arXiv:1907.03652v3 [math.MG] 23 Jul 2019

Maximally Dense Disc Packings on the Plane

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July 24, 2019

Abstract

Suppose one has a collection of disks of various sizes with disjoint interiors, a packing, in the plane, and suppose the ratio of the smallest radius divided by the largest radius lies between 1 and q. In his 1964 book *Regular Figures* [6], László Fejes Tóth found a series of packings that were his best guess for the maximum density for any $1 > q > 0.2$. Meanwhile Gerd Blind in [3, 4] proved that for $1 \geq q > 0.72$, the most dense packing possible is $\pi/\sqrt{12}$, which is when all the disks are the same size. In [6], the upper bound of the ratio q such that the density of his packings greater than $\pi/\sqrt{12}$ that Fejes Tóth found was 0.6457072159... Here we improve that upper bound to 0.6585340820... Both bounds were obtained by perturbing a packing that has the property that the graph of the packing is a triangulation, which L. Fejes Tóth called a *compact* packing, and we call a *triangulated* packing. Previously all of L. Fejes Tóth's packings that had a density greater than $\pi/\sqrt{12}$ and $q > 0.35$ were based on perturbations of packings with just two sizes of disks, where the graphs of the packings were triangulations. Our new packings are based on a triangulated packing that have three distinct sizes of disks, found by Fernique, Hashemi, and Sizova, [9], which is something of a surprise.

We also point out how the symmetries of a triangulated doubly periodic packing can by used to create the actual packing that is guaranteed by a famous result of Thurston, Andreev, and Andreeson [16].

Keywords: Compact packing, triangulated packing, disc packing, density, symmetry, orbifold.

1 Introduction

A disc packing is called *compact* or *triangulated* if its contact graph is triangulated, i.e. the graph formed by connecting the centers of every adjacent disc consists only of triangular faces. We are interested in packings on the flat torus, which are equivalent to doubly periodic packings in the plane. The problem is to find and classify all compact packings of order n on the torus, meaning the packing uses n different sized discs. For $n = 1$, only a single triangulated packing exists, the hexagonal lattice, where each disk touches six others of the same size. For $n = 2$, nine packings exist, found by Kennedy [13]. For $n = 3$, there are 164 such packings, found by Fernique, Hashemi, and Sizova [9]. (This is much less than the upper bound of 11,462 packings given by Messerschmidt [14].)

In 1890, Thue gave the first proof that the hexagonal lattice is the densest single size disc packing. However, some considered his proof to be incomplete. In 1940, L. Fejes Tóth provided the

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first rigorous proof. This caused Fejes Tóth to wonder about the densest possible packings with multiple disc sizes. The density of any such packing must be strictly greater than $\pi/\sqrt{12}$, which is the density of the hexagonal lattice. Here, in Theorem 2.4 we provide a simple proof.

For any disc packing, let $0 < q \leq 1$ be the ratio between the radii of the smallest and largest circles used. Florian derived a formula for an upper bound for the density of a packing depending on its value of q:

$$s(q) = \frac{\pi q^2 + 2(1-q^2)\sin^{-1}\left(\frac{q}{1+q}\right)}{2q\sqrt{1+2q}} \tag{1.1}$$

Theorem 1.1 (Florian [10]). *If δ is the density of a packing in the plane with radii between* 1 *and* q*, then $\delta \leq s(q)$.*

This function tends to one as q tends to zero, since arbitrarily small discs can fill up any gaps in a packing. As q approaches one, the bound decreases monotonically to $\pi/\sqrt{12}$, recovering the hexagonal lattice.

This formula is mentioned in Fejes Tóth's 1964 book *Regular Figures*. Additionally, Fejes Tóth provides guesses for the densest possible packings with radius ratio greater than or equal to a given q (Figure 1.1). Although none of the guesses exactly reach Florian's bound, some of them come quite close, while others are noticeably lower.

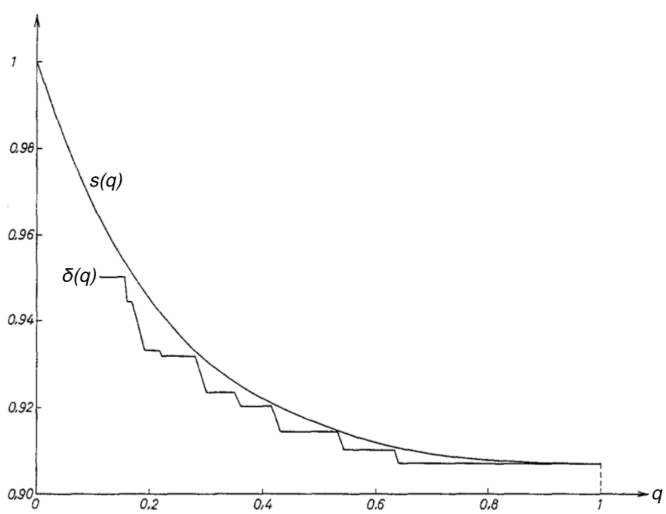

Figure 1.1: Florian's bound $s(q)$ along with $\delta(q)$, the density of Fejes Tóth's best guess as a function of q reproduced from page 189 of Fejes Tóth's book [6]. The final three piecewise sections of $\delta(q)$ will be defined exactly in Section 2. [4]

The problem of finding all triangulated packings of order with n different radii is interesting because we can scour these packings for ones which improve Fejes Tóth's guesses and come closer to Florian's bound. In fact, with Fernique's recently-discovered three disc packings in [9] we can do exactly that, which we explain in Section 3. A knowledge of how close we can get to Florian's bound is important because it helps us with a more general question: Given an arbitrary set of discs with radii between q and 1, what is the densest possible way to arrange them into a periodic packing?

Another interesting thing to do with these newly-found packings is to find out which plane symmetry groups each of them belong to. This is important because the orbifold of a given symmetry group can allow us to systematically construct new packings of any order, although the methods of Kennedy [13] and Fernique [9] are sufficient to find all examples for two sizes and three sizes of disks, respectively.

2 Multiple Size Packings

If the graph of a packing is a triangulation of the plane, the density of the packing can be calculated by taking an appropriate weighted average of the densities of the packing restricted to each triangle. Florian's bound (1.1) is the density of a packing of three circular disks in mutual contact, one of radius r_1 and two of radius $r_2 \leq r_1$, where $q = r_2/r_1$, in a triangle formed by their centers. In general, the density of a packing of 3 disks of radius $r_i = \tan(\theta_i)$, for $i = 1, 2, 3$ in the triangle formed by the centers normalizes so that the incenter of the triangle is 1, is the following function $\delta(\theta_1, \theta_2, \theta_3)$,

$$\delta(\theta_1, \theta_2, \theta_3) = \frac{(\pi/2 - \theta_1)\tan^2(\theta_1) + (\pi/2 - \theta_2)\tan^2(\theta_2) + (\pi/2 - \theta_3)\tan^2(\theta_3)}{\tan(\theta_1) + \tan(\theta_2) + \tan(\theta_3)} \tag{2.1}$$

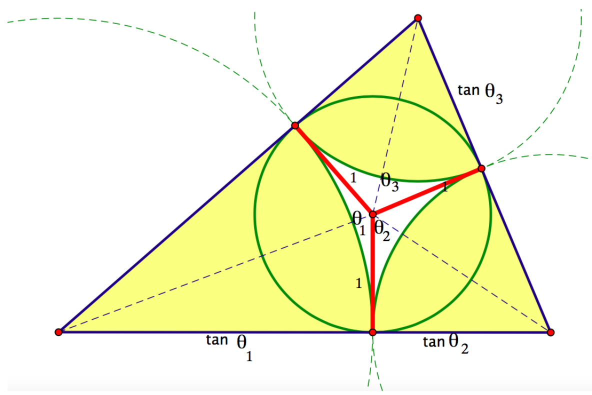

Figure 2.1: Diagram of three mutually tangent circles (dashed green) and the triangle with vertices at their centers (solid blue). There are also the triangle's angle bisectors (dashed blue) and its inscribed circle (solid green).

The diagram is normalized so that the radius of the incircle is 1. The red segments are radii of the incircle which are orthogonal to the sides of the triangle. θ_1, θ_2, and θ_3 are each angles between one of the angle bisectors and an adjacent red segment and each $\tan\theta_i = r_i$, the i-th radius. The density of the packing in the triangle, $\delta(\theta_1, \theta_2, \theta_3)$, is the ratio of the yellow area to the area of the entire triangle. We will show that δ is minimized when the radii of the three dashed circles are equal.

One can check that when $q = r_2/r_1 = r_3/r_1 = \tan(\theta_2)/\tan(\theta_1) = \tan(\theta_3)/\tan(\theta_1)$, then $s(q) = \delta(\theta_1, \theta_2, \theta_3)$. In this case $\theta_1 = \pi - 2\theta_2 = \sin^{-1}(\frac{q}{1+q})$ and $\tan(\theta_1) = \sqrt{\frac{1+2q}{q}}$, $\tan(\theta_2) = \sqrt{1+2q}$.

2.1 Triangulated Packing's Minimum Density

Here we show that the *minimum* density of all triangulated packings is when all the radii of all the disks are equal.

Let the area of the union of the yellow sectors be $A(\theta_1, \theta_2, \theta_2)$ as in Figure 2.1. So

$$\begin{aligned} A(\theta_1, \theta_2, \theta_3) &= a(\theta_1) + a(\theta_2) + a(\theta_3) \\ &= (\pi/2 - \theta_1)\tan^2(\theta_1) + (\pi/2 - \theta_2)\tan^2(\theta_2) + (\pi/2 - \theta_3)\tan^2(\theta_3). \end{aligned} \tag{2.2}$$

Let $t(\theta)$ be twice the area of a right triangle of side length 1, and angle θ adjacent to that unit length. Then

$$t(\theta) = \tan(\theta)$$

Let $T(\theta_1, \theta_2, \theta_2)$ be the area of the triangle as in Figure 2.1. Its area is the sum of the areas of the six smaller right triangles, so

$$T(\theta_1, \theta_2, \theta_3) = t(\theta_1) + t(\theta_2) + t(\theta_2) = \tan(\theta_1) + \tan(\theta_2) + \tan(\theta_3). \tag{2.3}$$

Thus overall the density of the covered portion of the triangle as in Figure 2.1 is

$$\begin{aligned} \delta(\theta_1, \theta_2, \theta_3) &= A(\theta_1, \theta_2, \theta_3)/T(\theta_1, \theta_2, \theta_3) \\ &= \frac{(\pi/2 - \theta_1)\tan^2(\theta_1) + (\pi/2 - \theta_2)\tan^2(\theta_2) + (\pi/2 - \theta_3)\tan^2(\theta_3)}{\tan(\theta_1) + \tan(\theta_2) + \tan(\theta_3)} \end{aligned} \tag{2.4}$$

Here we assume that $\theta_1 + \theta_2 + \theta_3 = \pi$ and each $0 < \theta_i < \pi/2$ so that the angles come from the situation of Figure 2.1.

We are mainly interested in the following:

Theorem 2.1. *The minimum value of δ is $\pi/\sqrt{12}$, and is achieved only when $\theta_1 = \theta_2 = \theta_3 = \pi/3$.*

In other words, this is achieved only when the radii of the touching circles, the $\tan(\theta_i)$ for our normalization are equal. In order to simplify the calculations, instead of calculating the critical minimum density directly, we will compute the complimentary maximum density

$$\bar{\delta} = 1 - \delta = (T(\theta_1, \theta_2, \theta_3) - A(\theta_1, \theta_2, \theta_3))/T(\theta_1, \theta_2, \theta_3.$$

This is the ratio of the curvilinear triangle in the unit circle over the area of the larger triangle that contains the unit circle. This result follows from the next theorem.

Theorem 2.2. *Subject to $\theta_1 + \theta_2 + \theta_3 = \pi$ and each $0 < \theta_i < \pi/2$ the maximum value of $T(\theta_1, \theta_2, \theta_3) - A(\theta_1, \theta_2, \theta_3)$ is achieved only when $\theta_1 = \theta_2 = \theta_3 = \pi/3$ and the minimum value of $T(\theta_1, \theta_2, \theta_3)$ is achieved only when $\theta_1 = \theta_2 = \theta_3 = \pi/3$.*

To do this fix one of the angles, say θ_3, and then regard θ_2 as a function of θ_1, where $\theta_2 = \pi - \theta_1 - \theta_3$. Since $t(\theta_3)$ and θ_3 are constant, we have the following:

Lemma 2.3. *The maximum of $t(\theta) - a(\theta_1) + t(\pi - \theta_1 - \theta_3) - a(\pi - \theta_1 - \theta_3)$ and the minimum of $t(\theta_1) + t(\pi - \theta_1 - \theta_3)$ occur only when $\theta_1 = \theta_2 = \pi - \theta_1 - \theta_3$.*

Proof. First the $t(\theta)$ case. For θ_3 fixed, it is clear that $\theta_1 = \theta_2 = \pi - 2\theta_1$ is a critical point. We calculate the derivatives for $0 < \theta < \pi/2$,

$$\begin{aligned} t'(\theta) &= 1 + \tan^2(\theta) \\ t''(\theta) &= 2\tan(\theta)(1 + \tan^2(\theta) > 0. \end{aligned}$$

Thus $(\theta_1, \theta_1, \pi - 2\theta_1)$ is the unique minimum point for T when $\theta_1 + \theta_2 + \theta_3 = \pi$ and each $0 < \theta_i < \pi/2$.

For $a(\theta)$ the argument is similar.

$$\begin{aligned} a'(\theta) &= 1 + 2\tan^2(\theta) - (\pi - 2\theta)\tan(\theta)(1 + \tan(\theta)^2) \\ a''(\theta) &= \frac{(2\pi - 4\theta)\cos^2(\theta) + 6\sin(\theta)\cos(\theta) - 3\pi + 6\theta}{\cos^4(\theta)} < 0. \end{aligned}$$

The last inequality is verified by Maple. Applying this to each pair of θ_i at a time, we get that the only overall minimum point for δ is when $\theta_1 = \theta_2 = \theta_3 = \pi/3$. □

Theorem 2.4. *The density of a triangulated/compact doubly periodic disc packing, with at least two distinct sizes of disks, is strictly greater than* $\pi/\sqrt{12} = 0.9068996821...$

By Theorem 2.1 the density of any packing, restricted to any triangle in that packing is at least $\pi/\sqrt{12}$ and is strictly greater unless all the radii of the triangle are the same. Since the density of the whole packing is a weighted average of the densities of each triangle, when at least two radii are used, the overall density is strictly greater than $\pi/\sqrt{12}$.

2.2 Density in Terms of Radii

Expression (2.1) for the density of three disks in a triangle is in terms of the angles of the bounding triangle which useful for Theorem 2.4, but it is also useful to write the same density in terms of the three radii that determine the triangle. From Heron's formula for a triangle, the area of the triangle is

$$T_r = T_r(r_1, r_2, r_3) = \sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)} = R(r_1 + r_2 + r_3),$$

where R is the inradius (that was assumed to be 1 in Figure 2.1). Thus the inradius is

$$R = R(r_1, r_2, r_3) = \sqrt{r_1 r_2 r_3 / (r_1 + r_2 + r_3)}.$$

Thus the density of three disks in a triangle as in Figure (2.1) from Equation (2.4) is

$$\delta_r(r_1, r_2, r_3) = \frac{(\pi/2 - \tan^{-1}(r_1/R))r_1^2 + (\pi/2 - \tan^{-1}(r_2/R))r_2^2 + (\pi/2 - \tan^{-1}(r_3/R))r_3^2}{T_r} \quad (2.5)$$

Then one can check that Florian's bound Equation (1.1) for $0 < q \leq 1$ is

$$s(q) = \delta_r(1, q, q).$$

2.3 Comments about the Florian Bound

Part of Florian's bound is that if there are two sizes of disks, large and small, and one puts three disks in contact as in Figure 2.1, there are three ways to do it, all the same size, which has density, $\pi/\sqrt{12}$, or large-large-small, or large-small-small. Theorem 2.1 shows that when all three have the same size, the density is the smallest of the three cases. The large-small-small case always has the largest density. Figure 2.2 shows a typical case for the density in a triangle of $\delta_r(r_0, r, 1)$, where $0 \leq r_0 \leq r \leq 1$.

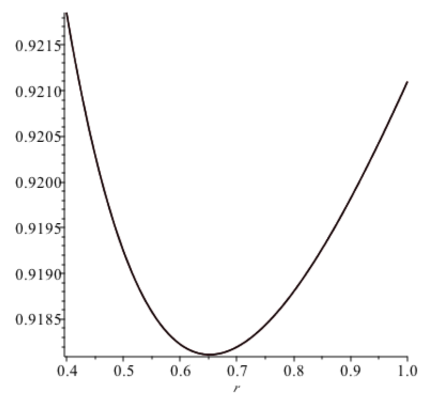

Figure 2.2: This shows the density $\delta_r(r_0, r, 1)$ of a packing in a triangle, where the smallest radius $r_0 = 0.4$ and largest 1 are fixed, with the intermediate radius $r_0 = 0.4 \leq r \leq 1$ varying between the two. Note that the ends of the interval $r = r_0$ and $r = 1$ have the largest density locally, with the large-small-small case $1, r_0, r_0$ having the largest density globally.

On the other hand, if one compares the density of the two ends of the interval in Figure 2.2, the ratio of the two densities is very close to 1.

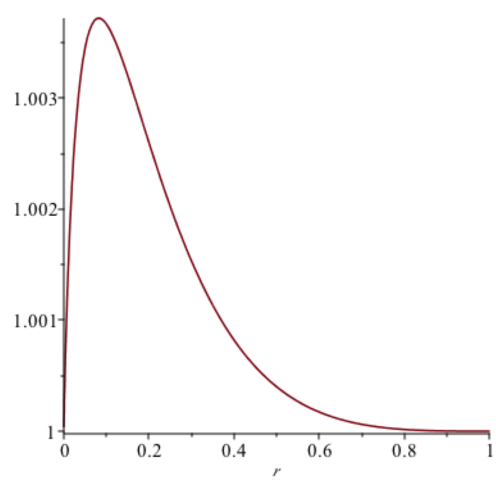

Figure 2.3: This shows the ratio of $\delta_r(1, r, r)/\delta_r(1, 1, r)$, for $0 < r < 1$, and it appears that for all r, the ratio is less than 1.00372119. Although, the large-small-small case always has higher density, the density $\delta_r(1, 1, r)$ is quite close to $\delta_r(1, r, r)$.

2.4 The Florian Bound in Never Achieved

In [12] Aladár Heppes said "*The upper bounds given by L. Fejes Tóth and Molnár [FM] for the least upper bound $\delta(1, r)$ of the density of a packing of unit discs and discs of radius $r < 1$ have been sharpened by Florian [Fl1], who proved that the density cannot exceed the packing density within a triangle determined by the centers of mutually touching circles of radius $1, r$ and r. Unfortunately, such packings do not tile the plane for any value of r, thus this general bound is never sharp.*" We explain that last statement here.

Here we assume that the packing is periodic with a finite number of packing disks per fundamental region, say. Equivalently, this means that the packing is a collection of circular disks with disjoint interiors in a flat torus, which is determined by some lattice with two independent generators. For any such packing, normalize the largest radius of a packing to be 1, and suppose that the smallest radius of the packing is $r_0 < 1$. Let the other radii of any triangle that contains the radius 1 be $r_1 \le r_2 \le 1$ and $r_0 \le r_1$ of course. Then

$$\delta_r(r_1, r_2, 1) \le \delta_r(r_1, r_1, 1) = s(r_1) \le \delta_r(r_0, r_0, 1) = s(r_0), \tag{2.6}$$

from Figure 2.2 and the monotone decreasing property of Florian's bound Figure 1.1. Furthermore if either of the inequalities in (2.6) is strict, Florian's overall bound for the packing will never be equality for a doubly periodic packing, say. We prove the following:

Theorem 2.5. *If δ is the density of a doubly periodic triangulated packing in the plane with radii between 1 and q, then $\delta < s(q)$.*

Proof. Assume that Florian's bound in Theorem 1.1 is attained, and we look for a contradiction. Let r_0 be the smallest radius of a disk, and 1 the largest radius. Choose any disk of radius 1. From the discussion above, each of its adjacent disks must have radius r_0 as well. Similarly, the disks in order around any r_0, must be alternately $1, r_0, 1, r_0, \dots$, for an even number of adjacent disks. Otherwise, we would have three adjacent disks with radii $r_0, r, 1$, with $r_0 < r \le 1$, where the triangle of centers would have density strictly less than $s(r_0)$ contradicting our assumption. Continuing this way, we see that all the triangles of the triangulation correspond to packing disks with radii, $r_0, r_0, 1$. Not only that, but the number of disks of size 1 will be adjacent to exactly, say $n \ge 3$ other disks of radius r_0, and each disk of radius r_0 will be adjacent to exactly $2m \ge 4$ other disks, m with radius 1, m with radius r_0. In particular, there will only be two sizes of disks. All triangulated packings with just two sizes of disks have been found by Kennedy [13], see Figure 5.3, and they all have at least one triangle that is either equilateral or that corresponds to the $1, 1, r_0$. Alternatively, one can use the following Lemma 2.6 that finds all triangulated packings with two sizes of disks, where each disk has radii of size a, b, b, where $a \ne b$ are positive radii, and the large, small, small case never appears. □

Lemma 2.6. *Suppose we have a triangulated packing of a flat torus, where the disks corresponding to each triangle have radii a, b, b. Then the only possible packings are Packings $4, 8$ in Figure 5.3 and the equal radius packing when $a = b$.*

Proof. Because the shape of each triangle in the triangulation is the same, the number of disks adjacent to disk with the a radius is the same, say $n \ge 3$, and similarly the number of disks adjacent to one with the b radius is the same even number, say $2m$ for $m \ge 2$, because the neighbors have to alternate as in the proof of Theorem 2.5. Let α be the half angle at the center of the a radius disk in one of its triangles as in Figure 2.4. Then the angle for the b radius disk in the same triangle is $\pi/2 - \alpha$. So $2\alpha n = 2\pi = 2m(\pi/2 - \alpha)$. Then $\pi/n + \pi/m = \pi/2$, or more simply $1/n + 1/m = 1/2$. Thus the only solutions are $n = 3, m = 6$, corresponding Packing 8; $n = 4, m = 4$ corresponding to Packing 4; n=6, m=3 corresponding to when $a = b$, all in Kennedy's list, Figure 5.3, from [13]. □

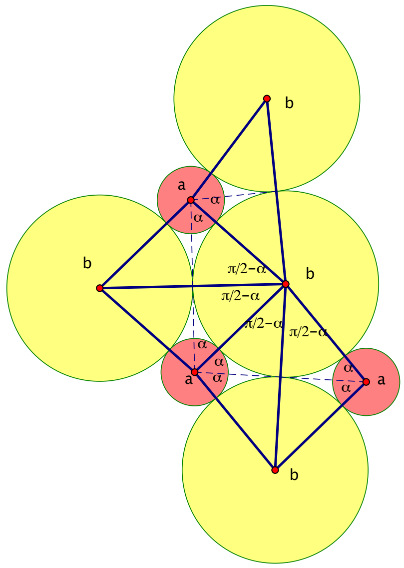

Figure 2.4: If there is a triangulated packing with only two sizes of disks, this shows how the angles in the triangle must be so the triangulation fits together.

Notice that the packings of Lemma 2.6 are of the big, big, small type, which does not have the maximum density for given radius ratio, nevertheless have densities that are very close to Florian's bound in Theorem 1.1. Interestingly, Aladar Heppes in [12] proved that for two sizes of disks in the ratio $\sqrt{2}-1$, as in Packing 4 in Figure 5.3, that its density $\pi(2-\sqrt{2})/2 = .9201511858..$ is the maximum possible, while Florian's bound is 0.9208355993...

3 Fejes Tóth's Packings

Let $q_1 = 0.6375559772\ldots$ be the radius ratio of the packing in Fejes Tóth's book [6], (Figure 3.2 left here), which is the same as Kennedy's first two-disc packing in Figure 5.3.

Let $q_2 = 0.6457072159\ldots$ be defined such that $\delta_{FT}(q_2) = \pi/\sqrt{12}$, where $\delta_{FT}(q)$ is defined in Equation (3.1) coming from the middle packing of Figure 3.2.

For $q_1 < q \leq q_2$, Fejes Tóth's guess is a version of Kennedy's packing with the radius of the smaller circle increased slightly so that the new ratio is equal to q. This, however, causes the packing to no longer be triangulated/compact.

With some work (See the Appendix), it is possible to write the density of this packing in terms of q:

$$\delta_{FT}(q) = \frac{\pi(q^2+1)(q+1)^4\sqrt{1+2q}}{4q(2q^2+5q+\sqrt{2q^3+5q^2+2q}+2)(q+\sqrt{2q^3+5q^2+2q})} \tag{3.1}$$

This is exactly the second-to-last piece of $\delta(q)$ shown in Figure 1.1. So $\delta_{FT}(q_1) = 0.9106832003\ldots$ recovers the density of the unaltered packing, and the density function $\delta_{FT}(q)$ strictly decreases to $\delta_{FT}(q_2) = 0.9068996827\ldots = \pi/\sqrt{12}$.

This is shown in Figure 3.1 with the piece-wise linear blue line.

For $q_2 < q \leq 1$, Fejes Tóth's guess is simply the hexagonal lattice, with density $\delta_{FT}(q) = \pi/\sqrt{12}$ (Figure 3.2, right). Note that the packing with ratio q_2 is distinct from the hexagonal lattice, despite having the same density.

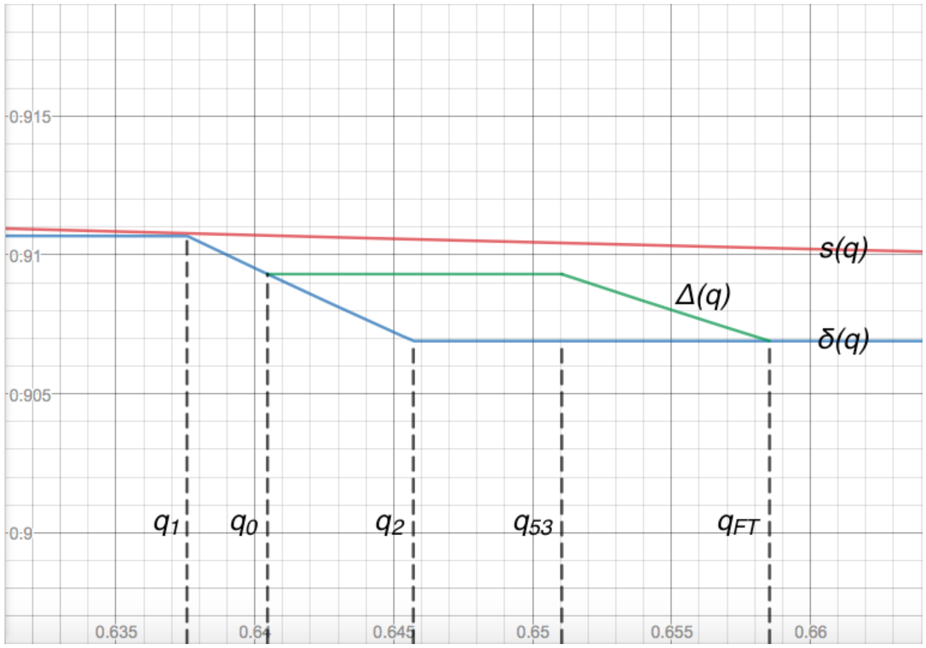

Figure 3.1: This graph is a magnified version of the one shown in Figure 1.1, with $s(q)$ in red and $\delta_{FT}(q)$ in blue. The green function $\Delta(q)$ shows the improvement to $\delta_{FT}(q) = \delta(q)$. The important values of q are marked by the vertical dashed lines.

Let

$$q_B = \sqrt{\frac{\sqrt{12} - 7\tan(\pi/7)}{5\tan(\pi/5) - \sqrt{12}}} = 0.7429909632\ldots$$

For $q_B \leq q \leq 1$, it is known that the hexagonal lattice is the best possible guess. This was shown by Blind in the 1960s. [3, 4]

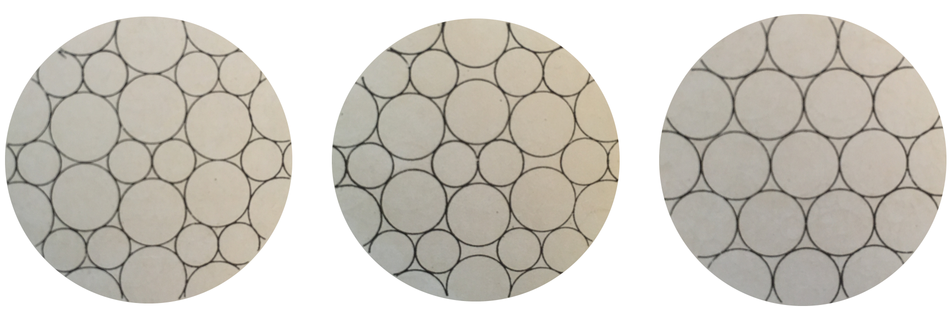

Figure 3.2: Fejes Tóth's best guesses, photographed directly from a copy of *Regular Figures*. (Left: $q \lesssim q_1$, Center: $q_1 < q \leq q_2$, Right: $q_2 < q \leq 1$) [6]

4 Fernique's Packings

Packing number 53 in Fernique's list [9], (Figure 4.1, left), is of special interest because it has the highest radius ratio of all triangulated/compact disc packings known, aside from the hexagonal lattice. More importantly, Packing 53 improves Fejes Tóth's guess for the densest packing with radius ratio q for $0.6404568491\ldots < q \leq 0.6585340820\ldots$.

Let $\delta_{53} = 0.9093065016\ldots$ be the density of Packing 53 in Fernique's list, (Figure 4.1, left).

Let $q_0 = 0.6404568491\ldots$ be defined such that $\delta_{FT}(q_0) = \delta_{53}$, where δ_{FT} is the density of the class of packings defined by Fejes Tóth as above.

Let $q_{53} = 0.6510501858\ldots$ be the radius ratio of Packing 53 (Figure 4.1, left).

Define $\Delta(q) = \delta_{53}$ for $q_0 < q \leq q_{53}$. Since $\Delta(q) \geq \delta(q)$ for those values of q, Packing 53 is an improvement on Fejes Tóth's guesses for this range.

For $q_{53} < q < q_B$, Packing 53 is no longer a valid guess because one of the disks in Packing 53 is smaller tha q. However, it is possible to create an altered version of Packing 53 using Fejes Tóth's technique in order to make an improved guess for some $q > q_{53}$. We will modify it by increasing the medium radius p and the small radius q according to the following constraint which is satisfied by the unaltered packing (See the Appendix for a derivation):

$$2p^4 + (4q+3)p^3 + (2q^2 - 2q + 1)p^2 - (5q^2 + 6q)p + q^2 = 0 \tag{4.1}$$

This is to ensure that the medium sized discs remain in contact with each other (Figure (4.1), right). Using the quartic formula to solve for p, we can write the density of the altered packing entirely in terms of q (Again this will be explained in the Appendix):

$$\Delta(q) = \frac{\pi(1+p^2+q^2)(p+q)^4(1+p)^2(1+q)^2}{32pq(1+p+q)\big(8p^2q^2 - (p^2 - 6pq + q^2)(1+p+q-pq)\big)\sqrt{pq(1+p+q)}} \tag{4.2}$$

Also, q is the radius ratio of this packing since the largest disc is normalized to have radius 1. $\Delta(q_{53}) = \delta_{53}$, and the function Δ strictly decreases for $q \geq q_{53}$.

Let $q_{FT} = 0.6585340820\ldots$ be defined such that $\Delta(q_{FT}) = \pi/\sqrt{12}$. For $q_{53} < q \leq q_{FT}$, $\Delta(q) \geq \delta_{FT}(q)$. Therefore, the altered Packing 53 is an improvement on the guess of the hexagonal lattice in this range.

These packings and their densities are shown as the green line in Figure 3.1.

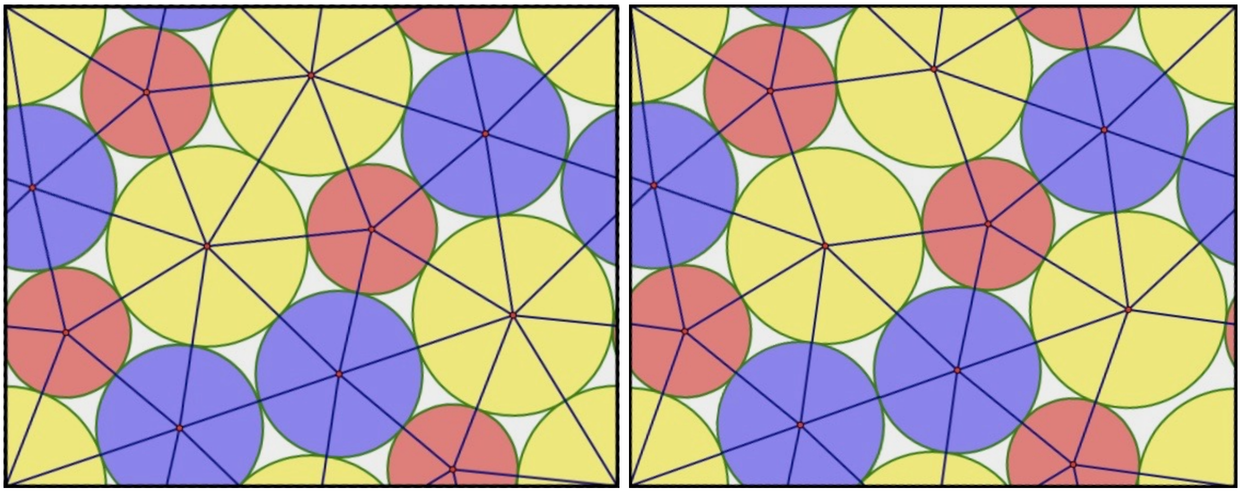

Figure 4.1: Packing 53 (left) and its altered version (right), along with their contact graphs. The rectangular borders of each packing are also their fundamental regions.

For $q_{FT} < q < q_B$, the altered packing is no longer a valid guess because $\Delta(q)$ goes below $\pi/\sqrt{12}$. More packings will need to be discovered and studied, if they exist, in order to make improvements in this range.

5 Symmetry Groups

Any periodic structure in the plane is guaranteed to be represented by one of 17 symmetry groups, known as the wallpaper groups. This fact was proven by Fedorov in 1891 and again by Pólya in 1924. The wallpaper group of a two-dimensional pattern can be determined by identifying its rotational, reflectional, and glide reflectional symmetries. For example, the hexagonal lattice belongs to the group *p6m*, as it is the only group with both reflectional and six-fold rotational symmetry. Grünbaum and Shephard give an excellent treatment of the wallpaper groups in their book *Tilings and Patterns.* [11]

Definition 1. *A **fundamental region** of a two-dimensional pattern is a smallest area of it that can be replicated to produce the entire pattern using only translations.*

Definition 2. *An **orbifold** of a two-dimensional pattern is a smallest area of it that can be replicated to produce the entire pattern using translations, reflections, and rotations.*

The more symmetries a pattern has, the smaller its orbifold is relative to its fundamental region. For example, for the group *p1*, the orbifold and fundamental region are the same size. On the other hand, for the group *p6m*, the orbifold is one-twelfth the size of the fundamental region.

It is possible to use the orbifolds of some symmetry groups to construct disc packings belonging to those groups. For example, to create an order n disc packing with *p6m* symmetry, simply place n discs onto the 30-60-90 triangle orbifold. One must decide whether to place the center of each disc on a vertex, an edge, or the face of the orbifold.

However, this method may not generate every order n disc packing with a given symmetry. It may be necessary to place more than one disc of the same size in different locations on the orbifold. This is the case when a packing has two or more discs of the same size that cannot be mapped to each other through translations, reflections, and rotations.

For example, the hexagonal lattice has one unique disc, and so its orbifold also contains a single disc (Figure 4.1, left). On the other hand, Kennedy's fifth packing is the only two disc packing with two distinct discs of the same size. As a result, its orbifold is the only one out of the two disc packings which contains three different discs (Figure 4.1, right).

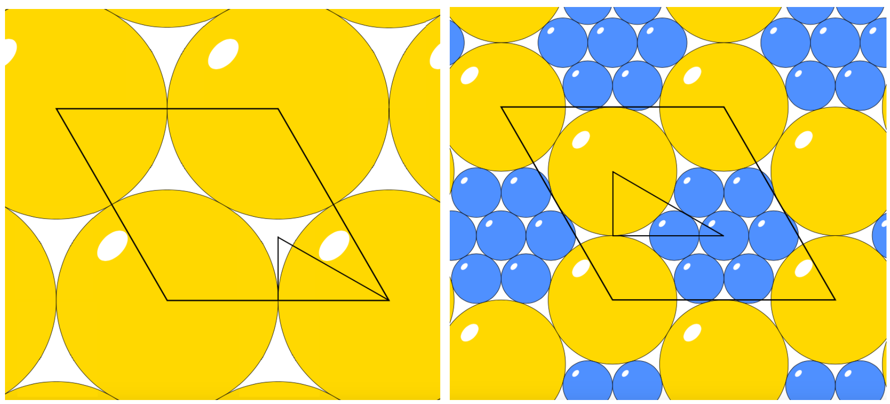

Figure 5.1: The hexagonal lattice (left) and Kennedy's fifth packing (right). Both packings belong to the symmetry group *p6m*. The rhombi are the fundamental regions, and the 30-60-90 triangles are the orbifolds. [8]

Many of Fernique's three disc packings have a similar property. However, two of them are especially noteworthy. Packing 154 is the only one with two sets of two distinct discs of the same size, while Packing 159 is the only one with three distinct discs of the same size.

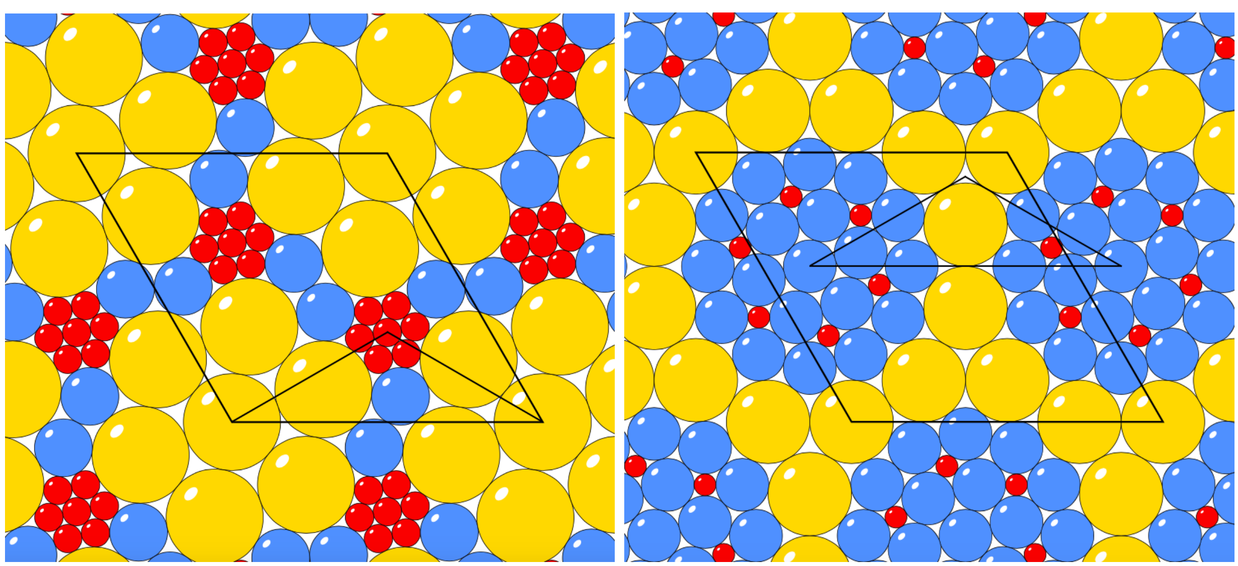

Figure 5.2: Packing 154 (left) and Packing 159 (right). Both packings belong to the symmetry group *p6*. The rhombi are the fundamental regions, and the isosceles triangles are the orbifolds. [9]

5.1 Wallpaper Groups of Compact Two Disc Packings

1	2	3	4	5	6	7	8	9
cmm	*p31m*	*cmm*	*p4m*	*p6m*	*p6m*	*cmm*	*p6m*	*p6m*

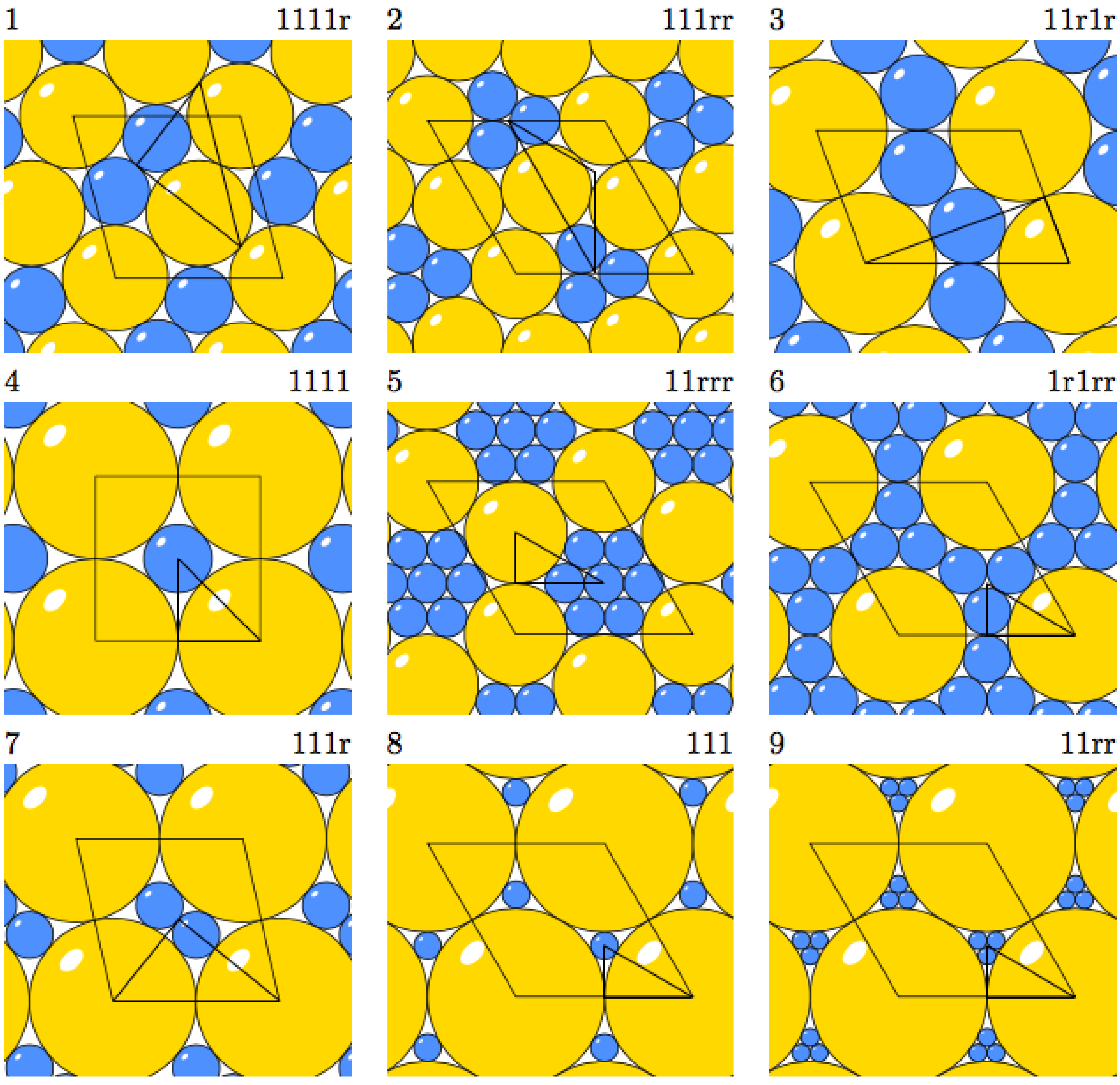

Figure 5.3: All 9 triangulated/compact 2-disc packings on the torus. The table above indicates the symmetry type for each of the packings in Figure. The large shapes in each picture are the fundamental regions, while the smaller shapes are the orbifolds. Notice that the more symmetric packings have smaller orbifolds relative to their fundamental regions. [13]

5.2 Wallpaper Groups of Compact Three Disc Packings

The numbering scheme for these packings is taken from Fernique, Hashemi, and Sizova. [9]

1	2	3	4	5	6	7	8	9	10
pmm	*cmm*	*p31m*	*pmg*	*pmm*	*pmm*	*p2*	*p2*	*p2*	*p3m1*

11	12	13	14	15	16	17	18	19	20
pmm	*cmm*	*p31m*	*pmg*	*pmm*	*pmm*	*p6*	*cm*	*p6m*	*cmm*

21	22	23	24	25	26	27	28	29	30
cmm	*cmm*	*p6m*	*p31m*	*p6m*	*p6m*	*p4m*	*cmm*	*p31m*	*cmm*

31	32	33	34	35	36	37	38	39	40
p6m	*p6m*	*cmm*	*p6m*	*p6m*	*p6m*	*p31m*	*p6m*	*p6m*	*p6m*

41	42	43	44	45	46	47	48	49	50
p31m	*p6m*	*p6m*	*p6m*	*cmm*	*p2*	*pmg*	*p31m*	*p6*	*p31m*

51	52	53	54	55	56	57	58	59	60
cmm	*p6*	*pgg*	*p31m*	*p2*	*p2*	*p6*	*p6*	*cmm*	*p31m*

61	62	63	64	65	66	67	68	69	70
pmm	*cmm*	*p31m*	*pmm*	*p6m*	*cmm*	*cmm*	*cm*	*p2*	*pmg*

71	72	73	74	75	76	77	78	79	80
cmm	*p31m*	*p6*	*p31m*	*cmm*	*cmm*	*p2*	*p6m*	*pmm*	*cmm*

81	82	83	84	85	86	87	88	89	90
cmm	*pmm*	*p2*	*p6*	*cmm*	*cmm*	*cmm*	*cmm*	*p6m*	*p6m*

91	92	93	94	95	96	97	98	99	100
p6m	*cmm*	*p4*	*cmm*	*cmm*	*pmm*	*p2*	*cmm*	*p6*	*cmm*

101	102	103	104	105	106	107	108	109	110
cmm	*cmm*	*cmm*	*cmm*	*p6m*	*p6m*	*p6m*	*p4m*	*cmm*	*p6m*

111	112	113	114	115	116	117	118	119	120
p6	*p3*	*p31m*	*p6m*	*p3m1*	*cmm*	*p6m*	*pmm*	*pmm*	*p6m*

121	122	123	124	125	126	127	128	129	130
cmm	*p6*	*pmg*	*p31m*	*cmm*	*p6*	*p2*	*p6*	*pmg*	*p2*

131	132	133	134	135	136	137	138	139	140
cmm	*cmm*	*cmm*	*pmm*	*cmm*	*p2*	*p6*	*cmm*	*cmm*	*cmm*

141	142	143	144	145	146	147	148	149	150
p6m	*p6m*	*p6m*	*pmg*	*cmm*	*cmm*	*p6m*	*p6*	*p31m*	*p6m*

151	152	153	154	155	156	157	158	159	160
p3m1	*p31m*	*p31m*	*p6*	*p31m*	*p6m*	*p3m1*	*p6m*	*p6*	*p4m*

161	162	163	164
p4m	*p6m*	*cmm*	*p6m*

Thirteen of the seventeen wallpaper groups are represented by the three disc packings. *p1*, *pg*, *pm*, and *p4g* are the only ones missing. The lack of the first three is not too surprising, as they each have very few symmetries. However, the fact that no packing belongs to *p4g* is rather interesting.

cm	*p2*	*pgg*	*pmg*	*pmm*	*cmm*	*p3*	*p31m*	*p3m1*	*p4*	*p4m*	*p6*	*p6m*	*Total*
2	13	1	7	14	46	1	19	4	1	4	16	36	164

Additionally, three of the groups are represented only once each: *pgg* (Packing 53), *p3* (Packing 112), and *p4* (Packing 93). Packing 53 will be discussed in further detail in the next section.

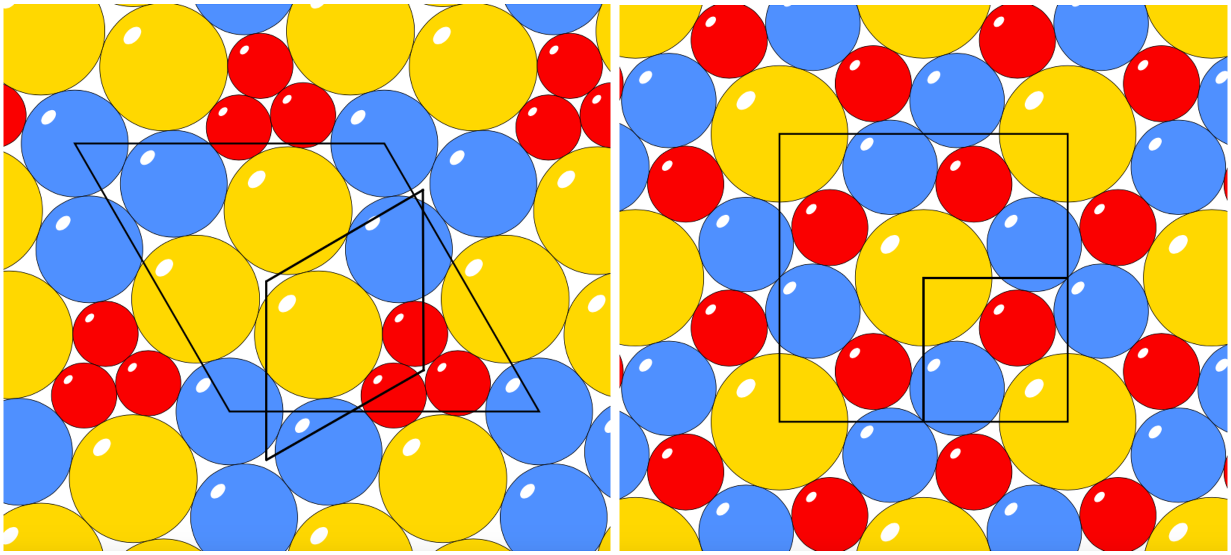

Figure 5.4: Packing 112 (left) and Packing 93 (right). The large shapes are the fundamental regions, while the smaller shapes are the orbifolds. [9]

6 Conjecture

Presented here is a far-fetched conjecture that the number of compact disc packings of order k is equal to the $(k+1)^{\text{st}}$ term in the OEIS sequence A086759:

$$0, 1, 9, 164, 5050, 227508, 14064519, 1146668608, 119249333028, 15400125776000... \quad [15]$$

This sequence is the permanent of the Cayley addition table of $\mathbb{Z}_n$.

7 Appendix

7.1 László Fejes Tóth's Packings

This is a derivation of Formula 3.1 for L. Fejes Tóth's middle packing in Figure 3.2. Figure 7.1 shows a displaced version of L. Fejes Tóth's packing, concentrating on the upper right trapezoid, which is a fourth of the whole fundamental region and which reflects into the whole fundamental region. The coordinates of the vertices of trapezoid are shown.

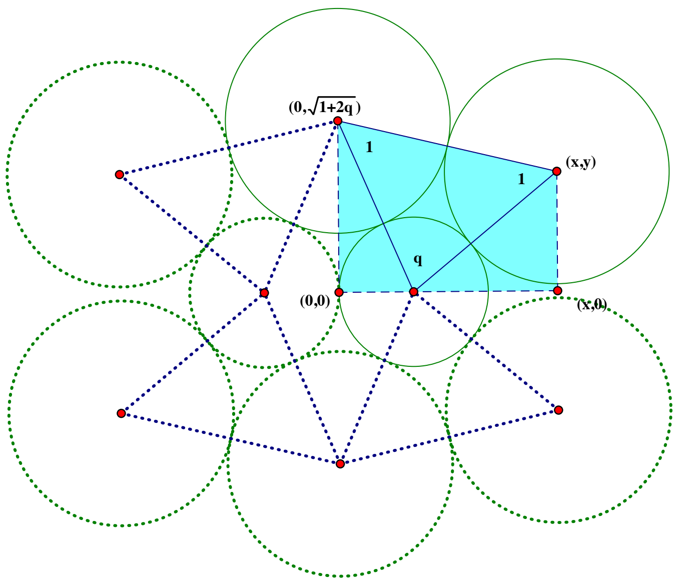

Figure 7.1: Diagram of fundamental region of at packing, with the corner trapezoid, constituting one-fourth of the fundamental region.

There are two circles of radius 1, and two circles of radius q per fundamental region in this Figure 7.1. The following equations show the edge length constraints:

$$\begin{aligned} |(x,y)-(0,\sqrt{1+2q})| &= 2 \\ |(x,y)-(q,0)| &= 1+q, \end{aligned}$$

which translates to $x^2+y^2 = 2y\sqrt{1+2q}-(1+2q)+4 = 2xq+1+2q$.

$$\begin{aligned} x^2+(y-\sqrt{1+2q})^2 &= 4 \\ (x-q)^2+y^2 &= (1+q)^2. \end{aligned}$$

So we get:

$$x^2+y^2 = 2y\sqrt{1+2q}-(1+2q)+4 = 2xq+1+2q.$$

Solving these two equations for x and y in terms of q we get:

$$\begin{aligned} x &= 2\frac{q+\sqrt{2q^3+5q^2+2q}}{(q+1)^2} \\ y &= \frac{2q^3+7q^2+4q\sqrt{2q^3+5q^2+2q})-1}{\sqrt{1+2q}(q+1)^2} \end{aligned}$$

Then the density of this packing in terms of q is

$$\delta = \delta(q) = \frac{\pi(q^2+1)}{(\sqrt{1+2q}+y(q))x(q)}.$$

Note that $y(0.6375559772\ldots) = 1$ which means that the configuration is as in Fejes Tóth's Figure 3.2 on the left, and Kennedy's Figure 1 of Figure 5.3, the triangulated packing. In Section 3 $q_1 = 0.6375559772\ldots$. Note also that when the ratio $q = 1$, $y(1) = \sqrt{3}$, and $x(1) = 2$, showing that the configuration is the ordinaty hexagonal packing, where two radius q disks come together and touch. Note that $\delta(0.6375559772) = 0.9106832003\ldots > \pi/\sqrt{12}$. When the q disk radius is expanded to $q = 0.6457072159\ldots = q_2$, then $\delta(q+2) = \pi/\sqrt{12}$. So $q_2 = 0.6457072159\ldots$ is the limit of the largest q radius for Fejes Tóth's packings.

$q_1 = 0.6375559772\ldots$ is the root of the following polynomial:

$$x^4 - 10x^2 - 8x + 9 = 0 \qquad [13]$$

$q_2 = 0.6457072159\ldots$ is the root of the following polynomial:

$$9x^{15} + 81x^{14} + 369x^{13} + 1161x^{12} + 2757x^{11} + 4749x^{10} + 5805x^9 + 5445x^8$$

$$+3643x^7 + 1235x^6 + 243x^5 - 1029x^4 - 969x^3 - 369x^2 - 81x - 9 = 0$$

This was calculated by solving $\delta(q) = \pi/\sqrt{12}$ in Wolfram Alpha.

7.2 Fernique's Packing

For the evaluation of q_{53} and q_{FT} we have the following:

$q_{53} = 0.6510501858\ldots$ is the root of the following polynomial:

$$89x^8 + 1344x^7 + 4008x^6 - 464x^5 - 2410x^4 + 176x^3 + 296x^2 - 96x + 1 = 0 \qquad [7]$$

$q_{FT} = 0.6585340820\ldots$ is the root of the following polynomial:

$$82944x^{31} + 2073600x^{30} + 25449984x^{29} + 204553728x^{28} + 1214611776x^{27} + 5674077504x^{26}$$

$$+21595717440x^{25} + 68441069376x^{24} + 183725780496x^{23} + 423619513104x^{22} + 846900183408x^{21}$$

$$+1474917242352x^{20} + 2239664278028x^{19} + 2959314640332x^{18} + 3384242724844x^{17} + 3313803241196x^{16}$$

$$+2719452571159x^{15} + 1783910866439x^{14} + 815514300847x^{13} + 88889109343x^{12} - 279883089565x^{11}$$

$$-346836129933x^{10} - 256274678853x^9 - 138435598005x^8 - 57157331979x^7 - 18283967739x^6$$

$$-4571655651x^5 - 892845459x^4 - 132201675x^3 - 14152347x^2 - 985635x - 33075 = 0$$

This was calculated by solving $\Delta(q) = \pi/\sqrt{12}$ in Wolfram Alpha.

For the calculation of the density of the perturbed (and unperturbed) Fernique packing 53, we consider the following portion of the packing as follows:

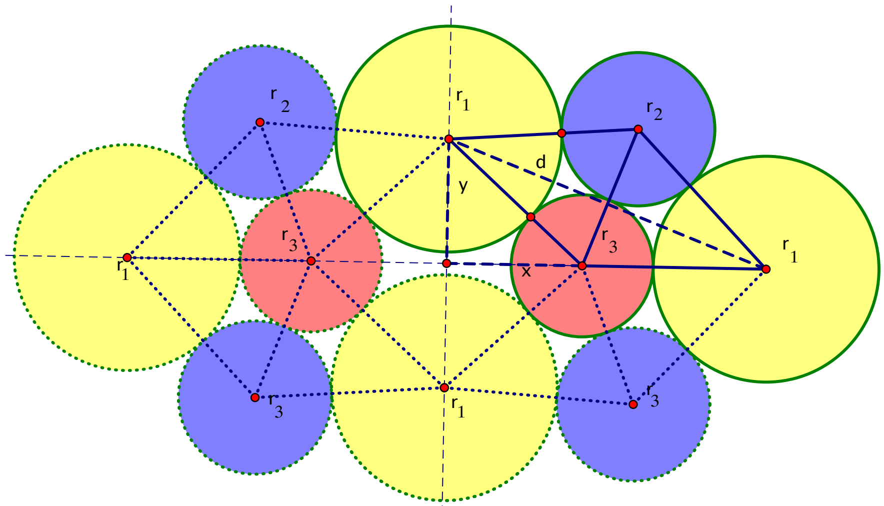

Figure 7.2: Diagram of a portion of Fernique's Packing 53.The center point is surrounded symmetrically by four circles with radii, r_1, r_2, r_1, r_2 in order with the two r_1 disks moved apart slightly.

The distance from the r_1 disks to the center of symmetry is defined to be y, and the distance from the center of the r_1 disk to the center of symmetry is x. The distance d is the distance between the centers of the r_1 disks and perpendicular to the line through the r_2, r_3 disk centers as shown.

As before the area of the triangle formed by the centers of the r_1, r_2, r_3 disks is

$$A_\Delta = \sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}.$$

The distance d can be calculated because of the symmetry about the line through the centers of the r_2 and r_3 disks.

$$\frac{d}{2}(r_2 + r_3)\frac{1}{2} = A_\Delta.$$

So d can be calculated in terms of the radii.

$$d^2 = \frac{16 r_1 r_2 r_3 (r_1 + r_2 + r_3)}{(r_2 + r_3)^2}.$$

Then using the right triangle formed by the two r_1 circle centers and the center of symmetry,

$$(x + r_1 + r_3)^2 + y^2 = d^2 = \frac{16 r_1 r_2 r_3 (r_1 + r_2 + r_3)}{(r_2 + r_3)^2}. \tag{7.1}$$

Similarly using the using the right triangle formed by the r_1, r_3 circle centers and the center of symmetry,

$$x^2 + y^2 = (r_1 + r_3)^2. \tag{7.2}$$

Substituting this into Equation (7.1) we get:

$$2(r_1 + r_3)^2 + 2x(r_1 + r_3) = \frac{16 r_1 r_2 r_3 (r_1 + r_2 + r_3)}{(r_2 + r_3)^2}.$$

Solving for x as the following explicit rational function of the radii, we get:

$$x(r_1, r_2, r_3) = \frac{8r_1r_2r_3(r_1 + r_2 + r_3)}{(r_2 + r_3)^2(r_1 + r_3)} - (r_1 + r_3). \tag{7.3}$$

Using Equation (7.2) we find y as a function of the radii:

$$y(r_1, r_2, r_3) = \sqrt{(r_1 + r_3)^2 - x(r_1, r_2, r_3)^2}. \tag{7.4}$$

Note that to get a packing (i.e. without overlap we must have $y \geq r_1$ and $x \geq r_3$, and we can switch the roles of r_2 and r_3 with the disks switching in Figure 7.2. Indeed as the yellow disks move apart and the red disk move together and touch, the packing deforms to the standard hexagonal packing.

Assume that $r_1 = 1, r_2 = p, r_3 = q$, and $q < p < 1$. From Figure (4.1) we see that each fundamental region has 4 disks of each size, so the total area covered by the disks is:

$$A(p, q) = 4\pi(1 + p^2 + q^2).$$

This formula is independent of the relative sizes of the disks.

We know that area of one of the triangles determined by the centers of 3 different triangles is:

$$A_\Delta(p, q) = \sqrt{pq(1 + p + q)}.$$

The area of a quadrilateral (a rhombus) determined by 2 yellow disk centers and 2 red disk centers in Figure 7.2 is:

$$QUAD(p, q) = 2x(1, p, q)y(1, p, q).$$

And the area of a quadrilateral (a rhombus) determined by 2 yellow disk centers and 2 blue disk centers in Figure 7.2 is:

$$QUAD(q, p) = 2x(1, q, p)y(1, q, p).$$

From Figure (4.1), left that there are 4 triangles determined by 2 yellow disks and a red disk corresponding to 2 QUAD regions. There are 4 triangles determined by 2 yellow disks and 2 blue disks corresponding to 2 other QUAD regions. Then there are 16 triangles determined by 3 different disks. Putting all these regions together we see that the total area of the torus is:

$$A_T(p, q) = 16A_\Delta(p, q) + 2QUAD(p, q) + 2QUAD(q, p).$$

Altogether we get that overall density of the packing is:

$$\Delta(p, q) = \frac{A(p, q)}{A_T(p, q)}.$$

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