On general position sets in Cartesian products

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Abstract

The general position number $\mathrm{gp}(G)$ of a connected graph G is the cardinality of a largest set S of vertices such that no three pairwise distinct vertices from S lie on a common geodesic; such sets are refereed to as gp-sets of G. A formula for the number of gp-sets in $P_r \,\square\, P_s$, $r, s \geq 2$, is determined. The general position number of cylinders $P_r \,\square\, C_s$ is deduced, while $\mathrm{gp}(C_r \,\square\, C_s)$ is bounded from the below by 6, whenever $r \geq s \neq 4$ and $r \geq 6$. It is proved that $\mathrm{gp}(P_\infty \,\square\, P_\infty \,\square\, P_\infty) \geq 14$. A probabilistic lower bound on the general position number of Hamming graphs and of Cartesian graph powers is also achieved.

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1 Introduction

The general position problem was independently introduced in [9, 14], the present terminology and formalism are from [9]. If $G = (V(G), E(G))$ is a graph, then $S \subseteq V(G)$ is a *general position set* if $d_G(u, v) \neq d_G(u, w) + d_G(w, v)$ holds for every $\{u, v, w\} \in \binom{S}{3}$, where $d_G(x, y)$ denotes the shortest-path distance in G. Equivalently, no three vertices lie on a common geodesic. We also say that the vertices from S *lie in a general position*. The *general position problem* is to find a largest general position set of G, the order of such a set is the *general position number* $\mathrm{gp}(G)$ of G. A general position set of G of order $\mathrm{gp}(G)$ will be shortly called a *gp-set*.

Following the seminal papers, the general position problem has been investigated in a sequence of papers [1, 4, 7, 10, 12]. As it happens, in the special case of hypercubes, the general position problem has been studied back in 1995 by Körner [8]. In this paper, asymptotic lower and upper bounds were proved on the gp-number of hypercubes, and several closely related problems (cf. Section 5) were considered. The lower bound from [8] was improved in [11].

The results from [10] on the general position problem in interconnection networks with the emphasize on grid graphs were a starting motivation for the present study. One of the main results of [10] asserts that if P_∞ denotes the two-way infinite path, then $\mathrm{gp}(P_\infty \,\square\, P_\infty) = 4$, and consequently $\mathrm{gp}(P_r \,\square\, P_s) = 4$ for $r, s \geq 3$. The non-trivial part of this result (that $\mathrm{gp}(P_r \,\square\, P_s) \leq 4$ holds) was proved using the so-called Monotone Geodesic Lemma which was in turn derived from the celebrated Erdös-Szekeres theorem, cf. [2, Theorem 1.1]. In order to get more insight into gp-sets of grid graphs, in the first main result of this paper (Theorem 2.1) we determine the number of gp-sets in $P_r \,\square\, P_s$ for every $r, s \geq 2$, and implicitly describe the structure of such sets. Then, in Section 3, we determine $\mathrm{gp}(P_r \,\square\, C_s)$ for every $r \geq 2$ and $s \geq 3$. In the subsequent section we prove that if $r \geq 6$ and $4 \neq s \leq r$, then $\mathrm{gp}(C_r \,\square\, C_s) \geq 6$. We also discuss upper bounds on $\mathrm{gp}(C_r \,\square\, C_s)$ and get the equality $\mathrm{gp}(C_3 \,\square\, C_s) = 6$, $s \geq 6$. In Section 5 we first prove that $\mathrm{gp}(P_\infty \,\square\, P_\infty \,\square\, P_\infty) \geq 14$, thus improving the earlier known bound 10 from [10]. Motivated by the results of [8], we consider at the end of the paper how to apply the probabilistic method to obtain asymptotic lower bounds on the gp-number of Cartesian powers of graphs.

1.1 Preliminaries

For a positive integer k we will use the notation $[k] = \{1, \ldots, k\}$ and $[k]_0 = \{0, \ldots k-1\}$. If $X \subseteq V(G)$, the subgraph of G induced by X is denoted $\langle X \rangle$.

The *Cartesian product* $G \,\square\, H$ of graphs G and H has the vertex set $V(G \,\square\, H) = V(G) \times V(H)$, vertices (g, h) and (g', h') are adjacent if either $gg' \in E(G)$ and $h = h'$, or $g = g'$ and $hh' \in E(H)$. If $h \in V(H)$, then the subgraph of $G \,\square\, H$ induced by the vertices (g, h), $g \in V(G)$, is a *G-layer* and is denoted by G^h. Analogously H-layers ${}^gH$ are defined. G-layers and H-layers are isomorphic to G and to H, respectively. If $X \subseteq V(G \,\square\, H)$, then the *projection* $p_G(X)$ of X to G is the set $\{g \in V(G) : (g, h) \in X \text{ for some } h \in V(H)\}$. Analogously the projection $p_H(X)$ of X to H is defined. The k-tuple Cartesian product of of a graph G by itself, alias Cartesian power of G, will be denoted by $G^{\square, n}$. This is well-defined since the Cartesian product operation is associative. For more on the Cartesian product see [5]. As stated above, the following result was the primary motivation for the present paper.

Theorem 1.1 [10] *If $r \geq 3$ and $s \geq 3$, then* $\mathrm{gp}(P_r \,\square\, P_s) = 4$.

A subgraph H of a graph G is *isometric* if $d_H(u, v) = d_G(u, v)$ holds for all $u, v \in V(H)$. A set of subgraphs $\{H_1, \ldots, H_k\}$ of a graph G is an *isometric cover* of G if each

H_i, $i \in [k]$, is isometric in G and $\bigcup_{i=1}^k V(H_i) = V(G)$.

Theorem 1.2 [9, Theorem 3.1] *If* $\{H_1, \ldots, H_k\}$ *is an isometric cover of* G*, then*

$$\mathrm{gp}(G) \le \sum_{i=1}^{k} \mathrm{gp}(H_i)\,.$$

If G is a connected graph, $S \subseteq V(G)$, and $\mathcal{P} = \{S_1, \ldots, S_p\}$ a partition of S, then $\mathcal{P}$ is *distance-constant* (alias "distance-regular" [6, p. 331]) if for any $i, j \in [p]$, $i \neq j$, the distance $d_G(u, v)$, where $u \in S_i$ and $v \in S_j$, is independent of the selection of u and v. This distance is then the distance $d_G(S_i, S_j)$ between the parts S_i and S_j. A distance-constant partition $\mathcal{P}$ is *in-transitive* if $d_G(S_i, S_k) \neq d_G(S_i, S_j) + d_G(S_j, S_k)$ holds for pairwise different indices $i, j, k \in [p]$.

Theorem 1.3 [1, Theorem 3.1] *Let* G *be a connected graph. Then* $S \subseteq V(G)$ *is a general position set if and only if the components of* $\langle S \rangle$ *are complete subgraphs, the vertices of which form an in-transitive, distance-constant partition of* S*.*

Suppose that G is a connected bipartite graph and a general position set S contains two adjacent vertices x and y. Then Theorem 1.3 implies that $|S| = 2$, because no other vertex of G can be at the same distance to x and y. We state this observation for later use.

Corollary 1.4 *If* G *is a bipartite graph with* $\mathrm{gp}(G) \ge 3$*, then every gp-set of* G *is an independent set.*

2 Enumeration of gp-sets in grids

If G is a graph, then let $\#\mathrm{gp}(G)$ be the number of gp-sets of G. For instance, $\#\mathrm{gp}(K_n) = 1$, for every $n \ge 1$, and $\#\mathrm{gp}(P_n) = \binom{n}{2}$ for every $n \ge 2$. The main result of this section reads as follows.

Theorem 2.1 *If* $2 \le r \le s$*, then*

$$\#\mathrm{gp}(P_r \,\square\, P_s) = \begin{cases} 6; & r = s = 2\,, \\ \dfrac{s(s-1)(s-2)}{3}; & r = 2, s \ge 3\,, \\ \dfrac{rs(r-1)(r-2)(s-1)(s-2)(r(s-3)-s+7)}{144}; & r, s \ge 3\,. \end{cases}$$

Proof. Set $V(P_n) = [n]$. If $r = s = 2$, then the assertion is clear since $P_2 \,\square\, P_2 = C_4$. Let next $r = 2$ and $s \ge 3$. It is straightforward to see that $\mathrm{gp}(P_2 \,\square\, P_s) = 3$. Moreover, if X is a gp-set of $P_2 \,\square\, P_s$, then X has one vertex in one of the two P_s-layers and two vertices in the other P_s-layer, say $X = \{(1,i),(2,j),(2,k)\}$, where $j < k$. Since X is a gp-set we infer that $j < i < k$. From this it follows that

$$\#\mathrm{gp}(P_2 \,\square\, P_s) = 2 \cdot \sum_{i=1}^{s} (i-1)(s-i) = \frac{s(s-1)(s-2)}{3}\,.$$

Suppose in the rest that $r, s \geq 3$, so that $\mathrm{gp}(P_r \,\square\, P_s) = 4$ by Theorem 1.1. Hence by Corollary 1.4, every gp-set is an independent set (of cardinality 4). Let X be an arbitrary such set and assume first that $|p_{P_s}(X)| = 2$. Then, clearly, X has two vertices in one P_r-layer and two vertices in another P_r-layer. Let $(i,j) \in X$ be a vertex that has the smallest first coordinate among the vertices of X. Then (i,j) and the two vertices of X from the P_r-layer not containing (i,j) lie on a common geodesic. Analogously, X cannot be a general position set if $|p_{P_r}(X)| = 2$. Since also $|p_{P_r}(X)| = 1$ or $|p_{P_s}(X)| = 1$ are not possible, we only need to distinguish the following two cases.

Case 1: $|p_{P_r}(X)| = 4$ and $|p_{P_s}(X)| = 4$.
Let $p_{P_r}(X) = \{a,b,c,d\}$, where $a < b < c < d$, and let $p_{P_s}(X) = \{a',b',c',d'\}$, where $a' < b' < c' < d'$. Then in the set $p_{P_r}(X) \times p_{P_s}(X)$ there are $4!$ different 4-sets of vertices that project onto both $p_{P_r}(X)$ and $p_{P_s}(X)$. They can be described with permutations π of $p_{P_s}(X)$. That is, if $\pi : p_{P_s}(X) \to p_{P_s}(X)$ is a bijection, then the corresponding gp-set of vertices of $P_r \,\square\, P_s$ is $S_\pi = \{(a,\pi(a')),(b,\pi(b')),(c,\pi(c')),(d,\pi(d'))\}$. Now, by the metric structure of $P_r \,\square\, P_s$ (cf. [10]), S_π is a general position set if and only if the sequence $(\pi(a'),\pi(b'),\pi(c'),\pi(d'))$ contains no monotone subsequence of length 3. By a direct inspection we find that if $\pi(a') = a'$ or if $\pi(a') = d'$, then we get no general position sets. If $\pi(a') = b'$, then exactly the sequences (b',a',d',c') and (b',d',a',c') yield general position sets. Symmetrically, if $\pi(a') = c'$, then exactly the sequences (c',a',d',b') and (c',d',a',b') yield general position sets. Hence, if $|p_{P_r}(X)| = 4$ and $|p_{P_s}(X)| = 4$, then there are exactly $4\binom{r}{4}\binom{s}{4}$ gp-sets.

Case 2: $|p_{P_r}(X)| = 3$ (and $|p_{P_s}(X)| = 3$ or $|p_{P_s}(X)| = 4$).
Let $p_{P_r}(X) = \{a,b,c\}$, where two vertices from X project to a, say $(a,a'),(a,b') \in X$, where $a' < b'$. Let (x,x') be a vertex of $P_r \,\square\, P_s$, where $x' \leq a'$ and $(x,x') \neq (a,a')$. Then $d((x,x'),(a,b')) = d((x,x'),(a,a')) + d((a,a'),(a,b'))$ which means that $(x,x') \notin X$. Similarly, if (x,x') is a vertex of $P_r \,\square\, P_s$ with $x' \geq b'$ and $(x,x') \neq (a,b')$, then also $(x,x') \notin X$. We have thus shown that

$$X \cap ([r] \times \{1,\ldots,a'\}) = \{(a,a')\} \text{ and } X \cap ([r] \times \{b',\ldots,s\}) = \{(a,b')\}\,.$$

Let $X = \{(a,a'),(a,b'),(x,x'),(y,y')\}$, where $x \neq y$. By a similar argument as above we see that, without loss of generality,

$$\begin{aligned}
&(x,x') \in \{1,\ldots,a-1\} \times \{a'+1,\ldots,b'-1\} \text{ and} \\
&(y,y') \in \{a+1,\ldots,r\} \times \{a'+1,\ldots,b'-1\}\,.
\end{aligned}$$

Since the vertices (x,x') and (y,y') are arbitrary vertices from $\{1,\ldots,a-1\} \times \{a'+1,\ldots,b'-1\}$ and $\{a+1,\ldots,r\} \times \{a'+1,\ldots,b'-1\}$, respectively, for fixed a, a', b' we obtain precisely

$$[(b'-a'-1)(a-1)]\cdot[(b'-a'-1)(r-a)] = (b'-a'-1)^2(a-1)(r-a)$$

gp-sets. Consequently, the number of gp-sets in Case 2 is

$$\sum_{a=1}^{r}\sum_{a'=1}^{s}\sum_{b'=a'+1}^{s}[(b'-a'-1)^2(a-1)(r-a)] = \frac{rs(r^2-3r+2)(s^3-4s^2+5s-2)}{72}\,.$$

By the above two cases, if $r,s\geq 3$, then

$$\begin{aligned}\#\mathrm{gp}(P_r\,\square\,P_s) &= 4\binom{r}{4}\binom{s}{4} + \frac{rs(r^2-3r+2)(s^3-4s^2+5s-2)}{72}\\ &= \frac{rs(r-1)(r-2)(s-1)(s-2)(r(s-3)-s+7)}{144}\,,\end{aligned}$$

which is the claimed expression. $\square$

If $r=3$ and $s\geq 3$, then Theorem 2.1 yields

$$\#\mathrm{gp}(P_3\,\square\,P_s) = \frac{s(s-2)(s-1)^2}{12}\,,$$

which, after substituting s with $s+1$ gives the sequence A002415 from OEIS [13]. In addition, the case $r=2$ and $s\geq 3$ yields the sequence A007290.

3 Cylinders

In this section we determine the general position number of cylinders. For this task, the following function will be useful. If G is a connected graph and $X\subseteq V(G)$ a general position set, then

$$F(X) = \{u\in V(G)-X:\ X\cup\{u\} \text{ is not a general position set}\}\,.$$

If $X=\{x,y\}$, we will simplify the notation $F(\{x,y\})$ to $F(x,y)$.

Set $V(P_r)=[r]_0$ and $V(C_s)=[s]_0$. From now on, operations with the integers in $V(C_s)$ are done modulo s.

Lemma 3.1 *Let $r\geq 2$, $s\geq 3$, and let S be a general position set of the cylinder graph $P_r\,\square\,C_s$. Then the following assertions hold.*

(i) If $|S|\geq 5$, then S is an independent set.

(ii) If $|S|\geq 4$, then $|S\cap V({}^iC_s)|\leq 2$ for every $i\in[r]_0$.

(iii) If $|S\cap V({}^iC_s)|=2$ for some $i\in[r]_0$, then $|S|\leq 4$.

(iv) If $r\geq 6$, $|S|=5$, and $|S\cap V({}^iC_s)|\leq 1$ for every $i\in[r]_0$, then $\mathrm{gp}(P_5\,\square\,C_s)\geq 5$.

Proof. (i) Suppose S is not independent. If $(i,k),(i+1,k)\in S$, then we observe that $F((i,k),(i+1,k)) = V(P_r \,\Box\, C_s) - \{(i,k),(i+1,k)\}$, which means that $S = \{(i,k),(i+1,k)\}$, a contradiction. On the other hand, if $(i,k),(i,k+1)\in S$, then either $F((i,k),(i,k+1)) = V(P_r \,\Box\, C_s) - \{(i,k),(i,k+1)\}$ (when s is even), or $F((i,k),(i,k+1)) = V(P_r \,\Box\, C_s) - (\{(i,k),(i,k+1)\} \cup (V(P_r)\times\{j\}))$ (when s is odd), where j is the vertex of C_s diametral with k and $k+1$. The first possibility directly leads to a contradiction. For the second one, since every P_r-layer, being an isometric subgraph, contributes at most two vertices to a general position set of $P_r \,\Box\, C_s$, it follows that $|S|\le 4$, which is again a contradiction. Consequently S must be an independent set.

(ii) The result follows directly from the following fact. If $\{(i,k_1),(i,k_2),(i,k_3)\}\subseteq S$, then $F(\{(i,k_1),(i,k_2),(i,k_3)\}) = V(P_r \,\Box\, C_s) - \{(i,k_1),(i,k_2),(i,k_3)\}$, which means that $|S|=3$.

(iii) Let $i\in[r]_0$ be such that $|S\cap V({}^iC_s)|=2$. We may assume without loss of generality that $S\cap V({}^iC_s) = \{(i,0),(i,j)\}$, where $j\le\lfloor s/2\rfloor$. Then $F((i,0),(i,j)) = [r]_0\times\{j-\lfloor s/2\rfloor,\ldots,\lfloor s/2\rfloor\} - \{(i,0),(i,j)\}$. Suppose now that $|S|\ge 5$. Then, without loss of generality, S contains at least two elements in $([i]_0\times V(C_s)) - F((i,0),(i,j))$, say (i',j') and (i'',j''), where $i'\le i''<i$. If $j'\le j''$, then $F((i',j'),(i'',j'')) = [r]_0\times\{j''-\lfloor s/2\rfloor,\ldots,j'+\lfloor s/2\rfloor\} - \{(i',j'),(i'',j'')\}$. But then $(i,0)\in F((i',j'),(i'',j''))$, a contradiction. Similarly, if $j''\le j'$, then we get the contradiction $(i,j)\in F((i',j'),(i'',j''))$.

(iv) Let $S=\{(i_k,j_k):\ k\in[5]_0\}$. Since $|S\cap V({}^iC_s)|\le 1$, the coordinates i_k are pairwise different, hence we may assume without loss of generality that $i_0<i_1<i_2<i_3<i_4$. Set $S'=\{(k,j_k):\ k\in[5]_0\}$. We claim that S' is a general position of $G_5 = P_5 \,\Box\, C_s$. Assume on the contrary that

$$d_{G_5}((p,j_p),(r,j_r)) = d_{G_5}((p,j_p),(q,j_q)) + d_{G_5}((q,j_q),(r,j_r))$$

for some $p,q,r\in[5]_0$, $p<q<r$. Since the distance function in Cartesian products is additive, we get that

$$d_{P_5}(p,r) + d_{C_s}(j_p,j_r) = d_{P_5}(p,q) + d_{C_s}(j_p,j_q) + d_{P_5}(q,r) + d_{C_s}(j_q,j_r)\,.$$

Since $d_{P_5}(p,r) = d_{P_5}(p,q) + d_{P_5}(q,r)$, we thus have

$$d_{C_s}(j_p,j_r) = d_{C_s}(j_p,j_q) + d_{C_s}(j_q,j_r)\,.$$

From this we get that in $G_r = P_r \,\Box\, C_s$,

$$\begin{aligned} d_{G_r}((i_p,j_p),(i_r,j_r)) &= d_{P_r}(i_p,i_r) + d_{C_s}(j_p,j_r) \\ &= [d_{P_r}(i_p,i_q) + d_{P_r}(i_q,i_r)] + [d_{C_s}(j_p,j_q) + d_{C_s}(j_q,j_r)] \\ &= [d_{P_r}(i_p,i_q) + d_{C_s}(j_p,j_q)] + [d_{P_r}(i_q,i_r) + d_{C_s}(j_q,j_r)] \\ &= d_{G_r}((i_p,j_p),(i_q,j_q)) + d_{G_r}((i_q,j_q),(i_r,j_r))\,. \end{aligned}$$

This contradiction proves that S' is a general position set of $P_5 \,\Box\, C_s$. We conclude that $\mathrm{gp}(P_5 \,\Box\, C_s)\ge 5$. $\square$

Note that Lemma 3.1(iv) allows us to map a general position set of cardinality 5 in long cylinders to a general position set of the same cardinality in cylinders over P_5.

Theorem 3.2 *If $r \geq 2$ and $s \geq 3$, then*

$$\mathrm{gp}(P_r \,\square\, C_s) = \begin{cases} 3; & r = 2, s = 3\,, \\ 5; & r \geq 5, \text{ and } s = 7 \text{ or } s \geq 9\,, \\ 4; & \text{otherwise}\,. \end{cases}$$

Proof. First, it is easy to verify that $\mathrm{gp}(P_2 \,\square\, C_3) = 3$.

Assume next that $r \leq 4$ and suppose that there exists a general position set S with $|S| \geq 5$. If $r = 2$, this is not possible by Lemma 3.1(ii). Let next $r \in \{3, 4\}$. Then there exists a C_s-layer ${}^iC_s$ with $|V({}^iC_s) \cap S| \geq 2$. The case $|V({}^iC_s) \cap S| > 2$ is not possible by Lemma 3.1(ii), while the case $|V({}^iC_s) \cap S| = 2$ is excluded by Lemma 3.1(iii). Hence $\mathrm{gp}(P_r \,\square\, C_s) \leq 4$ for $r \in \{2, 3, 4\}$. It is straightforward to see that the set $\{(0, 0), (1, 1), (0, \lfloor s/2 \rfloor), (1, \lfloor s/2 \rfloor + 1)\}$ is a general position set of $P_r \,\square\, C_s$ for $r \geq 2$ and $s \geq 4$. Moreover, if $s = 3$, then the set $\{(0, 1), (1, 0), (1, 2), (2, 1)\}$ is a general position set of $P_r \,\square\, C_3$ for $r \geq 3$. Hence $\mathrm{gp}(P_r \,\square\, C_s) \geq 4$ for $r \in \{2, 3, 4\}$ and so $\mathrm{gp}(P_r \,\square\, C_s) = 4$ for $r \in \{2, 3, 4\}$.

The general position set $\{(0, 0), (1, 2), (2, 4), (3, 6), (4, 1)\}$ of $P_5 \,\square\, C_7$ demonstrates that $\mathrm{gp}(P_5 \,\square\, C_7) \geq 5$.

Suppose next that for some $r \geq 6$ the cylinder $P_r \,\square\, C_8$ contains a general position set S with $|S| = 5$. From Lemma 3.1(iii) it follows that $|S \cap V({}^iC_8)| \leq 1$ for every $i \in [r]_0$. Hence the assumptions of Lemma 3.1(iv) are fulfilled which implies that $\mathrm{gp}(P_5 \,\square\, C_8) \geq 5$. Since we have checked by computer that $\mathrm{gp}(P_5 \,\square\, C_8) = 4$, we have a contradiction. Therefore, $\mathrm{gp}(P_r \,\square\, C_8) \leq 4$ for $r \geq 5$. Since clearly $\mathrm{gp}(P_r \,\square\, C_8) \geq 4$, we conclude that $\mathrm{gp}(P_r \,\square\, C_8) = 4$ for $r \geq 5$.

Suppose now that $r = 5$, $s \geq 9$, and consider the set

$$S = \{u_0 = (0, 1), u_1 = (1, 4), u_2 = (2, \lfloor s/2 \rfloor + 2), u_3 = (3, 0), u_4 = (4, 3)\}\,.$$

We claim that S is a general position set. Note first that the vertices u_0, u_1, u_3, u_4 lie in a general position. Further,

$$\begin{aligned} d(u_2, u_0) &= (s - \lfloor s/2 \rfloor - 1) + 2 = s - \lfloor s/2 \rfloor + 1, \\ d(u_2, u_1) &= \lfloor s/2 \rfloor - 1, \\ d(u_2, u_3) &= s - \lfloor s/2 \rfloor - 1, \\ d(u_2, u_4) &= \lfloor s/2 \rfloor + 1. \end{aligned}$$

Then $d(u_0, u_2) = s - \lfloor s/2 \rfloor + 1 < \lfloor s/2 \rfloor + 3 = d(u_0, u_1) + d(u_1, u_2)$. Similarly we see that u_2 is not on a geodesic containing three vertices of S. Hence, S is a general position set and thus $\mathrm{gp}(P_5 \,\square\, C_s) \geq 5$ for $s \geq 9$.

Note finally that the general position set for $P_5 \,\square\, C_7$ and the general position set for $P_5 \,\square\, C_s$, $s \geq 9$, are also general position sets for $P_r \,\square\, C_7$, $r \geq 6$, and for $P_r \,\square\, C_s$, $r \geq 6$, respectively. We conclude that $\mathrm{gp}(P_r \,\square\, C_s) \geq 5$ for $r \geq 5$ and $s \geq 7$, $s \neq 8$.

It remains to prove that the above constructed general position sets of cardinality 5 are gp-sets. Hence let S be a gp-set of $P_r \,\square\, C_s$, where $|S| \geq 5$. Note that S is an independent set, by Lemma 3.1(i). Then make a partition of $V(P_r \,\square\, C_s)$ into two sets

A_1 and A_2 inducing two grids that are isometric subgraphs of $P_r \,\square\, C_s$. Without loss of generality, we may assume $A_1 = V(P_r) \times [\lfloor s/2 \rfloor]_0$ and $A_2 = V(P_r) \times ([s]_0 - [\lfloor s/2 \rfloor]_0)$. Now, let $S_1 = S \cap A_1$ and $S_2 = S \cap A_2$. Since $\mathrm{gp}(P_r \,\square\, C_s) \geq 5$, it follows $|S_1| \geq 3$ or $|S_2| \geq 3$. Moreover, since A_1 and A_2 induce isometric grid graphs, Theorem 1.1 implies that $|S_1| \leq 4$ and $|S_2| \leq 4$. To simplify notation, we write $s_d = \lfloor s/2 \rfloor$. We consider two cases.

Case 1: $|S_1| = 4$.
Let $S_1 = \{(a', a), (b', b), (c', c), (d', d)\}$. Note that S_1 is then a gp-set of the grid induced by A_1. Since the structure of every gp-set of a grid graph is known from the proof of Theorem 2.1 we can assume without loss of generality the following facts: $a < b < d$, $a < c < d$, and either ($c' < a' < b'$ and $c' < d' < b'$) or ($b' < a' < c'$ and $b' < d' < c'$). Examples of such sets are shown in Fig. 1. From the presentation purposes, in this and the subsequent figures an orientation is selected such that C_s-layers are drawn horizontally and P_r-layers vertically.

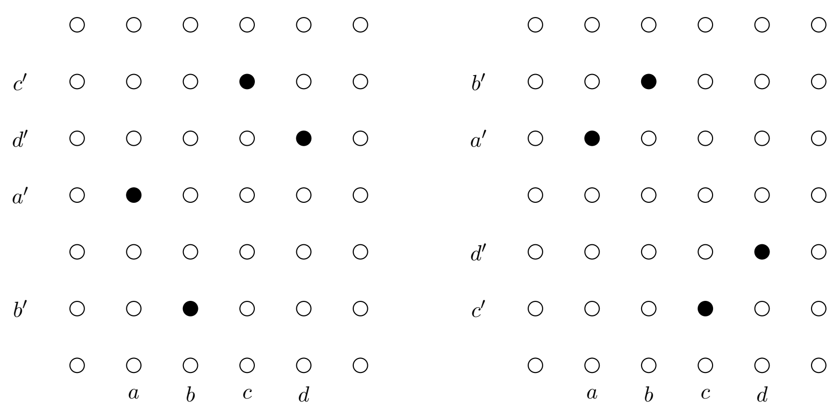

Figure 1: Two possible configurations of the set S_1 (edges of the grid have not been drawn).

We now consider the set $F(S_1)$ in $V(P_r \,\square\, C_s)$. Fig. 2 shows an example of the forbidden area generated by only two vertices of S_1 (($c', c)$ and (d', d) in this case). Since it is not necessary for our purposes, we do not look for the whole such set, but just a significant part of it.

We detail now the case $a < b < c < d$ and $c' < a' < d' < b'$, see Fig. 3. Observe that:

$$\begin{aligned}
\{a', \ldots, r-1\} \times \{a, a-1, \ldots, c - s_d\} &\subset F((c', c), (a', a)),\\
\{0, \ldots, a'\} \times \{a, a-1, \ldots, b - s_d\} &\subset F((b', b), (a', a)),\\
\{d', \ldots, r-1\} \times \{d, d+1, \ldots, c + s_d\} &\subset F((c', c), (d', d)),\\
\{0, \ldots, d'\} \times \{d, d+1, \ldots, b + s_d\} &\subset F((b', b), (d', d)).
\end{aligned}$$

Notice that there is a set of vertices $A \subset A_2$ such that $A = A_2 - F(S_1)$. Such a set could be empty under some distributions of the vertices $(a', a), (b', b), (c', c), (d', d)$ (for

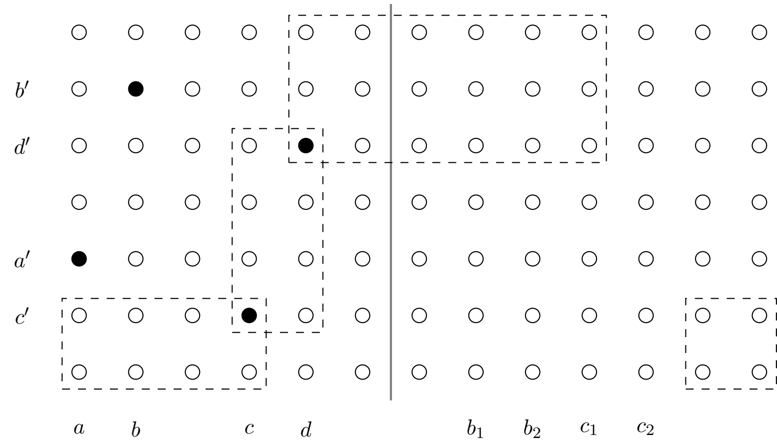

Figure 2: The forbidden area $F((c', c), (d', d))$ appears surrounded by the dashed rectangles. Vertices b_1 and b_2 of C_s are diametral with b while c_1 and c_2 are diametral with c. Here $b_1 = b + s_d$, $b_2 = b - s_d$, $c_1 = c + s_d$, and $c_2 = c - s_d$. Similar convention will be used in the subsequent figures. Notice that if the cycle would have an even order, then $b_1 = b_2$ and $c_1 = c_2$.

instance if $a' = d' - 1$ and maintaining the remaining assumptions). Fig. 3 shows an example of this where the set A is not empty.

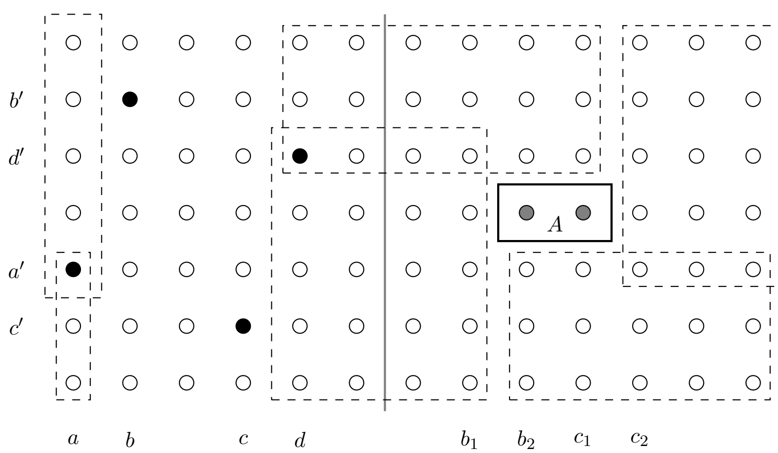

Figure 3: Part of the forbidden area of the bolded set of vertices appears in dashed rectangles. The two gray vertices of the thick rectangle (denoted by A) do not belong to the forbidden area of the bolded vertices.

In consequence, it must happen that $S_2 \subseteq A$, since otherwise we get a contradiction with S being a gp-set. If $|S_2| \geq 2$, then let $(x', x), (y', y) \in S_2$. It is then not difficult to observe that either $(a', a), (x', x), (y', y)$ or $(d', d), (x', x), (y', y)$ lie in a same geodesic of $P_r \,\Box\, C_s$, which is not possible. Thus $|S_2| \leq 1$.

By using similar reasoning, we deduce the same conclusion for any other relationship of a, b, c, d and a', b', c', d' (from those ones that allow to obtain a gp-set of the grid induced by A_1). As a consequence of the whole deduction of this case, we obtain that $\mathrm{gp}(P_r \,\square\, C_s) = |S| = |S_1| + |S_2| \leq 5$.

Case 2: $|S_1| = 3$ or $|S_2| = 3$. Assume $|S_1| = 3$, and let $S_1 = \{(a', a), (b', b), (c', c)\}$. Clearly, the three elements of S_1 cannot lie simultaneously in a same ${}^iC_s$-layer, or in a same $P_r{}^j$-layer. Moreover, it cannot happen that $a' \leq b' \leq c'$ and $a \leq b \leq c$ at the same time, or any other similar double monotone sequence. This means that, for instance, if $a' \leq b' \leq c'$, then either ($b < a$ and $b < c$) or ($b > a$ and $b > c$).

We may assume now that $a' \leq b' \leq c'$, $b < a$ and $b < c$. Fig. 4 shows an example of this.

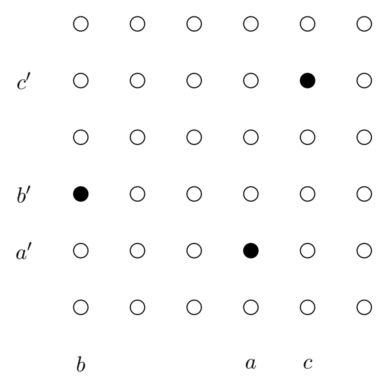

Figure 4: An example of a configuration for the set S_1.

We now consider the set $F(S_1)$ in $V(P_r \,\square\, C_s)$, and observe the following. Recalling that $s_d = \lfloor s/2 \rfloor$ we have:

$$\begin{aligned}
\{c', \ldots, r-1\} \times \{c, c+1, \ldots, a+s_d\} &\subset F((c', c), (a', a)), \\
\{0, \ldots, a'\} \times \{a, a-1, \ldots, c-s_d\} &\subset F((c', c), (a', a)), \\
\{b', \ldots, r-1\} \times \{b, b-1, \ldots, a-s_d\} &\subset F((b', b), (a', a)), \\
\{0, \ldots, a'\} \times \{a, a+1, \ldots, b+s_d\} &\subset F((b', b), (a', a)), \\
\{c', \ldots, r-1\} \times \{c, c+1, \ldots, b+s_d\} &\subset F((c', c), (b', b)), \\
\{0, \ldots, b'\} \times \{b, b-1, \ldots, c-s_d\} &\subset F((c', c), (b', b)).
\end{aligned}$$

See Fig. 5 for an example of the situations above.

Observe now that there are four sets, say B_1, B_2, B_3, and B_4, such that $B_1 \cup B_2 \cup B_3 \cup B_4 = A_2 - F(S_1)$, and satisfying the following:

$$\begin{aligned}
B_1 &= \{a'+1, \ldots, c'-1\} \times \{s_d+1, \ldots, b+s_d\}, \\
B_2 &= \{a'+1, \ldots, c'-1\} \times \{b+s_d+1, \ldots, a+s_d\}, \\
B_3 &= \{0, \ldots, a'\} \times \{b+s_d+1, \ldots, a+s_d\}, \\
B_4 &= \{0, \ldots, a'\} \times \{a+s_d+1, \ldots, c+s_d\}.
\end{aligned}$$

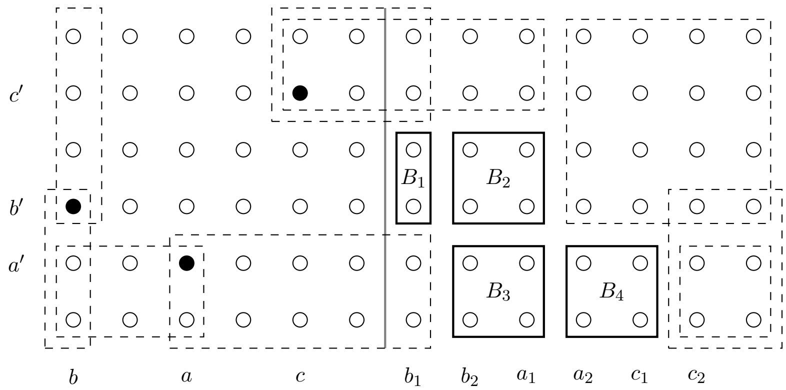

Figure 5: A significant part of the set $F(S_1)$ appears surrounded by dashed rectangles. For $x \in \{a, b, c\}$, the vertices x_1 and x_2 from C_s ($x_1 = x + s_d$ and $x_2 = x - s_d$) are diametral vertices with x. Note that if C_s is an even cycle, then $x_1 = x_2$.

Note that some of these sets could be empty, or could have non-empty intersection, depending on the parity of s and on the structure of the set S_1.

If $|S_2 \cap B_i| \geq 2$ for some $i \in [4]$, then we shall find an isometric subgraph of $P_r \,\square\, C_s$ isomorphic to a grid graph such that it contains four vertices of the set S. Hence, we can change the partition given by A_1 and A_2 from the beginning, to a new one, and proceed as in Case 1, to prove that $\mathrm{gp}(P_r \,\square\, C_s) \leq 5$. That is, if $|S_2 \cap B_i| \geq 2$ for some $i \in [3]$, then we can use the partition $A'_1 = [r]_0 \times \{a, a+1, \ldots, a+s_d\}$ and $A'_2 = V(P_r \,\square\, C_s) - A'_1$, and if $|S_2 \cap B_4| \geq 2$, then we can use the partition $A'_1 = [r]_0 \times \{a+s_d+1, a+s_d+2, \ldots, a\}$ (note that $a + s_d + 1 = a - s_d$) and $A'_2 = V(P_r \,\square\, C_s) - A'_1$. In concordance, we may assume that $|S_2 \cap B_i| \leq 1$ for every $i \in [4]$.

We consider now the three sets B_1, B_2 and B_3. If at least two of them contain one element from S_2, then, as above, we can find a different partition of $V(P_r \,\square\, C_s)$ and proceed like in Case 1. Thus, $|(B_1 \cup B_2 \cup B_3) \cap S_1| \leq 1$.

Finally, we deduce that $\mathrm{gp}(P_r \,\square\, C_s) = |S| = |S_1| + |S_2| = |S_1| + |(B_1 \cup B_2 \cup B_3) \cap S_1| + |B_4 \cap S_1| \leq 5$. By using similar arguments, we can again obtain a similar conclusion for any other possible relationship between a, b, c and a', b', c'. This completes the proof of this case, and therefore, of the whole theorem. $\square$

4 Torus graphs

Knowing $\mathrm{gp}(P_r \,\square\, P_s)$ and $\mathrm{gp}(P_r \,\square\, C_s)$, it is natural to consider the *torus graphs* $C_r \,\square\, C_s$, $r, s \geq 3$, where we keep the convention that $V(C_n) = [n]_0$. In contrast to the former two cases, for the torus graphs we are not able to give exact results but only the following lower bound.

Theorem 4.1 *If* $r \geq 6$ *and* $4 \neq s \leq r$, *then* $\mathrm{gp}(C_r \,\square\, C_s) \geq 6$.

Proof. The condition $s \neq 4$ assures that $S_s = \{0, \lfloor \frac{s}{3} \rfloor, \lfloor \frac{2s}{3} \rfloor\}$ is a gp-set of C_s. The condition that $r \geq 6$ assures that $\lfloor r/6 \rfloor \geq 1$. Consider now the set

$$S = \{(0,0), (\lfloor r/2 \rfloor, 0), (\lfloor r/6 \rfloor, \lfloor s/3 \rfloor), (\lfloor r/6 \rfloor + \lfloor r/2 \rfloor, \lfloor s/3 \rfloor),$$
$$(\lfloor (2r)/6 \rfloor, \lfloor (2s)/3 \rfloor), (\lfloor (2r)/6 \rfloor + \lfloor r/2 \rfloor, \lfloor (2s)/3 \rfloor)\}.$$

In Fig. 6 the set S is shown for the case $C_6 \,\square\, C_3$.

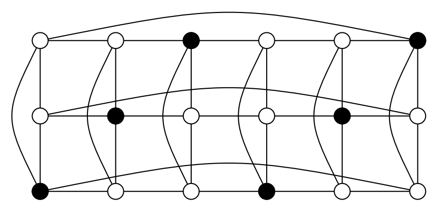

Figure 6: The set S in $C_6 \,\square\, C_3$ appears in bold.

We claim that S is a general position set. Since C_p-layers are isometric subgraphs, no other vertex is on a geodesic between the pair of vertices with the same second coordinate. Hence we only need to consider the triples of vertices from S with pairwise different second coordinates. We do this for the vertices $x_1 = (0,0)$, $x_2 = (\lfloor r/6 \rfloor, \lfloor s/3 \rfloor)$, and $x_3 = (\lfloor (2r)/6 \rfloor, \lfloor (2s)/3 \rfloor)$, the other cases are treated similarly. Since x_1, x_2, x_3 lie in a subgraph of $C_r \,\square\, C_s$ isomorphic to $P_{\lfloor r/3 \rfloor + 1} \,\square\, C_s$ which is an isometric subgraph, it suffices to show that $d(x_1, x_3) < d(x_1, x_2) + d(x_2, x_3)$. This can be verified using the facts $d(x_1, x_3) = \lfloor r/3 \rfloor + (s - \lfloor (2s)/3 \rfloor)$, $d(x_1, x_2) = \lfloor r/6 \rfloor + \lfloor s/3 \rfloor$, and $d(x_2, x_3) \geq \lfloor r/6 \rfloor + \lfloor s/3 \rfloor$. $\square$

Since $V(C_r \,\square\, C_s)$ can be partitioned into two parts such that each of them induces an isometric cylinder, Theorem 3.2 and Theorem 1.2 yield upper bounds that are twice the values from Theorem 3.2. In general we can always conclude that $\mathrm{gp}(C_r \,\square\, C_s) \leq 10$, and in many cases we can improve it by 1 or 2. For instance, $\mathrm{gp}(C_{16} \,\square\, C_s) \leq 8$, $\mathrm{gp}(C_{15} \,\square\, C_s) \leq 9$, and $\mathrm{gp}(C_{17} \,\square\, C_s) \leq 9$. Note also that $\mathrm{gp}(C_3 \,\square\, C_s) = 6$ for $s \geq 6$, which follows from the fact that in every C_s-layer we can have at most two vertices from a gp-set.

5 Additional Cartesian products

In this section we consider the general position number of 3-dimensional grids, Hamming graphs, and Cartesian powers. For the latter two classes we obtain asymptotically exponential lower bounds using a probabilistic approach.

In [10, Proposition 3.5] it was proved that

$$10 \leq \mathrm{gp}(P_\infty \,\square\, P_\infty \,\square\, P_\infty) \leq 16 \,. \tag{1}$$

To prove the lower bound, a set S of cardinality 10 was constructed such that $\{d(u,v) : \ u,v \in S, u \neq v\} = \{3,4,5\}$, from which it immediately follows that S is a general position set. Consider now $P_\infty \,\square\, P_\infty \,\square\, P_\infty$ embedded into the plane in the natural way, that is, the vertices being triples (x,y,z), $x,y,z \in \mathbb{Z}$. Let S be a general position set with $|S| = k$, where we may assume without loss of generality that $x_1 < \cdots < x_k$. By [10, Theorem 2.3], if there would be a monotone subsequnce of size 5 in the y-coordinate, then there would exists a monotone subsequnce of size 3 in the z-coordinate. But this would imply that S is not a general position set. Hence, to produce a general position set, we have to check all possible sequences of k elements with longest monotone subsequences of cardinality 4 both in the y-coordinate and the z-coordinate. In this way we were able to find the following sequences:

$$
\begin{aligned}
x &= (1,2,3,4,5,6,7,8,9,10,11,12,13,14),\\
y &= (8,11,5,3,13,14,9,6,1,2,12,10,4,7),\\
z &= (6,10,13,4,3,8,14,1,7,12,11,2,5,9).
\end{aligned}
$$

It can be now checked by hand or by computer that this yields a general position set. In summary, the above construction improves (1) as follows.

Proposition 5.1 $14 \le \mathrm{gp}(P_\infty \,\square\, P_\infty \,\square\, P_\infty) \le 16.$

The n-dimensional hypercube Q_n is defined as $K_2^{\square,n}$. In particular, $Q_1 = K_2$, $Q_2 = C_4$, and Q_3 is the graph of the 3-D cube. Cartesian products of complete graphs, known as Hamming graphs, form a natural generalization of hypercubes. In [4] it was proved that if $k \ge 2$ and $n_1,\ldots,n_k \ge 2$, then

$$\mathrm{gp}(K_{n_1} \,\square \cdots \square\, K_{n_k}) \ge n_1 + \cdots + n_k - k\,. \tag{2}$$

Moreover, this lower bound is sharp on products of two complete graphs, that is, $\mathrm{gp}(K_{n_1} \,\square\, K_{n_2}) = n_1 + n_2 - 2$.

The situation above changes dramatically as k grows. Körner [8] obtained a probabilistic construction of general position sets in Q_n of size $\frac{1}{2}\frac{2^n}{\sqrt{3^n}}$. He also pointed out that the problem of finding the size of the largest point set in general position in Q_n is equivalent to finding the largest size of what is called a $(2,1)$-separating system in coding theory. (For more on separating systems, see [3].) Körner was interested in

$$\alpha = \limsup_{n\to\infty} \frac{\log_2 \mathrm{gp}(Q_n)}{n}\,.$$

His probabilistic lower bound gives $\alpha \ge 1 - \frac{1}{2}\log_2 3$ and he also proved $\alpha \le 1/2$. Later, Randriambololona [11] improved the lower bound to $\alpha \ge \frac{3}{50}\log_2 11$ with an explicit construction.

The first moment method can be applied in a general setting to obtain large general position sets. For any graph G, let $p(G)$ denote the probability that if one picks a triple $(x,y,z) \in V(G)^3$ uniformly at random, then $d_G(y,z) = d_G(y,x) + d_G(x,z)$ holds.

Note that this is never the case if $x \neq y = z$, so $p(G) \leq 1 - \frac{|V(G)|-1}{|V(G)|^2} < 1$. Let $H = H_1 \,\Box \cdots \Box\, H_k$. Observe that for the triple $\mathbf{x} = (x_1, \ldots, x_k), \mathbf{y} = (y_1, \ldots, y_k), \mathbf{z} = (z_1, \ldots, z_k) \in V(H)$ we have $d_H(\mathbf{y}, \mathbf{z}) = d_H(\mathbf{y}, \mathbf{x}) + d_H(\mathbf{x}, \mathbf{z})$ if and only if $d_{H_i}(y_i, z_i) = d_{H_i}(y_i, x_i) + d_{H_i}(x_i, z_i)$ holds for all $i \in [k]$. So if we pick M vertices uniformly at random with repetition from $V(H)$, then the expected value of the number $X = X(M)$ of unordered triples on a geodesic will be $3\binom{M}{3} \prod_{i=1}^{k} p(H_i)$. If $X \leq M/2$, then removing one vertex from every bad triple will leave us a general position set of size at least $M/2$. As there is always an instance for which $X \leq \mathbb{E}(X)$ holds, we obtain a general position set of size $M/2$ provided $3\binom{M}{3} \prod_{i=1}^{k} p(H_i) \leq M/2$ holds. Therefore, it seems to be interesting to examine

$$\mathrm{gp}_{\Box}(G) := \limsup_{n\to\infty} \frac{\log_{|V(G)|} \mathrm{gp}(G^{\Box,n})}{n}\,.$$

Clearly, we have $\mathrm{gp}_{\Box}(G) \leq 1$ and the above reasoning yields the following theorem.

For a graph G one can consider its Cartesian power $G^{\Box,n}$. Then the required inequality is $3\binom{M}{3}p(G)^n \leq M/2$ which is equivalent to $(M-1)(M-2) \leq p(G)^{-n}$. Thus there exists a general position set in $G^{\Box,n}$ of size $\frac{1}{2}p(G)^{-n/2}$. This and the inequality $p(G) \leq 1 - \frac{|V(G)|-1}{|V(G)|^2}$ yields the following statement.

Theorem 5.2 *If G is a graph, then*

$$\mathrm{gp}_{\Box}(G) \geq \log_{|V(G)|} p(G)^{-1/2} \geq 1 - \log_{|V(G)|}(|V(G)|^2 - |V(G)| + 1)\,.$$

Let us calculate $p(G)$ for some graphs. First of all, $p(K_n) = \frac{2n-1}{n^2}$ as in K_n the equality $d(y,z) = d(y,x) + d(x,z)$ holds if and only if $x = y$ or $x = z$. (The case $p(K_2) = \frac{3}{4}$ in Theorem 5.2 is just Körner's result.) For even cycles we have $p(C_{2k}) = \frac{k(k+3)-1}{4k^2}$. If the vertices are $\{-(k-1), -(k-2), \ldots, 0, , \ldots, k-1, k\}$ in this cyclic order, then by symmetry we can assume $x = 0$. There are $4k-1$ triples with $x = y$ or $x = z$ that form bad triples. If $y = k$ or $z = k$, then there are no other bad triples, otherwise for any y, there are $k - |y|$ ways to choose z to obtain a bad triple. Similarly, one can verify $p(C_{2k+1}) = \frac{k(k+3)+1}{(2k+1)^2}$. Finally, consider the star S_k with k leaves. Then conditioning on whether x is the center or not one obtains $p(S_k) = \frac{1}{k+1} + \frac{k}{k+1}\frac{2k+1}{(k+1)^2}$. Observe that if one picks uniformly at random only among the leaves of S_k, then the probability of picking a bad triple is $p'(S_k) = \frac{2k-1}{k^2}$ which for large enough ks is roughly 2/3 of $p(S_k)$, so in this way one obtains the better bound $\mathrm{gp}_{\Box}(S_k) \geq \log_2 p'(S_k)^{-1/2}$.

To conclude the paper, we list a couple of open problems that are explicitly or implicitly related to the results of this paper. First, since we have determined $\mathrm{gp}(P_r \,\Box\, C_s)$ for all r and s, it would be a natural next step to enumerate the corresponding gp-sets. However, in view of the rather complex proof of Theorem 3.2, this task seems to be very demanding. But in some special cases it would still be of interest to find $\#\mathrm{gp}(P_r \,\Box\, C_s)$ and check whether one gets some known integer sequences. Next, in view of Theorem 4.1 we pose:

Problem 5.3 *Determine* $\mathrm{gp}(C_r \,\square\, C_s)$ *for every* $r, s \geq 3$.

Concerning $\mathrm{gp}_{\square}(G)$, can one write limit instead of limit superior in the definition of $\mathrm{gp}_{\square}(G)$? Moreover, by the above we have $\lim_{k\to\infty} p(C_k) = \frac{1}{4}$.

Proposition 5.4 *Decide whether* $\liminf_{k\to\infty} \frac{\mathrm{gp}_{\square}(C_k)}{\log_k 2} > 1$ *holds.*

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