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THE COHOMOLOGY RINGS OF HOMOGENEOUS SPACES

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Abstract. Let G be a compact connected Lie group and K a closed connected subgroup. Assume that the order of any torsion element in the integral cohomology of G and K is invertible in a given principal ideal domain $\Bbbk$. It is known that in this case the cohomology of the homogeneous space G/K with coefficients in $\Bbbk$ and the torsion product of $H^*(BK)$ and $\Bbbk$ over $H^*(BG)$ are isomorphic as $\Bbbk$-modules. We show that this isomorphism is multiplicative and natural in the pair (G, K) provided that 2 is invertible in $\Bbbk$. The proof uses homotopy Gerstenhaber algebras in an essential way. In particular, we show that the normalized singular cochains on the classifying space of a torus are formal as a homotopy Gerstenhaber algebra.

Contents

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1. Introduction

In 1950, H. Cartan gave the first uniform description of the cohomology of homogeneous spaces of Lie groups. Using a differential-geometric approach, he established the following result for a compact connected Lie group G and a closed connected subgroup $K \subset G$ [4, Thm. 5].

Theorem 1.1 (H. Cartan)**.** *There is an isomorphism of graded algebras*

$$H^*(G/K;\mathbb{R}) \cong \operatorname{Tor}_*^{H^*(BG;\mathbb{R})}\bigl(\mathbb{R}, H^*(BK;\mathbb{R})\bigr).$$

A topological way to look at this formula is the following: One has a fibre bundle

$$G/K \hookrightarrow EG/K = BK \to BG, \tag{1.1}$$

and there is an associated Eilenberg–Moore spectral sequence

$$E_2 = \operatorname{Tor}_*^{H^*(BG;\mathbb{R})}\bigl(\mathbb{R}, H^*(BK;\mathbb{R})\bigr) \;\Rightarrow\; H^*(G/K). \tag{1.2}$$

In this language, Cartan's result says that the spectral sequence collapses at the second page and that the product on that page agrees with the one on $H^*(G/K)$.

The real cohomology of the classifying space of a connected Lie group is a polynomial algebra on even degree generators. An obvious question is whether a result analogous to Cartan's holds for other principal ideal domains $\Bbbk$ for which $H^*(BG)$ and $H^*(BK)$ have this property. An equivalent condition is that the orders of the torsion subgroups of $H^*(G;\mathbb{Z})$ and $H^*(K;\mathbb{Z})$ are invertible in $\Bbbk$, and we assume this throughout. It holds in many cases, for example for $U(n)$, $SU(n)$ and $Sp(n)$ over any $\Bbbk$, and for $SO(n)$ and $Spin(n)$ if 2 is invertible in $\Bbbk$.

A partial result in this direction was achieved in 1968 by Baum, who proved that for field coefficients, the Eilenberg–Moore spectral sequence again collapses at the second page under a certain 'deficiency condition' [2, Thm. 7.4]. This yields an additive isomorphism

$$H^*(G/K) \cong \operatorname{Tor}_*^{H^*(BG)}\bigl(\Bbbk, H^*(BK)\bigr). \tag{1.3}$$

Baum's result caused a flurry of activities in the early 1970's: Wolf [26, Thm. B] removed the deficiency condition, and Husemoller–Moore–Stasheff [13, Thm. IV.8.2] proved the collapse of the Eilenberg–Moore spectral sequence for any $\Bbbk$. Gugenheim–May [11, Thm. A] and Munkholm [17, Thm.] additionally solved the extension problem, which gives the following result.

Theorem 1.2. *If $H^*(BG)$ and $H^*(BK)$ are polynomial algebras on even-degree generators, then there is an isomorphism of graded $\Bbbk$-modules*

$$H^*(G/K) \cong \operatorname{Tor}_*^{H^*(BG)}\bigl(\Bbbk, H^*(BK)\bigr).$$

Apart from one special case [2, Cor. 7.5], the product structure is not addressed in any of the works mentioned. In their introduction [11, p. viii], Gugenheim and May remark:

> Multiplicatively, however, we are left with an extension problem; our results will compute the associated graded algebra[] of $H^*(G/K)$ [...] with respect to suitable filtrations. Refinements of our algebraic theory could conceivably yield precise procedures for the computation of these cohomology algebras. When $\Bbbk = \mathbb{Z}_2$, there are examples where the extensions are non-trivial. There are no such examples known when $\Bbbk$ is a field of characteristic $\neq 2$.[1]

[1] We have aligned the original notation with ours.

The examples alluded to are the projective unitary groups $PU(n) = U(n)/U(1)$ for $n \equiv 2 \pmod 4$, see Remark 12.7. To the author's knowledge, no progress on the multiplicative structure has been made since these words were written. In the present paper we prove the following:

Theorem 1.3. *Assume that 2 is invertible in $\Bbbk$. If $H^*(BG)$ and $H^*(BK)$ are polynomial algebras, then there is an isomorphism of graded $\Bbbk$-algebras*

$$H^*(G/K) \cong \operatorname{Tor}_*^{H^*(BG)}\big(\Bbbk, H^*(BK)\big),$$

natural with respect to maps of pairs $(G, K) \to (G', K')$.

The central difficulty one faces when proving an isomorphism of the form (1.3) is the lack of commutativity of the singular cochain algebra. At some point one has to pass from cochains to cohomology, and unlike in the case of differential forms, the assignment of representatives $a_i \in C^*(BG)$ to generators $x_i \in H^*(BG)$ does not extend to a morphism of differential graded algebras (dgas). To address this, all approaches after Baum resorted to some 'up to homotopy' structure, as suggested by Stasheff–Halperin [23, p. 575].

Munkholm for example further develops the idea of strongly homotopy commutative algebras introduced by Stasheff–Halperin. The only additional ingredient he then needs is that both BG and BK have polynomial cohomology, and his result holds more generally for the fibre of bundles where both the total space and the base have this property.

In contrast to this, Husemoller–Moore–Stasheff, Gugenheim–May and Wolf rely on the existence of a maximal torus $T \subset K$ to reduce the problem to that of a homogeneous space G/T. This was already done by Baum [2], who observed that $H^*(G/K)$ injects into $H^*(G/T)$. A crucial result in this direction, also used by Wolf, is the following [11, Thm. 4.1].

Theorem 1.4 (Gugenheim–May)**.** *There is a quasi-isomorphism of dgas $C^*(BT) \to H^*(BT)$ annihilating all $\cup_1$-products.*

We are going to extend Theorem 1.4 to homotopy Gerstenhaber algebras (hgas), which were introduced by Voronov–Gerstenhaber [25]. An hga structure on a dga A is essentially a family of operations $E_k \colon A^{\otimes(k+1)} \to A$ that allow to define a product on the bar construction $\mathbf{B}A$ compatible with the coalgebra structure. Based on a result of Baues [1], the former authors also noted that singular cochain algebras are endowed with this structure [8]. In this case, the first hga operation E_1 is the usual $\cup_1$-product, up to sign. We strengthen the Gugenheim–May result as follows.

Theorem 1.5. *There is a quasi-isomorphism of dgas $C^*(BT) \to H^*(BT)$ annihilating all hga operations. In particular, $C^*(BT)$ is formal as an hga.*

See Theorem 9.6. This seems to be the first time that the hga formality of a non-trivial space is established. The quasi-isomorphism from Theorem 1.5 actually annihilates even more operation, see Proposition 10.1. This includes the ones identified by Kadeishvili [14] to construct a $\cup_1$-product on $\mathbf{B}C^*(BT)$. The only exception is the $\cup_2$-product on $C^*(BT)$, but we can show that also $\cup_2$-products of cocycles are in the kernel of the formality map provided that 2 is invertible in $\Bbbk$ (Proposition 9.7). We call an hga having a $\cup_2$-product as well as the other additional operations "extended".

The following result from the companion paper [7] allows us to combine Theorem 1.5 with Munkholm's techniques, see Theorem 6.3.

Theorem 1.6. *Any extended hga is canonically an shc algebra in the sense of Munkholm.*

In a nutshell, our strategy to prove Theorem 1.3 is the following: By the Eilenberg–Moore theorem, $H^*(G/K)$ is naturally isomorphic to the differential torsion product

$$\operatorname{Tor}^{C^*(BG)}\bigl(\Bbbk, C^*(BK)\bigr). \tag{1.4}$$

Kadeishvili–Saneblidze [15] observed that the hga structure on cochains permits to define a product on the one-sided bar construction underlying (1.4); the Eilenberg–Moore isomorphism then becomes multiplicative. Imitating mostly Munkholm, we first construct a $\Bbbk$-module isomorphism

$$H^*(\Theta)\colon\ \operatorname{Tor}^{H^*(BG)}\bigl(\Bbbk, H^*(BK)\bigr) \to \operatorname{Tor}^{C^*(BG)}\bigl(\Bbbk, C^*(BK)\bigr) \tag{1.5}$$

where we use the shc algebra structure given by Theorem 1.6. In order to show that our map is multiplicative and natural, we look at the composition

$$\begin{aligned} \operatorname{Tor}^{C^*(HG)}\bigl(\Bbbk, H^*(BK)\bigr) &\xrightarrow{H^*(\Theta)} \operatorname{Tor}^{C^*(BG)}\bigl(\Bbbk, C^*(BK)\bigr) \\ &\hookrightarrow \operatorname{Tor}^{C^*(BG)}\bigl(\Bbbk, C^*(BT)\bigr) \xrightarrow{\cong} \operatorname{Tor}^{C^*(BG)}\bigl(\Bbbk, H^*(BT)\bigr). \end{aligned} \tag{1.6}$$

The last map involves the quasi-isomorphism from Theorem 1.5 in the same way as Wolf applied the formality map constructed by Gugenheim–May. This leads to a dramatic simplification of the formulas and allows us to complete the proof of Theorem 1.3, see Section 12.

Along the way we exhibit an explicit homotopy between the two possible definitions of a tensor product of two A_∞-maps (Proposition 4.1).

Acknowledgements. Maple and Sage [18] were used to derive the formulas in Sections 4 and 9. The connection between tensor products of A_∞-maps and hypercubes (Remark 4.2) was discovered by consulting the OEIS [21].

2. Twisting cochains

We work over a fixed commutative ring $\Bbbk$ with unit, which will be assumed to be a principal ideal domain from Section 8.5 on. Since we will mostly deal with cohomological complexes, we assume a cohomological grading throughout this review section. The identity map on a complex M is denoted 1_M. The suspension map on a complex is denoted by $\mathbf{s}$ and the desuspension by $\mathbf{s}^{-1}$. All tensor products are over $\Bbbk$ unless otherwise indicated.

Given two $\mathbb{Z}$-graded complexes A and B, the complex $\operatorname{Hom}(A, B)$ consists in degree $n \in \mathbb{Z}$ of all linear maps $f\colon A \to B$ raising degrees by n. The differential of such a map is

$$d(f) = d\,f - (-1)^n f\,d. \tag{2.1}$$

We write

$$T = T_{A,B}\colon A \otimes B \to B \otimes A, \qquad a \otimes b \mapsto (-1)^{|a||b|}\, b \otimes a \tag{2.2}$$

for the transposition of factors in a tensor product. This illustrates the Koszul sign rule, another incarnation of which is the definition

$$f \otimes g\colon A \otimes B \to C \otimes D, \qquad a \otimes b \mapsto (-1)^{|g||a|}\, f(a) \otimes g(b) \tag{2.3}$$

of the tensor product of two maps $f\colon A \to C$ and $g\colon B \to D$. To unclutter formulas, we often write "$\overset{\varkappa}{=}$" to indicate that we suppress the Koszul sign as described above. For example, the formula

$$F(a,b,c) \overset{\varkappa}{=} f(c) \otimes g(a,b) \tag{2.4}$$

means

$$F(a,b,c) = (-1)^{(|a|+|b|+|g|)|c|}\, f(c) \otimes g(a,b). \tag{2.5}$$

In other words, we only specify the sign in the endomorphism operad explicitly. As a convention, we distribute composition of maps over tensor products: The formula

$$F(a,b) \overset{\varkappa}{=} f_1(f_2(a)) \otimes g_1(g_2(b)) \tag{2.6}$$

stands for the identity

$$F = f_1\, f_2 \otimes g_1\, g_2 = (-1)^{|f_2||g_1|}\,(f_1 \otimes g_1)\,(f_2 \otimes g_2) \tag{2.7}$$

in the endomorphism operad. Another shorthand notation will be introduced in equation (3.5) in Section 3.

We refer to [17, §§1.1, 1.2, 1.11] for the definitions of differential graded algebras (dgas) and dga maps as well as for differential graded coalgebras (dgcs), dgc maps and coalgebra homotopies. By an *ideal* $\mathfrak{a}$ of a dga A, we mean a two-sided differential ideal $\mathfrak{a} \lhd A$. We write augmentations as ε and coaugmentations as η; the augmentation ideal of a dga A is denoted by $\bar{A}$. A dga A is connected if it is $\mathbb{N}$-graded and $\eta_A\colon \Bbbk \to A^0$ is an isomorphism; it is simply connected if additionally $A^1 = 0$. A connected or simply connected dgc C is defined similarly.

For $n \geq 0$, we write

$$\mu_A^{[n]}\colon A^{\otimes n} \to A \tag{2.8}$$

for the iterated multiplication of a dga A, so that $\mu_A^{[0]} = \eta_A$, $\mu_A^{[1]} = 1_A$ and $\mu_A^{[2]} = \mu_A$. The iterations $\Delta^{[n]}$ are defined analogously. A dgc C is cocomplete if for any $c \in C$ there is an $n \geq 0$ such that $(1_C - \varepsilon_C)^{\otimes n}\Delta^{[n]}(c) = 0$. Any connected dgc is cocomplete.

Given two ideals $\mathfrak{a} \lhd A$ and $\mathfrak{b} \lhd B$ where A and B are dgas, we define the ideal

$$\mathfrak{a} \boxtimes \mathfrak{b} = \mathfrak{a} \otimes B + A \otimes \mathfrak{b} \lhd A \otimes B. \tag{2.9}$$

It is then clear how the ideal $\mathfrak{a}^{\boxtimes n} \lhd A^{\otimes n}$ is defined for $n \geq 1$; $\mathfrak{a}^{\boxtimes 0} = 0$.

We will make heavy use of the (reduced) bar construction

$$\mathbf{B}A = \bigoplus_{k \geq 0} \mathbf{B}_k A, \qquad \mathbf{B}_k A = (\mathbf{s}^{-1}\bar{A})^{\otimes k} \tag{2.10}$$

of an augmented dga A, which is a cocomplete coaugmented dgc, connected if A is simply connected, see [13, Sec. II.3] or [17, §1.6]. The canonical map

$$t_A\colon \mathbf{B}A \to \mathbf{B}_1 A = \mathbf{s}^{-1}\bar{A} \xrightarrow{\mathbf{s}} \bar{A} \hookrightarrow A \tag{2.11}$$

is a twisting cochain in the sense of the following definition.

For an augmented dga A and a coaugmented dgc C, the complex $\operatorname{Hom}(C, A)$ is an augmented dga with cup product

$$f \cup g = \mu_A\,(f \otimes g)\,\Delta_C, \tag{2.12}$$

unit element $\eta_A\,\varepsilon_C$ and augmentation $\varepsilon(f)=(\varepsilon_A\,f\,\eta_C)(1)$. Note that for f, g as before and any dgc map $k\colon B\to C$ we have

$$(f\cup g)\,k=(f\,k)\cup(g\,k). \tag{2.13}$$

A *twisting cochain* is an element $t\in\operatorname{Hom}_1(C,A)$ such that

$$d(t)=t\cup t, \tag{2.14}$$

$$\varepsilon_A\,t=0 \qquad\text{and}\qquad t\,\eta_C=0. \tag{2.15}$$

Example 2.1. Let A and B be augmented dgas. The shuffle map

$$\nabla=\nabla_{A,B}\colon \mathbf{B}A\otimes\mathbf{B}B\to\mathbf{B}(A\otimes B) \tag{2.16}$$

is the dgc map with associated twisting cochain $t_A\otimes\eta_B\,\varepsilon_{\mathbf{B}B}+\eta_A\,\varepsilon_{\mathbf{B}A}\otimes t_B$,

$$[a_1|\dots|a_k]\otimes[b_1|\dots|b_l]\mapsto\begin{cases}a_1\otimes 1 & \text{if } k=1 \text{ and } l=0,\\ 1\otimes b_1 & \text{if } k=0 \text{ and } l=1,\\ 0 & \text{otherwise.}\end{cases} \tag{2.17}$$

The shuffle map is associative and also commutative in the sense that the diagram

$$\begin{array}{ccc}\mathbf{B}A\otimes\mathbf{B}B & \xrightarrow{\nabla_{A,B}} & \mathbf{B}(A\otimes B)\\ {\scriptstyle T_{\mathbf{B}A,\mathbf{B}B}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{B}T_{A,B}}\\ \mathbf{B}B\otimes\mathbf{B}A & \xrightarrow{\nabla_{B,A}} & \mathbf{B}(B\otimes A)\end{array} \tag{2.18}$$

commutes.

If A is commutative, then the composition

$$\mu_{\mathbf{B}A}=\mathbf{B}\mu_A\,\nabla_{A,A}\colon \mathbf{B}A\otimes\mathbf{B}A\to\mathbf{B}A \tag{2.19}$$

turns $\mathbf{B}A$ into a *dg bialgebra*, that is, into a coaugmented dgc with an associative product that is a morphism of dgcs.

An element $h\in\operatorname{Hom}_0(C,A)$ is a *twisting cochain homotopy* from the twisting cochain $t\colon C\to A$ to the twisting cochain $u\colon C\to A$, in symbols $h\colon t\simeq u$, if

$$d(h)=t\cup h-h\cup u, \tag{2.20}$$

$$\varepsilon_A\,h=\varepsilon_C \qquad\text{and}\qquad h\,\eta_C=\eta_A. \tag{2.21}$$

Let A be an augmented dga. For any cocomplete coaugmented dgc C, the assignment $f\mapsto t_Af$ sets up a bijection between the dgc maps $C\to\mathbf{B}A$ and the twisting cochains $C\to A$. A map $h\colon C\to\mathbf{B}A$ is a coalgebra homotopy from a dgc map $f\colon C\to\mathbf{B}A$ to another dgc map g if and only if $1+t_A\,h\in\operatorname{Hom}(C,A)$ is a twisting cochain homotopy from $t_A\,f$ to $t_A\,g$.

Let $h\colon C\to A$ be a twisting cochain homotopy, and let $\mathfrak{a}\lhd A$ be an ideal. If h is congruent to $1=\eta_A\,\varepsilon_C$ modulo $\mathfrak{a}\lhd A$, we say that h as well as the associated coalgebra homotopy $C\to\mathbf{B}A$ is $\mathfrak{a}$-*trivial*. By the first normalization condition (2.21) any twisting cochain homotopy $h\colon C\to A$ is $\bar{A}$-trivial.

Lemma 2.2. *Let $\mathfrak{a}\lhd A$ be an ideal, and let C be a cocomplete dgc. Being related by an $\mathfrak{a}$-trivial homotopy is an equivalence relation among twisting cochains $C\to A$. More precisely:*

(i) *Let $h\colon t\simeq u$ and $k\colon u\simeq v$ be $\mathfrak{a}$-trivial twisting cochain homotopies. Then $h\cup k$ is an $\mathfrak{a}$-trivial homotopy from t to v.*

(ii) *Let $h\colon t \simeq u$ be an $\mathfrak{a}$-trivial twisting cochain homotopy Then h is invertible in* $\operatorname{Hom}_0(C, A)$*, and its inverse*

$$h^{-1} = \sum_{n=0}^{\infty} (1-h)^{\cup n} \colon C \to A$$

is an $\mathfrak{a}$-trivial homotopy from u to t.

In particular, we may unambiguously speak of an "$\mathfrak{a}$-trivial homotopy between twisting cochains t and u" without specifying the direction of the homotopy.

Proof. The first part follows immediately from the definition of the cup product. Apart from the obvious $\mathfrak{a}$-triviality, the second claim is [17, §1.12]. □

3. Strongly homotopy multiplicative maps

Our discussed is based on the treatment in [17, §3.1] and [26, Sec. 1 (c)].

Let A and B be augmented dgas. By definition, a *strongly homotopy multiplicative (shm) map*[2] $f\colon A \Rightarrow B$ is a twisting cochain $f\colon \mathbf{B}A \to B$. We write the corresponding dgc map as $\mathbf{B}f\colon \mathbf{B}A \to \mathbf{B}B$. Following Munkholm [17, Appendix], we define for $n \geq 0$ the map[3]

$$f_{(n)}\colon \bar{A}^{\otimes n} \xrightarrow{(\mathbf{s}^{-1})^{\otimes n}} \mathbf{B}_n A \xrightarrow{f} B \tag{3.1}$$

of degree $1-n$ and extend it to $A^{\otimes n}$ by setting

$$f_{(1)}(1) = 1, \tag{3.2}$$

$$f_{(n)}(a_1 \otimes \cdots \otimes a_n) = 0 \quad \text{if } n \geq 2 \text{ and } a_k = 1 \text{ for some } k. \tag{3.3}$$

The twisting cochain conditions (2.14) and (2.15) for f translate into

$$f_{(0)} = \varepsilon_B\, f_{(n)} = 0, \tag{3.4}$$

$$d(f_{(n)})(a_\bullet) \overset{\varkappa}{=} \sum_{k=1}^{n-1} (-1)^k \left(f_{(k)}(a_\bullet)\, f_{(n-k)}(a_\bullet) - f_{(n-1)}(a_\bullet, a_k a_{k+1}, a_\bullet) \right) \tag{3.5}$$

for all $n \geq 1$. In (3.5) we have used the symbol $\overset{\varkappa}{=}$ to indicate the Koszul sign and also the notation $a_\bullet$ to denote a (possibly empty) sequence of a-variables, ordered by their indices. The length of the sequence is to be inferred from the context. For instance, $f_{(n-1)}(a_\bullet, a_k a_{k+1}, a_\bullet)$ stands for $f_{(n-1)}(a_1, \ldots, a_{k-1}, a_k a_{k+1}, a_{k+2}, \ldots, a_n)$.

We call a family of multilinear functions

$$f_{(n)}\colon A^{\otimes n} \to B \tag{3.6}$$

of degree $1-n$ satisfying (3.2)–(3.5) a *twisting family.* Twisting families correspond bijectively to shm maps $A \Rightarrow B$. Note that $f_{(1)}\colon A \to B$ is a chain map which is multiplicative up to homotopy since

$$d(f_{(2)}) = f_{(1)}\, \mu_A - \mu_B\, (f_{(1)} \otimes f_{(1)}). \tag{3.7}$$

Given an shm map $f\colon A \Rightarrow B$, we define

$$H^*(f) = H^*(f_{(1)})\colon H^*(A) \to H^*(B). \tag{3.8}$$

It is a morphism of graded algebras.

[2]We prefer the term "shm map" used by Munkholm over the nowadays more popular terminology "A_∞-map" because it pairs better with the "shc algebras" to be introduced in Section 5.

[3]This definition leads to a sign convention different from Wolf's [26].

Any dga morphism $f\colon A \to B$ induces an shm map $\tilde f\colon A \Rightarrow B$ with $\tilde f_{(1)} = f$ and $\tilde f_{(n)} = 0$ for $n \ge 2$. We call such an shm map *strict*. Note that $H^*(\tilde f) = H^*(f)$ in this case. We will not distinguish between a dga map and its induced strict shm map.

More generally, we say that an shm map $f\colon A \Rightarrow B$ is $\mathfrak{b}$-*strict* for some $\mathfrak{b} \lhd B$ if

$$f_{(n)} \equiv 0 \pmod{\mathfrak{b}} \qquad \text{for all } n \ge 2. \tag{3.9}$$

Then f is 0-strict if and only if it is strict, and every $f\colon A \Rightarrow B$ is $\bar B$-strict. Any $\mathfrak{b}$-strict shm map $f\colon A \Rightarrow B$ induces a strict map $A \to B/\mathfrak{b}$.

A twisting cochain homotopy $h\colon f \simeq g$ from an shm map $f\colon A \Rightarrow B$ to another shm map $g\colon A \Rightarrow B$ is called an *shm homotopy*. Based on h we define the maps

$$h_{(n)} = h\,(\mathbf{s}^{-1})^{\otimes n}\colon \bar A^{\otimes n} \to B \tag{3.10}$$

of degree $-n$ for $n \ge 0$ and extend them to $A^{\otimes n}$ by

$$h_{(n)}(a_1 \otimes \cdots \otimes a_n) = 0 \quad \text{if } a_k = 1 \text{ for some } k. \tag{3.11}$$

The normalization conditions (2.21) mean

$$h_{(0)} = \eta_B \qquad \text{and} \qquad \varepsilon_B\, h_{(n)} = 0 \quad \text{for } n \ge 1, \tag{3.12}$$

and condition (2.20) is equivalent to

$$\begin{aligned} d(h_{(n)})(a_\bullet) &\overset{\varkappa}{=} \sum_{k=1}^{n-1} (-1)^k\, h_{(n-1)}(a_\bullet, a_k a_{k+1}, a_\bullet) \\ &\quad + \sum_{k=0}^{n} \Bigl(f_{(k)}(a_\bullet)\, h_{(n-k)}(a_\bullet) - (-1)^k\, h_{(k)}(a_\bullet)\, g_{(n-k)}(a_\bullet) \Bigr) \end{aligned} \tag{3.13}$$

for all $n \ge 0$.

We call a family of multilinear functions

$$h_{(n)}\colon A^{\otimes n} \to B \tag{3.14}$$

of degree $-n$ satisfying (3.11)–(3.13) a *twisting homotopy family* from the twisting family $f_{(n)}$ to $g_{(n)}$. Twisting homotopy families correspond bijectively to homotopies between twisting cochains. We also write $Bh\colon \mathbf{B}A \to \mathbf{B}B$ for the coalgebra homotopy induced by the twisting homotopy $h\colon \mathbf{B}A \to B$.

The twisting homotopy family $h_{(n)}$ as above is called $\mathfrak{b}$-trivial for some $\mathfrak{b} \lhd B$ if the twisting homotopy $\mathbf{B}A \to B$ is so. Equivalently,

$$h_{(n)} \equiv 0 \pmod{\mathfrak{b}} \qquad \text{for all } n \ge 1. \tag{3.15}$$

Let $f\colon A \Rightarrow B$ and $g\colon B \Rightarrow C$ be shm maps. We define the composition

$$g \circ f\colon A \Rightarrow C \tag{3.16}$$

to be the twisting cochain $g\,\mathbf{B}f$ associated to the dgc map $\mathbf{B}g\,\mathbf{B}f\colon \mathbf{B}A \to \mathbf{B}C$. The corresponding twisting cochain family is given by

$$(g \circ f)_{(n)}(a_\bullet) \overset{\varkappa}{=} \sum_{k \ge 1} \sum_{i_1 + \cdots + i_k = n} (-1)^{\varepsilon}\, g_{(n)}\bigl(f_{(i_1)}(a_\bullet), \dots, f_{(i_k)}(a_\bullet)\bigr) \tag{3.17}$$

for $n \ge 0$ where the second sum is over all decompositions of n into k positive integers and

$$\varepsilon = \sum_{s=1}^{k} (k-s)(i_s - 1). \tag{3.18}$$

The composition of an shm map and an shm homotopy is similarly defined as the shm homotopy associated to the composition of the corresponding maps between bar constructions.

Lemma 3.1.

(i) *Let* $f\colon A \Rightarrow B$ *be a* $\mathfrak{b}$*-strict shm map, and let* $g\colon B \Rightarrow C$ *a* $\mathfrak{c}$*-strict shm map. If* $g_{(1)}(\mathfrak{b}) \subset \mathfrak{c}$*, then* $g \circ f$ *is* $\mathfrak{c}$*-strict.*
(ii) *Let* $h\colon C \to A$ *be an* $\mathfrak{a}$*-trivial twisting cochain homotopy, and let* $f\colon A \to B$ *be a* $\mathfrak{b}$*-strict shm map. If* $f_{(1)}(\mathfrak{a}) \subset \mathfrak{b}$*, then* $f \circ h$ *is* $\mathfrak{b}$*-trivial.*
(iii) *Let* $h\colon C \to A$ *be an* $\mathfrak{a}$*-trivial twisting cochain homotopy, and let* $g\colon D \to C$ *be a map of coaugmented dgcs. Then* $h \circ g$ *is* $\mathfrak{a}$*-trivial.*

Proof. The first two claims are readily verified, and the last one is trivial. □

4. Tensor products of shm maps

In this section, A, B, A' and B' denote augmented dgas. We write $a_\bullet \otimes b_\bullet$ for a sequence $a_1 \otimes b_1$, $a_2 \otimes b_2$, … in $A \otimes B$ whose length is given by the context.

Let $f\colon A \Rightarrow A'$ be an shm map, and let $g\colon B \to B'$ be a dga map. Then

$$(f \otimes g)_{(n)}(a_\bullet \otimes b_\bullet) \stackrel{\asymp}{=} f_{(n)}(a_\bullet) \otimes g\,\mu^{[n]}(b_\bullet) \tag{4.1}$$

is a twisting cochain family, hence defines an shm map

$$f \otimes g\colon A \otimes B \Rightarrow A' \otimes B'. \tag{4.2}$$

If h is an $\mathfrak{a}$-trivial homotopy from f to another shm map $\tilde{f}$, then

$$(h \otimes g)_{(n)}(a_\bullet \otimes b_\bullet) \stackrel{\asymp}{=} h_{(n)}(a_\bullet) \otimes g\,\mu^{[n]}(b_\bullet) \tag{4.3}$$

defines an $\mathfrak{a} \otimes B$-trivial shm homotopy $h \otimes g$ from $f \otimes g$ to $\tilde{f} \otimes g$.

Similarly, if $f\colon A \to A'$ is a dga map and $g\colon B \Rightarrow B'$ an shm map, then

$$(f \otimes g)_{(n)}(a_\bullet \otimes b_\bullet) \stackrel{\asymp}{=} f\,\mu^{[n]}(a_\bullet) \otimes g_{(n)}(b_\bullet) \tag{4.4}$$

defines an shm map

$$f \otimes g\colon A \otimes B \Rightarrow A' \otimes B'. \tag{4.5}$$

If h is a $\mathfrak{b}$-trivial homotopy from g to another shm map $\tilde{g}$, then

$$(f \otimes h)_{(n)}(a_\bullet \otimes b_\bullet) \stackrel{\asymp}{=} f\,\mu^{[n]}(a_\bullet) \otimes h_{(n)}(b_\bullet) \tag{4.6}$$

defines an $A \otimes \mathfrak{b}$-trivial shm homotopy $f \otimes h$ from $f \otimes g$ to $f \otimes \tilde{g}$.

Now let both $f\colon A \Rightarrow A'$ and $g\colon B \Rightarrow B'$ be shm maps. Then the two shm maps

$$(f \otimes 1_{B'}) \circ (1_A \otimes g) \qquad \text{and} \qquad (1_{A'} \otimes g) \circ (f \otimes 1_B) \tag{4.7}$$

are not equal in general. In fact, for any $n \geq 0$ one has

$$\bigl((f \otimes 1) \circ (1 \otimes g)\bigr)_{(n)}(a_\bullet \otimes b_\bullet) \stackrel{\asymp}{=} \sum_{l \geq 1} \sum_{j_1 + \dots + j_l = n} (-1)^\varepsilon\, F \otimes G \tag{4.8}$$

where the sum is over all decompositions of n into l positive integers and

$$F = f_{(l)}\bigl(\mu^{[j_1]}(a_\bullet), \dots, \mu^{[j_l]}(a_\bullet)\bigr), \tag{4.9}$$

$$G = \mu^{[l]}\bigl(g_{(j_1)}(b_\bullet), \dots, g_{(j_l)}(b_\bullet)\bigr), \tag{4.10}$$

$$\varepsilon = \sum_{t=1}^{l} (l - t)(j_t - 1), \tag{4.11}$$

compare (3.17) and (3.18), while

$$\big((1\otimes g)\circ(f\otimes 1)\big)_{(n)}(a_\bullet\otimes b_\bullet)\overset{\varkappa}{=}\sum_{k\geq 1}\sum_{i_1+\cdots+i_k=n}(-1)^{\varepsilon}\,F\otimes G \tag{4.12}$$

where the sum is analogously over all decompositions of n into k positive integers and

$$F=\mu^{[k]}\left(f_{(i_1)}(a_\bullet),\dots,f_{(i_k)}(a_\bullet)\right), \tag{4.13}$$

$$G=g_{(k)}\left(\mu^{[i_1]}(b_\bullet),\dots,\mu^{[i_k]}(b_\bullet)\right), \tag{4.14}$$

$$\varepsilon=\sum_{s=1}^{k}(s-1)(i_s-1). \tag{4.15}$$

Note that if f or g is strict, then (4.8) and (4.12) coincide and agree with the formulas given previously. Following Munkholm [17, Prop. 3.3] we define

$$f\otimes g=(f\otimes 1)\circ(1\otimes g) \tag{4.16}$$

in the general case. We compare it to the other composition.

Proposition 4.1. *Assume that f is $\mathfrak{a}$-strict for some $\mathfrak{a}\lhd A'$ and that g is $\mathfrak{b}$-strict for some $\mathfrak{b}\lhd B'$. Then the two shm maps*

$$f\otimes g=(f\otimes 1)\circ(1\otimes g)\qquad\text{and}\qquad(1\otimes g)\circ(f\otimes 1)$$

are homotopic via an $\mathfrak{a}\otimes\mathfrak{b}$-trivial homotopy. In particular, if f or g is strict, then the two compositions agree.

Proof. Instead of using Munkholm's theory of trivialized extensions [17, Sec. 2], we exhibit an explicit homotopy from $(1\otimes g)\circ(f\otimes 1)$ to $(f\otimes 1)\circ(1\otimes g)$. It is given by $h_{(0)}=\eta_{A'}\otimes\eta_{B'}$ and

$$h_{(n)}(a_\bullet\otimes b_\bullet)\overset{\varkappa}{=}\sum_{k,l\geq 1}\sum_{\substack{i_1+\cdots+i_k+\\ j_1+\cdots+j_l=n}}(-1)^{\varepsilon}\,F\otimes G \tag{4.17}$$

for $n\geq 1$, where the second sum is over all decompositions of n into $k+l$ positive integers,

$$F=\mu^{[k]}\Big(f_{(i_1)}(a_\bullet),\dots,f_{(i_{k-1})}(a_\bullet),f_{(i_k+l)}\big(a_\bullet,\mu^{[j_1]}(a_\bullet),\dots,\mu^{[j_l]}(a_\bullet)\big)\Big), \tag{4.18}$$

$$G=\mu^{[l]}\Big(g_{(k+j_1)}\big(\mu^{[i_1]}(b_\bullet),\dots,\mu^{[i_k]}(b_\bullet),b_\bullet\big),g_{(j_2)}(b_\bullet),\dots,g_{(j_l)}(b_\bullet)\Big), \tag{4.19}$$

$$\varepsilon=\sum_{s=1}^{k}s\,(i_s-1)+\sum_{t=1}^{l}(l-t)(j_t-1)+k\,(l-1)+1. \tag{4.20}$$

Verifying that h is a homotopy as claimed is lengthy, but elementary, see Appendix A. That the homotopy is $\mathfrak{a}\otimes\mathfrak{b}$-trivial follows from the assumptions on f and g and the inequalities $i_k+l\geq 2$ and $k+j_1\geq 2$. In particular, h takes values in $\bar{A}'\boxtimes\bar{B}'\supset\bar{A}'\otimes\bar{B}'$ since f and g are $\bar{A}$-strict and $\bar{B}$-strict, respectively.

Let us verify the normalization condition (3.11). Assume that $a_i=b_i=1$ for some i and consider a term $F\otimes G$ of the sum (4.17). Let m be the index such that a_i appears is the m-th f-term of F. If $i_s>1$ or $s=k$, this term vanishes by (3.5). Otherwise, the product inside $g_{(k+j_1)}$ containing b_m is b_m itself, so that this term vanishes again by (3.5). In any case we have $F\otimes G=0$.

The last part of the statement is the special case $\mathfrak{a} = 0$ or $\mathfrak{b} = 0$ and already known from the explicit formulas gives earlier. □

Formula (4.17) looks as follows in small degrees.

$$h_{(1)} = 0, \tag{4.21}$$

$$h_{(2)} \overset{\varkappa}{=} -f_{(2)}(a_1, a_2) \otimes g_{(2)}(b_1, b_2), \tag{4.22}$$

$$\begin{aligned} h_{(3)} \overset{\varkappa}{=} &-f_{(1)}(a_1)\, f_{(2)}(a_2, a_3) \otimes g_{(2)}(b_1, b_2, b_3) \\ &+ f_{(3)}(a_1, a_2, a_3) \otimes g_{(2)}(b_1, b_2)\, g_{(1)}(b_3) \\ &+ f_{(3)}(a_1, a_2, a_3) \otimes g_{(2)}(b_1\, b_2, b_3) \\ &+ f_{(2)}(a_1, a_2\, a_3) \otimes g_{(3)}(b_1, b_2, b_3). \end{aligned} \tag{4.23}$$

Remark 4.2. The summands appearing in (4.8) and (4.12) are in bijection with the vertices of an $(n-1)$-dimensional cube. For example, the vertex of $[0,1]^{n-1}$ corresponding to the decomposition $i_1 + \cdots + i_k = n$ is given by

$$\bigl(\overbrace{0, \ldots, 0, 1}^{i_1}, \ldots, \overbrace{0, \ldots, 0, 1}^{i_{k-1}}, \overbrace{0, \ldots, 0}^{i_k - 1}\bigr). \tag{4.24}$$

Similarly, the summands appearing in (4.17) are in bijection with the edges of an $(n-1)$-dimensional cube. Here the summand corresponding to the decomposition $i_1 + \cdots + i_k + j_1 + \cdots + j_l = n$ is identified with the edge

$$\bigl(\overbrace{0, \ldots, 0, 1}^{i_1}, \ldots, \overbrace{0, \ldots, 0, 1}^{i_{k-1}}, \overbrace{0, \ldots, 0}^{i_k - 1}, *, \underbrace{0, \ldots, 0}_{j_1 - 1}, \underbrace{1, 0, \ldots, 0}_{j_2}, \ldots, \underbrace{1, 0, \ldots, 0}_{j_l}\bigr), \tag{4.25}$$

where "$*$" denotes the free parameter.

Corollary 4.3. *Let $f_1 \colon A_0 \Rightarrow A_1$, $f_2 \colon A_1 \Rightarrow A_2$, $g_1 \colon B_0 \Rightarrow B_1$ and $g_2 \colon B_1 \Rightarrow B_2$ be shm maps. Assume that f_1 is $\mathfrak{a}_1$-trivial, f_2 $\mathfrak{a}_2$-trivial and g_2 $\mathfrak{b}_2$-trivial and that $(f_2)_{(1)}(\mathfrak{a}_1) \subset \mathfrak{a}_2$ for ideals $\mathfrak{a}_1 \lhd A_1$, $\mathfrak{a}_2 \lhd A_2$ and $\mathfrak{b}_2 \lhd B_2$. Then the two shm maps*

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) \qquad \text{and} \qquad (f_2 \circ f_1) \otimes (g_2 \circ g_1)$$

are homotopic via an $\mathfrak{a}_2 \otimes \mathfrak{b}_2$-trivial homotopy. If f_1 or g_2 are strict, then the two maps agree.

Proof. This follows by writing the maps as

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) = (f_2 \otimes 1) \circ (1 \otimes g_2) \circ (f_1 \otimes 1) \circ (1 \otimes g_1) \tag{4.26}$$

$$(f_2 \circ f_1) \otimes (g_2 \circ g_1) = (f_2 \otimes 1) \circ (f_1 \otimes 1) \circ (1 \otimes g_2) \circ (1 \otimes g_1) \tag{4.27}$$

and applying Proposition 4.1 and Lemma 3.1. The second equality above is a consequence of the formulas (4.1), (4.4) and (3.17). □

Lemma 4.4. *The shuffle map is natural with respect to shm maps. In other words, the diagram*

$$\begin{array}{ccc} \mathbf{B}A \otimes \mathbf{B}B & \xrightarrow{\nabla} & \mathbf{B}(A \otimes B) \\ {\scriptstyle \mathbf{B}f \otimes \mathbf{B}g}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{B}(f \otimes g)} \\ \mathbf{B}A' \otimes \mathbf{B}B' & \xrightarrow{\nabla} & \mathbf{B}(A' \otimes B') \end{array}$$

commutes for all shm maps $f \colon A \Rightarrow A'$ and $g \colon B \Rightarrow B'$.

Proof. Since all morphisms involved are dgc maps and the bar construction cocomplete, it suffices to compare the associated twisting cochains. Let $\boldsymbol{a}\otimes\boldsymbol{b} = [a_1|\dots|a_k]\otimes[b_1|\dots|b_l]\in \mathbf{B}_k A\otimes\mathbf{B}_l B$.

Assume $g = 1_B$. Then both twisting cochains vanish on $\boldsymbol{a}\otimes\boldsymbol{b}$ if $k\ge 1$ and $l\ge 1$. For $l=0$ both twisting cochains yield $f(\boldsymbol{a})\otimes 1$, and for $k=0$ they give $1\otimes b_1$ if $l=1$ and 0 otherwise, compare Example 2.1.

The case $f = 1_A$ is analogous, and the general case follows by combining the two and using the definition (4.16). □

Now let $f_i\colon A_i \Rightarrow B_i$ be a family of shm maps, $1\le i\le m$. Generalizing (4.16), we define the shm map

$$\text{(4.28)}\quad f_1\otimes\dots\otimes f_m = (f_1\otimes 1\otimes\dots\otimes 1)\circ(1\otimes f_2\otimes 1\otimes\dots\otimes 1)\circ\dots\circ(1\otimes\dots\otimes 1\otimes f_m).$$

If one of the maps is instead an shm homotopy $f_i = h$, we use the same definition. The resulting map is an shm homotopy in this case. We observe that this convention is compatible with the definitions (4.3) and (4.6).

Lemma 4.5. *Let $h\colon A\to B$ be an shm homotopy.*

(i) *For any dga map $f\colon A'\to B'$ we have*

$$h\otimes f = (1_B\otimes f)\circ(h\otimes 1_{A'}) \qquad\textit{and}\qquad f\otimes h = (1_{B'}\otimes h)\circ(f\otimes 1_A).$$

(ii) *For any shm map $g\colon C\to A$ and any dga D we have*

$$(h\otimes 1_D)\circ(g\otimes 1_D) = (h\circ g)\otimes 1_D.$$

Proof. The first part follows from inspection of the formulas (4.3) and (4.6). The second claim additionally uses that formula (3.17) remains valid for the shm homotopy h instead of the shm map f. □

5. Strongly homotopy commutative algebras

Let A be an augmented dga. According to Stasheff–Halperin [23, Def. 8], A is a *strongly homotopy commutative (shc) algebra* if

(i) the multiplication map $\mu_A\colon A\otimes A\to A$ extends to an shm morphism

$$\Phi\colon A\otimes A\to A.$$

Munkholm [17, Def. 4.1] additionally requires the following:

(ii) The map η_A is a unit for Φ, that is,

$$\Phi\circ(1_A\otimes\eta_A) = \Phi\circ(\eta_A\otimes 1_A) = 1_A\colon A\Rightarrow A.$$

(iii) The shm map Φ is homotopy associative, that is,

$$\Phi\circ(\Phi\otimes 1_A)\simeq\Phi\circ(1_A\otimes\Phi)\colon A\otimes A\otimes A\Rightarrow A.$$

(iv) The map Φ is homotopy commutative, that is,

$$\Phi\circ T_{A,A}\simeq\Phi\colon A\otimes A\Rightarrow A.$$

Whenever we speak of an shc algebra, we mean one satisfying all four properties unless otherwise indicated. Any commutative dga is canonically an shc algebra.

Let A and B be shc algebras, and let $\mathfrak{b} \lhd B$. A *morphism of shc algebras* is an shm map $f\colon A \Rightarrow B$ such that the diagram

$$\begin{array}{ccc} A\otimes A & \overset{f\otimes f}{\Longrightarrow} & B\otimes B \\ {\scriptstyle\Phi_A}\Downarrow & & \Downarrow{\scriptstyle\Phi_B} \\ A & \overset{f}{\Longrightarrow} & B \end{array} \tag{5.1}$$

commutes up to homotopy.[4] It is called $\mathfrak{b}$-*strict* if it is so as an shm map, and $\mathfrak{b}$-*natural* if there is a $\mathfrak{b}$-trivial homotopy making (5.1) commute.

Recall from [17, Prop. 4.2] that the tensor product of two shc algebras A and B is again an shc algebra with structure map

$$\Phi_{A\otimes B}\colon A\otimes B\otimes A\otimes B \xrightarrow{1\otimes T_{B,A}\otimes 1} A\otimes A\otimes B\otimes B \overset{\Phi_A\otimes\Phi_B}{\Longrightarrow} A\otimes B. \tag{5.2}$$

The following is a variant of the result just cited.

Lemma 5.1. *Let $f_i\colon A_i \to B_i$ be strict $\mathfrak{b}_i$-natural shc maps, $i = 1, 2$. Then $f_1 \otimes f_2\colon A_1\otimes A_2 \to B_1\otimes B_2$ is a strict $\mathfrak{b}_1 \boxtimes \mathfrak{b}_2$-natural shc map.*

Proof. We have to show that there is a $\mathfrak{b}_1 \boxtimes \mathfrak{b}_2$-trivial homotopy for the diagram (5.1), which in the present setting reads

$$\begin{array}{ccc} A_1\otimes A_2\otimes A_1\otimes A_2 & \xrightarrow{f_1\otimes f_2\otimes f_1\otimes f_2} & B_1\otimes B_2\otimes B_1\otimes B_2 \\ \downarrow{\scriptstyle 1\otimes T\otimes 1} & & \downarrow{\scriptstyle 1\otimes T\otimes 1} \\ A_1\otimes A_1\otimes A_2\otimes A_2 & \xrightarrow{f_1\otimes f_1\otimes f_2\otimes f_2} & B_1\otimes B_1\otimes B_2\otimes B_2 \\ \downarrow{\scriptstyle 1\otimes 1\otimes \Phi} & & \downarrow{\scriptstyle 1\otimes 1\otimes\Phi} \\ A_1\otimes A_1\otimes A_2 & \xrightarrow{f_1\otimes f_1\otimes f_2} & B_1\otimes B_1\otimes B_2 \\ \downarrow{\scriptstyle \Phi\otimes 1} & & \downarrow{\scriptstyle\Phi\otimes 1} \\ A_1\otimes A_2 & \xrightarrow{f_1\otimes f_2} & B_1\otimes B_2. \end{array} \tag{5.3}$$

Since f_1 and f_2 are strict, the top square commutes. If h_i denotes a $\mathfrak{b}_i$-trivial naturality homotopy for f_i, then $f_1\otimes f_1\otimes h_2$ is a $B_1\otimes B_1\otimes\mathfrak{b}_2$-natural homotopy making the middle diagram commute, and $h_1\otimes f_2$ is a $\mathfrak{b}_1\otimes B_2$-natural one for the bottom square. Hence the cup product of

$$(\Phi\otimes 1)\circ(f_1\otimes f_1\otimes h_2)\circ(1\otimes T\otimes 1) \tag{5.4}$$

and

$$(h_1\otimes f_2)\circ(1\otimes 1\otimes\Phi)\circ(1\otimes T\otimes 1) \tag{5.5}$$

yields the required homotopy by Lemmas 3.1 and 2.2 (i). □

Let A be an shc algebra with structure map $\Phi\colon A\otimes A \Rightarrow A$. Following [17, p. 30], we define the shm map

$$\Phi^{[n]}\colon A^{\otimes n} \Rightarrow A \tag{5.6}$$

for $n \ge 0$ by

$$\Phi^{[0]} = \eta_A, \qquad \Phi^{[1]} = 1_A, \qquad \Phi^{[n+1]} = \Phi\circ(\Phi^{[n]}\otimes 1_A) \tag{5.7}$$

---

[4] Munkholm also requires the identity $f\circ\eta_A = \eta_B$. Given the normalization condition (2.15), this holds automatically as both maps necessarily represent 0 as twisting cochains $\Bbbk = \mathbf{B}\Bbbk \to B$.

for $n \geq 1$. Note that

$$\big(\Phi^{[n]}\big)_{(1)} = \mu_A^{[n]}. \tag{5.8}$$

If Φ is $\mathfrak{a}$-strict for some ideal $\mathfrak{a} \lhd A$, then so is $\Phi^{[n]}$ for any $n \geq 0$ by Lemma 3.1 (i).

For the next result, compare [17, Prop. 4.6].

Lemma 5.2. *Let A and B be shc algebras with ideals $\mathfrak{a} \lhd A$ and $\mathfrak{b} \lhd B$. Assume that Φ_A is $\mathfrak{a}$-strict and Φ_B $\mathfrak{b}$-strict. Let $f\colon A \Rightarrow B$ be a $\mathfrak{b}$-strict and $\mathfrak{b}$-natural map of shc algebras such that $f_{(1)}(\mathfrak{a}) \subset \mathfrak{b}$. Then the diagram*

$$\begin{array}{ccc} A^{\otimes n} & \overset{f^{\otimes n}}{\Longrightarrow} & B^{\otimes n} \\ {\scriptstyle \Phi^{[n]}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \Phi^{[n]}} \\ A & \overset{f}{\Longrightarrow} & B \end{array}$$

commutes up to a $\mathfrak{b}$-trivial homotopy for any $n \geq 0$.

Proof. The claim is trivial for $n \leq 2$. Assume it proven for n and consider the diagram

$$\begin{array}{ccccc} A^{\otimes n} \otimes A & \overset{1^{\otimes n} \otimes f}{\Longrightarrow} & A^{\otimes n} \otimes B & \overset{f^{\otimes n} \otimes 1}{\Longrightarrow} & B^{\otimes n} \otimes B \\ {\scriptstyle \Phi_A^{[n]} \otimes 1}\Big\Downarrow & & {\scriptstyle \Phi_A^{[n]} \otimes 1}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \Phi_B^{[n]} \otimes 1} \\ A \otimes A & \overset{1 \otimes f}{\Longrightarrow} & A \otimes B & \overset{f \otimes 1}{\Longrightarrow} & B \otimes B \\ {\scriptstyle \Phi_A}\Big\Downarrow & & & & \Big\Downarrow{\scriptstyle \Phi_B} \\ A & & \overset{f}{\Longrightarrow} & & B. \end{array} \tag{5.9}$$

Since $\Phi_A^{[n]}$ is $\mathfrak{a}$-strict, the top left square commutes up to an $\mathfrak{a} \otimes B$-trivial homotopy by Proposition 4.1. The composition of this homotopy with $\Phi_B \circ (f \otimes 1)$ is $\mathfrak{b}$-trivial by Lemma 3.1 because Φ_B and f are $\mathfrak{b}$-strict and $f_{(1)}(\mathfrak{a}) \subset \mathfrak{b}$. By induction, the top right square commutes up to a $\mathfrak{b} \otimes B$-trivial homotopy, whose composition with Φ_B is $\mathfrak{b}$-trivial. The bottom rectangle finally commutes up to a $\mathfrak{b}$-trivial homotopy since f is $\mathfrak{b}$-natural. The claim follows. □

6. Homotopy Gerstenhaber algebras

6.1. **Definition of an hga.** Let A be an augmented dga. We say that A is a *homotopy Gerstenhaber algebra* (homotopy G-algebra, *hga*) if it is equipped with operations

$$E_k\colon A \otimes A^{\otimes k} \to A, \qquad a \otimes b_1 \otimes \cdots \otimes b_k \mapsto E_k(a; b_1, \ldots, b_k) \tag{6.1}$$

of degree $|E_k| = -k$ for $k \geq 1$. To state the properties they satisfy, it is convenient to use the additional operation $E_0 = 1_A$. All E_k with $k \geq 1$ take values in the augmentation ideal $\bar{A}$ and vanish if any argument is equal to 1. For $k \geq 1$ and all a, b_1, …, $b_k \in A$ one has

$$\begin{aligned} d(E_k)(a; b_\bullet) \overset{\varkappa}{=} b_1\, E_{k-1}(a; b_\bullet) + \sum_{m=1}^{k-1} (-1)^m\, E_{k-1}(a; b_\bullet, b_m b_{m+1}, b_\bullet) \\ + (-1)^k\, E_{k-1}(a; b_\bullet)\, b_k. \end{aligned} \tag{6.2}$$

For $k \geq 0$ and all $a_1, a_2, b_1, \ldots, b_k \in A$ one has

$$E_k(a_1 a_2; b_\bullet) \overset{\varkappa}{=} \sum_{k_1+k_2=k} E_{k_1}(a_1; b_\bullet)\, E_{k_2}(a_2; b_\bullet) \tag{6.3}$$

where the sum is over all decompositions of k into two non-negative integers. Finally, for $k, l \geq 0$ and all $a, b_1, \ldots, b_k, c_1, \ldots, c_l \in A$ one has

$$E_l(E_k(a; b_\bullet); c_\bullet) \overset{\varkappa}{=} \sum_{\substack{i_1+\dots+i_k+\\ j_0+\dots+j_k=l}} (-1)^\varepsilon\, E_n\bigl(a; \underbrace{c_\bullet}_{j_0}, E_{i_1}(b_1; c_\bullet), \underbrace{c_\bullet}_{j_1}, \ldots, \underbrace{c_\bullet}_{j_{k-1}}, E_{i_k}(b_k; c_\bullet), \underbrace{c_\bullet}_{j_k}\bigr), \tag{6.4}$$

where the sum is over all decompositions of l into $2k+1$ non-negative integers,

$$n = k + \sum_{t=0}^{k} j_t \tag{6.5}$$

and

$$\varepsilon = \sum_{s=1}^{k} i_s \Bigl(k + \sum_{t=s}^{k} j_t\Bigr) + \sum_{t=1}^{k} t\, j_t. \tag{6.6}$$

A *morphism of hgas* is a morphism $f\colon A \to B$ of augmented dgas that is compatible with the hga operations in the obvious way.

Given an hga A, we can define

$$\mathbf{E}_{kl}\colon \mathbf{B}_k A \otimes \mathbf{B}_l A = (\mathbf{s}^{-1} A)^{\otimes(k+l)} \to A \tag{6.7}$$

for $k, l \geq 0$ by

$$\mathbf{E}_{kl}\,(\mathbf{s}^{-1})^{\otimes(k+l)} = \begin{cases} 1_A & \text{if } k = 0 \text{ and } l = 1, \\ E_l & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

The functions $\mathbf{E}_{kl}$ assemble to a map

$$\mathbf{E}\colon \mathbf{B}A \otimes \mathbf{B}A \to A, \tag{6.9}$$

which is a twisting cochain by (6.2) and (6.3) together with the normalization conditions. Moreover, the identity (6.4) implies that the induced dgc map

$$\mu_{\mathbf{B}A}\colon \mathbf{B}A \otimes \mathbf{B}A \to \mathbf{B}A \tag{6.10}$$

is associative and therefore turns $\mathbf{B}A$ into a dg bialgebra. Conversely, a dg bialgebra structure on $\mathbf{B}A$ whose associated twisting cochain $\mathbf{E}$ is of the form (6.8) defines an hga structure on A with operations E_k.

Remark 6.1. Our hga operations are related to the braces originally defined by Voronov and Gerstenhaber [25, §8], [8, Sec. 1.2], [24, Sec. 3.2] by the identity

$$a\{b_1, \ldots, b_k\} = \mathbf{E}_{1k}([a] \otimes [b_1|\ldots|b_k]) = (-1)^\varepsilon\, E_k(a; b_1, \ldots, b_k) \tag{6.11}$$

for $k \geq 0$ where

$$\varepsilon = k\,|a| + \sum_{m=1}^{k} (k-m)\,|b_m|. \tag{6.12}$$

Our grading agrees with [24]; in [25] and [8] the degrees of the desuspended arguments are used.[5]

We observe that the $\cup_1$-*product*

$$a \cup_1 b = -E_1(a;b) \tag{6.13}$$

is a homotopy from the product with commuted factors to the standard one,

$$d(a \cup_1 b) + da \cup_1 b + (-1)^{|a|}\, a \cup_1 db = ab - (-1)^{|a||b|}\, ba \tag{6.14}$$

satisfying the *Hirsch formula*

$$ab \cup_1 c = (-1)^{|a|}\, a(b \cup_1 c) + (-1)^{|b||c|}(a \cup_1 c)\, b \tag{6.15}$$

for a, b, $c \in A$. Hence the cohomology $H^*(A)$ is (graded) commutative and in fact a Gerstenhaber algebra with bracket

$$\begin{aligned} \bigl\{[a],[b]\bigr\} &= \bigl[E_1(a;b) - (-1)^{(|a|-1)(|b|-1)} E_1(b;a)\bigr] \\ &= (-1)^{|a|-1}\bigl[\, a \cup_1 b + (-1)^{|a||b|}\, b \cup_1 a\,\bigr] \end{aligned} \tag{6.16}$$

for a, $b \in A$, see [25, §10].

The main examples of hgas are the cochains on a simplicial set, see Section 8.2, and the Hochschild cochains of an algebra, see the references given below. Any commutative dga is canonically an hga by setting $E_k = 0$ for all $k \geq 1$. The induced multiplication on $\mathbf{B}A$ then is the shuffle product discussed in Example 2.1.

We say that an hga A is *formal* if it is quasi-isomorphic to its cohomology $H^*(A)$, considered as an hga.

6.2. **Extended hgas.** In his study of $\cup_i$-products on $\mathbf{B}A$ for $i \geq 1$, Kadeishvili introduced operations E^i_{kl} for an hga A defined over $\Bbbk = \mathbb{Z}_2$ [14]. He called an hga equipped with these operations an 'extended hga'. We will only need the family $F_{kl} = E^1_{kl}$, but for coefficients in any $\Bbbk$. We therefore say that an hga is *extended* if it has a family of operations

$$F_{kl}\colon A^{\otimes k} \otimes A^{\otimes l} \to A \tag{6.17}$$

of degree $|F_{kl}| = -(k+l)$ for k, $l \geq 1$, satisfying the following conditions. All operations F_{kl} take values in the augmentation ideal $\bar{A}$ and vanish if any argument equals $1 \in A$. Their differential is given by

$$d(F_{kl})(a_\bullet; b_\bullet) = A_{kl} + (-1)^k\, B_{kl} \tag{6.18}$$

for all $a_1, \dots, a_k, b_1, \dots, b_l \in A$, where

$$A_{1l} = E_l(a_1; b_\bullet), \tag{6.19}$$

$$\begin{aligned} A_{kl} &\overset{\varkappa}{=} a_1\, F_{k-1,l}(a_\bullet; b_\bullet) + \sum_{i=1}^{k-1} (-1)^i\, F_{k-1,l}(a_\bullet, a_i a_{i+1}, a_\bullet; b_\bullet) \\ &\quad + \sum_{j=1}^{l} (-1)^k\, F_{k-1,j}(a_\bullet; b_\bullet)\, E_{l-j}(a_k; b_\bullet) \end{aligned} \tag{6.20}$$

for $k \geq 2$, and

$$B_{k1} \overset{\varkappa}{=} -E_k(b_1; a_\bullet), \tag{6.21}$$

[5]The signs given in eqs. (6) and (7) of [8] seem to be incorrect.

$$B_{kl} \stackrel{\approx}{=} \sum_{i=0}^{k-1} E_i(b_1; a_\bullet)\, F_{k-i,l-1}(a_\bullet; b_\bullet) + \sum_{j=1}^{l-1} (-1)^j\, F_{k,l-1}(a_\bullet; b_\bullet, b_j b_{j+1}, b_\bullet) + (-1)^l\, F_{k,l-1}(a_\bullet; b_\bullet)\, b_l \tag{6.22}$$

for $l \ge 2$, compare [14, Def. 2].

In particular, the operation $\cup_2 = -F_{11}$ is a $\cup_2$-*product* for A in the sense that

$$d(\cup_2)(a; b) = a \cup_1 b + (-1)^{|a||b|}\, b \cup_1 a \tag{6.23}$$

for all a, $b \in A$. This implies that the Gerstenhaber bracket in $H^*(A)$ is trivial.

A morphism of extended hgas is a morphism of hgas that commutes with all operations F_{kl}, k, $l \ge 1$.

The following observation will be used in Section 9.3.

Lemma 6.2. *Let $f\colon A \to B$ be a morphism of hgas where A is extended and B a commutative graded algebra, for example $B = H^*(A)$. Then for any cocycles a, $b \in A$, the value $f(a \cup_2 b)$ depends only on the cohomology classes of a and b.*

Proof. We have to show that $f(a \cup_2 b)$ vanishes if one cocycle is a coboundary. If $a = dc$, then

$$a \cup_2 b = d\big(c \cup_2 b\big) - a \cup_1 b - (-1)^{|a||b|} b \cup_1 a \tag{6.24}$$

maps to $0 \in B$ since f vanishes on coboundaries and on $\cup_1$-products. The same argument works for b. □

6.3. **Extended hgas as shc algebras.** We will need the following result.

Theorem 6.3. *Let A be an extended hga, and let $\mathfrak{a} \lhd A$ be the ideal generated by the values of all operations E_k with $k \ge 1$ as well as those of all operations F_{kl} with $(k, l) \ne (1, 1)$.*

(i) *The extended hga A is canonically an shc algebra. The structure maps Φ, h^a and h^c commute with morphisms of extended hgas.*
(ii) *The shm map Φ is $\mathfrak{a}$-strict. More generally, all iterations $\Phi^{[n]}$ with $n \ge 0$ as well as the composition $\Phi \circ (1 \otimes \Phi)$ are $\mathfrak{a}$-strict.*
(iii) *The homotopy h^a is $\mathfrak{a}$-trivial.*
(iv) *Modulo $\mathfrak{a}$, we have for any $n \ge 0$ and any $a_\bullet$, $b_\bullet \in A$ the congruence*

$$h^c_{(n)}(a_\bullet \otimes b_\bullet) \equiv \begin{cases} 1 & \text{if } n = 0, \\ \pm b_1\,(a_2 \cup_2 b_1)\, a_2 & \text{if } n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The shc structure is constructed explicitly in the companion paper [7]. Inspection of the definition of Φ there shows that it is $\mathfrak{k}$-strict. The case $n = 0$ of the iteration is void, and for $n \ge 2$ it is a consequence of Lemma 3.1 (i) (observed already in Section 5), as is the case of the other composition. The statements about h^a and h^c follow again by looking at their definitions in [7]. □

7. Twisted tensor products

Let A be an augmented dga and C a coaugmented dgc. For any $f \in \operatorname{Hom}(C, A)$ we set

$$\delta_f = (1_C \otimes \mu_A)\,(1_C \otimes f \otimes 1_A)\,(\Delta_C \otimes 1_A)\colon C \otimes A \to C \otimes A. \tag{7.1}$$

If $t \in \operatorname{Hom}(C, A)$ is a twisting cochain, then

$$d_\otimes - \delta_t = \bigl(d_C \otimes 1_A + 1_C \otimes d_A\bigr) - \delta_t \tag{7.2}$$

is a differential on $C \otimes A$. The resulting complex is called a *twisted tensor product* and denoted by $C \otimes_t A$, compare [13, Def. II.1.4] or [12, Sec. 1.3].

Lemma 7.1. *Let $\mathfrak{a} \lhd A$, and let $h\colon C \to A$ be an $\mathfrak{a}$-trivial homotopy from the twisting cochain $t\colon C \to A$ to $\tilde{t}\colon C \to A$. Then the map*

$$\delta_h\colon C \otimes_{\tilde{t}} A \to C \otimes_t A$$

is an isomorphism of complexes, congruent to the identity map modulo $C \otimes \mathfrak{a}$.

Proof. The inverse of δ_h is given by $\delta_{h^{-1}}$, see [12, Cor. 1.4.2]. The congruence to the identity map follows directly from the $\mathfrak{a}$-triviality. □

Lemma 7.2. *Let $t\colon C \to A$ be a twisting cochain, and let $g\colon C' \to C$ be a map of dgcs. Then $t \circ g\colon C' \to A$ is a twisting cochain and*

$$g \otimes 1_A\colon C' \otimes_{t \circ g} A \to C \otimes_t A$$

is a chain map.

Proof. This follows directly from the definitions. □

Let $f\colon A \to B$ be a map of augmented dgas. Then $f \circ t_A\colon \mathbf{B}A \to B$ is a twisting cochain. The associated twisted tensor product

$$\mathbf{B}(\Bbbk, A, B) = \mathbf{B}A \otimes_{f \circ t_A} B \tag{7.3}$$

is the *one-sided bar construction*. Usually, the map f will be understood from the context and not indicated. We write the cohomology of the one-sided bar construction as the differential torsion product

$$\operatorname{Tor}^A(\Bbbk, B) = H^*\bigl(\mathbf{B}(\Bbbk, A, B)\bigr). \tag{7.4}$$

Note that this is just a notation; we are not concerned with whether the bar construction leads to a proper projective resolution in case $\Bbbk$ is not a field. However, if A and B have zero differentials, then (7.4) is the usual torsion product.

Given an shm map $g\colon B_1 \Rightarrow B_2$, we define

$$\Gamma_g\colon \mathbf{B}(\Bbbk, A, B_1) = \mathbf{B}A \otimes_{t_A} B_1 \to \mathbf{B}A \otimes_{g \circ t_A} B_2, \tag{7.5}$$

$$\Gamma_g\bigl([a_1|\dots|a_k] \otimes b\bigr) = \sum_{m=0}^{k} [a_1|\dots|a_m] \otimes \mathfrak{g}([a_{m+1}|\dots|a_k] \otimes b) \tag{7.6}$$

where for any $k \geq 0$ the map $\mathfrak{g}$ of degree 0 is defined as the composition

$$\mathfrak{g}\colon \mathbf{B}_k B_1 \otimes B_1 \xrightarrow{1^{\otimes k} \otimes \mathbf{s}^{-1}} \mathbf{B}_{k+1} B_1 \xrightarrow{g} B_2. \tag{7.7}$$

The following is essentially taken from [26, Thm. 7], where also a version of Lemma 7.2 for two-sided bar constructions is given.

Lemma 7.3. *Assume that $g\colon B_1 \Rightarrow B_2$ is $\mathfrak{b}$-strict for some $\mathfrak{b} \lhd B_2$. Then Γ_g as defined above is a chain map, congruent to $1_{\mathbf{B}A} \otimes g_{(1)}$ modulo $\mathbf{B}A \otimes \mathfrak{b}$.*

Proof. This is a direct computation. □

Remark 7.4. If all complexes involved are torsion-free over the principal ideal domain $\Bbbk$ and (including the bar constructions) bounded below and if the map g is a quasi-isomorphism, then the resulting maps in Lemmas 7.2 and 7.3 are quasi-isomorphisms. This follows from the Künneth theorem and a standard spectral sequence argument, compare the proof of Proposition 12.2 (i) below.

Assume now that $A \to A'$ is a morphism of hgas. It is convenient to introduce the map

$$\mathfrak{E}\colon A' \otimes \mathbf{B}A' \to A', \qquad a \otimes \boldsymbol{b} \mapsto \mathbf{E}([a], \boldsymbol{b}) \tag{7.8}$$

of degree 0. Following [15], we can then define the map

$$\circ\colon \mathbf{B}(\Bbbk, A, A') \otimes \mathbf{B}(\Bbbk, A, A') \to \mathbf{B}(\Bbbk, A, A'), \tag{7.9}$$

$$(\boldsymbol{a} \otimes a) \circ (\boldsymbol{b} \otimes b) \overset{\varkappa}{=} \sum_{m=0}^{l} \bigl(\boldsymbol{a} \circ [b_1|\dots|b_m]\bigr) \otimes \mathfrak{E}(a; [b_{m+1}, \dots, b_l])\, b$$

where $\boldsymbol{a} = [a_1|\dots|a_k]$, $\boldsymbol{b} = [b_1|\dots|b_l] \in \mathbf{B}A$ and a, $b \in A'$. Observe that the summand for $m = l$ is the componentwise product

$$(-1)^{|a||\boldsymbol{b}|}\, \boldsymbol{a} \circ \boldsymbol{b} \otimes a\, b. \tag{7.10}$$

Proposition 7.5 (Kadeishvili–Saneblidze)**.** *Assume the notation introduced above. Then $\mathbf{B}(\Bbbk, A, A')$ is naturally an augmented dga with unit $1_{\mathbf{B}A} \otimes 1_{A'}$, augmentation $\varepsilon_{\mathbf{B}A} \otimes \varepsilon_{A'}$ and product (7.9).*

Proof. In [15, Cor. 6.2, 7.2] this is only stated for simply connected hgas.[6] It is, however, a formal consequence of the defining properties of any hga. □

8. Simplicial sets

Our basic reference for this material is [16]. We write $[n] = \{0, 1, \dots, n\}$.

8.1. **Preliminaries.** Let X be a simplicial set. We call X *reduced* if X_0 is a singleton and 1-*reduced* if X_1 is a singleton. We abbreviate repeated face and degeneracy operators as

$$\partial_i^j = \partial_i \circ \dots \circ \partial_j, \qquad \partial_i^{i-1} = \mathrm{id}, \qquad s_I = s_{i_m} \circ \dots \circ s_{i_1} \tag{8.1}$$

for $i \le j$ and $I = \{i_1 < \dots < i_m\}$.

We write $C(X)$ and $C^*(X)$ for the normalized chain and cochain complex of X with coefficients in $\Bbbk$. Then $C(X)$ is a dgc with the Alexander–Whitney map as diagonal and augmentation induced by the unique map $X \to *$, and $C^*(X)$ is a dga with product $\Delta^*_{C(X)}$.

We say X has *polynomial cohomology* (with respect to the chosen coefficient ring $\Bbbk$) if $H^*(X)$ is a polynomial algebra on finitely many generators of positive even degrees. Note that X is of finite type over $\Bbbk$ in this case.

For $0 \le k \le n$ we define the "partial diagonal"

$$P^n_k\colon C_n(X) \to C_k(X) \otimes C_{n-k}(X), \tag{8.2}$$

$$\sigma \mapsto \partial^n_{k+1}\sigma \otimes \partial^{k-1}_0\sigma = \sigma(0, \dots, p) \otimes \sigma(p, \dots, n)$$

[6]Note that the definition of an hga in [15, Def. 7.1] uses Baues' convention (see Footnote 7) and differs from ours (as does the definition of the differential on the bar construction [15, p. 208]). This results in a product on the one-sided bar construction $\mathbf{B}(A', A, \Bbbk)$.

so that

$$\Delta c = \sum_{k=0}^{n} P^n_k(c) \tag{8.3}$$

for any $c \in C_n(X)$. Note that each P^n_k is well-defined on normalized chains.

For simplicial sets X and Y, the shuffle map

$$\nabla = \nabla^{X,Y} \colon C(X) \otimes C(Y) \to C(X \times Y) \tag{8.4}$$

is a map of dgcs and additionally commutative in the sense that the diagram

$$\begin{array}{ccc} C(X) \otimes C(Y) & \xrightarrow{\nabla^{X,Y}} & C(X \times Y) \\ {\scriptstyle T_{C(X),C(Y)}}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{X,Y}} \\ C(Y) \otimes C(X) & \xrightarrow{\nabla^{Y,X}} & C(Y \times X) \end{array} \tag{8.5}$$

commutes, where $\tau_{X,Y} \colon X \times Y \to Y \times X$ swaps the factors, *cf.* [6, Sec. 3.2].

8.2. **The extended hga structure on cochains.** Gerstenhaber and Voronov [8, Sec. 2.3] have constructed an hga structure on the non-normalized cochain complex of a simplicial set X, which descends to the normalized cochain complex $C^*(X)$. It can be given in terms of the interval cut operations

$$E_k = AW^*_{e_k} \tag{8.6}$$

corresponding to the surjections

$$e_k = (1, 2, 1, 3, 1, \ldots, 1, k+1, 1), \tag{8.7}$$

cf. [3, §1.6.6, Sec. 2]. The operations E_k vanish for $k \geq 1$ if any argument is of degree 0 and never return a non-zero cochain of degree 0. This implies that the normalization condition (2.15) is satisfied independently of the chosen augmentation $C^*(X) \to \Bbbk$. This hga structure generalizes the multiplication on $BC^*(X)$ previously defined by Baues [1, §IV.2] for 1-reduced X.[7]

Kadeishvili [14][8] observed that $C^*(X)$ is an extended hga with operations F_{kl} corresponding to the surjections

$$\begin{aligned} f_{kl} = (&k+1, 1, k+1, 2, k+1, \ldots, k+1, k, \\ &k+1, k, k+2, k, \ldots, k, k+l, k) \end{aligned} \tag{8.8}$$

for k, $l \geq 1$. The associated $\cup_2$-product is $\cup_2 = -AW^*_{(2,1,2,1)}$.

8.3. **Simplicial groups.** Let G be a simplicial group with multiplication μ. We write $1_p \in G$ for the identity element of the group of p-simplices. A *loop* in G is a 1-simplex $g \in G$ such that $\partial_0 g = \partial_1 g = 1_0$.

The dgc $C(G)$ is a dg bialgebra with unit given by the identity element of G and multiplication

$$C(G) \otimes C(G) \xrightarrow{\nabla^{G,G}} C(G \times G) \xrightarrow{\mu_*} C(G). \tag{8.9}$$

If G is commutative, then so is $C(G)$.

---

[7] More precisely, Baues' multiplication is obtained by transposing the factors of the product, so that $\mathbf{E}_{kl}$ vanishes for $l \neq 1$, except for $\mathbf{E}_{10}$. This also affects the components of the homotopy $\mathbf{F}$ from [7, Cor. 6.2].

[8] Kadeishvili's choice for f_{kl} [14, pp. 116, 123] does not lead to the formula (6.18) (or [14, Def. 2]) for $d(F_{kl})$, but to the one with a-variables and b-variables interchanged.

Similarly, if G acts on the simplicial set X, then $C(G)$ acts on $C(X)$. We write this action as $a * c$ for $a \in C(G)$ and $c \in C(X)$. If the G-action is trivial, then the $C(G)$-action factors through the augmentation $\varepsilon\colon C(G) \to \Bbbk$. (Remember that we use normalized chains.)

For any loop $g \in G$ and any $0 \le m \le n$ we define the map

$$A^g_m\colon C_n(X) \to C_{n+1}(X), \qquad \sigma \mapsto (s_{[n]\setminus m}\, g) \cdot s_m\sigma \tag{8.10}$$

(which is again well-defined on normalized chains). By the definition of the shuffle map we can write the action of the loop $g \in C(G)$ on $\sigma \in C(X)$ as

$$g * \sigma = \sum_{m=0}^{n} (-1)^m\, A^g_m(\sigma). \tag{8.11}$$

The diagonal of $C(X)$ is known to be $C(G)$-equivariant, *cf.* [6, Prop. 3.5]. For loops, a more refined statement is the following.

Lemma 8.1. *Assume that $g \in G$ is a loop, and let $\sigma \in X_n$. Then*

$$P^{n+1}_k\big(A^g_m(\sigma)\big) = \begin{cases} (-1)^k\,(1 \otimes A^g_{m-k})\, P^n_k(\sigma) & \text{if } k \le m, \\ (A^g_m \otimes 1)\, P^n_{k-1}(\sigma) & \text{if } k > m. \end{cases}$$

for any $0 \le m \le n$ and $0 \le k \le n+1$.

Proof. We have

$$\begin{aligned} P^{n+1}_k(A^g_m(\sigma)) &= \partial^{n+1}_{k+1}\, A^g_m(\sigma) \otimes \partial^{k-1}_0\, A^g_m(\sigma) \\ &= \big(\partial^{n+1}_{k+1}\, s_{[n]\setminus m}\, g\big) \cdot \big(\partial^{n+1}_{k+1}\, s_m\, \sigma\big) \otimes \big(\partial^{k-1}_0\, s_{[n]\setminus m}\, g\big) \cdot \big(\partial^{k-1}_0\, s_m\, \sigma\big). \end{aligned} \tag{8.12}$$

If $k \le m$, then

$$\partial^{n+1}_{k+1}\, s_{[n]\setminus m}\, g = 1 \in G_k, \tag{8.13}$$

hence

$$\begin{aligned} P^{n+1}_k(A^g_m(\sigma)) &= \partial^{n+1}_{k+1}\, s_m\, \sigma \otimes \big(\partial^{k-1}_0\, s_{[n]\setminus m}\, g\big) \cdot \big(\partial^{k-1}_0\, s_m\, \sigma\big) \\ &= \partial^{n}_{k+1}\sigma \otimes \big(s_{[n-k]\setminus m-k}\, g\big) \cdot \big(\partial^{k-1}_0\, s_m\, \sigma\big) \\ &= (-1)^k\,(1 \otimes A^g_{m-k})\, P^n_k(\sigma). \end{aligned} \tag{8.14}$$

In the case $k > m$ we similarly find

$$\partial^{k-1}_0\, s_{[n-1]\setminus m}\, g = 1 \in G_{n-k} \tag{8.15}$$

and

$$\begin{aligned} P^{n+1}_k(A^g_m(\sigma)) &= \big(\partial^{n+1}_{k+1}\, s_{[n]\setminus m}\, g\big) \cdot \big(\partial^{n+1}_{k+1}\, s_m\, \sigma\big) \otimes \partial^{k-1}_0\, s_m\, \sigma \\ &= \big(s_{[k]\setminus m}\, g\big) \cdot \big(s_m\, \partial^n_k \sigma\big) \otimes \partial^{k-1}_0\, s_m\, \sigma \\ &= (A^g_m \otimes 1)\, P^n_{k-1}(\sigma), \end{aligned} \tag{8.16}$$

as claimed. ∎

8.4. **Universal bundles.** The standard reference for this material is [16, §21], where the notation $BG = \overline{W}(G)$ and $EG = W(G)$ is used.

Let G be a simplicial group. Its classifying space is the simplicial set BG whose p-simplices are elements of the Cartesian product

$$[g_{p-1}, \dots, g_0] \in G_{p-1} \times \dots \times G_0 = BG_p. \tag{8.17}$$

It is always reduced (with unique vertex $b_0 \coloneqq [] \in BG_0$) and 1-reduced in case G is reduced. The simplices in the total space of the universal G-bundle $\pi\colon EG \to BG$ are similarly given by

$$e = \bigl(g_p, [g_{p-1}, \dots, g_0]\bigr) \in G_p \times BG_p = EG_p\,; \tag{8.18}$$

the map π is the obvious projection. We write $e_0 = (1, b_0) \in EG_0$ for the canonical basepoint, which projects onto b_0. See [16, pp. 71, 87] for the face and degeneracy maps of EG and BG. We consider EG as a left G-space via

$$h \cdot \bigl(g_p, [g_{p-1}, \dots, g_0]\bigr) = \bigl(g_p\, h^{-1}, [g_{p-1}, \dots, g_0]\bigr) \tag{8.19}$$

for $h \in G_p$.

There is a canonical map $S\colon EG \to EG$ of degree 1 given by

$$S\bigl(g_p, [g_{p-1}, \dots, g_0]\bigr) = \bigl(1_{p+1}, [g_p, g_{p-1}, \dots, g_0]\bigr), \tag{8.20}$$

cf. [16, p. 88]. For all $e \in EG_p$ one has

$$\partial_0 Se = e, \tag{8.21}$$

$$\partial_1 Se = e_0 \qquad \text{if } p = 0, \tag{8.22}$$

$$\partial_k Se = S\partial_{k-1} e \qquad \text{if } p > 0 \text{ and } k > 0. \tag{8.23}$$

This implies that S induces a chain homotopy on $C(EG)$, again called S, between the projection to e_0 and the identity on EG,

$$(dS + Sd)(e) = \begin{cases} e - e_0 & \text{if } p = 0, \\ e & \text{if } p > 0, \end{cases} \tag{8.24}$$

for any simplex $e \in EG$, and that it additionally satisfies

$$SS = 0 \qquad \text{and} \qquad Se_0 = 0, \tag{8.25}$$

compare [5, Prop. 2.7.1] or [6, Sec. 3.7].

Lemma 8.2. *Let $c \in C(EG)$ be of degree n.*

(i) *For any $0 \le k \le n+1$ one has*

$$P^{n+1}_k(Sc) = \begin{cases} e_0 \otimes Sc & \text{if } k = 0, \\ (S \otimes 1)\, P^n_{k-1}(c) & \text{if } k > 0. \end{cases}$$

(ii) *One has*

$$\Delta\, Sc = (S \otimes 1)\, \Delta c + e_0 \otimes Sc.$$

Proof. The first statement is immediate if $c = e_0$ or $k = 0$. For $n > 0$ and $k > 0$ it follows from the identities (8.21)–(8.23). Combining it with (8.3) gives the second claim, *cf.* [5, Prop. 2.7.1] or [6, Prop. 3.8]. □

8.5. **An Eilenberg–Moore theorem.** For this rest of this article we assume that $\Bbbk$ is a principal ideal domain.

The following result is suggested by work of Kadeishvili–Saneblidze [15, Cor. 6.2].

Proposition 8.3. *Let $F \overset{\iota}{\hookrightarrow} E \to B$ be a simplicial fibre bundle. If B is 1-reduced and of finite type over $\Bbbk$, then the map*

$$\mathbf{B}(\Bbbk, C^*(B), C^*(E)) \to C^*(F), \quad [\gamma_1|\dots|\gamma_k] \otimes \gamma \mapsto \begin{cases} \iota^*(\gamma) & \text{if } k = 0, \\ 0 & \text{otherwise} \end{cases} \tag{8.26}$$

is a quasi-isomorphism of dgas. In particular, there is an isomorphism of graded algebras

$$H^*(F) \cong \operatorname{Tor}^{C^*(B)}\bigl(\Bbbk, C^*(E)\bigr).$$

Proof. By the usual Eilenberg–Moore theorem, the map is a quasi-isomorphism of complexes. For field coefficients, we can refer to [22, Thm. 3.2]. For general $\Bbbk$, it follows by dualizing the homological quasi-isomorphism [9, Sec. 6]

$$C(F) \to \mathbf{\Omega}\bigl(\Bbbk, C(B), C(E)\bigr) \tag{8.27}$$

where the target is the one-sided cobar construction.

Let us recall the argument: If we write G for the structure group of the bundle $E \to B$, then $C(F)$ is a left $C(G)$-module. By the twisted Eilenberg–Zilber theorem [9, Sec. 4], there is a twisting cochain $t\colon C(B) \to C(G)$ and a homotopy equivalence

$$C(E) \simeq C(B) \otimes_t C(F) \tag{8.28}$$

of left $C(B)$-comodules. Under this isomorphism, the map $\iota_*\colon C(F) \to C(E)$ corresponds to the canonical inclusion of $C(F)$ into the twisted tensor product with the unique base point of B as first factor.

We therefore get a homotopy equivalence of complexes

$$\begin{aligned} \mathbf{\Omega}\bigl(\Bbbk, C(B), C(E)\bigr) &\simeq \mathbf{\Omega}\bigl(\Bbbk, C(B), C(B) \otimes_t C(F)\bigr) \\ &= \mathbf{\Omega}\bigl(\Bbbk, C(B), C(B)\bigr) \otimes_t C(F) \end{aligned} \tag{8.29}$$

between the one-sided cobar constructions, where we consider $\mathbf{\Omega}(\Bbbk, C(B), C(B))$ as a right $C(B)$-comodule, *cf.* [13, Def. II.5.1].

The canonical inclusion $\Bbbk \hookrightarrow \mathbf{\Omega}(\Bbbk, C(B), C(B))$ is a homotopy equivalence [13, Prop. II.5.2], and δ_t vanishes on its image. Because B is 1-reduced, a spectral sequence argument shows that the map $C(F) \to \mathbf{\Omega}(\Bbbk, C(B), C(B)) \otimes_t C(F)$ is a quasi-isomorphism of complexes, hence so is the natural map

$$C(F) \to \mathbf{\Omega}\bigl(\Bbbk, C(B), C(E)\bigr), \qquad c \mapsto 1 \otimes 1_{\mathbf{\Omega}C(B)} \otimes \iota_*(c). \tag{8.30}$$

Since B is of finite type, the canonical map

$$(\mathbf{s}^{-1}\bar{C}^*(B))^{\otimes k} \otimes C^*(E) \to \bigl(\Bbbk \otimes (\mathbf{s}^{-1}\bar{C}(B))^{\otimes k} \otimes C(E)\bigr)^* \tag{8.31}$$

is a quasi-isomorphisms for any $k \geq 0$ by the universal coefficient theorem, hence so is the composition

$$\mathbf{B}\bigl(\Bbbk, C^*(B), C^*(E)\bigr) \to \mathbf{\Omega}\bigl(\Bbbk, C(B), C(E)\bigr)^* \to C^*(F). \tag{8.32}$$

A look at Proposition 7.5 finally shows that the quasi-isomorphism is multiplicative because any cochain on B of positive degree restricts to 0 on F. (Recall that we are working with normalized cochains.) $\square$

If we define an increasing filtration on $\mathbf{B}(\Bbbk, C^*(B), C^*(E))$ by the length of elements, then we get an (Eilenberg–Moore) spectral sequence of algebras converging to $H^*(F)$ because the deformation terms in the product formula given in Proposition 7.5 lower the filtration degree. By the Künneth theorem, the second page of this spectral sequence is of the form

$$E_2 = \operatorname{Tor}^{H^*(B)}\bigl(\Bbbk, H^*(E)\bigr) \tag{8.33}$$

with the usual product on Tor, provided that $H^*(B)$ is free over $\Bbbk$.

Remark 8.4. Assume that the base B has polynomial cohomology, say $H^*(B) = \Bbbk[y_1, \ldots, y_n]$. Let $b_1, \ldots, b_n \in C^*(B)$ be representatives of the generators, and let

$$\bigwedge(x_1, \ldots, x_n) \tag{8.34}$$

be the exterior algebra on generators x_i of degrees $|x_i| = |y_i| - 1$. Since $\mathbf{B}C^*(Y)$ is a dg bialgebra and the elements $[b_i] \in \mathbf{B}C^*(Y)$ primitive, the assignment

$$\bigwedge(x_1, \ldots, x_n) \to \mathbf{B}C^*(Y), \qquad x_{i_1} \wedge \cdots \wedge x_{i_k} \mapsto [b_{i_1}] \circ \cdots \circ [b_{i_k}] \tag{8.35}$$

is a dgc map (but not multiplicative in general) and in fact a quasi-isomorphism. Evaluating the product from the left to the right shows that the associated twisting cochain t_{GM} is of the form $t_{\mathrm{GM}}(x_i) = b_i$ and

$$\begin{aligned} t_{\mathrm{GM}}(x_{i_1} \wedge \cdots \wedge x_{i_k}) &= E_1(\cdots E_1(E_1(b_{i_1}; b_{i_2}); b_{i_3}); \cdots; b_{i_k}) \\ &= (-1)^{k-1} \bigl(((b_{i_1} \cup_1 b_{i_2}) \cup_1 b_{i_3}) \cup_1 \cdots\bigr) \cup_1 b_{i_k} \end{aligned} \tag{8.36}$$

for $k \geq 2$ and $i_1 < \cdots < i_k$. A standard spectral sequence argument then implies that the twisted tensor product

$$\bigwedge(x_1, \ldots, x_n) \otimes_{t_{\mathrm{GM}}} C^*(E) \tag{8.37}$$

is quasi-isomorphic to $\mathbf{B}(\Bbbk, C^*(B), C^*(E))$ as a complex, hence computes $H^*(F)$ as a graded $\Bbbk$-module by the Eilenberg–Moore theorem. We thus recover the model constructed by Gugenheim–May [11, Example 2.2 & Thm. 3.3].

Lemma 8.5. *Let G be a connected simplicial group and $K \subset G$ a connected subgroup. Write $\check{G} \subset G$ for the reduced subgroup of simplices lying over $1 \in G_0$, and define $\check{K} \subset K$ analogously. Then the inclusion $\check{G}/\check{K} \hookrightarrow G/K$ is a homotopy equivalence, natural in the pair (G, K).*

Proof. The inclusions $\check{G} \hookrightarrow G$ and $\check{K} \hookrightarrow K$ are homotopy equivalences, compare [16, Thm. 12.5]. The long exact sequence of homotopy groups implies that the map $\check{G}/\check{K} \hookrightarrow G/K$ is also a homotopy equivalence. Injectivity follows from the identity $\check{K} = K \cap \check{G}$, and naturality is clear. □

Proposition 8.6. *Let G be a reduced simplicial group and K a reduced subgroup. There is an isomorphism of graded algebras*

$$H^*(G/K) \cong \operatorname{Tor}^{C^*(BG)}\bigl(\Bbbk, C^*(BK)\bigr),$$

natural with respect to maps of pairs (G, K).

Proof. The map $\pi\colon EG/K \to BG$ is a fibre bundle with fibre G/K. By Proposition 8.3, the dgas $C^*(G/K)$ and $\mathbf{B}(\Bbbk, C^*(BG), C^*(EG/K))$ are naturally quasi-isomorphic. The homotopy equivalence $BK = EK/K \to EG/K$ is a map over BG and induces a quasi-isomorphism

$$\mathbf{B}\big(\Bbbk, C^*(BG), C^*(EG/K)\big) \to \mathbf{B}\big(\Bbbk, C^*(BG), C^*(BK)\big), \tag{8.38}$$

which is multiplicative by the naturality of the hga structure on cochains. □

9. Homotopy Gerstenhaber formality of BT

9.1. **Dga formality.** Let T be a simplicial torus of rank n. By this we mean a commutative simplicial group T such that $H(T)$ is an exterior algebra on generators x_1, …, x_n of degree 1. For example, T can be the compact torus $(S^1)^n$, the algebraic torus $(\mathbb{C}^\times)^n$ or the simplicial group $B\mathbb{Z}^n$.

As mentioned in the introduction, Gugenheim–May [11, Thm. 4.1] have constructed a quasi-isomorphism of dgas

$$C^*(BT) \to H^*(BT) \tag{9.1}$$

annihilating all $\cup_1$-products. An alternative approach was given by the author in his doctoral dissertation [5, Prop. 2.2], see also [6, Prop. 5.3]. The goal of this section is to promote the latter construction to a quasi-isomorphism of hgas, that is, one that annihilates all operations E_k with $k \geq 1$. We will see that also all operations F_{kl} with the exception of the $\cup_2$-product are send to 0.

We write $\mathbf{\Lambda} = H(T)$ and $\mathbf{S} = H(BT)$. The latter is the commutative coalgebra on generators $y_i \in \mathbf{S}_2$ that correspond to the x_i's under transgression. The y_i's define a $\Bbbk$-basis y_α of $\mathbf{S}$ index by multi-indices $\alpha \in \mathbb{N}^n$. We also write $y_0 = 1$.

Let $t\colon \mathbf{S} \to \mathbf{\Lambda}$ be the (homological) twisting cochain that sends each y_i to x_i and vanishes in other degrees. The twisted tensor product

$$\mathbf{K} = \mathbf{\Lambda} \otimes_t \mathbf{S} \tag{9.2}$$

is the Koszul complex. It is a dgc with $\mathbf{\Lambda}$-equivariant diagonal given by the componentwise diagonals. For $a \in \mathbf{\Lambda}$ and $c \in \mathbf{S}$ we write $a \cdot c \in \mathbf{K}$ instead of $a \otimes c$, reflecting the $\mathbf{\Lambda}$-action. The differential on $\mathbf{K}$ is given by

$$d(a \cdot y_\alpha) = (-1)^{|\alpha|} \sum_i a \wedge x_i \cdot y_{\alpha|i} = \sum_i x_i \wedge a \cdot y_{\alpha|i} \tag{9.3}$$

where the sum runs over all i such that $\alpha_i > 0$, and $\alpha|i$ means that the i-th component of α is decreased by 1.

Let c_1, …, $c_n \in C_1(T)$ be linear combinations of loops in G representing the generators x_i. They define a quasi-isomorphism of dg bialgebras

$$\varphi\colon \mathbf{\Lambda} \to C(T), \qquad x_{i_1} \wedge \cdots \wedge x_{i_k} \mapsto c_{i_1} * \cdots * c_{i_k} \tag{9.4}$$

for $i_1 < \cdots < i_k$. Moreover, let $\pi\colon ET \to BT$ be the universal T-bundle. Note that $C(ET)$ is an $\mathbf{\Lambda}$-module via φ.

Our map (9.1) will be the transpose of a quasi-isomorphism $f\colon \mathbf{S} \to C(BT)$. The construction of the latter is based on a map

$$F\colon \mathbf{K} \to C(ET) \tag{9.5}$$

recursively defined by

$$F(1) = e_0, \tag{9.6}$$

$$F(a\cdot c) = a * F(c) \qquad \text{if } |a|>0, \tag{9.7}$$

$$F(c) = S\,F(dc) \qquad \text{if } |c|>0 \tag{9.8}$$

for $c \in \mathbf{S}$ and $a \in \mathbf{\Lambda}$, where S is the homotopy defined in (8.20).

Proposition 9.1. *The map F is a $\mathbf{\Lambda}$-equivariant quasi-isomorphism of dgcs.*

For the convenience of the reader, we adapt the proof given in [6, Prop. 4.3] to our slightly more general setting.[9]

Proof. It is clear from the definition that F commutes with the $\mathbf{\Lambda}$-action. To show that it is a chain map, we proceed by induction on the degree of $a\cdot y \in \mathbf{K}$. For $a \cdot y = 1$ this is obvious. For $|a|>0$ we have by equivariance and induction

$$\begin{aligned} d\,F(a\cdot y) &= d\big(\varphi(a) * F(y)\big) = \varphi(da) * F(y) + (-1)^{|a|}\,\varphi(a) * dF(y) \\ &= F\big(da\cdot y + (-1)^{|a|} a \cdot dy\big) = F\,d(a\cdot y). \end{aligned} \tag{9.9}$$

For $|y|>0$ we have by (8.24) and induction

$$d\,F(y) = d\,S\,F(dy) = F(dy) - S\,d\,F(dy) = F(dy). \tag{9.10}$$

To show that f is a map of coalgebras, we proceed once more by induction on $|a\cdot y|$, the case $a\cdot y = 1$ being trivial. If $|a|>0$, then again by equivariance and induction we have

$$\begin{aligned} \Delta F(a\cdot y) &= \Delta\big(\varphi(a) * F(y)\big) = \Delta\varphi(a) * \Delta F(y) = \Delta\varphi(a) * (F\otimes F)\Delta y \\ &= (F\otimes F)(\Delta a \cdot \Delta y) = (F\otimes F)\Delta(a\cdot y). \end{aligned} \tag{9.11}$$

For $\alpha \ne 0$ we therefore have by Lemma 8.2 (ii) that

$$\begin{aligned} \Delta\,F(y_\alpha) &= \Delta\,S\,F(dy_\alpha) = (S\otimes 1)\,\Delta\,F(dy_\alpha) + e_0 \otimes S\,F(dy_\alpha) \\ &= \sum_i (S\otimes 1)\,\Delta F(x_i \cdot y_{\alpha|i}) + F(1)\otimes F(y_\alpha), \end{aligned} \tag{9.12}$$

where the sum runs over the indices i such that $\alpha_i \ne 0$. Using again the equivariance of the Alexander–Whitney map and induction, we get

$$\begin{aligned} \Delta\,F(y_\alpha) &= \sum_i (S\otimes 1)\,\Delta c_i * \Delta\,F(y_{\alpha|i}) + F(1)\otimes F(y_\alpha) \\ &= \sum_i \sum_{\beta+\gamma=\alpha|i} (S\otimes 1)\,\Delta c_i * \big(F(y_\beta)\otimes F(y_\gamma)\big) + F(1)\otimes F(y_\alpha). \end{aligned} \tag{9.13}$$

Now $\Delta c_i = c_i \otimes 1 + 1 \otimes c_i$, and $S\,F(y_\gamma) = 0$ by (8.25), hence

$$\Delta F(y_\alpha) = \sum_i \sum_{\beta+\gamma=\alpha|i} S(c_i * F(y_\beta)) \otimes F(y_\gamma) + F(1)\otimes F(y_\alpha). \tag{9.14}$$

We reorder the summands. For each $\gamma \ne \alpha$ whose components are all less than or equal to those of α, we have one term of the form $c_i * F(y_{\beta|i})$ for each $\beta = \alpha - \gamma$ and each i such that $\beta_i \ne 0$. This gives

$$\Delta\,F(y_\alpha) = \sum_{\substack{\beta+\gamma=\alpha \\ \gamma\ne\alpha}} \sum_i S(c_i * F(y_{\beta|i}) \otimes F(y_\gamma) + F(1)\otimes F(y_\alpha) \tag{9.15}$$

[9] Using [5, eq. (2.12c)] or [6, eq. (3.29a)], one can see that our new construction coincides with the previous one if each c_i lies entirely in the i-th factor of a circle decomposition of T.

$$= \sum_{\substack{\beta+\gamma=\alpha\\ \gamma\neq\alpha}} F(y_\beta)\otimes F(y_\gamma) + F(1)\otimes F(y_\alpha)$$
$$= \sum_{\beta+\gamma=\alpha} F(y_\beta)\otimes F(y_\gamma),$$

as was to be shown.

That F induces an isomorphism in homology is trivial. □

Since $C(G)$ acts trivially on BT, the composition $\pi_* F\colon \mathbf{K}\to C(BT)$ descends to a map of dgcs

$$f\colon \mathbf{S} = \Bbbk\otimes_{\mathbf{\Lambda}}\mathbf{K}\to C(BT). \tag{9.16}$$

Proposition 9.2. *The transpose $f^*\colon C^*(BT)\to \mathbf{S}^*$ is a quasi-isomorphism of dgas.*

Proof. We only have to show that f is a quasi-isomorphism. We start with a general remark. All complexes in this proof are homological and $\mathbb{N}$-graded.

Recall that for any dga A the canonical map

$$\mathbf{B}(\Bbbk, A, A)\to\Bbbk \tag{9.17}$$

is a homotopy equivalence of complexes [13, Prop. II.5.2]. This implies that for any twisting cochain $t\colon C\to A$ and any left C-comodule M the chain map

$$\mathbf{B}(\Bbbk, A, A\otimes_t M)\to M = \Bbbk\underset{A}{\otimes}(A\otimes_t M) \tag{9.18}$$

is a quasi-isomorphism.

Now consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{B}(\Bbbk,\mathbf{\Lambda},\mathbf{K}) & \longrightarrow & \mathbf{B}\big(\Bbbk, C(T), C(ET)\big) & & \\ \downarrow & & \downarrow & & \\ \mathbf{S} = \Bbbk\underset{\mathbf{\Lambda}}{\otimes}\mathbf{K} & \longrightarrow & \Bbbk\underset{C(T)}{\otimes} C(ET) & \longrightarrow & C(BT). \end{array} \tag{9.19}$$

The composition along the bottom row equals f. The top arrow is a quasi-isomorphism by a standard spectral sequence argument, and the left vertical arrow is one by the remark above.

Recall from the twisted Eilenberg–Zilber theorem [9, Lemmas 4.3, 4.5$_*$] (see also [20, § II.4]) that $C(ET)$ and $C(T)\otimes_t C(BT)$ (with a suitable twisting cochain t) are homotopic as $C(T)$-modules. It follows that the bottom right arrow above is a homotopy equivalence of complexes and, again by the remark above, that the right vertical arrow is a quasi-isomorphism.

Putting everything together, we see that f is a quasi-isomorphism. □

9.2. **Hga formality.** We say that a (non-degenerate) simplex $\sigma\in ET$ *appears* in an element of $C(ET)$ if its coefficient in this chain is non-zero; an analogous definition applies to tensor products of chain complexes.

Lemma 9.3. *Let $0\le k\le n+1$, $a\in\mathbf{\Lambda}_1$ and $c\in\mathbf{S}_n$. For any simplex $\sigma\in(ET)_{n+1}$ appearing in $F(a\cdot c)$ we have*

$$(S\otimes S)\,P^{n+1}_k(\sigma) = 0.$$

Proof. By construction and formula (8.11), the simplex σ is of the form $A^g_m(\tau)$ for some loop $g \in T_1$, some $0 \le m \le n$ and some n-simplex τ appearing in $F(c)$.

If $n = 0$, then $\tau = e_0$. Hence

$$P^1_0(\sigma) = e_0 \otimes g * e_0 \qquad \text{and} \qquad P^1_1(\sigma) = g * e_0 \otimes e_0 \tag{9.20}$$

by Lemma 8.1, and our claim follows from the second identity in (8.25).

Now consider the case $n > 0$. The definition of the map F implies that τ is of the form $S\rho$ where ρ is a simplex appearing in $F(\tilde{a} \cdot \tilde{c})$ with $\tilde{a} \in \mathbf{\Lambda}_1$ and $\tilde{c} \in \mathbf{S}_{n-2}$.

Assume $k \le m$. Then

$$P^{n+1}_k(\sigma) = P^{n+1}_k(A^g_m(S\rho)) = (-1)^k\,(1 \otimes A^g_{m-k})\,P^n_k(S\rho) \tag{9.21}$$

where we have once again used Lemma 8.1. In the case $k = 0$ we obtain

$$(S \otimes S)\,P^{n+1}_0(\sigma) = Se_0 \otimes S\,A^g_m(S\rho) = 0 \tag{9.22}$$

by Lemma 8.2 (i) and (8.25). If $k > 0$, then

$$(S \otimes S)\,P^{n+1}_k(\sigma) = (-1)^k\,(S\,S \otimes S\,A^g_{m-k})\,P^{n-1}_{k-1}(\rho) = 0 \tag{9.23}$$

again by Lemma 8.2 (i) and the first identity in (8.25).

In the case $k > m$, we have

$$P^{n+1}_k(\sigma) = (A^g_m \otimes 1)\,P^n_{k-1}(S\rho). \tag{9.24}$$

For $k = 1$ this gives

$$(S \otimes S)\,P^{n+1}_1(\sigma) = -S\,A^g_m(e_0) \otimes S\,S\rho = 0. \tag{9.25}$$

If $k > 1$, we finally get

$$\begin{aligned}(S \otimes S)\,P^{n+1}_k(\sigma) &= (S\,A^g_m\,S \otimes S)\,P^{n-1}_{k-2}(\rho) \\ &= (S\,A^g_m \otimes 1)\,(S \otimes S)\,P^{n-1}_{k-2}(\rho) = 0\end{aligned} \tag{9.26}$$

by induction. □

For $0 \le k < l \le n$ we define

$$\begin{aligned}Q^n_{k,l}\colon C_n(ET) &\to C_{n-l+k+1}(ET) \otimes C_{l-k}(BT), \\ \sigma &\mapsto \partial^{l-1}_{k+1}\sigma \otimes \pi_*\,\partial^{k-1}_0\,\partial^n_{l+1}\sigma \\ &\qquad = \sigma(0,\dots,p,l,\dots,n) \otimes \pi_*\sigma(k,\dots,l).\end{aligned} \tag{9.27}$$

This operation is related to the $\cup_1$-product since for $\sigma \in C_n(ET)$ we have

$$(1 \otimes \pi_*)\,AW_{(1,2,1)}(\sigma) = \sum_{0 \le k < l \le n} (-1)^{(n-l)(l-k)+k}\,Q^n_{k,l}(\sigma), \tag{9.28}$$

compare [3, §2.2.8].

Lemma 9.4. *Let $0 \le k < l \le n$ and $a \cdot c \in \mathbf{K}_n$. For any n-simplex $\sigma \in ET$ appearing in $F(a \cdot c)$ we have*

$$Q^n_{k,l}(\sigma) = 0.$$

Proof. We proceed by induction on n, the case $n = 0$ being void. For the induction step from n for $n+1$, we start by considering the case $|a| = 0$, which entails $n \ge 1$. The definition of F then implies that σ is of the form $\sigma = S\tau$ for some n-simplex $\tau \in ET$ that appears in $F(\tilde{a} \cdot \tilde{c})$ for some $\tilde{a} \in \mathbf{\Lambda}_1$ and some $\tilde{c} \in \mathbf{S}_{n-1}$.

If $1 \le k < l \le n+1$, we get

$$Q^{n+1}_{k,l}(\sigma) = \partial^{l-1}_{k+1}\,S\tau \otimes \pi_*\,\partial^{k-1}_0\,\partial^{n+1}_{l+1}\,S\tau \tag{9.29}$$

$$= S\,\partial_k^{l-2}\tau \otimes \pi_*\,\partial_0^{k-1}\,S\,\partial_l^n\tau \tag{9.30}$$

$$= S\,\partial_k^{l-2}\tau \otimes \pi_*\,\partial_0^{k-2}\,\partial_l^n\tau \tag{9.31}$$

$$= (S\otimes 1)\,Q^n_{k-1,l-1}(\tau) = 0 \tag{9.32}$$

by induction.

For $0 < l \le n+1$ we have

$$Q^{n+1}_{0,l}(\sigma) = \partial_1^{l-1}\,S\tau \otimes \pi_*\,\partial_{l+1}^{n+1}\,S\tau \tag{9.33}$$

$$= S\,\partial_0^{l-2}\tau \otimes \pi_*\,\partial_{l+1}^{n+1}S\tau \tag{9.34}$$

$$= S\,\partial_0^{l-1}S\tau \otimes \pi_*\,\partial_{l+1}^{n+1}S\tau \tag{9.35}$$

$$= \pm(1\otimes\pi_*)\,T\,(1\otimes S)\,P_l^{n+1}(S\tau) \tag{9.36}$$

where T denotes the transposition of factors,

$$= \mp(1\otimes\pi_*)\,T\,(S\otimes S)\,P^n_{l-1}(\tau) = 0 \tag{9.37}$$

by Lemmas 8.2 (i) and 9.3.

Now we turn to the case $|a| > 0$. Then a simplex appearing in $F(a\cdot c)$ is of the form $\sigma = A^g_m(\tau)$ for some loop $g \in T_1$, some n-simplex $\tau \in ET$ appearing in $F(c)$ and some $0 \le m \le n$. We have

$$\begin{aligned} Q^{n+1}_{k,l}(\sigma) &= Q^{n+1}_{k,l}\bigl(A^g_m(\tau)\bigr) = Q^{n+1}_{k,l}\bigl(s_{[n]\setminus m}\,g\cdot s_m\tau\bigr) \\ &= \partial_{k+1}^{l-1}\bigl(s_{[n]\setminus m}\,g\cdot s_m\tau\bigr)\otimes\pi_*\,\partial_0^{k-1}\,\partial_{l+1}^{n+1}\,s_m\tau. \end{aligned} \tag{9.38}$$

Assume $l > m$. Then

$$\partial_0^{k-1}\,\partial_{l+1}^{n+1}\,s_m\tau = \partial_0^{k-1}\,s_m\,\partial_l^n\tau = \begin{cases} s_{m-k}\,\partial_0^{k-1}\,\partial_l^n\tau & \text{if } k\le m,\\ \partial_0^{k-2}\,\partial_l^n\tau & \text{if } k > m.\end{cases} \tag{9.39}$$

In the first case we obtain a degenerate simplex, so that (9.38) vanishes. In the second case we have

$$\begin{aligned} Q^{n+1}_{k,l}(\sigma) &= \bigl(s_{[n-l+k+1]\setminus m}\,g\bigr)\cdot\bigl(s_m\,\partial_k^{l-2}\tau\bigr)\otimes\pi_*\,\partial_0^{k-2}\,\partial_l^n\tau \\ &= (A^g_m\otimes 1)\Bigl(\partial_k^{l-2}\tau\otimes\pi_*\,\partial_0^{k-2}\,\partial_l^n\tau\Bigr) \\ &= (A^g_m\otimes 1)\,Q^n_{k-1,l-1}(\tau) = 0 \end{aligned} \tag{9.40}$$

by induction.

Finally consider $l \le m$. Then

$$\begin{aligned} Q^{n+1}_{k,l}(\sigma) &= \bigl(\partial_{k+1}^{l-1}\,s_{[n]\setminus m}\,g\bigr)\cdot\bigl(\partial_{k+1}^{l-1}\,s_m\tau\bigr)\otimes\pi_*\,\partial_0^{k-1}\,\partial_{l+1}^{n+1}\,s_m\tau \\ &= \bigl(s_{[n-l+k+1]\setminus m-l+k+1}g\bigr)\cdot\bigl(s_{m+k-l+1}\,\partial_{k+1}^{l-1}\tau\bigr)\otimes\pi_*\,\partial_0^{k-1}\,\partial_{l+1}^n\tau \\ &= A^g_{m-l+k+1}\bigl(\partial_{k+1}^{l-1}\tau\bigr)\otimes\pi_*\,\partial_0^{k-1}\,\partial_{l+1}^n\tau \\ &= (A^g_{m-l+k+1}\otimes 1)\,Q^n_{k,l}(\tau) = 0 \end{aligned} \tag{9.41}$$

by induction. This completes the induction step and the proof. □

We write $\underline{n} = \{1,\dots,n\}$. We say that a surjection $u\colon \underline{k+l}\to\underline{l}$ *has an enclave* at some position $1 < i < k+l$ if $u(i-1) = u(i+1)$ and if the value $u(i)$ does not appear elsewhere in the surjection. For example, the surjection $(1,2,1,3,4,1,5)$ has exactly one enclave at position 2.

Proposition 9.5. *If the surjection* $u\colon \underline{k+l} \to \underline{l}$ *has an enclave, then*

$$AW_u\, f = 0.$$

Proof. We start with a general observation. Let σ be a simplex in some simplicial set. If u has an enclave, then it follows from the definition of interval cut operations [3, Sec. 2.2] that any tensor product of simplices appearing in $AW_u(\sigma)$ can be obtained from a term $\tau \otimes \rho$ appearing in $AW_{(1,2,1)}(\sigma)$ by applying an interval cut to τ (at one choice of positions, not at all positions as in [3, §2.2.6]) and permuting the factors of the result.

Now let $c \in \mathbf{S}_n$ for some $n \geq 0$. By definition and naturality we have

$$AW_u\, f(c) = AW_u\, \pi_*\, F(c) = (\pi_* \otimes \pi_*)\, AW_u\, F(c). \tag{9.42}$$

Our previous remarks together with (9.28) show that it suffices to prove that $Q^n_{k,l}(\sigma)$ vanishes for any $\sigma \in ET$ appearing in $F(c)$ and any $0 \leq k < l \leq n$. But this has been done in Lemma 9.4. □

Theorem 9.6. *The map* $f^*\colon C^*(BT) \to H^*(BT)$ *is a quasi-isomorphism of hgas that additionally annihilates all extended hga operations* F_{kl} *with* $(k,l) \neq (1,1)$*. In particular,* $C^*(BT)$ *is formal as an hga.*

Proof. We know from Proposition 9.2 that f^* is a quasi-isomorphism of dgas. The hga operations E_k with $k \geq 1$ as well as the operations F_{kl} with $(k,l) \neq (1,1)$ are defined by surjections having enclaves, see (8.7) and (8.8). Hence the claim follows by dualizing Proposition 9.5. □

9.3. **The case where 2 is invertible.** It would greatly simplify the discussion of the next sections if the formality map f^* also annihilated the operation $F_{11} = -\cup_2$. However, this is impossible to achieve for the transpose of a quasi-isomorphism $f\colon H(BT) \to C(BT)$, independently of the coefficient ring $\Bbbk$. This can be seen as follows.

Take a non-zero $y \in H_2(BT)$ and set $w = f(y) \in C_2(BT)$. Choose a cochain $a \in C^2(BT)$ such that $a(w) \neq 0$. Let σ be a 2-simplex appearing in w with coefficient $w_\sigma \neq 0$ and such that $a(\sigma) \neq 0$. Define $b \in C^2(BT)$ by $b(\sigma) = 1$ and $b(\tau) = 0$ for $\tau \neq \sigma$. Then

$$(a \cup_2 b)(w) = \sum_\tau w_\tau\, a(\tau)\, b(\tau) = w_\sigma\, a(\sigma) \neq 0, \tag{9.43}$$

where we have used the identity $(a \cup_2 b)(\sigma) = a(\sigma)\, b(\sigma)$, *cf.* [3, §2.2.8]. Hence $f^*(a \cup_2 b) \neq 0$, and analogously $f^*(b \cup_2 a) \neq 0$. Note that a may be a cocycle, but b is not. (If $\sigma = [\, g \,|\, 1_0\,]$ for a loop $1_1 \neq g \in T_1$, then $b\big(d\,[\, s_0 g^{-1} \,|\, g \,|\, 1_0\,]\big) \neq 0$.)

In general one cannot even expect f^* to annihilate all $\cup_2$-products of cocycles as they are related to Steenrod squares. For $\Bbbk = \mathbb{Z}_2$ and any non-zero $[a] \in H^2(BT)$ one has

$$[a] = \mathrm{Sq}^0[a] = [a \cup_2 a] \neq 0. \tag{9.44}$$

The situation changes if we can invert 2.

Proposition 9.7. *Assume that* 2 *is invertible in* $\Bbbk$*. Then one can choose representatives* (c_i) *such that* f^* *additionally annihilates all* $\cup_2$*-products of cocycles.*

Proof. Let $\iota\colon T \to T$ be the group inversion. Being a morphism of groups, it induces involutions of ET and BT, which we denote by the same letter. Recall that ι_* changes the sign of all generators $x_i \in H_1(T)$ and all cogenerators $y_i \in H_2(BT)$. Starting from any set of representatives (c_i), we set

$$\tilde{c}_i = \tfrac{1}{2}\, c_i - \tfrac{1}{2}\, \iota_* c_i, \tag{9.45}$$

so that $\iota_*\tilde{c}_i = -\tilde{c}_i$. We construct F and f based on these representatives. The equivariance of F with respect to the involutions follows inductively from the recursive definition, and it entails that of f.

Now let a and b be cocycles. By Lemma 6.2, the value $f^*(a \cup_2 b)$ only depends on the cohomology classes of a and b. In particular, we may assume that a is of even degree $2k$ and b of degree $2l$. Then $a \cup_2 b$ is of degree $2(k+l-1)$, whence

$$\iota^* f^*(a \cup_2 b) = -(-1)^{k+l} f^*(a \cup_2 b). \tag{9.46}$$

On the other hand, we have

$$\iota^*(a \cup_2 b) = \iota^*(a) \cup_2 \iota^*(b) \tag{9.47}$$

by naturality. Now $\iota^*(a)$ is cohomologous to $(-1)^k\, a$ and $\iota^*(b)$ cohomologous to $(-1)^l\, a$, which implies that

$$f^*(\iota^*(a \cup_2 b)) = (-1)^{k+l}\, f^*(a \cup_2 b). \tag{9.48}$$

Since 2 is invertible in $\Bbbk$, this can only happen if the $\cup_2$-product vanishes. □

10. The kernel of the formality map

Let T be a simplicial torus, and let $f^*\colon C^*(BT) \to H^*(BT)$ be the formality map constructed in the previous section for some choice of representatives c_i. We need to study its kernel $\mathfrak{k} = \ker f^*$. We summarize what we know so far.

Proposition 10.1. *The kernel $\mathfrak{k} \lhd C^*(BT)$ of f^* contains the following elements:*

1. *all elements of odd degree,*
2. *all coboundaries,*
3. *the images of the interval cut operations AW_u^* if the surjection u has an enclave. This includes the images of all hga operations E_k with $k \ge 1$ and of all extended hga operations F_{kl} with $(k,l) \ne (1,1)$.*
4. *If 2 is invertible in $\Bbbk$, then $\mathfrak{k}$ contains all $\cup_2$-products of cocycles.*

Proof. The first two claims hold because $H^*(BT)$ is concentrated in even degrees. The other two are Theorem 9.6 and Proposition 9.7. □

Let us write

$$[a,b] = ab - (-1)^{|a||b|}\, ba \tag{10.1}$$

for the commutator of a, $b \in C^*(BT)$.

Lemma 10.2. *For all a, $b \in C^*(BT)$,*

$$[a,b] \equiv 0 \pmod{\mathfrak{k}}.$$

Proof. This follows from Proposition 10.1 together with the identity

$$d(\cup_1)(a;b) = [a,b], \tag{10.2}$$

or from the fact that $\mathfrak{k}$ is the kernel of a map to a commutative dga. □

Lemma 10.3. *The $\cup_1$-product is a right derivation of the commutator. That is,*

$$[a,b]\cup_1 c = (-1)^{|a|}[a, b\cup_1 c] + (-1)^{|b||c|}[a\cup_1 c, b]$$

for all a, b, $c \in C^*(BT)$.

Proof. This is a consequence of the Hirsch formula (6.15). □

Lemma 10.4. *Let a, b, $c \in C^*(BT)$.*

(i) *Modulo $\mathfrak{k}$, the $\cup_2$-product is both a left and a right derivation of the commutator. That is,*

$$a\cup_2 (b\,c) \equiv (a\cup_2 b)\,c + (-1)^{|a||b|}\,b\,(a\cup_2 c) \pmod{\mathfrak{k}},$$
$$(a\,b)\cup_2 c \equiv (-1)^{|b||c|}\,(a\cup_2 c)\,b + a\,(b\cup_2 c) \pmod{\mathfrak{k}}.$$

(ii) *One has*

$$a\cup_2 [b,c] \equiv [a,b]\cup_2 c \equiv 0 \pmod{\mathfrak{k}}.$$

Proof. The first part follows from the identities

$$d(F_{12})(a;b,c) \overset{\varkappa}{=} E_2(a;b,c) - F_{11}(a;b)\,c + F_{11}(a, b\,c) - F_{11}(a;c), \tag{10.3}$$

$$d(F_{21})(a,b;c) \overset{\varkappa}{=} a\,F_{11}(b;c) - F_{11}(a\,b;c) + F_{11}(a;c)\,b - E_2(c;a,b). \tag{10.4}$$

It implies the formulas

$$a\cup_2 [b,c] \equiv [a\cup_2 b, c] + (-1)^{|a||b|}[b, a\cup_2 c] \pmod{\mathfrak{k}}, \tag{10.5}$$

$$[a,b]\cup_2 c \equiv (-1)^{|b||c|}[a\cup_2 c, b] + [a, b\cup_2 c] \pmod{\mathfrak{k}}, \tag{10.6}$$

which together with Lemma 10.2 entail the second claim. □

Lemma 10.5. *Let a, b, c_1, …, $c_k \in C^*(BT)$ with $k \geq 2$. Then*

$$a\cup_2 E_k(b;c_1,\dots,c_k) \equiv E_k(b;c_1,\dots,c_k)\cup_2 a \equiv 0 \pmod{\mathfrak{k}}.$$

Proof. When the surjection e_k with $k \geq 2$ is split into two, then at least one of them will have an enclave. By the composition rule in the surjection operad, this implies that each surjection appearing in $f_{11}\circ_2 e_k$ or $f_{11}\circ_1 e_k$ again has an enclave. Together with Proposition 10.1 this gives the desired identities. □

Lemma 10.6. *Let a, b, $c \in C^*(BT)$. If a is cocycle of degree $|a| \leq 2$, then*

$$a\cup_2 (b\cup_1 c) \equiv (b\cup_1 c)\cup_2 a \equiv 0 \pmod{\mathfrak{k}}.$$

Proof. We consider the element $g_{12} = (2,3,1,3,1,2,1)$ in the surjection operad, following Kadeishvili [14, Rem. 2].[10] It satisfies

$$d\,g_{12} = e_1\circ_1 f_{11} + (1\,2)\cdot(e_1\circ_2 f_{11}) - f_{11}\circ_2 e_1 - f_{12} + (2\,3)\cdot f_{12} \tag{10.7}$$

and the corresponding interval cut operation therefore

$$d(G_{12})(a,b,c) \equiv \mp a\cup_2 (b\cup_1 c) \pmod{\mathfrak{k}}. \tag{10.8}$$

Because there are three 1's appearing in g_{12}, the homological interval cut operation on a simplex σ has the property that the first simplex $\sigma_{(1)}$ in the resulting tensor product involves three intervals, which each contributes at least one vertex.

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[10] Kadeishvili takes $g_{12} = (1,2,1,3,1,3,2)$ and $g_{21} = 1,2,3,2,3,1,3)$ instead. Assuming his definition of E^1_{pq} (see Footnote 8), this gives the formula for $d(G_{12})$ stated in [14, Rem. 2] with the term $(a\cup_2 c)\cup_1 b$ replaced by $(a\cup_2 b)\cup_1 c$. In the formula for $d(G_{21})$, the double $\cup_2$-product should read $(a\cup_2 c)\cup_1 b$.

Now the first occurrence of 1 in g_{12} is surrounded by two occurrences of 3. Hence the associated interval for this 1 must involve at least two vertices for otherwise the simplex $\sigma_{(3)}$ made up of the 3-intervals would contain twice the same vertex and therefore be degenerate. Hence $\sigma_{(1)}$ is of dimension at least 3.

Dually, $G_{12}(a, b, c)$ vanishes for $|a| \leq 2$, which implies that the left-hand side of (10.8) is a coboundary if a is additionally a cocycle. This proves that the first term in the statement is congruent to 0.

The second part follows analogously by looking at $g_{21} = (3, 1, 3, 2, 3, 2, 1)$, which satisfies

$$d\,g_{21} = (2\,3) \cdot (e_1 \circ_1 f_{11}) + e_1 \circ_2 f_{11} - f_{11} \circ_1 e_1 + f_{21} - (1\,2) \cdot f_{21}. \tag{10.9}$$ □

For elements $b_0, \ldots, b_k \in C^*(BT)$ we write the repeated $\cup_1$-product as

$$U_0(b_0) = b_0 \tag{10.10}$$

for $k = 0$ and

$$U_k(b_0, \ldots, b_k) = (-1)^{k-1} \big(((b_0 \cup_1 b_1) \cup_1 b_2) \cup_1 \cdots \big) \cup_1 b_k \tag{10.11}$$

for $k \geq 1$, compare the Gugenheim–May twisting cochain (8.36).

Proposition 10.7. *For all cocycles a, $b_0, \ldots, b_k \in C^*(BT)$, $k \geq 1$, we have*

$$a \cup_2 U_k(b_0, \ldots, b_k) \equiv U_k(b_0, \ldots, b_k) \cup_2 a \equiv 0 \pmod{\mathfrak{k}}.$$

Proof. We show that the first term in the statement lies in $\mathfrak{k}$; the proof for the second is analogous.

Write $b = U_k(b_0, \ldots, b_k)$ and assume first that $a = dc$ is a coboundary. Then

$$d(c \cup_2 b) = dc \cup_2 b \pm c \cup_2 db \pm c \cup_1 b \pm b \cup_1 c, \tag{10.12}$$

hence

$$a \cup_2 b \equiv \mp c \cup_2 db \pmod{\mathfrak{k}}. \tag{10.13}$$

Since $b_1, \ldots, b_k$ are cocycles, we have

$$d\,b = \sum_{i=1}^{k} \pm U_{k-i}\big(\big[U_{i-1}(b_0, \ldots, b_{i-1}), b_i\big], b_{i+1}, \ldots, b_k \big) \tag{10.14}$$

By a repeated application of Lemma 10.3 we see that each term on the right-hand side of (10.14) is a sum of commutators, so that the right-hand side of (10.13) vanishes by Lemma 10.4 (ii). This proves the claim for $a = dc$.

As a consequence, we may replace a by any cocycle cohomologous to it. Because $H^*(BT)$ is generated in degree 2, we may in particular assume that a is the product of cocycles of degree 2. By Lemma 10.4 (i), it is enough to consider the case where a is a single degree-2 cocycle, where Lemma 10.6 applies. □

11. Spaces and shc maps

Let T be a simplicial torus and let $\kappa\colon BT \to X$ be a map of simplicial sets. We write $\mathfrak{k} = \mathfrak{k}_X \lhd C^*(X)$ for the kernel of the composition

$$C^*(X) \xrightarrow{\kappa^*} C^*(BT) \xrightarrow{f^*} H^*(BT) \tag{11.1}$$

where f^* denotes the formality map constructed in the previous section. We want to relate $\mathfrak{k}$ to the the canonical shc structure on $C^*(X)$. Combining Theorem 6.3 with Proposition 10.1 we see that the structure map $\Phi\colon C^*(X) \otimes C^*(X) \Rightarrow C^*(X)$

is $\mathfrak{k}$-strict. Moreover, the associativity homotopy $h^a\colon \Phi\circ(\Phi\otimes 1)\simeq\Phi\circ(1\otimes\Phi)$ is $\mathfrak{k}$-trivial, but the commutativity homotopy $h^c\colon \Phi\circ T_{A,A}\simeq\Phi$ is not in general. This failure requires extra attention.

We introduce the following terminology: Let A and B be dgas, $\mathfrak{b}\lhd B$ and $m\ge 0$. An shm homotopy $h\colon A^{\otimes m}\to B$ is called *$\mathfrak{k}$-trivial on cocycles* if

$$h_{(n)}(a_{11}\otimes\cdots\otimes a_{1m},\dots,a_{n1}\otimes\cdots\otimes a_{nm})\equiv 0\pmod{\mathfrak{b}} \tag{11.2}$$

for all $n\ge 1$ and all cocycles $a_{11},\dots,a_{nm}\in A$. Similarly, an shc map $f\colon A^{\otimes m}\Rightarrow B$ is called *$\mathfrak{k}$-natural on cocycles* if there is a homotopy $h\colon A^{\otimes 2m}\to B$ that is $\mathfrak{b}$-trivial on cocycles and makes the diagram (5.1) commute.

Lemma 11.1. *Let h, $k\colon A^{\otimes m}\to B$ be shm homotopies, $\mathfrak{b}$-trivial on cocycles.*

(i) *The shm homotopies $h\cup k$ and h^{-1} are again $\mathfrak{b}$-trivial on cocycles.*
(ii) *If $f\colon B\to C$ is a $\mathfrak{c}$-strict shm map such that $f_{(1)}(\mathfrak{b})\subset\mathfrak{c}$, then $f\circ h$ is $\mathfrak{c}$-trivial on cocycles.*
(iii) *If $T\colon A^{\otimes m}\to A^{\otimes m}$ is some permutation of the factors, then $h\circ g$ is $\mathfrak{b}$-trivial.*

Proof. The first claim follows from the definition of the cup product and the formula for the inverse given in Lemma 2.2 (ii). The second part is analogous to Lemma 3.1 (ii), and the last claim is trivial. □

We assume from now on that 2 is invertible in $\Bbbk$. Then $\mathfrak{k}$ contains all $\cup_2$-products of cocycles by Proposition 9.7, so that both h^c and the homotopy $k^c=h^c\circ T$ in the other direction become $\mathfrak{k}$-trivial on cocycles. We need to extend this observation.

Lemma 11.2. *For any $n\ge 0$, the shm homotopy*

$$h^c\circ\bigl(1\otimes\Phi^{[n]}\bigr)\colon C^*(X)\otimes C^*(X)^{\otimes n}\to C^*(X)$$

is $\mathfrak{k}$-trivial on cocycles, and the same holds with k^c instead of h^c.

Proof. By naturality we may assume that κ is the identity map of $X=BT$.

Let $l\ge 0$, and let $b_1,\dots,b_l\in C^*(BT)^{\otimes n}$ with $b_i=b_{i,1}\otimes\cdots\otimes b_{i,n}$ where all $b_{i,j}$ are cocycles. We claim that $\Phi^{[n]}_{(l)}(b_1,\dots,b_l)$ is a linear combination of products of terms of the following two kinds: Repeated $\cup_1$-products $U_k(c_0,\dots,c_k)$ of cocycles with $k\ge 0$, or E_k-terms with $k\ge 2$ (and not necessarily cocycles as arguments).

This follows by induction: $\Phi^{[n]}_{(0)}=0$, and $\Phi^{[n]}_{(1)}(b_1)=b_{1,1}\cdots b_{1,n}$ is a product of cocycles $b_{1,j}=U_0(b_{1,j})$. For the induction step, we observe from the formula for Φ and the composition formula (3.17) for shm maps that we get products of terms $E_m(\dots)$ with some value $\Phi^{[n]}_{(l)}(b_1,\dots,b_l)$ plugged into the first argument and cocycles into the remaining arguments.

We consider each factor E_m separately. For $m=0$ the induction hypothesis applies and for $m\ge 2$ there is nothing to show. So assume $m=1$. By induction and the Hirsch formula (6.15), we may assume that the first argument is a repeated $\cup_1$-product of cocycles or a term E_k with $k\ge 2$. In the former case we get another repeated $\cup_1$-product of cocycles. In the latter case the identity (6.4) shows that we end up with a sum of terms $E_{k'}$ with $k'\ge k\ge 2$. This completes the proof of the claim.

Now consider $h^c_{(m)}$ for $m\ge 1$, or rather its description modulo $\mathfrak{k}$ given in Theorem 6.3 (iv). We have to plug cocycles into the first arguments and values $\Phi^{[n]}_{(l)}(b_1,\dots,b_l)$ as above into the second arguments. Because the $\cup_2$-product is a derivation modulo $\mathfrak{k}$ by Lemma 10.4 (i), we only have to consider terms of the following two kinds in light of our previous discussion: firstly, $\cup_2$-products $a \cup_2 b$ where a is a cocycle and b a repeated $\cup_1$-product of cocycles, and secondly, $\cup_2$-products where the second argument is an E_k-term with $k \geq 2$. The second case is covered by Lemma 10.5 and the first by Proposition 10.7 if we have at least one $\cup_1$-product. Finally, all $\cup_2$-product of cocycles are contained in $\mathfrak{k}$ as remarked above.

The proof for k^c is analogous. □

Proposition 11.3. *For any $n \geq 0$ the iteration $\Phi^{[n]}\colon C^*(X)^{\otimes n} \Rightarrow C^*(X)$ is an shc map that is $\mathfrak{k}$-natural on cocycles.*

Proof. Munkholm [17, Prop. 4.5] has shown that $\Phi^{[n]}$ is an shc map. The non-trivial part of the proof is to construct a homotopy

$$h^{[n]}\colon \Phi \circ \left(\Phi^{[n]} \otimes \Phi^{[n]}\right) \simeq \Phi^{[n]} \circ \Phi^{\otimes n} \circ T_n \tag{11.3}$$

where $T_n\colon A^{\otimes n} \otimes A^{\otimes n} \to (A \otimes A)^{\otimes n}$ is the reordering of the $2n$ factors $A = C^*(X)$ corresponding to the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n & n+1 & n+2 & \dots & 2n \\ 1 & 3 & \dots & 2n-1 & 2 & 4 & \dots & 2n \end{pmatrix}. \tag{11.4}$$

We follow Munkholm's arguments and verify that in our setting they lead to a homotopy that is $\mathfrak{k}$-trivial on cocycles. There is nothing to show for $n \leq 1$.

We start with the case $n = 2$, see [17, p. 31]. The homotopies labelled h_1, h_2, h_4 and h_5 by Munkholm are $\mathfrak{k}$-trivial because $\Phi \circ (1 \otimes \Phi)$ is $\mathfrak{k}$-strict and the homotopy h^a is $\mathfrak{k}$-trivial, see Lemmas 6.3, 11.1 (ii) and 3.1. So consider the homotopy

$$h_3 = \Phi \circ (1_A \otimes \Phi) \circ (1_A \otimes k^c \otimes 1_A). \tag{11.5}$$

We have remarked above that k^c is $\mathfrak{k}$-trivial on cocycles. Together with the $\mathfrak{k}$-strictness of $\Phi \circ (1 \otimes \Phi)$ this implies by Lemma 11.1 that h_3 is $\mathfrak{k}$-trivial on cocycles, too, and therefore also the sought-after homotopy

$$h^{[2]} = h_1 \cup h_2 \cup h_3 \cup h_4 \cup h_5. \tag{11.6}$$

For the induction step we have another set of homotopies h_1 to h_5, see [17, p. 32]. The homotopies h_1 and h_4 are $\mathfrak{k}$-trivial by Corollary 4.3 and Lemma 3.1 because Φ is $\mathfrak{k}$-strict. The homotopy h_3 is actually not needed. In fact, the identity

$$\left(\Phi^{[n]} \otimes 1\right) \circ T_{A,A^{\otimes n}} = T_{A,A} \circ \left(1 \otimes \Phi^{[n]}\right) \tag{11.7}$$

(see [17, §3.6 (iii)]) and [17, Prop. 3.3 (ii)] (or Corollary 4.3) imply that

$$\begin{aligned} \left(1_A \otimes T_{A,A} \otimes 1_A\right) \circ \left(\Phi^{[n]} \otimes 1_A \otimes \Phi^{[n]} \otimes 1_A\right) = \\ \left(\Phi^{[n]} \otimes \Phi^{[n]} \otimes 1_A \otimes 1_A\right) \circ \left(1_{A^{\otimes n}} \otimes T_{A,A^{\otimes n}} \otimes 1_A\right), \end{aligned} \tag{11.8}$$

which means that the homotopy relation labelled "$\overset{3}{\simeq}$" in [17, p. 32] is an equality. That the homotopy h_5 is $\mathfrak{k}$-trivial on cocycles uses that so is $h^{[n]}$ by induction, that Φ is $\mathfrak{k}$-strict and also Lemma 11.1.

To show that

$$h_2 = h^{[2]} \circ \left(\Phi^{[n]} \otimes 1_A \otimes \Phi^{[n]} \otimes 1_A\right) \tag{11.9}$$

is $\mathfrak{k}$-trivial on cocycles, we may by (2.13) and Lemma 11.1 (i) consider the composition with each factor in (11.6) separately. (Recall that $h \circ f = h\,\mathbf{B}f$ for an shm homotopy h and an shm map f.) The homotopies h_1, h_2, h_4 and h_5 for the case $n=2$ are $\mathfrak{k}$-trivial on all arguments and therefore pose no problem.

It remains to look at the homotopy

$$\begin{aligned} k_1 &= h_3 \circ \big(\Phi^{[n]} \otimes 1_A \otimes \Phi^{[n]} \otimes 1_A\big) \\ &= \Phi \circ (1_A \otimes \Phi) \circ (1_A \otimes k^c \otimes 1_A) \circ \big(\Phi^{[n]} \otimes 1_A \otimes \Phi^{[n]} \otimes 1_A\big). \end{aligned} \tag{11.10}$$

We want to compare it to the homotopy

$$k_2 = \Phi \circ (1_A \otimes \Phi) \circ \Big(\Phi^{[n]} \otimes \big(k^c \circ \big(1_A \otimes \Phi^{[n]}\big)\big) \otimes 1_A\Big). \tag{11.11}$$

Denoting reduction modulo $\mathfrak{k}$ by a bar above a map, we have

$$\bar{k}_1 = \mu^{[3]}_{A/\mathfrak{k}} \circ (1_{A/\mathfrak{k}} \otimes \bar{k}^c \otimes 1_{A/\mathfrak{k}}) \circ \big(\bar{\Phi}^{[n]} \otimes 1_A \otimes \Phi^{[n]} \otimes \bar{1}_A\big), \tag{11.12}$$

$$\bar{k}_2 = \mu^{[3]}_{A/\mathfrak{k}} \circ \Big(\bar{\Phi}^{[n]} \otimes \big(\bar{k}^c \circ \big(1_A \otimes \Phi^{[n]}\big)\big) \otimes \bar{1}_A\Big). \tag{11.13}$$

Because $\bar{\Phi}^{[n]}$ is a dga map, $\bar{k}_1$ and $\bar{k}_2$ agree, see Lemma 4.5. Moreover, the homotopy $\bar{k}^c \circ (1 \otimes \Phi^{[n]})$ is 0-trivial (that is, trivial) on cocycles by Lemma 11.2, which together with Lemma 11.1 (ii) implies that $\bar{k}_2$ has the same property. Putting these facts together, we obtain that k_1 is $\mathfrak{k}$-trivial on cocycles. This completes the proof. □

Let $n \ge 0$ and choose cocycles $a_1, \ldots, a_n \in C^*(X)$ of even positive degrees. We write $\boldsymbol{a} = (a_1, \ldots, a_n)$ and consider the shm map

$$\Lambda_{\boldsymbol{a}} \colon \Bbbk[\boldsymbol{x}] := \Bbbk[x_1] \otimes \cdots \otimes \Bbbk[x_n] \xrightarrow{\lambda_{\boldsymbol{a}}} C^*(X)^{\otimes n} \overset{\Phi^{[n]}}{\Longrightarrow} C^*(X) \tag{11.14}$$

where $\lambda_{\boldsymbol{a}}$ is the tensor product of the dga maps sending each x_i to a_i.

Remark 11.4. The map $\Lambda_{\boldsymbol{a}}$ can be expressed in terms of the hga operations on $C^*(X)$. It is not the same as Wolf's explicit shm map [26, Sec. 3], which only uses $\cup_1$-products.

Proposition 11.5. *The map $\Lambda_{\boldsymbol{a}} \colon \Bbbk[\boldsymbol{x}] \to C^*(X)$ is a $\mathfrak{k}$-strict and $\mathfrak{k}$-natural shc map.*

Proof. Since $\Phi^{[n]}$ is $\mathfrak{k}$-strict, so is $\Lambda_{\boldsymbol{a}}$ by Lemma 3.1 (i). It remains to consider the diagram

$$\begin{array}{ccc} \Bbbk[\boldsymbol{x}] \otimes \Bbbk[\boldsymbol{x}] & \xrightarrow{\quad\mu^{[n]}\quad} & \Bbbk[\boldsymbol{x}] \\ {\scriptstyle \lambda_{\boldsymbol{a}} \otimes \lambda_{\boldsymbol{a}}}\downarrow & & \downarrow{\scriptstyle \lambda_{\boldsymbol{a}}} \\ C^*(X)^{\otimes n} \otimes C^*(X)^{\otimes n} & \underset{\Phi^{\otimes n} \circ T_n}{\Longrightarrow} & C^*(X)^{\otimes n} \\ {\scriptstyle \Phi^{[n]} \otimes \Phi^{[n]}}\Downarrow & & \Downarrow{\scriptstyle \Phi^{[n]}} \\ C^*(X) \otimes C^*(X) & \underset{\Phi}{\Longrightarrow} & C^*(X). \end{array} \tag{11.15}$$

Each dga map $\Bbbk[x_i] \to C^*(X)$, $x_i \mapsto a_i$ is in fact a $\mathfrak{k}$-natural shc map because we can choose $b = -\frac{1}{2}\, a_i \cup_2 a_i \in \mathfrak{k}$ in the statement of [7, Prop. 7.2]. Then the shc map $\lambda_{\boldsymbol{a}}$ is $\mathfrak{k}^{\boxtimes n}$-natural by Lemma 5.1 and induction. Because $\Phi^{[n]}$ is $\mathfrak{k}$-strict, its composition h_1 with the homotopy making the top diagram commute is $\mathfrak{k}$-trivial by Lemma 3.1 (ii).

Composed with the top left arrow, the homotopy making the bottom square commute is $\mathfrak{k}$-trivial by Proposition 11.3. The cup product of this composed homotopy h_2 with h_1 then is $\mathfrak{k}$-trivial as well by Lemma 2.2 (i). This proves the claim since

$$(\lambda_{\boldsymbol{a}} \otimes \lambda_{\boldsymbol{a}}) \circ \left(\Phi^{[n]} \otimes \Phi^{[n]}\right) = (\lambda_{\boldsymbol{a}} \circ \Phi^{[n]}) \otimes (\lambda_{\boldsymbol{a}} \circ \Phi^{[n]}) = \Lambda_{\boldsymbol{a}} \otimes \Lambda_{\boldsymbol{a}} \tag{11.16}$$

by Corollary 4.3. □

If $H^*(X) \cong \Bbbk[\boldsymbol{x}]$ is polynomial and each a_i represents x_i under this isomorphism, then $\Lambda_{\boldsymbol{a}}$ is a quasi-isomorphism. Note that it depends both on the choice of the generators x_i and of their representatives a_i.

Theorem 11.6. *Let $\varphi\colon Y \to X$ be a map of simplicial sets with polynomial cohomology. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be representatives of some generators of $H^*(X)$ and $H^*(Y)$, respectively. Then the diagram*

$$\begin{array}{ccc} H^*(X) & \xrightarrow{H^*(\varphi)} & H^*(Y) \\ {\scriptstyle\Lambda_{\boldsymbol{a}}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle\Lambda_{\boldsymbol{b}}} \\ C^*(X) & \xrightarrow{\varphi^*} & C^*(Y) \end{array}$$

commutes up to a $\mathfrak{k}_Y$-trivial homotopy.

The corresponding result in [17, Sec. 7] is the heart of Munkholm's paper, and for our proof of Theorem 1.3 in the next section Theorem 11.6 will also be crucial.

Proof. We write $f = \varphi^*$, $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\mathfrak{k} = \mathfrak{k}_Y$. By assumption, we have $H^*(X) = \Bbbk[x_1, \dots, x_n]$. We define $\tilde{\boldsymbol{a}} = (H^*(f)(a_1), \dots, H^*(f)(a_n))$ and consider the diagram

$$\begin{array}{ccccc} & \lambda_{\boldsymbol{a}} \swarrow & H^*(X) & \searrow \lambda_{\tilde{\boldsymbol{a}}} & \\ C^*(X)^{\otimes n} & \xrightarrow[f^{\otimes n}]{} & C^*(Y)^{\otimes n} & \xLeftarrow[\Lambda_{\boldsymbol{b}}^{\otimes n}]{} & H^*(Y)^{\otimes n} \\ \Big\Downarrow{\scriptstyle\Phi_X^{[n]}} & & \Big\Downarrow{\scriptstyle\Phi_Y^{[n]}} & & \Big\downarrow{\scriptstyle\mu^{[n]}} \\ C^*(X) & \xrightarrow[f]{} & C^*(Y) & \xLeftarrow[\Lambda_{\boldsymbol{b}}]{} & H^*(Y). \end{array} \tag{11.17}$$

The composition from $H^*(X)$ to $C^*(X)$ equals $\Lambda_{\boldsymbol{a}}$, and the one from $H^*(X)$ to $H^*(Y)$ is $H^*(f)$. The left square commutes strictly by the naturality of the hga structure. Lemma 5.2 implies that the right square commutes up to a $\mathfrak{k}$-trivial homotopy since $\Lambda_{\boldsymbol{b}}$ is $\mathfrak{k}$-strict and $\mathfrak{k}$-natural by Proposition 11.5.

The composition

$$\Bbbk[x_i] \longrightarrow H^*(Y) \xRightarrow{\Lambda_{\boldsymbol{b}}} C^*(Y) \tag{11.18}$$

is a $\mathfrak{k}$-strict shm map, and the composition

$$\Bbbk[x_i] \longrightarrow C^*(X) \xrightarrow{f} C^*(Y) \tag{11.19}$$

is the dga map sending x_i to $f(a_i)$. Since both $(\Lambda_{\boldsymbol{b}})_{(1)}(\tilde{a}_i)$ and $f(a_i)$ represent the even-degree element $\tilde{a}_i \in H^*(B)$ and $\mathfrak{k}$ contains all elements of odd degree, these two maps are homotopic via a $\mathfrak{k}$-trivial shm homotopy by [7, Prop. 7.1].

The two ways to go from $H^*(X)$ to $C^*(Y)^{\otimes n}$ in the diagram represent the tensor products of the shm maps just discussed. This implies by induction and Lemma 3.1 that also the triangle commutes up to a $\mathfrak{k}^{\boxtimes n}$-trivial homotopy. Its composition with $\Phi_Y^{[n]}$ is $\mathfrak{k}$-trivial by Lemma 3.1 as $\Phi_Y^{[n]}$ is $\mathfrak{k}$-strict. Lemma 2.2 concludes the proof. □

12. Homogeneous spaces

We are now ready to prove Theorem 1.3. We continue to assume that 2 is invertible in $\Bbbk$.

Let G be a compact connected Lie group and $\iota\colon K \hookrightarrow G$ a closed connected subgroup such that the order of the torsion subgroup of $H^*(G;\mathbb{Z})$ is invertible in $\Bbbk$ and analogously for K. This implies that BG and BK have polynomial cohomology over $\Bbbk$ (and in fact is equivalent to it), see [13, Rem. IV.8.1]. By Lemma 8.5 we may assume both BG and BK to be 1-reduced. For simplicity, we denote the induced maps $C^*(BG) \to C^*(BK)$ and $H^*(BG) \to H^*(BK)$ both by ι^*.

Our goal is to construct an isomorphism of graded algebras

$$H^*(G/K) \cong \operatorname{Tor}^{H^*(BG)}\bigl(\Bbbk, H^*(BK)\bigr), \tag{12.1}$$

natural in the pair (G, K). Recall from Proposition 8.6 that there is a natural isomorphism of graded algebras

$$H^*(G/K) \cong \operatorname{Tor}^{C^*(BG)}\bigl(\Bbbk, C^*(BK)\bigr), \tag{12.2}$$

It suffices therefore to connect the two bar constructions underlying the torsion products in (12.1) and (12.2). We start by establishing an isomorphism of graded $\Bbbk$-modules, proceeding in a way similar to Munkholm [17, §7.4]. Remember that we have defined one-sided bar constructions as twisted tensor products in (7.3).

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be representatives of generators of $H^*(BG)$ and $H^*(BK)$, respectively. We write the induced shm quasi-isomorphism $\Lambda_{\boldsymbol{a}}\colon H^*(BG) \Rightarrow C^*(BG)$ defined in (11.14) as Λ^G and analogously $\Lambda^K = \Lambda_{\boldsymbol{b}}\colon H^*(BK) \Rightarrow C^*(BK)$.

We define the map

$$\Theta_{G,K}\colon \mathbf{B}(\Bbbk, H^*(BG), H^*(BK)) \to \mathbf{B}(\Bbbk, C^*(BG), C^*(BK)) \tag{12.3}$$

as the composition of the chain maps

$$\begin{array}{c} \mathbf{B}H^*(BG) \otimes_{\iota^*\circ t_{H^*(BG)}} H^*(BK) \\ \big\downarrow \Gamma_{\Lambda^K} \\ \mathbf{B}H^*(BG) \otimes_{\Lambda^K\circ\iota^*\circ t_{H^*(BG)}} C^*(BK) \\ \big\downarrow \delta_h \\ \mathbf{B}H^*(BG) \otimes_{\iota^*\circ\Lambda^G\circ t_{H^*(BG)}} C^*(BK) \\ \big\downarrow \mathbf{B}\Lambda^G\otimes 1 \\ \mathbf{B}C^*(BG) \otimes_{\iota^*\circ t_{C^*(BG)}} C^*(BK), \end{array} \tag{12.4}$$

given by Lemmas 7.3, 7.1 and 7.2, respectively, where the twisting cochain homotopy h in the second step comes from Theorem 11.6. Note that $\Theta_{G,K}$ depends on the chosen representative cocycles $\boldsymbol{a}$ and $\boldsymbol{b}$.

Let $\kappa\colon T \to K$ be a morphism of simplicial groups where T is some torus, and choose a formality map $f^*\colon C^*(BT) \to H^*(BT)$ as in Proposition 9.7. As before, we write $\mathfrak{k}_{BK}$ for the kernel of the composition

$$C^*(BK) \xrightarrow{\kappa^*} C^*(BT) \xrightarrow{f^*} H^*(BT). \tag{12.5}$$

Then the homotopy h mentioned in the previous paragraph is $\mathfrak{k}_{BK}$-trivial.

Lemma 12.1. *Modulo* $\mathbf{B}C^*(BG) \otimes \mathfrak{k}_{BK}$ *we have*

$$\Theta_{G,K} \equiv \mathbf{B}\Lambda^G \otimes \Lambda^K_{(1)}.$$

Recall that $\Lambda^K_{(1)}$ is the quasi-isomorphism of complexes

$$H^*(BK) \cong \Bbbk[y_1, \dots, y_n] \to C^*(BK), \qquad y_1^{k_1} \cdots y_n^{k_n} \mapsto b_1^{k_1} \cdots b_n^{k_n}. \tag{12.6}$$

Proof. The congruence follows from Lemmas 7.1, 7.2 and 7.3, given that Λ^K is a $\mathfrak{k}_{BK}$-strict shm map and h a $\mathfrak{k}_{BK}$-trivial homotopy. □

Proposition 12.2.

(i) *The map*

$$H^*(\Theta_{G,K})\colon\ \operatorname{Tor}^{H^*(BG)}\bigl(\Bbbk, H^*(BK)\bigr) \to \operatorname{Tor}^{C^*(BG)}\bigl(\Bbbk, C^*(BK)\bigr)$$

is an isomorphism of graded $\Bbbk$*-modules.*

(ii) *The Eilenberg–Moore spectral sequence for the fibration* $G/K \hookrightarrow BK \to BG$ *collapses at the second page.*

Proof. Both Λ^G and Λ^K are quasi-isomorphisms, and so is $\mathbf{B}\Lambda^G$. It follows from Lemma 12.1 as in Remark 7.4 that $\Theta_{G,K}$ induces an isomorphism between the second pages of these spectral sequences and therefore between the torsion products.

Because the spectral sequence for $\mathbf{B}(\Bbbk, H^*(BG), H^*(BK))$ collapses at this stage, so does the one for $\mathbf{B}(\Bbbk, C^*(BG), C^*(BK))$, which is the Eilenberg–Moore spectral sequence of the fibration. □

We now turn to the multiplicativity and naturality of $H^*(\Theta_{G,K})$. Here our approach is inspired by Wolf [26, p. 331]. Based on κ and f^* we define the map

$$\Psi_\kappa\colon \mathbf{B}(\Bbbk, C^*(BG), C^*(BK)) \to \mathbf{B}(\Bbbk, C^*(BG), H^*(BT)) \tag{12.7}$$

as the composition

$$\begin{array}{c} \mathbf{B}C^*(BG) \otimes_{\iota^* \circ t_{C^*(BG)}} C^*(BK) \\ \big\downarrow{\scriptstyle 1\otimes\kappa^*} \\ \mathbf{B}C^*(BG) \otimes_{\kappa^*\iota^* \circ t_{C^*(BG)}} C^*(BT) \\ \big\downarrow{\scriptstyle 1\otimes f^*} \\ \mathbf{B}C^*(BG) \otimes_{f^*\kappa^*\iota^* \circ t_{C^*(BG)}} H^*(BT) \end{array} \tag{12.8}$$

Lemma 12.3.

(i) Ψ_κ *is a morphism of dgas.*

(ii) *If* κ *is the inclusion of a maximal torus into* K*, then* $H^*(\Psi_\kappa)$ *is injective, hence so is the map* $H^*(G/K) \to H^*(G/T)$*.*

(iii) *The composition* $\Psi_\kappa\,\Theta_{G,K}$ *is the map*

$$\begin{array}{c}\mathbf{B}H^*(BG)\otimes_{\iota^*\circ t_{H^*(BG)}} H^*(BK)\\ \big\downarrow{\scriptstyle \mathbf{B}\Lambda^G\otimes\kappa^*}\\ \mathbf{B}C^*(BG)\otimes_{f^*\,\kappa^*\iota^*\circ t_{C^*(BG)}} H^*(BT).\end{array}$$

The idea of reducing to a maximal torus goes back to Baum [2, Lemma 7.2].

Proof. The first map in (12.8) is a dga map by naturality and the second one by inspection of the product formula (7.9). This proves the first claim.

If $T\subset K$ is a maximal torus, then $H^*(K/T)$ is concentrated in even degrees, as is $H^*(BK)$ by assumption. Hence the Serre spectral sequence for the fibration $K/T\to BT\to BK$ degenerates at the second page. By the Leray–Hirsch theorem, this implies that $H^*(BT)$ is a free module over $H^*(BK)$ with $\kappa^*(H^*(BK))$ being a direct summand.

As a consequence, the induced map

$$\operatorname{Tor}^{H^*(BG)}\big(\Bbbk,H^*(BK)\big)\xrightarrow{\operatorname{Tor}^1(1,\kappa^*)}\operatorname{Tor}^{H^*(BG)}\big(\Bbbk,H^*(BT)\big)\tag{12.9}$$

is injective. This is the map between the second pages of the Eilenberg–Moore spectral sequences for G/K and G/T, respectively. Because these spectral sequences degenerate at this level by Proposition 12.2 (ii), this implies that the map $1\otimes\kappa^*$ in (12.8) is injective in cohomology.

Another standard spectral sequence argument shows that the map $1\otimes f^*$ in (12.8) is a quasi-isomorphism since f^* is so. Together with the naturality of the isomorphism (12.2) this shows the second claim.

The last part is a consequence of Lemma 12.1. □

Theorem 12.4. *The isomorphism* $H^*(\Theta_{G,K})$ *is multiplicative.*

Proof. Let $\kappa\colon T\hookrightarrow K$ be the inclusion of a maximal torus. By Lemma 12.3 it suffices to prove that the composition $\Psi_\kappa\,\Theta_{G,K}=\mathbf{B}\Lambda^G\otimes\kappa^*$ is multiplicative up to homotopy. Clearly, κ^* is multiplicative.

We claim that $\mathbf{B}\Lambda^G$ is multiplicative up to a coalgebra homotopy

$$h\colon\mathbf{B}H^*(BG)\otimes\mathbf{B}H^*(BG)\to\mathbf{B}C^*(BG).\tag{12.10}$$

To see this, consider the diagram

$$\begin{array}{ccccc}\mathbf{B}H^*(BT)\otimes\mathbf{B}H^*(BT) & \xrightarrow{\nabla} & \mathbf{B}\big(H^*(BT)\otimes H^*(BT)\big) & \xrightarrow{\mathbf{B}\mu} & \mathbf{B}H^*(BT)\\ \big\downarrow{\scriptstyle\mathbf{B}\Lambda^G\otimes\mathbf{B}\Lambda^G} & & \big\downarrow{\scriptstyle\mathbf{B}(\Lambda^G\otimes\Lambda^G)} & & \big\downarrow{\scriptstyle\mathbf{B}\Lambda^G}\\ \mathbf{B}C^*(BT)\otimes\mathbf{B}C^*(BT) & \xrightarrow{\nabla} & \mathbf{B}\big(C^*(BT)\otimes C^*(BT)\big) & \xrightarrow{\mathbf{B}\Phi} & \mathbf{B}C^*(BT).\end{array}\tag{12.11}$$

The composition along the top row is the multiplication in $H^*(BT)$, and by [7, Prop. 4.2] the one along the bottom row equals the product in $C^*(BT)$. The left square commutes by naturality of the shuffle map (Lemma 4.4). The right square commutes up to a $\mathfrak{k}_{BG}$-trivial coalgebra homotopy h because Λ^G is an shc map by Proposition 11.5. This implies that it induces a homotopy

$$\tilde h\colon\mathbf{B}(\Bbbk,H^*(BG),H^*(BK))^{\otimes 2}\longrightarrow\mathbf{B}(\Bbbk,C^*(BG),C^*(BK))\tag{12.12}$$

such that $\mathbf{B}\Lambda^G\otimes\kappa^*$ is multiplicative up to $\tilde h$. □

Theorem 12.5. *Let $\varphi\colon (G,K) \to (G',K')$ be a map of pairs, both satisfying our assumptions, and choose representatives $\boldsymbol{a}'$ and $\boldsymbol{b}'$ for generators of $H^*(BG')$ and $H^*(BK')$, respectively. Then the following diagram commutes.*

$$\begin{array}{ccc} \operatorname{Tor}^{H^*(BG')}\big(\Bbbk, H^*(BK')\big) & \xrightarrow{\operatorname{Tor}^{\varphi^*}(1,\varphi^*)} & \operatorname{Tor}^{H^*(BG)}\big(\Bbbk, H^*(BK)\big) \\ \downarrow & & \downarrow \\ H^*(G'/K') & \longrightarrow & H^*(G/K) \end{array}$$

Proof. Let $T \subset K$ again be a maximal torus. We consider the diagram

$$\begin{array}{ccc} \mathbf{B}(\Bbbk, H^*(BG'), H^*(BK')) & \xrightarrow{\mathbf{B}(1,\varphi^*,\varphi^*)} & \mathbf{B}(\Bbbk, H^*(BG), H^*(BK)) \\ \downarrow{\scriptstyle \Theta_{G',K'}} & & \downarrow{\scriptstyle \Theta_{G,K}} \\ \mathbf{B}(\Bbbk, C^*(BG'), C^*(BK')) & \xrightarrow{\mathbf{B}(1,\varphi^*,\varphi^*)} & \mathbf{B}(\Bbbk, C^*(BG), C^*(BK)) \\ \downarrow{\scriptstyle 1\otimes\kappa^*\varphi^*} & & \downarrow{\scriptstyle 1\otimes\kappa^*} \\ \mathbf{B}(\Bbbk, C^*(BG'), C^*(BT)) & \xrightarrow{\mathbf{B}(1,\varphi^*,1)} & \mathbf{B}(\Bbbk, C^*(BG), C^*(BT)) \\ \downarrow{\scriptstyle 1\otimes f^*} & & \downarrow{\scriptstyle 1\otimes f^*} \\ \mathbf{B}(\Bbbk, C^*(BG'), H^*(BT)) & \xrightarrow{\mathbf{B}(1,\varphi^*,1)} & \mathbf{B}(\Bbbk, C^*(BG), H^*(BT)). \end{array} \tag{12.13}$$

We have to show that the top square in the diagram commutes in cohomology. By Lemma 12.3 (ii), it suffices to consider the prolongations of the maps in question to $\mathbf{B}(\Bbbk, C^*(BG), H^*(BT))$.

The composition along the path via $\mathbf{B}(\Bbbk, H^*(BG), H^*(BK))$ gives the map

$$\mathbf{B}\Lambda^G\,\mathbf{B}\varphi^* \otimes \kappa^*\varphi^* \tag{12.14}$$

by Lemma 12.3 (iii). Since the middle square in (12.13) commutes by naturality and the bottom square by construction, the same result shows that the path via $\mathbf{B}(\Bbbk, C^*(BG'), C^*(BK'))$ gives

$$\mathbf{B}\varphi^*\,\mathbf{B}\Lambda_{\boldsymbol{a}'} \otimes \kappa^*\,\varphi^*. \tag{12.15}$$

By Theorem 11.6 there is a $\mathfrak{k}_{BG}$-trivial homotopy h between the shm maps $\varphi^* \circ \Lambda_{\boldsymbol{a}'}$ and $\Lambda^G \circ \varphi^*$. In other words, $\mathbf{B}h$ is a $\mathfrak{k}_{BG}$-trivial coalgebra homotopy between $\mathbf{B}\Lambda^G\,\mathbf{B}\varphi^*$ and $\mathbf{B}\varphi^*\,\mathbf{B}\Lambda_{\boldsymbol{a}'}$. This implies that $\mathbf{B}h\otimes 1$ is a homotopy between the maps (12.14) and (12.15) and completes the proof. □

Corollary 12.6. *The isomorphism* (12.1) *does not depend on the chosen representatives $\boldsymbol{a}$ and $\boldsymbol{b}$.*

Proof. Take $\varphi\colon (G,K) \to (G,K)$ to be the identity map in Theorem 12.5. □

Remark 12.7. Baum [2, Ex. 4] has observed that there is no multiplicative isomorphism of the form (12.1) for the projective unitary group $PU(n) = U(n)/U(1)$ with $n \equiv 2 \pmod 4$ and $\Bbbk = \mathbb{Z}_2$. This is readily verified for $PU(2)$. Recall that the torsion product of graded commutative algebras is bigraded with the Tor-degree being non-positive. In the case at hand one obtains

$$\begin{array}{ccc|c} & \mathbb{Z}_2 & & 4 \\ & \mathbb{Z}_2 & \mathbb{Z}_2 & 2 \\ & & \mathbb{Z}_2 & 0 \\ \hline -2 & -1 & 0 & \end{array}\,. \tag{12.16}$$

Because the product respects bidegrees, the non-zero element in bidegree $(-1,2)$ squares to 0. This does of course not happen for the generator $x \in H^1(PU(2))$ as $PU(2) \cong SO(3) \approx \mathbb{RP}^3$.

The same counterexample shows that one cannot expect the isomorphism (12.1) to be natural if 2 is not invertible in $\Bbbk$. Consider the diagonal map

$$PU(2) = U(2) \,/\, U(1) \to \big(U(2)\times U(2)\big) \,/\, \big(U(1)\times U(1)\big) = PU(2)\times PU(2), \tag{12.17}$$

which induces the cup product in cohomology. Naturality of the isomorphism (12.1) would predict that the image of $x \otimes x$ in $H^2(PU(2))$ vanishes, which again is not the case.

Appendix A. Proof of Proposition 4.1

We only give a sketch of the proof. The computation is elementary, but somewhat lengthy because of the many cases to consider. In each case, we indicate whether the corresponding terms cancel against other terms or contribute to the final result. The notation "$\mathbf{X} \to \mathbf{Y}$" means that the terms $\mathbf{X}$ cancel with or result in the terms $\mathbf{Y}$.

We say that a term is split at position m if it is split between the m-th and the $(m+1)$-st argument. Similarly, two arguments are multiplied at position m if the arguments at positions m and $m+1$ are multiplied.

Terms produced by $d(h_{(n)})$

- **1.** Splitting of a term $f_{(i_s)}$
  - **1.1.** Term $f_{(i_s)}$, $1 \le s < k$, at any position (if $k \ge 2$) $\to$ **4.1.**
  - **1.2.** Term $f_{(i_k+l)}$ at position $1 \le m < i_k$ (if $i_k \ge 2$) $\to$ **4.2.**
  - **1.3.** Term $f_{(i_k+l)}$ at position $m = i_k$
    - **1.3.1.** $j_1 = 1$ and $l = 1 \to$ **10.1.**
    - **1.3.2.** $j_1 = 1$ and $l > 1 \to$ **2.2.2.**
    - **1.3.3.** $j_1 > 1 \to$ **3.3.2.**
  - **1.4.** Term $f_{(i_k+l)}$ at position $i_k + 1 \le m < i_k + l$ (if $l \ge 2$) $\to$ **12.**
- **2.** Splitting of a term $g_{(j_t)}$
  - **2.1.** Term $g_{(k+j_1)}$ at position $1 \le m < k$ (if $k \ge 2$) $\to$ **9.**
  - **2.2.** Term $g_{(k+j_1)}$ at position $m = k$
    - **2.2.1.** $i_k = 1$ and $k = 1 \to$ **11.1.**
    - **2.2.2.** $i_k = 1$ and $k > 1 \to$ **1.3.2.**
    - **2.2.3.** $i_k > 1 \to$ **4.3.2.**
  - **2.3.** Term $g_{(k+j_1)}$ at position $k + 1 \le m < k + j_1$ (if $j_1 \ge 2$) $\to$ **3.4.**
  - **2.4.** Term $g_{(j_t)}$, $1 < t \le l$, at any position (if $l \ge 2$) $\to$ **3.5.**
- **3.** Multiplication of two arguments of a term $f_{(i_s)}$
  - **3.1.** Term $f_{(i_s)}$, $1 \le s < k$, at any position $\to$ **5.**
  - **3.2.** Term $f_{(i_k+l)}$ at position $1 \le m < i_k \to$ **5.**
  - **3.3.** Term $f_{(i_k+l)}$ at position $m = i_k$
    - **3.3.1.** $i_k = 1$ and $k = 1 \to$ **11.2.**
    - **3.3.2.** $i_k = 1$ and $k > 1 \to$ **1.3.3.**
    - **3.3.3.** $i_k > 1 \to$ **4.3.3.**
  - **3.4.** Term $f_{(i_k+l)}$ at position $m = i_k + 1$ (if $l \ge 2$) $\to$ **2.3.**
  - **3.5.** Term $f_{(i_k+l)}$ at position $i_k + 1 < m < i_k + l$ (if $l \ge 3$) $\to$ **2.4.**
- **4.** Multiplication of two arguments of a term $g_{(j_t)}$
  - **4.1.** Term $g_{(k+j_1)}$ at position $1 \le m < k - 1$ (if $k \ge 3$) $\to$ **1.1.**

4.2. Term $g_{(k+j_1)}$ at position $m = k-1$ (if $k \geq 2$) $\to$ **1.2.**
4.3. Term $g_{(k+j_1)}$ at position $m = k$
4.3.1. $j_1 = 1$ and $l = 1 \to$ **10.2.**
4.3.2. $j_1 = 1$ and $l > 1 \to$ **2.2.3.**
4.3.3. $j_1 > 1 \to$ **3.3.3.**
4.4. Term $g_{(k+j_1)}$ at position $k+1 \leq m < k+j_1 \to$ **7.**
4.5. Term $g_{(j_t)}$, $1 < t \leq l$, at any position $\to$ **8.**

Terms appearing in $h_{(n-1)}(\dots, a_m a_{m+1} \otimes b_m b_{m+1}, \dots)$

5. $m \leq i_1 + \dots + i_{k-1}$ (if $k \geq 2$) $\to$ **3.1.**
6. $i_1 + \dots + i_{k-1} < m \leq i_1 + \dots + i_k \to$ **3.2.**
7. $i_1 + \dots + i_k < m \leq i_1 + \dots + i_k + j_1 \to$ **4.4.**
8. $i_1 + \dots + i_k + j_1 > m$ (if $l \geq 2$) $\to$ **4.5.**

Terms appearing in $\big((1 \otimes g)(f \otimes 1)\big)_{(k)} \cdot h_{(n-k)}$

9. $k < n \to$ **2.1.**
10. $k = n$
10.1. $i_n = 1 \to$ **1.3.1.**
10.2. $i_n > 1 \to$ **4.3.1.**

Terms appearing in $h_{(k)} \cdot \big((f \otimes 1)(1 \otimes g)\big)_{(n-k)}$

11. $k = 0$
11.1. $j_1 = 1 \to$ **2.2.1.**
11.2. $j_1 > 1 \to$ **3.3.1.**
12. $k > 0 \to$ **1.4.**

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