with respect to some other test, expresses the fact that quantum probability is not about hidden variables. Or, to make the statement in a positive form, it expresses quantum vitality. Thus the inequality

$$\langle p_1, p_2 \rangle > 0 \tag{29}$$

looks small but packs a big punch.

There may be many possible reference measurements. In fact, in quantum theory, the set thereof is rather richly structured [25, 26]. However, the *canonical* choice, the analogue of a Carnot engine or a light clock, is a measurement that expresses vitality in the crispest possible way. Therefore, we will take as canonical reference measurement one for which there is a single constant value of overlap that corresponds to perfect distinguishability:

$$\langle p_1, p_2 \rangle \geq L. \tag{30}$$

Imagine building up a qplex  $\mathcal{P}$  one point at a time. If we include a point  $p$ , we automatically exclude all points that are too far away from  $p$ , that is, all vectors  $v$  such that  $\langle p, v \rangle < L$ . Are there points that are *never* excluded during this process? Such probability distributions would be a necessary part of *any* qplex, for without them, the set could never be maximal. The points that cannot be excluded are those  $p$  for which  $\langle p, v \rangle \geq L$  for all  $v$  in the probability simplex, or in other words, the polar of the probability simplex. Thus, we know that all qplexes must include the convex hull of the basis distributions. They tumble out once we introduce the idea of taking a polar, which follows directly from our way of expressing that intrinsic hidden variables don't exist!

Let  $\mu_E : \mathcal{P} \rightarrow [0, 1]$  be the function that computes the probability of event  $E$  given a probability vector over the outcomes of the reference measurement. Suppose that  $p_1, p_2 \in \mathcal{P}$  are two valid states. Because the qplex  $\mathcal{P}$  is a maximal consistent set, the convex combination

$$p_\lambda = \lambda p_1 + (1 - \lambda) p_2 \tag{31}$$

must also belong to  $\mathcal{P}$ . The reflection principle gives this state meaning: It is my belief now about what a reference measurement might yield in the future, provided that I believe now that some intermediate action will lead me to update my gambling commitment either to  $p_1$  or to  $p_2$  with probabilities  $\lambda$  and  $1 - \lambda$  respectively. Define

$$q_1(E) = \mu_E(p_1), \quad q_2(E) = \mu_E(p_2), \quad q_\lambda(E) = \mu_E(p_\lambda). \tag{32}$$

If I believe now that my future gambling commitment about the event  $E$  will be either  $q_1(E)$  or  $q_2(E)$ , then my van Fraassen reflection probability for  $E$  is

$$q_\lambda(E) = \lambda q_1(E) + (1 - \lambda) q_2(E). \tag{33}$$

We begin to see why, morally,  $\mu_E$  must be a linear function on the qplex  $\mathcal{P}$ . Specifically, if I am willing to treat  $p_\lambda$  just like any other probability vector for the reference measurement, even though I know I calculated it by taking a weighted average, then I can say

$$q_\lambda(E) = \mu_E(p_\lambda), \quad (34)$$

and equating our two ways of writing  $q_\lambda(E)$  shows that  $\mu_E$  must preserve convex combinations:

$$\mu_E \left( \sum_k \lambda_k p_k \right) = \sum_k \lambda_k \mu_E(p_k), \quad \sum_k \lambda_k = 1. \quad (35)$$

This means that  $\mu_E$  must be an affine function, that is, a linear function of  $p$  possibly combined with a constant offset. Suppose we could make that constant vanish. Then  $\mu_E$  would be a linear, nonnegative function on the qplex. But we know what all such functions look like: We can write each one as an inner product of  $p$  with some vector, and we have the full set of vectors whose inner product with  $p$  is nonnegative for all  $p \in \mathcal{P}$ , thanks to self-polarity. Thus, we would have

$$\mu_E(p) = \kappa_E \langle s_E, \Phi p \rangle, \quad (36)$$

for some constant  $\kappa_E$  and some vector  $s_E \in \mathcal{P}$ .

We know that the offset has to vanish at least some of the time, because the reference measurement is itself a valid experiment. ‘‘Computing’’ the probabilities for the outcomes of the reference measurement in terms of themselves amounts to saying  $\mu_E(p) = p(i)$  for some  $i = 1, \dots, N$ . In other words,  $\mu_E$  just plucks out a particular element of  $p$ .

To make further progress, let’s combine the probability vector  $p$  and the conditional probability matrix  $r$  into a matrix of joint probabilities:

$$[P]_{ji} = P(j, i) = r(j|i)p(i). \quad (37)$$

If Alice were doing classical probability and treating  $p$  as the probability vector over the outcomes of a classical reference measurement, then she would find her probabilities for the outcomes of any other experiment by marginalizing  $P$  over the index  $i$ . But Alice is doing quantum physics, and so if she does not physically perform the reference measurement, then she must find her probabilities  $q$  by a ‘‘quantum marginalization’’ rule:

$$q = M(P). \quad (38)$$

The discrepancy between quantum and classical marginalization indicates what it means that Alice isn’t just ignoring the reference outcome, but *refraining from calling it into being*.

Suppose that Alice believes now that some future event will cause her to adopt either the joint probabilities  $P_1$  or alternatively  $P_2$ . Then ‘‘ $ji$ ’’ is just a label like ‘‘ $i$ ’’ was before, and Alice can write the van Fraassen reflection state

$$P_\lambda = \lambda P_1 + (1 - \lambda) P_2. \quad (39)$$

As above, if Alice is willing to treat  $P_\lambda$  in the same way as  $P_1$  and  $P_2$  even though she knows it is a van Fraassen reflection state, then

$$\lambda M(P_1) + (1 - \lambda) M(P_2) = M(\lambda P_1 + (1 - \lambda) P_2), \quad (40)$$

and  $M$  must be an affine map from matrices to vectors. The matrix of joint probabilities is, speaking formally, a bilinear function of  $r$  and  $p$ . Therefore, the “quantum marginalization” rule must be a bilinear map from pairs  $(r, p)$  to vectors, plus a constant vector offset. We can show that the offset is in fact the zero vector, because for the appropriate choice of  $r$ , we must have  $q = p$ ; that is, the “quantum marginalization” map simply becomes the identity, which has no offset at all. (For more on how to represent multi-affine maps in terms of multi-linear ones, see Mac Lane and Birkhoff [43].)

We aren’t actually *obligated* by coherence to treat a van Fraassen reflection state in exactly the same way we would treat the same state arrived at by another method. Alice may believe now that her belief tomorrow will be  $P_k$  for  $k = 1$  or  $2$ . Alice also believes now that she will have had the experience which will lead her to select her new gambling commitment. In other words, Alice believes that she will have an outcome  $k$  in hand. That is extra information which could affect what functions she is willing to apply to her matrix of joint probabilities. The crucial fact is that she is willing *sometimes* to treat a van Fraassen reflection state like any other. Sometimes, she will believe that she will have experienced a value of  $k$ , and that the consequences will be restrained. Modest, one might say. In practice, this seems to mean treating the event  $k$  as affecting one’s choice of  $p$  or of  $r$  but not both.

Probability theory is a very linear theory. In this project, we are trying to squeeze the difference between quantum physics and classical intuition into as small a corner as possible. Thus, when we introduce the idea of a reference measurement, we want the resulting theory to inherit as many of probability’s natural linearities and affinities as are feasible. To do otherwise would be to build a pathological theory, and that’s not what we’re interested in doing. We have arrived at a quantitative statement, that the best replacement for the classical Law of Total Probability is also a bilinear map, from the more conceptual consideration that a belief about beliefs can be treated as a belief itself. Drilling further down into this topic may require a deeper investigation into why and how probability theory itself is a viable theory for managing preferences, expectations and utilities.

The rows of any given conditional probability matrix  $r$ , which are labeled by the index  $j$ , won’t necessarily be normalized nicely, so let’s write them as

$$r(j|i) = N\gamma_j s_j(i), \quad (41)$$

where  $s_j$  is a probability vector. The way we’ve chosen for writing the prefactor will turn out to be convenient later. Linearity on both inputs means that

$$q(j) = N\gamma_j s_j^T A p \quad (42)$$

for some matrix  $A$ . We identify  $A$  with the  $\Phi$  we defined in Eq. (8) by identifying the basis distributions as the post-measurement states of the reference experiment. When we feed in a basis distribution  $e_k$ , we are setting  $p$  equal to a column of  $\Phi^{-1}$ , and so our result for  $q(j)$  just reads out  $r(j|k)$ . Self-polarity then implies that the set of valid  $s_j$  is the same as the set of valid  $p$ , that is to say, the qplex  $\mathcal{P}$ .

We have arrived at the *generalized urgleichung*:

$$q = r \Phi p. \quad (43)$$

The matrix  $\Phi$  preserves the flat probability distribution  $c$ . Consequently, we have an interpretation for  $\gamma_j$ : It is just the probability for obtaining the  $j^{\text{th}}$  outcome of the measurement  $r$ , given the state of complete indifference for the reference experiment.

The reflection principle has led us to linearity, and maximality has brought us to the fact that a qplex must be self-polar, which in turn allowed us to conclude that the set of valid measurement matrices,  $\mathcal{R}$ , is built from the set of valid preparations,  $\mathcal{P}$ . In other words, our mental model of a measurement is a set of preparations, possibly weighted. To explore this in more conceptual detail, we echo the discussion from [2] with minor edits:

What conditions must an object meet in order to qualify as a piece of laboratory apparatus? Classically, a bare minimum requirement is that the object has a set of distinguishable configurations in which it can exist. These might be positions of a pointer needle, heights of a mercury column, patterns of glowing lights and so forth. The essential point is that the system can be in different configurations at different times: A thermometer that always reports the same temperature is useless. We can label these distinguishable configurations by an index  $j$ . The *calibration* process for a laboratory instrument is a procedure by which a scientist assigns conditional probabilities  $\{r(j|i)\}$  to the instrument, relating the readout states  $j$  to the inputs  $i$ . In order to make progress, we habitually assume that nature is not so perverse that the results of the calibration phase become completely irrelevant when we proceed to the next step and apply the instrument to new systems of unknown character.

But what if nature *is* perverse? Not enough so to forbid the possibility of science, but enough to make life interesting. Quantitatively speaking, what if we must modify the everyday assumption that one can carry the results of a calibration process unchanged from one experimental context to another?

*The urgleichung is just such a modification.* The conditional probabilities  $\{r(j|i)\}$  do not become irrelevant when we move from the upper path in Figure 1 to the lower, but we do have to use them in a different way.

In quantum physics, we no longer treat “measurement” as a passive reading-off of a specified, pre-existing physical quantity. However, we do still have a counterpart for our classical notion of a system that can qualify as a laboratory apparatus. Instead of asking whether the system can exist in one of multiple possible classical states, we ask whether our overall mesh of beliefs allows us to consistently assign any one of multiple possible catalogues of expectations. That is, if an agent Alice wishes to use a system as a laboratory apparatus, she must be able to say now that she can conceive of ascribing any one of several states to it at a later time.

The analogue of classical uncertainty about where a pointer might be pointing is the convex combination of the states  $\{s_j\}$ . Therefore, our basic mental model of a laboratory apparatus is a polytope in  $\mathcal{P}$ , with the  $\{s_j\}$  as its vertices. The conclusion that we build  $\mathcal{R}$  from  $\mathcal{P}$  says that *Alice can pick up any such apparatus and use it as a “prosthetic hand” to enrich her experience of asking questions of nature.*

Instantly updating to a post-measurement state on the out-sphere of a qplex is like converting a massive body into light: We leap to an edge that more mundane transformations could not attain. *Maximal certainty* is what plays the role of the speed of light or absolute zero. Our third law, which tells us that reversible transformations can asymptotically approach maximal certainty, where the post-measurement states of the reference experiment live, brings homogeneity into the story.

It took me a long time to become accepting of homogeneity as a candidate axiom. Originally, it seemed like the kind of constraint that the mathematicians impose just so they have a set they can classify. Furthermore, the objects being classified did not seem particularly “natural” or well-motivated from our starting point. My first temptation to reconsider was Marcus Appleby’s construction of what I have termed qplectic cones [35]. Later still, building upon this, I realized that “doing as the mathematicians do” might be exactly what the project called for. Maximal symmetry and maximal abstraction can be, at the right juncture, an expression that the physical content lives at a different level.

Prior work has noted that given a qplex, the *urgleichung* is the natural way to calculate probabilities for other experiments in terms of the reference measurement, and in consequence, “posteriors from maximal ignorance are valid priors” [2]. Homogeneity is an extension of this idea. Not a trivial one, as the existence of highly asymmetrical qplexes will attest, but an amiable one. Instead of merely noting that any point  $p$  in our qplex  $\mathcal{P}$  is a valid post-measurement state for some experiment given the flat vector  $c$  as input, we posit an update rule, a map from the qplectic cone to itself that takes  $c$  to  $p$  (up to normalization). One justification for this elaboration comes from the convexity of the qplex. Because the flat vector  $c$  can be decomposed as a convex combination of other states (in many ways), we can regard  $c$  as a van Fraassen reflection state, an “expected future expectation”, and a valid transformation of  $c$  ought to be a valid transformation of the points into which we decompose it as well. (If an event leads me to update my state of belief to  $p$  if I currently believe  $c$ , what does it prompt me to do if I believe  $s$ ? Because  $s$  and  $p$  are taken from the qplex  $\mathcal{P}$ , this sets us on the path to defining a binary operation on  $\mathcal{P}$ .) Homogeneity is motivated by asking that these transformations be nontrivial and reversible whenever possible.

Another way to express this motivation is to note that we are trying to put an algebraic structure on the qplex, or on the qplectic cone, so we should consider where the notion of an *algebra* came from. As students, somewhere in our education, we learned to think of *multiplication* as *scaling*: The quantity “ $a$  times  $b$ ” is where  $b$  lands when we apply the transformation of space that sends 1 to  $a$ . (Usually, we might first meet this picture in the setting of complex numbers, though not always [44].) If we have an isomorphism of the qplectic cone that sends the order unit  $\mathcal{I}$  to a point  $a$ , we can define the product  $a \star b$  by finding where this transformation takes the point  $b$ . This is how we “treat points as numbers”, i.e., how we give the qplectic cone an algebraic structure.

A final justification for homogeneity of the qplectic cone is that homogeneity is a property that holds classically [45], and we are doing our best to localize the conceptual departure from classicality in a single place, the principle of quantum vitality. In classical probability theory, we can use the all-ones vector as our order unit, and obtain any point in the interior of the probability simplex trivially, by multiplying each entry by  $p(i)$ . The quantum analogue of this map is slightly more elaborate: The operation

$$L_\rho(X) = \rho^{1/2} X \rho^{1/2} \tag{44}$$

sends the identity matrix to  $\rho$  and is invertible whenever  $\rho$  is not singular.

Now that we are at least content enough with the idea of putting an algebra on a qplex to jump in and see where it goes, we can investigate what kinds of algebraic structure are consistent with the features that are common to all qplexes. In section VII, we will touch upon the possibility of an alternative path to establishing an algebra.

#### IV. NARROWING THE CHOICE OF ALGEBRAS

A Jordan algebra is an abstraction of some of the properties enjoyed by observables in quantum theory [45, 46]. In what follows, we will restrict ourselves to considering finite-dimensional algebras only. To build a Jordan algebra, we need a set of quantities that we can combine with a product operation, and this product must satisfy two conditions. First, it needs to be commutative:

$$x \circ y = y \circ x. \quad (45)$$

And second, rather than being so kind as to satisfy the associative law, the algebra satisfies the *Jordan identity*,

$$(x \circ y) \circ (x \circ x) = x \circ (y \circ (x \circ x)). \quad (46)$$

One way to build a Jordan algebra is to take a matrix algebra and define the Jordan product in terms of a symmetrized combination of matrix products:

$$x \circ y = \frac{1}{2}(xy + yx). \quad (47)$$

A Jordan algebra is *formally real* if a sum of squares only vanishes when each individual element being squared vanishes:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 0 \Rightarrow x_1 = x_2 = \cdots = x_n = 0. \quad (48)$$

An *ideal* in a Jordan algebra is a subspace that “absorbs multiplication”. A subspace  $B \subseteq A$  is an ideal of  $A$  if, for every  $b \in B$ , we have  $x \circ b \in B$  for all  $x \in A$ . We call a Jordan algebra *simple* if its only ideals are the empty set and the whole algebra itself. Any formally real Jordan algebra can be written as a direct sum of simple Jordan algebras. Thus, classifying all formally real Jordan algebras boils down to the task of classifying the simple ones. Jordan, von Neumann and Wigner did this, back in the day. There are three infinite sequences, defined by applying Eq. (47) to matrix algebras of  $d \times d$  self-adjoint matrices built from the real numbers, the complex numbers and the quaternions respectively. In addition, there is another infinite family built from the direct sum of vector spaces  $\mathbb{R}^d \oplus \mathbb{R}$ . An element of such an algebra is an ordered pair

$$(v, \alpha), \text{ with } v \in \mathbb{R}^d, \alpha \in \mathbb{R}. \quad (49)$$

The product rule for this algebra is

$$(v, \alpha) \circ (w, \beta) = (\alpha w + \beta v, v \cdot w + \alpha\beta). \quad (50)$$

This is the family of “spin factors”. Finally, there is an exceptional case, the *Albert algebra*, defined by applying Eq. (47) to  $3 \times 3$  self-adjoint matrices of octonions.

We will see in the next section that we can parameterize generalized qplex theories by relating the cardinality of the reference measurement,  $N$ , to the maximum size of an MMD set:

$$N = d + q \frac{d(d-1)}{2}, \quad (51)$$

where  $q$  is a nonnegative integer. The case  $q = 0$  collapses to classical probability theory: The qplex is just the probability simplex in  $\mathbb{R}^d$ , the lower bound  $L$  drops to zero, and the MMD states are Kronecker delta functions. When we set  $q = 2$ , we recover the quantum formula,

$N = d^2$ , while  $q = 1$  and  $q = 4$  reproduce the scaling relations of the real and quaternionic “foil theories” to quantum mechanics. The spin-factor theories have state spaces that are Euclidean balls, and so they can only exist for  $d = 2$ : Given any extremal point, there is exactly one antipodal point.

In a Jordan theory, the space of observables forms a simple Jordan algebra. The state space is the set of positive observables that have trace equal to 1. This generalizes quantum theory: In the quantum theory of a finite-dimensional system, the observables are self-adjoint complex matrices, and the states are the density operators, which are positive semidefinite matrices with unit trace. Generally, then, in a Jordan theory, any state is also an observable. We can also define effect operators in the analogous manner to quantum theory, and then we can construct the Jordan-theoretic version of POVMs, as collections of effects which sum to the identity.

We can see the general plan for the final step of a Jordan-theoretic reconstruction of quantum theory. One way or another, we wish to select the *complex* option out from among the real, the quaternionic, the octonionic and the spin-factorial. (In what follows, we will be expanding upon a suggestion made in [39, §7].)

The constant  $N$  is fixed by the number of real parameters needed to specify a density operator. If we cannot find  $N$  equiangular pure states within the state space of a Jordan theory, then we cannot satisfy the generalized urgleichung, Eq. (43). Conversely, if we assert that the generalized urgleichung is valid for all dimensions, then we rule out all classes of Jordan theories which do not admit a full set of equiangular pure states.

Why should we focus on *simple* algebras? Because an algebra splitting into a direct sum,  $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ , represents *classical ignorance*. This models the situation where our system of interest might be one for which we use the algebra  $\mathcal{A}_1$ , or it might be one for which we use the algebra  $\mathcal{A}_2$ , and the choice between them is a classical bit. Recall that we admitted the possibility of there being many possible reference measurements, of which the most conceptually illuminating — the analogues of Carnot engines or light clocks — are those for which perfect distinguishability corresponds to the constant overlap  $L$ . If the algebra of the qplectic cone factored, we could make a reference measurement whose elements are direct sums of operators in the separate factors, and states from different factors would have nonoverlapping probabilistic representations. The requirement that all reference measurements avoid this ensures that the algebra has just one factor. Another heuristic is that, if the basis distributions  $\{e_k\}$  map to algebra elements that are spread across multiple factors, then they represent both classical ignorance (the choice of factor) and quantum uncertainty, and so they would not be states of maximal confidence, which we know they are. So, the basis distributions must each map to a state localized within a factor, and since by construction they cannot be orthogonal states, they must all lie within the same factor, along with the entirety of the state space they span.<sup>1</sup>

While much remains unknown about the subject of equiangular lines, all indications to date are that the complex option — that is, ordinary quantum theory — is the only

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<sup>1</sup> Recall how Barnum, Müller and Ududec [47] obtain simplicity: by noting that there can’t be two different kinds of orthogonality. Two orthogonal pure states from the same factor define a Bloch ball (possibly not three-dimensional), while two pure states from different factors define a quasi-classical bit. Having two qualitatively different kinds of faces in this way is ruled out by their first two postulates (“classical decomposability” and “strong symmetry”).



possibility consistent with this requirement. For the real case, we would require

$$N_{\mathbb{R}} = \binom{d+1}{2} \quad (52)$$

equiangular lines for each value of the dimension  $d$ . This is an upper bound that one can deduce from linear algebra, but not every dimension can sustain a full set of  $N_{\mathbb{R}}$  equiangular lines [48–53]. In fact, the only known dimensions in which the bound is attainable are  $d = 2, 3, 7$  and  $23$ . One can prove that the only possibilities for having a full set are when  $d$  is 2, 3 or a value 2 less than the square of an odd number. (Such a mathematician’s statement!) The conjecture that the bound can be attained whenever  $d + 2$  is an odd square was disproved by an explicit counterexample for dimension 47, back in 2002 [53]. So, the real-vector-space analogue of quantum theory is ruled out: We cannot even formulate the generalized *urgleichung* in the EPR-type case,  $d = 4$ ; nor in the GHZ-type scenario,  $d = 8$ . By contrast, not only do we have SICs in  $\mathbb{C}^4$  and  $\mathbb{C}^8$ , but they even have rather remarkable properties above and beyond the symmetry implied by the definition [27, 54–57].

The quaternionic case is much less understood than either the real or the complex, but searches to date have failed to find a quaternionic SIC in any  $d > 3$  [58]. That is, we require for all dimensions  $d$  a set of

$$N_{\mathbb{H}} = d(2d - 1) \quad (53)$$

equiangular projection operators for a symmetric, informationally complete quaternionic measurement [59]. Granting that these measurements are probably not available in all dimensions, we rule out the quaternionic analogue of quantum theory. Moreover, there are arguments that the real and quaternionic theories should be considered together or not at all [60], and since we *know* that “real SICs” do not exist for all dimensions, we can set both  $\mathbb{R}$  and  $\mathbb{H}$  aside.

There *does* appear to be an “octonionic SIC” in  $d = 3$ , corresponding to the Albert algebra. Cohn, Kumar and Minton give an existence proof and a numerical solution, though not an exact construction [58]. However, we can say confidently that we do not want our theory to stop at three-level systems; this case may be more interesting for group theory than for quantum physics [61]. For example, it may be possible to think of each point in the Leech lattice as a “Hamiltonian” for an “octonionic qutrit”, since the Leech lattice can be embedded in the space of traceless self-adjoint  $3 \times 3$  matrices over the octonions [62]. The symmetry group of the generalized qplex for this unusual system would be what the specialists call  $F_4$ , and the symmetries of the set of Leech “Hamiltonians” would give the Conway groups. A similar claim could indeed be made for the equiangular lines in  $\mathbb{R}^d$ , since one construction of the maximal set in  $\mathbb{R}^7$  extracts it from the  $E_8$  lattice, and the maximal set in  $\mathbb{R}^{23}$  is derived from the Leech lattice, thereby connecting the “real SICs” with many other optimality problems [63, 64].

The real and octonionic “foil theories” may well be having a subtle liason with the Leech lattice, but for physics purposes, it appears that the complex option is the only theory left standing. We can articulate the lesson as follows:

Quantum theory is the maximally symmetric probabilistic theory that embraces vitality, and we pursue maximum symmetry because we want to present that vitality in the clearest manner possible.

## V. INTERLUDE: PAUSE FOR A MILKSHAKE

One summer in graduate school I was a student of [Edward] Condon’s. I remember vividly his account of being brought up before some loyalty review board:

“Dr. Condon, it says here that you have been at the forefront of a revolutionary movement in physics called” — and here the inquisitor read the words slowly and carefully — “quantum mechanics. It strikes this hearing that if you could be at the forefront of one revolutionary movement . . . you could be at the forefront of another.”

— Carl Sagan, *The Demon-Haunted World* [65]

Physics has, most likely, as morally checkered a past as most modern vocations. We can without contradiction observe simultaneously that physics suffered in the McCarthy era — with Melba Phillips being blacklisted for five years, for example [66] — and that the profession was in essence the long-term planning department of the military-industrial complex. It is in this context that we pause to note that Pascual Jordan was a morally repugnant human being. After publishing extremist political screeds in the 1920s under a pseudonym, he abandoned pretence in 1933, joining the Nazi Party and signing up to be a brownshirt. He tried to defend quantum mechanics from the charge of being “Jewish physics” . . . because he said it was an antidote to Bolshevik materialism. Perhaps many organizations can be changed from within, as Jordan apparently tried to justify himself to Bohr after the war [67, 68], but we doubt that any fascist parties are among them.

Now I know how specialists in Teichmüller spaces feel, if they have any conscience. (I recall that at one point, Oswald Teichmüller’s Wikipedia page described him bluntly as “a mathematician and Nazi”.)

But there is a flipside. Bloody Agamemnon lives by grace of Homer’s verse; Macbeth ruled Scotland for 17 years, but he is now forever the king who trafficked with witches who were the Fates. (For my generation, gargoyles were added to the mix.) In that vein, less grandiosely: These algebras will matter to the pop-science crowd because of a scruffy queer kid who votes for democratic socialists and gets mistaken for Jewish at the bus stop.

## VI. MORE DETAILS REGARDING RECONSTRUCTION

My sins my own, they belong to me. Me!  
 People say “beware!” But I don’t care.  
 The words are just rules and regulations  
 to me. Me!

— Patti Smith

It is worth investigating a little more deeply the idea that the inequality  $\langle p_1, p_2 \rangle \geq L$  is a statement of quantum vitality. We can do so thanks to a variant of Lemma 4 in the “MIC Facts” survey [26]. A *minimal informationally complete* measurement, or *MIC*, is what we get when we relax the requirement that our reference experiment be symmetric: It is a set of positive semidefinite operators  $\{E_i : i = 1, \dots, d^2\}$  that sum to the identity matrix. This furnishes a reference measurement for quantum states ascribed to  $d$ -dimensional systems.

Specifically, we can express any quantum state  $\rho$  as an expansion

$$\rho = \sum_i p(E_i) \tilde{E}_i, \quad (54)$$

where  $p(E_i)$  is the Born-rule probability  $\text{tr}(\rho E_i)$  and  $\{\tilde{E}_i\}$  is the dual basis of the MIC.

Assume for the moment that all the MIC elements  $\{E_i\}$  are proportional to rank-1 projectors. Can two orthogonal states  $\rho$  and  $\sigma$  have orthogonal probabilistic representations  $p_\rho$  and  $p_\sigma$ ? Not for qubits, that's for sure: When  $d = 2$ , we can have at most *one* zero in a probability vector. At most one of the  $\{E_i\}$  can be orthogonal to a given vector, and by linear independence, two different MIC elements cannot be proportional to each other. What about  $d > 2$ ? Let  $|\psi\rangle$  and  $|\phi\rangle$  be two orthogonal states. Together they define a qubit-sized subspace. Now, project the MIC into this subspace. If all the images of the MIC elements lined up with  $|\psi\rangle$  or  $|\phi\rangle$ , then the MIC would not be informationally complete on that subspace, because spanning requires four elements. Therefore, at least one of the  $\{E_i\}$  must be nonorthogonal to both  $|\phi\rangle$  and  $|\psi\rangle$ . The slot in the probability distributions corresponding to this MIC element must be nonzero for both. Consequently, the Euclidean inner product of these vectors cannot be zero.

More generally, we can drop the rank-1 condition. If  $\rho$  and  $\sigma$  are any two quantum states, then their inner product is

$$\text{tr}(\rho\sigma) = \sum_{ij} p_\rho(E_i) p_\sigma(E_j) \text{tr}(\tilde{E}_i \tilde{E}_j). \quad (55)$$

A convenient fact of frame theory has it that the Gram matrix of the dual basis is the inverse of the Gram matrix  $G$  of the original, so

$$\text{tr}(\rho\sigma) = \sum_{ij} p_\rho(E_i) p_\sigma(E_j) [G^{-1}]_{ij}. \quad (56)$$

Note that *if* the inverse Gram matrix  $G^{-1}$  could be the identity, then the Hilbert–Schmidt inner product of density matrices would reduce to the Euclidean inner product of probability vectors. In particular, orthogonality of matrices would correspond exactly to disjoint support of probability distributions. But we know that no MIC can ever exist for which  $G^{-1}$  is the identity matrix [25].

Trying to make orthogonal quantum states have disjoint probabilistic representations is asking for a self-dual basis, and no MIC can ever be self-dual. The closest any MIC can ever come is by being a SIC [25, 26].

Now that we understand the significance of  $L$  a bit more, let's return to the more general context, where the fundamental inequalities are

$$L \leq \langle p, s \rangle \leq U \quad (57)$$

for probability vectors  $p$  and  $s$  of Euclidean dimension  $N$ . As in the qplex paper [2], we consider mutually maximally distant (MMD) sets of probability distributions. An MMD set  $\{p_k\}$  satisfies

$$\langle p_k, p_k \rangle = U, \quad \langle p_k, p_{l \neq k} \rangle = L \quad (58)$$

for  $k, l = 1, \dots, m$ . Changing to barycentric coordinates

$$p'_k = p_k - c, \quad (59)$$

the shifted vectors satisfy

$$\langle p'_k, p'_k \rangle = U - \frac{1}{N} = r_{\text{out}}^2, \quad \langle p'_k, p'_{l \neq k} \rangle = L - \frac{1}{N} = -r_{\text{mid}}^2. \quad (60)$$

Construct the sum of all the  $\{p'_k\}$ :

$$V = \sum_{k=1}^m p'_k. \quad (61)$$

The norm of  $V$  is

$$\langle V, V \rangle = mr_{\text{out}}^2 + (m^2 - m)(-r_{\text{mid}}^2). \quad (62)$$

Because the norm of a vector is always nonnegative, we have that the size  $m$  of the MMD set satisfies the bound

$$m \leq 1 + \frac{r_{\text{out}}^2}{r_{\text{mid}}^2}. \quad (63)$$

When the value of  $m$  attains this upper bound, the norm of  $V$  is zero, and thus  $V$  itself must be the zero vector, meaning that

$$\sum_k p_k = mc. \quad (64)$$

The Gram matrix for the MMD set  $\{p_k\}$  is

$$G = (U - L)I_m + LJ_m. \quad (65)$$

Using the values familiar from quantum theory,  $m \leq d$  and

$$[G]_{jk} = \frac{1 + \delta_{jk}}{d(d+1)}. \quad (66)$$

Because the Gram matrix is invertible, the vectors  $\{p_k\}$  are linearly independent. Consequently, the only way to get  $p_1$  through  $p_m$  to sum up to the vector  $mc$  is to add them with equal weights, as above. We see that the only way the MMD set  $\{p_k\}$  can constitute a measurement matrix is to set

$$r(k|i) = \frac{N}{m} p_k(i). \quad (67)$$

Given the garbage state  $c$ , all outcomes of this measurement are necessarily assigned the same probability  $1/m$ .

Let  $d$  denote the bound we deduced above on the size of an MMD set in terms of  $N$  and  $L$ . By assuming that this bound is in fact an integer, we can show that

$$N = d + q \frac{d(d-1)}{2}, \quad (68)$$

where  $q$  is a nonnegative integer. Fixing a value of  $q$  fixes the constants  $N$ ,  $L$  and  $U$  in terms of the parameter  $d$ . Homogeneity of the qplectic cone, the assumption with which we have gradually made peace, leads us to set  $q = 2$ , so that  $N = d^2$  and  $U$  is exactly twice  $L$ :

$$\frac{1}{d(d+1)} \leq \langle p_1, p_2 \rangle \leq \frac{2}{d(d+1)}. \quad (69)$$

With this choice, the generalized *urgleichung* becomes the original *urgleichung* stated in earlier work [14]:

$$q(j) = \sum_i \left[ (d+1)p(i) - \frac{1}{d} \right] r(j|i). \quad (70)$$

I’ve often wanted to begin a lecture on some abstruse technical topic by intoning, in my best pompous announcer voice, “Since Man first looked up at the stars in wonder, he has asked himself, are all SIC-POVMs group covariant?” At this juncture, I would have better warrant to do so, because our relation between  $N$  and  $d$  is an instance of a formula that “Man” has known for quite a while. When we set  $q$  equal to 2, then we get that  $N$  is just the square of  $d$ ; in other words,  $N$  ranges over the square numbers. If instead we fix  $q$  equal to 1,  $N$  will range over the triangular numbers. And, for an arbitrary value of  $q$ , this formula says that  $N$  is the  $d^{\text{th}}$  *polygonal number*, where the polygons in question have  $q+2$  sides [69]. Polygonal numbers go back to Pythagorean number mysticism, and were studied as long ago as Hypsiclēs, who was active in the second century BCE.<sup>2</sup>

The Pythagorean number mystics could have arrived at the rule we express by Eq. (68) from such a simple starting point as arranging pebbles into nice shapes and then counting how many pebbles the shape contains. Shapes of larger and larger size are built up by adding more pebbles. The arrangement of pebbles added in each step is called the *gnomon*, and it has the form of  $q$  sides of a regular polygon with  $(q+2)$  sides total. For example, by starting with a single pebble, and then stacking a vertical line of two pebbles next to that, three pebbles next to that, and so on, we build up triangles, whose pebble populations are sums of the natural numbers:

$$1 + 2 + \dots + d = \frac{d(d+1)}{2}. \quad (71)$$

If we instead build our shape outwards by adding two sides of a square, we get the square numbers. If the gnomon has  $q$  sides, then the increment between successive polygonal numbers grows by  $q$  with each step. For a given value of  $q$ , the  $d^{\text{th}}$  polygonal number is the

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<sup>2</sup> When you go back that far, the history of mathematics and science becomes semi-legendary. The best one can typically do for “evidence” is a fragment of a lost book quoted in another book that happened to survive, and all of it dating to decades or centuries after the events ostensibly being chronicled. Did Pythagoras actually prove the theorem we named after him, or did he merely observe that it held true in a few special cases, like the 3–4–5 right triangle? Tough to say, but the latter would have been easier, and it would seem to appeal to a number mystic, for whom it’s all about the successive whole numbers. Pythagoras himself probably wrote nothing, and nothing in his own words survives. It’s not clear whether his contemporaries viewed him as a mathematician or primarily as a propounder of an ethical code. (Even only 150 years after the time he purportedly lived, the ancient authorities disagreed about whether Pythagoras was a vegetarian, with Aristoxenus saying no and Eudoxus yes [70].) Suppose that Pythagoras had never lived, and a cult had attributed their work to that name in ritual self-denial — like the Bourbaki collective [71], we might say, but more so. Their bibliographic practices would not be exactly the same as ours today. Where we’d say, “This idea comes from Egyptian mathematics, where it is stated in the Rhind papyrus,” they might have said, “Pythagoras learned this idea in Egypt.” Later, parables of this kind could have been taken for biography: “In his youth, Pythagoras visited Egypt.” The result of such a process would be hard to tell from the surviving historical evidence we have today.

sum of the first  $d$  terms of the sequence that starts with 1 and grows by  $q$  at each step:

$$N = \sum_{k=0}^{d-1} (1 + kq) = d + q \sum_{k=0}^{d-1} k. \quad (72)$$

Because the second term is just  $q$  times the  $(d-1)^{\text{th}}$  triangular number, we can also arrange our  $N$  pebbles as a line of  $d$ , plus  $q$  triangles of side length  $d-1$  pebbles each.<sup>3</sup>

Thus, the function  $N(d)$  for a given choice of  $q > 0$  would be the number of real parameters in a  $d \times d$  self-adjoint matrix wherein the elements have  $q-1$  distinct imaginary units. As is well known, “numbers” of such forms can only constitute normed division algebras if  $q$  equals 1, 2, 4 or 8. Pythagoreans would probably like the fact that the  $N(d)$  relations for real, complex and quaternionic quantum mechanics correspond to triangles, squares and hexagons, the three polygons that, when drawn regularly, tile the plane without gaps. The fact that these are the three possible regular tilings is another result whose original proof is lost to legend [76, pp. 209–10]. Despite much mulling over “Arnold trinitities” [77], I have been unable to invent a deep significance for this coincidence. Nor have I been able to find a solid connection with the *other* appearance of normed division algebras in this theory, the classification of the sporadic SICs [29].

## VII. (EN)TANGLED BANKS: CHARTING THE SHORELINES OF THE HILBERT QPLEXES

I first presented the result at an APS meeting a couple years ago. Charlie Bennett was in the audience and asked, “Is that a 7?” I said, “Yep, it’s really a 7.” Charlie said, “Well then, it’s the first 7 I’ve ever seen in quantum information.” And what else would you expect from a truly fundamental equation?! Indeed it is a 7, and well checked many times by myself and independently by my students. In fact, just the other day by the latest, Ryan Morris, who first found a 6 instead . . . but then ultimately found a 7.

— Chris Fuchs, in correspondence, 2008 [78]

The last technical discussion in this essay will be a more detailed treatment of the Hilbert qplexes. For a given dimension  $d$ , a Hilbert qplex lives in the space  $\mathbb{R}^{d^2}$ , and more specifically in the hyperplane through that space comprising the vectors whose entries sum to unity. Like all qplexes, the Hilbert qplexes are closed and convex. One way to specify a Hilbert qplex is to say it is the convex hull of those probability distributions that stand for maximal certainty. These distributions are those which satisfy two conditions, one quadratic and

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<sup>3</sup> The polygonal numbers, as a class, are not the sort of mathematics that physics habitually invokes. However, historians of science observe that Leonardo da Vinci wrote, “The freely falling body acquires with each unit of time a unit of motion, and with each unit of motion a unit of velocity” [72]. One reading of this is that Leonardo believed that the distance a body falls in successive time intervals goes as the positive integers. The *total* distance fallen after  $T$  time units is thus a triangular number. The step to Galileo’s law of falling bodies is to replace successive integers with successive odd numbers, making the total distance fallen go as the perfect squares. Amusingly, this is the same change that separates quantum mechanics from its most closely-studied foil theory [73–75].

the other cubic. Given my background, they look like conditions on diversity indices, or on expected scores in peculiar games [16, 79, 80]. The first is just the upper bound of the fundamental inequalities, and it demarcates a sphere:

$$\sum_j p(j)^2 = \frac{2}{d(d+1)}. \quad (73)$$

The second — we sometimes call it the QBic equation — carves away at that sphere:

$$\sum_{jkl} p(j)p(k)p(l)C_{jkl} = \frac{d+7}{(d+1)^3}, \quad (74)$$

where the tensor  $C_{jkl}$  brings in the SICs:

$$C_{jkl} = \text{Re tr} \Pi_j \Pi_k \Pi_l. \quad (75)$$

Perhaps the easiest way to see that peculiar 7 arise is to note that in any dimension  $d$ , we can always construct a *quasi-SIC*, a set of  $d^2$  Hermitian operators that sum to  $dI$  and satisfy

$$\text{tr} Q_j Q_k = \frac{d\delta_{jk} + 1}{d+1}, \quad (76)$$

while not necessarily being positive semidefinite [2]. Any quasi-SIC establishes a mapping from a qplex to operator space.

Substituting in one of the basis distributions

$$e_a(j) = \frac{1}{d(d+1)} + \frac{\delta_{aj}}{d+1}, \quad (77)$$

we find that

$$\sum_{jkl} e_a(j)e_a(k)e_a(l)\text{Re tr} Q_j Q_k Q_l = \frac{d+6 + \text{Re tr} Q_a^3}{(d+1)^3}. \quad (78)$$

The “first 7 in quantum information” occurs when  $\text{tr} Q_a^3$  is constant and maximized, which happens when the quasi-SIC is a genuine SIC.<sup>4</sup>

Asking that the contraction of  $C_{jkl}$  with the probability vectors always evaluates to the same thing is, it seems, a way of asking for any pure state to be an element in some SIC. That is, while we got here by imposing homogeneity on the *interior* points, the QBic equation is telling us about *transitivity* on the *extreme* points. The shoreline of a Hilbert qplex is made by the orbit of a single volumeless grain of sand.

The properties of the  $C_{jkl}$  tensor that we just invoked follow from the fact that the  $\{\Pi_j\}$  are Hermitian, and more particularly, that they are rank-1 projectors. We can say much more, actually: The *triple products*

$$T_{jkl} = \text{tr} \Pi_j \Pi_k \Pi_l \quad (79)$$

for any SIC necessarily have a rather rich algebraic personality [82, 83]. Furthermore, for all the SICs discovered to date, they also reach out to other areas of mathematics, sometimes

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<sup>4</sup> Incidentally, evaluating the sum in Eq. (77) appears to be one of the recurring common patterns that the ancient Egyptians relied upon in their arithmetic [81].

group theory and discrete geometry [56], and sometimes algebraic number theory [27, 28], and nobody knows why. I suspect there is even more to tell than that. All the SICs known so far enjoy a property called *group covariance* (and again, nobody knows why). That is, each SIC can be constructed by starting with an initial vector  $|\pi_0\rangle$  and taking its orbit under the action of a group, and so that group can serve to turn any  $|\pi_j\rangle$  into any other  $|\pi_k\rangle$  in the SIC. Moreover, in almost all cases, the group is the *Weyl–Heisenberg group* for dimension  $d$ ; the exception is a class of solutions in  $d = 8$  which use a close variant, and which relate, as so many mathematical exceptions seem to do, with the octonions [29]. The Weyl–Heisenberg group is of considerable technical and historical importance [31], and so its tight connection with SICs warrants much thinking upon. About the best we can currently say in general terms is Huangjun Zhu’s result that group covariance in prime dimension implies Weyl–Heisenberg covariance [84].

Let  $V$  be a Weyl–Heisenberg operator, and consider the *Clifford group*, the normalizer of the Weyl–Heisenberg group — that is, the set of unitaries that map the set of Weyl–Heisenberg operators to itself up to phase factors. Suppose that  $\{W(t)\}$  is a one-parameter family of Clifford unitaries, for convenience chosen such that no extraneous phase factors arise. The parameter  $t$  can be interpreted as the clock in a discrete time-evolution process. Then, the quantity

$$F(t) = \langle \pi_0 | W(t)^\dagger V^\dagger W(t) V | \pi_0 \rangle \quad (80)$$

will always be the inner product between  $|\pi_0\rangle$  and some SIC vector  $|\pi_j\rangle$  (possibly multiplied by an overall root of unity):

$$\langle \pi_j | \pi_0 \rangle = \frac{e^{i\theta_j}}{\sqrt{d+1}}, \quad j \neq 0. \quad (81)$$

These numbers are closely related to the triple products, because

$$\text{tr} \Pi_j \Pi_k \Pi_l = \langle \pi_j | \pi_k \rangle \langle \pi_k | \pi_l \rangle \langle \pi_l | \pi_j \rangle. \quad (82)$$

When all three indices are distinct, this will be a complex number of magnitude  $(d+1)^{-3/2}$  and phase  $e^{i\theta_{jkl}}$ , which we can interpret as a *geometric phase* factor [85]. And the quantity  $F(t)$  is, formally, equal to an *out-of-time-ordered correlator* (OTOC) for the Clifford time evolution  $W(t)$ . OTOCs have become of considerable interest in quantum chaos and thermodynamics [86, 87]. Loosely adapting the language one hears in APS talks about OTOCs, we might say that the geometric phases indicate how quantum information is redistributed as we go from one SIC vector to another.

The QBic equation (74) looks a little like differential geometry, a little like game theory; and it brings algebraic number theory into unusual proximity with quantum chaos.

## VIII. CONCLUSION

I have elsewhere devoted an excessive number of words to setting QBism in its historical context [88–90]. In this essay, I have tried to be more forward-looking, though this has involved bringing up old things to make them part of the new.

Borges once wrote an essay where he listed stories that he found Kafkaesque. The common denominator of these stories, going back to Zeno’s paradox of Achilles and the tortoise, was that they all felt like Kafka, but that he would never have noticed a shared thread between them if he had not read Kafka. So, Borges argued, Kafka invented his precursors [91].



Likewise, if there were ever a Niels Bohr whose philosophy was compatible with QBism, it would have to be a Bohr that QBism invented, as Kafka did for Zeno of Elea.

One of the talking heads in *The Creation of the Universe* was John Archibald Wheeler, who spoke of his belief that under it all there must lie, not a simple equation, but a compellingly simple idea. He gave a version of the parable that he used in his first-year classical mechanics course at Princeton [78], where he asked each student to write down what they thought were the most important equations of physics.

He gathered the papers up and placed them all side-by-side on the stage at the front of the classroom. Finally, he looked out at the students and said, “These pages likely contain all the fundamental equations we know of physics. They encapsulate all that’s known of the world.” Then he looked at the papers and said, “Now fly!” Nothing happened. He looked out at the audience, then at the papers, raised his hands high, and commanded, “Fly!” Everyone was silent, thinking this guy had gone off his rocker. Wheeler said, “You see, these equations can’t fly. But our universe flies. We’re still missing the single, simple ingredient that makes it all fly.”

Wheeler was once asked, “Is the Big Bang here?” Judging by his response, he found the question rather charming [92]. It is of course wholly orthodox to say that the Big Bang *was* here, as the theory of the metric expansion of space has detailed [93]. Withdraw all the air from an ideal balloon, and all the galaxies drawn on it come together, as we have each heard often enough. Wheeler took the question down a different path, stating, “Each elementary quantum phenomenon is an elementary act of ‘fact creation.’” He wondered, “Have we had the mechanism of creation before our eyes all this time without recognizing the truth?” QBism took this idea and ran with it: The Bang was here long before us, *is* here with us, *will be* here after us. Each quantum measurement is a personal sampling of it, a species of the ongoing creatia distinguished by the fact that one participant is an agent who bears expectations and the burden of choice.

This essay began with a probability simplex and a bilinear form, notions that are crisply geometrical and briefly stated. Yet the story of how the resulting structure of maximal symmetry fits into  $\mathbb{C}^d$  explodes into tottering piles of nested radicals — chaos within order [22]. Looking more deeply still, we find hints of an order within that chaos — the construction of superlatively symmetric measurements from number theory [27, 28].

I recall that years ago, before my adventure into QBism, I read an argument in what I think was an interview with Anton Zeilinger. In my memory, he said that measurement is an essential part of doing science, and so it should not be surprising that the next great advance in fundamental physics might require an improved theory of the measurement concept. I cannot find this interview, however, and it is possible that I am remembering the attribution incorrectly. In any case, whoever said it, I think it is a much more healthy attitude than rejecting the very idea that a theory of measurement can be a theory of physics. This latter position is, of course, an article of faith to many. I might well have had more sympathy myself for this creed in my younger days, had I spent much time thinking about the issue at all. For in those days, I was less experienced in the reality of physical practice, and I was temperamentally inclined to align myself with whoever proclaimed their love of physics the most loudly. My youthful affinity for this way of making judgments was, I’d say, very much akin to my unexamined belief that our profession is a meritocracy, naturally immune to discrimination by race or by gender [94–102]. I must plead my youth, not as excuse but only as explanation.

We physicists can be great hypocrites. For instance, it is common to complain about the imprecision of the word *measurement* in quantum mechanics, particularly as older texts use it [103, 104]. But you will seldom hear a physicist make a peep about the word *event* going undefined in a probability book, or for an example even more within our wheelhouse, in a text on special relativity. In a homework problem for special relativity, an *event* could be a single electron annihilating with a positron, or it could be the explosion of a supergiant star. This is, I'm sure you'll agree, a category with no well-defined boundaries. It is criminally vague. One wonders if we could somehow blame Niels Bohr for it.

As a profession, we have found it adequate to gloss an *event* in special relativity as a phenomenon whose spatial extent and temporal duration can be neglected for the purposes at hand [105]. To be a touch more technical about it, *events* are phenomena that we can associate with points in the conceptual contrivance we call Minkowski spacetime. (“A point,” we learned from Euclid, “is that which has no parts” [106]; yet a train pulling into a station or a clock striking noon definitely has parts in plenty.) Saying that a *measurement* on a quantum system is a process that we can associate with a set of positive semidefinite operators is no less respectable. The only reason we ever had to think otherwise was the historical accident that we could see how to deduce the Minkowski metric from Einstein's postulates [107] before we had an equally principled construction of the quantum formalism.  $\square$

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