ELEMENTARY PROOFS OF GENERALIZED CONTINUED FRACTION FORMULAE FOR e

ZHENTAO LU

ABSTRACT. In this short note we prove two elegant generalized continued fraction formulae

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{2}{3 + \frac{3}{4 + \cdots}}}}$$

and

$$e = 3 + \frac{-1}{4 + \frac{-2}{5 + \frac{-3}{6 + \frac{-4}{7 + \cdots}}}}$$

using elementary methods. The first formula is well-known, and the second one is newly-discovered in arXiv:1907.00205 [cs.LG]. We then explore the possibility of automatic verification of such formulae using computer algebra systems (CAS's).

1. Introduction

We write a (generalized) continued fraction in the form

(1)
$$z = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \frac{a_4}{b_4 + \cdots}}}},$$

with the convergents

$$z_0 = \frac{A_0}{B_0} = b_0, z_1 = \frac{A_1}{B_1} = \frac{b_1 b_0 + a_1}{b_1}, z_2 = \frac{A_2}{B_2} = \frac{b_2 (b_1 b_0 + a_1) + a_2 b_0}{b_2 b_1 + a_2}, \dots$$

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In general, as stated in [1], A_n and B_n satisfies

(3)
$$A_n = b_n A_{n-1} + a_n A_{n-2}, B_n = b_n B_{n-1} + a_n B_{n-2}, n \ge 1, A_{n-1} = 1, A_0 = b_0, B_{n-1} = 0, B_0 = 1.$$

Thus, there is a straightforward way to verify a proposed continued fraction formula. Given $\{a_n\}$, $\{b_n\}$, one first computes A_n and B_n , and then computes $\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{A_n}{B_n}$.

2. Proofs of two formulae of e

2.1. **The first formula.** We first compute

(4)
$$w = 2 + \frac{1}{1 + \frac{1}{2 + \frac{2}{3 + \frac{1}{4}}}} = 2 + \frac{1}{1 + \frac{\frac{1}{2}}{1 + \frac{\frac{1}{3}}{1 + \frac{\frac{1}{4}}{1 + \frac{1}{4}}}}}.$$

In this case we have

(5)
$$a_n = \frac{1}{n}, b_n = 1, n \ge 1.$$

Hence

(6)
$$A_{-1} = 1, A_0 = 2, B_{-1} = 0, B_0 = 1, A_n = A_{n-1} + \frac{1}{n} A_{n-2}, B_n = B_{n-1} + \frac{1}{n} B_{n-2}, n \ge 1.$$

It is trivial to verify by induction that

$$(7) A_n = n + 2, n > 0.$$

For B_n , we make the auxiliary sequence

(8)
$$k_n = \frac{B_{n-2}}{n} - \frac{B_{n-3}}{n-1}, n \ge 2.$$

Then we have $k_{n+2} = \frac{1}{(n+2)(n+1)}k_n$ by (14). And in turn this shows $k_n = (-1)^n \frac{1}{n!}$. Now by (8) and (14) we have

(9)
$$\sum_{i=2}^{n} k_i = \frac{B_{n-2}}{n},$$

hence

(10)
$$B_n = (n+2) \sum_{i=2}^{n+2} k_i = (n+2) \sum_{i=2}^{n+2} (-1)^i \frac{1}{i!},$$

hence

(11)
$$w = \lim_{n \to \infty} \frac{A_n}{B_n} = \lim_{n \to \infty} \frac{1}{\sum_{i=2}^{n+2} (-1)^i \frac{1}{i!}} = \frac{1}{e^{-1}} = e.$$

2.2. The second formula. Now we compute

(12)
$$v = 3 + \frac{-1}{4 + \frac{-2}{5 + \frac{-3}{6 + \frac{-4}{7 + \cdots}}}}.$$

This formula (v = e) is recently numerically discovered in [2] using machine learning techniques. In this case we have

(13)
$$a_n = -n, \quad b_n = n+3, \quad n \ge 1.$$

Hence

(14)

$$A_{-1} = 1, \quad A_0 = 3, \quad B_{-1} = 0, \quad B_0 = 1,$$

 $A_n = (n+3)A_{n-1} - nA_{n-2}, \quad B_n = (n+3)B_{n-1} - nB_{n-2}, \quad n \ge 1.$

This time it is trivial to verify by induction that $B_n = \frac{(n+1)^2}{n} B_{n-1}$, so

(15)
$$B_n = \frac{(n+1)!^2}{n!} = (n+1) \cdot (n+1)!, \quad n \ge 1.$$

For A_n , an observation of the first few terms suggests that it is a shifted version of the sequence A001339 in [3]. Then it is routine to verify that

(16)
$$A_n = \sum_{k=0}^{n+1} (k+1)! \binom{n+1}{k}, \quad n \ge 1,$$

by induction argument. ¹

With the expression of A_n and B_n known, we get

(18)
$$v_n = \frac{A_n}{B_n} = \sum_{k=0}^{n+1} \frac{(k+1)}{(n+1)\cdot(n+1-k)!} = \sum_{k=0}^{n+1} \frac{(n+2-k)}{(n+1)\cdot k!}.$$

(17)
$$A_n = \sum_{k=0}^{n+1} (n+2-k)! \binom{n+1}{n+1-k}.$$

¹We find that the inductive step is easier to carry out if using the equivalent expression

Hence

(19)
$$v = \lim_{n \to \infty} v_n = \lim_{n \to \infty} \frac{n+2}{n+1} \sum_{k=0}^{n+1} \frac{1}{k!} - \frac{1}{n+1} \sum_{k=0}^{n} \frac{1}{k!} = e.$$

3. Computer-aided verification of generalized continued fraction formulae

Given a generalized continued fraction formula, we propose the following work flow to verify it:

- Step 1. extract the $\{a_n\}, \{b_n\}$ terms.
- Step 2. Get the recursive formulae of A_n and B_n using (3).
- Step 3. Compute the first few terms of A_n and B_n and using WolframAlpha and OEIS to guess a closed-form of them.
- Step 4. Using mathematical induction to prove the closed-form expression.
- Step 5. Compute the limit and check against the proposed formula.

Contact: Zhentao Lu, zhentao@sas.upenn.edu

References

- 1. William B Jones and Wolfgang J Thron, Continued fractions: Analytic theory and applications, Cambridge University Press, 1984.
- 2. Gal Raayoni, George Pisha, Yahel Manor, Uri Mendlovic, Doron Haviv, Yaron Hadad, and Ido Kaminer, *The ramanujan machine: Automatically generated conjectures on fundamental constants*, arXiv preprint arXiv:1907.00205 (2019).
- 3. Neil JA Sloane et al., The on-line encyclopedia of integer sequences, 2003, Available at https://oeis.org/.