On duplicate representations as $2^x + 3^y$ for nonnegative integers x and y

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Abstract

We prove a conjecture posted in the Online Encyclopedia of Integer Sequences, namely that there are exactly five positive integers that can be written in more than one way as the sum of a nonnegative power of 2 and a nonnegative power of 3. The case for both powers being positive follows from a theorem of Bennett. We use elementary methods to prove the case where zero exponents are allowed.

1 Introduction

In the Online Encyclopedia of Integer Sequences (*OEIS*, Sloane [3]), sequence `A004050` comprises the integers of the form $2^x + 3^y$ for nonegative integers x and y. On this sequence's entry in the *OEIS*, it was remarked as a conjecture in September 2012 that only five of these integers can be so expressed in two different ways.

In fact, sequence `A085634` lists those very integers representable both as $2^x + 3^y$ and $2^a + 3^b$, with x, y, a, and b nonnegative integers and $x > a$. The five elements listed are

$$\begin{array}{lllll} 5 = 2^2 + 3^0, & 11 = 2^3 + 3^1, & 17 = 2^4 + 3^0, & 35 = 2^5 + 3^1, & 259 = 2^8 + 3^1, \\ 5 = 2^1 + 3^1, & 11 = 2^1 + 3^2, & 17 = 2^3 + 3^2, & 35 = 2^3 + 3^3, & 259 = 2^4 + 3^5. \end{array}$$

On the entry in the *OEIS* for sequence `A085634`, it was remarked in February 2005 that if n is in the sequence and $n > 259$, then $n > 10^{4000}$. In this note, we render this lower bound vacuously true by proving the conjecture: indeed, the five numbers listed above are the only elements of `A085634`.

We thus assume that

$$2^x + 3^y = 2^a + 3^b, \tag{1}$$

where x, y, a, and b are nonnegative integers, such that (without loss of generality) $x > a$ (whence $y < b$).

Equivalently,

$$2^x - 3^b = 2^a - 3^y. \tag{2}$$

This brings us to sequence A207079 in the *OEIS*, which is described in its entry as "the only nonunique differences between powers of 3 and 2." It is given as a finite sequence of five elements, namely 1, 5, 7, 13, and 23. It is commented that the finiteness of this sequence is due to Bennett [1], who, in fact, proved a more general result, from which the finiteness of A207079 follows directly. In his article, he gives a clear, precise history of the general problem of determining the number of solutions to the exponential Diophantine equation $|a^x - b^y| = c$, and we learn that the finiteness of the specific sequence A207079 was first proved in 1982. We state here, as a lemma, the special case of Bennett's result that applies most directly to (2).

Lemma 1. (Bennett) *There are precisely three integers of the form $2^x - 3^b$, with x and b natural numbers, that are also expressible as $2^a - 3^y$, with a and y natural numbers such that $x > a$. They are*

$$-1 = 2^3 - 3^2 = 2 - 3, \qquad 5 = 2^5 - 3^3 = 2^3 - 3, \qquad 13 = 2^8 - 3^5 = 2^4 - 3.$$

These are, respectively, the only two such representations for these three integers. All other integers have either a unique such representation, or none at all.

We apply Bennett's result to the cases of (1) and (2) where x, y, a, and b are all positive integers. This leaves us with the special case when $y = 0$; clearly (1) and (2) are impossible if $a = 0$. We prove the special case $y = 0$ by elementary methods, except for the one instance where we apply Lemma 1 to deduce that 1 has only the single representation $1 = 2^2 - 3$ (although it is not difficult to prove this fact independently).

2 The case when $y > 0$

Theorem 2. *There are precisely three solutions to* (1) *when $y > 0$. They are*

$$11 = 2^3 + 3 = 2 + 3^2, \qquad 35 = 2^5 + 3 = 2^3 + 3^3, \qquad 259 = 2^8 + 3 = 2^4 + 3^5.$$

Proof. Let $c = 2^a - 3^y$ in (2). By Lemma 1, if $c \notin \{-1, 5, 13\}$, then $x = a$, which contradicts the hypothesis $x > a$. Otherwise, $c \in \{-1, 5, 13\}$.

Suppose $c = -1$. By Lemma 1, we have the two representations, as in (2),

$$-1 = 2^3 - 3^2 = 2 - 3.$$

Thus, $x = 3$, $b = 2$, $a = 1$, and $y = 1$. This produces

$$2^3 + 3 = 2 + 3^2 = 11.$$

Suppose $c = 5$. Similarly,

$$5 = 2^5 - 3^3 = 2^3 - 3,$$

thus producing

$$2^5 + 3 = 2^3 + 3^3 = 35.$$

Suppose $c = 13$. Similarly,

$$13 = 2^8 - 3^5 = 2^4 - 3$$

produces

$$2^8 + 3 = 2^4 + 3^5 = 259.$$

□

3 The case when $y = 0$

For a prime p and a natural number n, we write $p \| n$ if $p \mid n$ but $p^2 \nmid n$. We denote the *p-valuation of* n by $v_p(n)$: i.e., $v_p(n) = k$ if $p^k \| n$.

Lemma 3. *If n is a natural number then*

$$v_2(3^n - 1) = \begin{cases} 1, & \text{if } 2 \nmid n; \\ 2 + v_2(n), & \text{if } 2 \mid n. \end{cases}$$

Lemma 4. *If n is a natural number then*

$$v_3(2^n - 1) = \begin{cases} 0, & \text{if } 2 \nmid n; \\ 1 + v_3(n), & \text{if } 2 \mid n. \end{cases}$$

Lemmata 3 and 4 follow easily from Theorems 94 and 95, Nagell [2].

Theorem 5. *There are precisely two solutions to* (1) *when* $y = 0$. *They are*

$$5 = 2^2 + 1 = 2 + 3, \qquad 17 = 2^4 + 1 = 2^3 + 3^2.$$

Proof. We are given

$$2^x + 1 = 2^a + 3^b, \tag{3}$$

where x, a, and b are natural numbers, where $x > a$. Let $s = x - a$. Thus,

$$2^a(2^s - 1) = 3^b - 1. \tag{4}$$

It is necessary by Lemma 4 that s is odd, as, by (4), $3 \nmid 2^s - 1$.

First, suppose b is odd. Then Lemma 3 implies $2 \| 3^b - 1$, hence, by (4), $a = 1$. Thus, by (3),

$$2^x - 3^b = 1.$$

Thus, by Lemma 1, $x = 2$ and $b = 1$. This produces the equation

$$2^2 + 1 = 2 + 3 = 5.$$

It remains to let b be even. Then $a = 2 + v_2(b)$ by Lemma 3. Suppose $2^2 \mid b$. Then $3^4 - 1 \mid 3^b - 1$, hence $5 \mid 3^b - 1$. Then (4) implies $5 \mid 2^s - 1$, hence $4 \mid s$, a contradiction as s is odd. Therefore $2 \| b$ and $a = 3$. Writing $b = 2c$ for an odd natural number c, we have by (4)

$$2^s - 1 = \frac{3^c - 1}{2} \cdot \frac{3^c + 1}{4}.$$

Letting

$$z = \frac{3^c + 1}{4},$$

then z is a natural number by Lemma 3, and we obtain the quadratic in z,

$$2^s - 1 = 2z^2 - z.$$

Completing the square yields

$$(4z - 1)^2 = 2^{s+3} - 7.$$

Writing $s = 2t + 1$ yields the difference of squares factorization

$$(2^{t+2} - 4z + 1)(2^{t+2} + 4z - 1) = 7.$$

Therefore

$$2^{t+2} - 4z + 1 = 1, \qquad 2^{t+2} + 4z - 1 = 7;$$

thus,

$$2^{t+2} = 4z = 4.$$

Therefore $t = 0$, $z = 1$; thus, $c = 1$. Hence $s = 1$ and $b = 2$. Recalling $a = 3$, we have $x = 4$. This produces the equation

$$2^4 + 1 = 2^3 + 3^2 = 17.$$

□

References

[1] M. A. Bennett, "Pillai's conjecture revisited," *J. Number Theory* **98** (2003) 228–235.

[2] T. Nagell, *Introduction to Number Theory,* Wiley Publishers, New York, 1951.

[3] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

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