The automorphism groups of some token graphs

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Abstract

In this paper we obtain the automorphism groups of the token graphs of some graphs. In particular we obtain the automorphism group of the k-token graph of the path graph P_n, for $n \neq 2k$. Also, we obtain the automorphism group of the 2-token graph of the following graphs: cycle, star, fan and wheel graphs.

1 Introduction.

Let Γ be a simple graph of order n. Let $1 \leq k \leq n-1$ be an integer. The *k-token graph* $F_k(\Gamma)$ of Γ is defined as the graph with vertex set all k-subsets of $V(\Gamma)$, where two vertices are adjacent in $F_k(\Gamma)$ whenever their symmetric difference is an edge of Γ. If $k \in \{1, n-1\}$, then $F_k(\Gamma)$ is isomorphic to Γ and in this case we say that $F_k(\Gamma)$ is a *trivial token graph* of Γ. In fact, if Γ is a graph of order n, then $F_k(\Gamma) \simeq F_{n-k}(\Gamma)$.

The token graphs have been redefined several times and with different names. When $k = 2$, this class of graphs are called *double vertex graphs* that were widely studied by Alavi et al. [1, 2, 3, 4] and are the same that the 2-subgraph graphs defined in a thesis of G. Johns [19]. In the work of Zhu et al. [31], the k-token graphs are named *n-tuple vertex graphs*. Later, T. Rudolph [29] redefined the double vertex graphs with the name of *symmetric powers of graphs* with the idea to study the graph isomorphism problem and some problems in quantum mechanics. There are several papers related with Rudolph's work, see, e. g., [6, 7, 8, 15], and the references therein. Some of them motivated by the connection between token graphs and the Heisenberg Hamiltonian (see, e. g., [26] and the references therein), that is related with the Heisenberg model [17], a quantum theory of ferromagnetism.

Finally, R. Fabila-Monroy, et. al. [14], in an independent way, reintroduce this concept but now with the name of token graphs and began a systematic study of several combinatorial properties of this graphs: connectivity, diameter, cliques, chromatic number and Hamiltonian paths. In the last years, several groups of authors have continued with this line of research (see, e.g., [5, 11, 12, 18, 22, 28]). For example, Carballosa et al. [10] studied the planarity and regularity of token graphs and Leaños and Trujillo-Negrete [22] proved a conjecture of Fabila-Monroy, et. al [14] about the connectivity of token graphs.

When Γ is the complete graph K_n, the k-token graph $F_k(\Gamma)$ is isomorphic to the Johnson graph $J(n, k)$ [20]. To the knowledge of the authors, the only results about the automorphism groups of token graphs are about Johnson graphs. It is known that if $n \neq 2k$, then $\mathrm{Aut}(F_k(K_n)) \simeq S_n$ and if $n = 2k$, then $\mathrm{Aut}(F_k(K_n)) \simeq S_2 \times S_n$, where S_n denotes the symmetric group on n symbols (see., e.g. [21, 25, 27]).

In this work, we study the automorphism group of other token graphs. Our main results can be stated as two theorems:

Theorem 1.1. *Let $n \neq 4$ be an integer. If $\Gamma \in \{C_n, K_{1,n}, A_{1,n}, W_{1,n}\}$, then*

$$\mathrm{Aut}(F_2(\Gamma)) = \mathrm{Aut}(\Gamma),$$

where $C_n, K_{1,n}, A_{1,n}$ and $W_{1,n}$, denotes the cycle, star, fan and wheel graphs, respectively.

Theorem 1.2. *Let P_n be the path graph of order $n \neq 2k$. Then $\mathrm{Aut}(F_k(P_n)) = \mathrm{Aut}(P_n)$.*

Theorem 1.1 is not true in general. For example, the automorphism group of the grid graph $G_{2,3}$ is of order 4 but $|\mathrm{Aut}(F_2(G_{2,3}))| = 8$.

In the proofs of our results, we use elementary group theory, as in [23, 24, 25], and properties of token graphs. For the case of the token graphs of path graphs we obtain a formula for the distance between pair of vertices in $F_k(P_n)$ that generalizes the one given by Beaula et al. [9].

Our interest in the properties of $F_k(P_n)$ began with the work of Gómez Soto et al. [16], were the authors showed that $F_2(P_n)$ is isomorphic to a graph defined by Sloane in the comments of sequence A085680 in the Online Encyclopedia of Integer Sequences (OEIS) [30]. The Sloane's graph is related with error-correcting codes. In fact Gómez Soto et al. [16] found the packing number of the 2-token graph of the path graph, that is equal to the size of largest binary code of length n and constant weight 2 that can correct a single adjacent transposition (sequence A085680 in [30]).

The outline of this paper is as follows. In Section 2 we present some definitions, notation and some preliminary results. We show that $\mathrm{Aut}(\Gamma)$ is a subgroup of $\mathrm{Aut}(F_k(\Gamma))$, for every graph Γ. Also, we show that if $n = 2k$, then $|\mathrm{Aut}(F_k(\Gamma))| \geq 2|\mathrm{Aut}(\Gamma)|$. The proof of Theorem 1.1 is worked for each case separately. In Section 3 we show that $\mathrm{Aut}(F_2(C_n)) = \mathrm{Aut}(C_n))$, for $n \neq 4$. In Section 4 we prove that if $\Gamma \in \{K_{1,n}, A_{1,n}, W_{1,n}\}$, then $\mathrm{Aut}(F_2(\Gamma)) = \mathrm{Aut}(\Gamma)$, for $n \neq 4$. In Section 5 we prove Theorem 1.2.

2   Preliminaries and first results

In this paper, all our graphs are simple and finite, that is, a *graph* Γ is a pair $(V(\Gamma), E(\Gamma))$ where $V(\Gamma)$ is a finite set and $E(\Gamma)$ is a subset of the set of all 2-subsets of $V(\Gamma)$. An edge of a graph Γ will be denoted by $\{u, v\}$ or uv, where $u, v \in V(\Gamma)$. We use $u \sim v$ to indicate that u and v are adjacent vertices, that is $uv \in E(\Gamma)$. The *neighborhood* of a vertex v is defined as $N(v) = \{u \in V(\Gamma) : uv \in E(\Gamma)\}$ and the *degree $d(v)$* of v is defined as $|N(v)|$. The neighborhood of a set of vertices X is defined as $N(X) = \bigcup_{x \in X} N(x) \setminus X$. Let U be a subset of $V(\Gamma)$, we will use $\Gamma\langle U \rangle$ to denote the subgraph of Γ induced by U. The graph difference $\Gamma - U$ is defined as the graph $\Gamma\langle V(\Gamma) \setminus U \rangle$.

Let Γ_1 and Γ_2 be two simple graphs. An *isomorphism* of Γ_1 onto Γ_2 is a bijection $\phi : V(\Gamma_1) \to V(\Gamma_2)$ such that $uv \in E(\Gamma_1)$ if and only if $\phi(u)\phi(v) \in E(\Gamma_2)$. An automorphism of a graph Γ is an isomorphism of Γ onto itself. The set of all automorphism of a graph Γ is a subgroup of $\mathrm{Sym}(V(\Gamma))$, the group of all permutations of $V(\Gamma)$, an is denoted by $\mathrm{Aut}(\Gamma)$. To obtain the automorphism group of graphs in general is a difficult problem. But it is possible to obtain this group for particular cases.

It is well-known that $\mathrm{Aut}(\Gamma)$ acts on $V(\Gamma)$. Let $v \in V(\Gamma)$, the *orbit* of v is defined as $O(v) = \{f(v) : f \in \mathrm{Aut}(\Gamma)\}$ and the *stabilizer* of v is $\mathrm{Stab}(x) = \{f \in \mathrm{Aut}(\Gamma) : f(x) = x\}$. The orbit-stabilizer theorem says that $|\mathrm{Aut}(\Gamma)| = |O(v)||\mathrm{Stab}(v)|$, for every $v \in V(\Gamma)$. Let $f \in \mathrm{Aut}(\Gamma)$, we say that a vertex $x \in V(\Gamma)$ is a *fixed point* of f if $f(x) = x$. We use $\mathrm{Fix}(f)$

to denote the set of fixed points of f. As usual, sometimes we write $\mathrm{Aut}(\Gamma) = G$ instead of $\mathrm{Aut}(\Gamma) \simeq G$. We use S_n to denote the symmetric group over $\{1, \ldots, n\}$.

The following observation (see [14]) will be used, sometimes without reference, when we compute the degree of vertices in $F_k(\Gamma)$.

Observation 2.1. *The degree of a vertex A in $F_k(\Gamma)$ is equal to the number of edges between A and $V(G) \setminus A$.*

In particular if $\{x, y\} \in V(F_2(\Gamma))$, then $d(\{x, y\}) = d(x) + d(y)$ if $x \nsim y$ and $d(\{x, y\}) = d(x) + d(y) - 2$ if $x \sim y$.

The following result, that appears in [10] and [14], will be useful in some of the proofs.

Proposition 2.2. *Let X be a subset of $V(\Gamma)$ and $\Gamma' = \Gamma - X$. Then $F_k(\Gamma')$ is isomorphic to the graph obtained from $F_k(\Gamma)$ by deleting the vertices A in $F_k(\Gamma)$ such that A has al least one element of X.*

2.1 First results

Our first result shows an important relation between $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(F_k(\Gamma))$.

Theorem 2.3. *Let Γ be a graph. Then $\mathrm{Aut}(\Gamma)$ is isomorphic to a subgroup of $\mathrm{Aut}(F_k(\Gamma))$. In fact, if $\theta \in \mathrm{Aut}(\Gamma)$, then the function $f_\theta \colon V(F_k(\Gamma)) \to V(F_k(\Gamma))$ defined as $f_\theta(\{v_1, \ldots, v_k\}) = \{\theta(v_1), \ldots, \theta(v_k)\}$ is an automorphism of $F_k(\Gamma)$.*

Proof. For $\phi \in \mathrm{Aut}(\Gamma)$ we define a function $f_\phi \colon V(F_k(\Gamma)) \to V(F_k(\Gamma))$ as follows: given $\{a_1, \ldots, a_k\} \in V(F_k(\Gamma))$, let

$$f_\phi(\{a_1, \ldots, a_k\}) = \{\phi(a_1), \ldots, \phi(a_k)\}.$$

We will show that f_ϕ is an automorphism of $F_k(\Gamma)$. The function f_ϕ is a bijection because $f_{\phi^{-1}}$ is the inverse for f_ϕ, where ϕ^{-1} is the inverse of ϕ. Since f_ϕ is a bijection, $f_\phi(A) \triangle f_\phi(B) = \{\phi(a), \phi(b)\}$ if and only if $A \triangle B = \{a, b\}$. Therefore, using that ϕ is an automorphism of Γ we have that AB is an edge of $F_k(\Gamma)$ if and only if $f_\phi(A)f_\phi(B)$ is an edge of $F_k(\Gamma)$. Now, let $\varphi \colon \mathrm{Aut}(\Gamma) \to \mathrm{Aut}(F_k(\Gamma))$ be the function defined by $\varphi(\phi) = f_\phi$, for every $\phi \in \mathrm{Aut}(\Gamma)$. It is an easy exercise to show that φ is an injective homomorphism of groups. Therefore $\mathrm{Aut}(\Gamma)$ is isomorphic to a subgroup of $\mathrm{Aut}(F_k(\Gamma))$. $\square$

For a given $\theta \in \mathrm{Aut}(\Gamma)$, the automorphism f_θ of $F_k(\Gamma)$ defined in previous theorem is called the *automorphism induced* by θ. When the context is clear, we use $\mathrm{Aut}(\Gamma)$ as the set of automorphism of Γ or as the subgroup of $\mathrm{Aut}(F_k(\Gamma))$ induced by the automorphisms of Γ. We write $\mathrm{Aut}(F_k(\Gamma)) = \mathrm{Aut}(\Gamma)$ to mean that every automorphism of $F_k(\Gamma)$ is induced by some automorphism of Γ.

Now, the following proposition shows that for $n = 2k$, $\mathrm{Aut}(F_k(\Gamma))$ has always more elements that $\mathrm{Aut}(\Gamma)$.

Theorem 2.4. *Let Γ be a graph of order n, with n even. The function $f_c \colon V(F_{n/2}(\Gamma)) \to V(F_{n/2}(\Gamma))$ defined as $f_c(A) = A^c$ is an automorphism, where $A^c = V(\Gamma) \setminus A$. Even more f_c is not an induced automorphism of any $\phi \in \mathrm{Aut}(\Gamma)$.*

Proof. The proof is exactly the same that the given in the proof of Theorem 3.5 in [25] for the case when G is the complete graph K_n (the graph $F_k(K_n)$ is isomorphic to the Johnson graph $J(n, k)$). $\square$

Clearly, the function f_c in previous theorem is a fixed point free involution.

Corollary 2.5. *Let $n \geq 4$ be an even integer. If Γ is a graph of order n, then*

$$|\mathrm{Aut}(F_{n/2}(\Gamma))| \geq 2|\mathrm{Aut}(\Gamma)|.$$

3   Automorphism group of the 2-token graph of cycle graphs

In this section we prove that $\mathrm{Aut}(F_2(C_n)) = \mathrm{Aut}(C_n)$, for $n \neq 4$. In Figure 1 we show $F_2(C_7)$. Let D_{2n} denote the dihedral group of $2n$ elements. It is well-known that $\mathrm{Aut}(C_n) = D_{2n}$. Using computer software, we obtain that $\mathrm{Aut}(F_2(C_4)) = S_2 \times D_8$. First we present some observations and results that will be useful. In this section, $V(C_n) = \{1, 2 \ldots, n\}$ and $E(C_n) = \{\{i, i+1\} \colon 1 \leq i \leq n-1\} \cup \{\{1, n\}\}$.
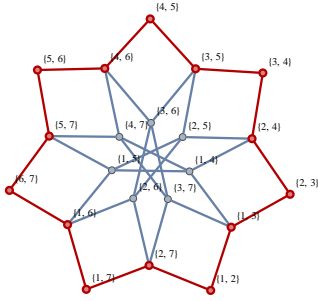

Figure 1: The 2-token graph of C_7, where $V(C_7) = \{1, \ldots, 7\}$. The red subgraph is induced by $L_1 \cup L_2$.
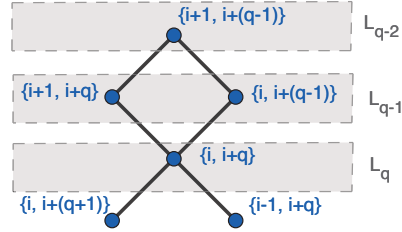
Figure 2: An illustration of Proposition 3.2(4).

Observation 3.1. *Let $n \geq 4$ be an integer.*

1. *If $v \in F_2(C_n)$, then $d(v) \in \{2, 4\}$.*

2. *For every pair of vertices $u, v \in F_2(C_n)$, $|N(u) \cap N(v)| \leq 2$*

We use $i \oplus j$ and $i \ominus j$ to denote the sum $(i + j) \bmod n$ and $(i - j) \bmod n$, respectively, with the convention that $n \equiv n \pmod{n}$. Let $r = \lfloor n/2 \rfloor$. We define the following subsets of $V(F_2(C_n))$.

$$L_q = \{\{i, i \oplus q\} \colon 1 \leq i \leq n\},$$

where $1 \leq q \leq r$. Note that $L_1 = \{w \in V(F_2(C_n)) \colon d(w) = 2\}$. For example in $F_2(C_7)$ (see Figure 1) we have that

$$
\begin{aligned}
L_1 &= \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,7\}, \{7,1\}\} \\
L_2 &= \{\{1,3\}, \{2,4\}, \{3,5\}, \{4,6\}, \{5,7\}, \{6,1\}, \{7,2\}\} \\
L_3 &= \{\{1,4\}, \{2,5\}, \{3,6\}, \{4,7\}, \{5,1\}, \{6,2\}, \{7,3\}\}.
\end{aligned}
$$

Proposition 3.2. *Let $n \geq 3$ be an integer and $r = \lfloor n/2 \rfloor$.*

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1. *If n is even, then $|L_{n/2}| = n/2$ and $|L_q| = n$, for $1 \le q < r$.*

2. *If n is odd, then $|L_q| = n$, for $1 \le q \le r$.*

3. *The set $L = \{L_1, \ldots, L_r\}$ is a partition of $V(F_2(C_n))$.*

4. *Let $n \ge 6$ and $3 \le q \le r$. If $\{i, i \oplus q\} \in L_q$, with $1 \le i \le n$, then two neighbors, say B, C, of $\{i, i \oplus q\}$ belongs to L_{q-1} and the vertex in $N(B) \cap N(C) \setminus \{\{i, i \oplus q\}\}$ belongs to L_{q-2}.*

5. *Let $v \in V(F_2(C_n))$, then $d(v) = 2$ if and only if $v \in L_1$.*

Proof. First we prove (1) and (2). Note that for n even, $L_{n/2} = \{\{i, i+n/2\} \colon 1 \le i \le n/2\}$, that is $|L_{n/2}| = n/2$. Now, if $i \ne j$, then the equality $\{i, i \oplus q\} = \{j, j \oplus q\}$ implies that $q = n/2$. Therefore, if n is even, then $|L_q| = n$, for $1 \le q < r$, and if n is odd, then $|L_q| = n$, for every $1 \le q \le r$.

Second we prove (3). If $n = 3$, then $L_1 = V(F_2(C_3))$. Let $n \ge 4$. First we prove, by contradiction, that L is a set of pairwise disjoint sets. Suppose that $L_p \cap L_q \ne \emptyset$, with $p \ne q$. Let $A \in L_p \cap L_q$. Then $A = \{i, i \oplus p\} = \{j, j \oplus q\}$, with $1 \le i, j \le n$. As $p \ne q$, then $i = j \oplus q$ and $i \oplus p = j$, which implies that $q + p = n$. but this means that one of p or q is greater that $\lfloor n/2 \rfloor$, a contradiction. Finally, we prove that $\bigcup_{i=1}^r L_i = V(F_2(C_n))$. It is clear that $\bigcup_{i=1}^r L_i \subseteq V(F_2(C_n))$. Let $\{x, y\} \in V(F_2(C_n))$. Let $a = \min\{x, y\}$ and $b = \max\{x, y\}$. Let $d = b - a$. If $d \le r$, then $\{x, y\} \in L_d$. If $d > r$, then $\{x, y\} \in L_{n-d}$ because $\{b, b \oplus (n - d)\} = \{b, b - d\} = \{b, a\} = \{x, y\}$.

Now we prove (4) (see Figure 2). If $q \in \{3, \ldots, r\}$ and $\{i, i \oplus q\} \in L_q$, then

$$N(\{i, i \oplus q\}) = \{\{i \ominus 1, i \oplus q\}, \{i, i \oplus (q+1)\}, \{i \oplus 1, i \oplus q\}, \{i, i \oplus (q-1)\}\},$$

By the hypothesis over q it follows that $\{\{i \oplus 1, i \oplus q\}, \{i, i \oplus (q-1)\}\} \subset L_{q-1}$. Note that $N(\{i \oplus 1, i \oplus q\}) \cap N(\{i, i \oplus (q-1)\}) = \{\{i \oplus 1, i \oplus (q-1)\}, \{i, i \oplus q\}\}$ with $\{i \oplus 1, i \oplus (q-1)\} \in L_{q-2}$.

The proof of (5) is easy.

$\square$

In Figure 1 we show the subgraph of $F_2(C_7)$ induced by $L_1 \cup L_2$.

Proposition 3.3. *Let $n \ge 6$. The subgraph of $F_2(C_n)$ induced by $L_1 \cup L_2$ is isomorphic to C_{2n}.*

Proof. First note that if $\{i, i \oplus 1\}$ in L_1, then

$$N(\{i, i \oplus 1\}) = \{\{i \ominus 1, i \oplus 1\}, \{i, i \oplus 2\}\},$$

and if $\{i, i \oplus 2\}$ in L_2, then

$$N(\{i, i \oplus 2\}) = \{\{i, i \oplus 1\}, \{i, i \oplus 3\}, \{i \oplus 1, i \oplus 2\}, \{i \ominus 1, i \oplus 2\}\}.$$

Since $n \ge 6$, then $N(\{i, i \oplus 1\}) \subset L_2$, $N(\{i, i \oplus 2\}) \cap L_1 = \{\{i, i \oplus 1\}, \{i \oplus 1, i \oplus 2\}\}$ and $N(\{i, i \oplus 2\}) \cap L_3 = \{\{i, i \oplus 3\}, \{i \ominus 1, i \oplus 2\}\}$. Thus, it is easy to check that the function $\phi \colon V(L_1 \cup L_2) \to V(C_{2n})$ given by $\phi(\{i, i \oplus 1\}) = 2i - 1$, for every $\{i, i \oplus 1\} \in L_1$, and $\phi(\{i, i \oplus 2\}) = 2i$, for every $\{i, i \oplus 2\} \in L_2$, is a graph isomorphism. $\square$

We also need the following well-known observation.

Proposition 3.4. *If $f \in Aut(C_n)$ fixes two adjacent vertices on C_n, then $f = id$.*

Now we present our main result in this section.

Theorem 3.5. *Let $n \geq 3$ be an integer. If $n \neq 4$, then $\text{Aut}(F_2(C_n)) = \text{Aut}(C_n)$.*

Proof. If $n = 3$, then $F_2(C_3) \simeq C_3$ and the result follows for this case. Suppose now that $n \geq 5$. Let $\Gamma = F_2(C_n)$. In the view of Theorem 2.3, it is enough to show that $|\text{Aut}(\Gamma)| \leq 2n$. Let $x \in V(\Gamma)$ be the vertex $\{1, n\}$ and let $\Gamma_1 = \Gamma \langle N(x) \rangle$. Since $N(x) = \{\{1, n-1\}, \{2, n\}\}$, then $|\text{Aut}(\Gamma_1)| = 2$. Let $\varphi \colon \text{Stab}(x) \to \text{Aut}(\Gamma_1)$ be the function defined by $\varphi(f) = f|_{\Gamma_1}$. As φ is a group homomorphism we have that $|\text{Stab}(x)| \leq |\text{Ker } \varphi||\text{Aut}(\Gamma_1)| \leq 2|\text{Ker } \varphi|$. We will prove that $\text{Ker } \varphi = \{id\}$.

Let $f \in \text{Ker } \varphi$. By Proposition 3.2(3) it is enough to show that $f(L_q) \subset \text{Fix}(f)$, for every $q \in \{1, \dots, \lfloor n/2 \rfloor\}$.

First we prove that $f(L_1 \cup L_2) \subset \text{Fix}(f)$. Let $\Gamma_2 = \langle L_1 \cup L_2 \rangle$. By Proposition 3.3 it follows that $\Gamma_2 \simeq C_{2n}$. Note that $f(L_1 \cup L_2) = L_1 \cup L_2$. Indeed, L_1 is equal to the set of vertices of degree 2 in Γ and the vertices in L_2 are the unique vertices in Γ that have two of its neighbors in L_1 (see Figure 1 for an example). Then $f|_{L_1 \cup L_2} \in \text{Aut}(\Gamma_2)$. Since $f \in \text{Ker } \varphi$ and $f \in \text{Stab}(x)$, we have that $f(\{1, n\}) = \{1, n\}$, $f(\{1, n-1\}) = \{1, n-1\}$ and $f(\{2, n\}) = \{2, n\}$. But $\{1, n\}$ and $\{2, n\}$ are adjacent vertices in Γ and then all the vertices in $L_1 \cup L_2$ are fixed by f (by Proposition 3.4). If $n = 5$ we are done. For $n \geq 6$ we will prove that $f(L_q) \subset \text{Fix}(f)$, for $3 \leq q \leq \lfloor n/2 \rfloor$. Suppose by induction that $f(L_j) \subset \text{Fix}(f)$, for $2 \leq j < q \leq \lfloor n/2 \rfloor$. Let $u \in L_q$, that is $u = \{i, i \oplus q\}$, for some $i \in \{1, \dots, n\}$. By Proposition 3.2(4), we have that two neighbors of u, say v and w, belongs to L_{q-1} and the vertex z in $N(v) \cap N(w) \setminus \{u\}$ belongs to L_{q-2} (see Figure 2). By hypothesis, $\{v, w, z\} \subset \text{Fix}(f)$ which implies that $f(u) \in N(v) \cap N(w)$ and hence $f(u) = u$ (we are using Observation 3.1 that shows that $|N(v) \cap N(w)| \leq 2$). Then $f(L_q) \subset \text{Fix}(f)$ as desired.

Therefore $\text{Ker } \varphi = \{id\}$ and hence $|\text{Stab}(x)| \leq 2$. Now, since $x \in L_1$, by Proposition 3.2(5) it follows that $|O(x)| \leq n$. Finally, by the orbit-stabilizer theorem we obtain $|\text{Aut}(\Gamma)| = |O(x)||\text{Stab}(x)| \leq 2n$ which concludes the proof of the theorem. $\square$

The following conjecture is based on experimental results obtained by computer.

Conjecture 3.6. *Let n be an integer and $3 \leq k \leq n/2$. If $k \neq n/2$ then $\text{Aut}(F_k(C_n)) = \text{Aut}(C_n)$ and if $k = n/2$, then $\text{Aut}(F_k(C_n)) = S_2 \times \text{Aut}(C_n)$.*

4 Automorphism groups of the 2-token graphs of star, fan and wheel graphs

In this section we obtain the automorphism groups of the 2-token graphs of star, fan and wheel graphs. In all such cases we have that if $|G| \geq 5$, then $\text{Aut}(F_2(G)) = \text{Aut}(G)$.

Star graphs

First we consider the case of the star graph $K_{1,n-1}$. In Figure 3 we show $F_2(K_{1,6})$. For small star graphs we have that $F_2(K_{1,2}) \simeq P_3$ and $F_2(K_{1,3}) \simeq C_6$ and hence $\text{Aut}(F_2(K_{1,2})) \simeq S_2$ and $\text{Aut}(F_2(K_{1,3})) \simeq D_6$. In this section $V(K_{1,n-1}) = \{1, 2, \dots, n\}$, where 1 is the vertex of degree $n - 1$.
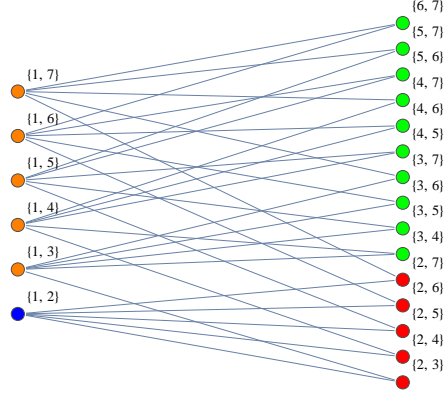
Figure 3: The 2-token graph of $K_{1,6}$ ($V(K_{1,6}) = \{1,\ldots,7\}$, with $d(1) = 7$). The set B, R, O, G defined in the proof of Theorem 4.1 are the set of blue, red, orange and green vertices, respectively.

Theorem 4.1. *Let $n \geq 5$ be an integer. Then* $\mathrm{Aut}(F_2(K_{1,n-1})) = \mathrm{Aut}(K_{1,n-1})$.

Proof. Let $\Gamma = F_2(K_{1,n-1})$. Since graph $K_{1,n-1}$ is bipartite, it follows that Γ is also bipartite ([14, Proposition 1]). In fact, a bipartition of $V(\Gamma)$ is $\{\mathcal{B}, \mathcal{R}\}$, where

$$\mathcal{B} = \{\{1,i\}\colon 2 \leq i \leq n\},$$

and $\mathcal{R} = V(\Gamma)) \setminus \mathcal{B}$. Note that if $x \in \mathcal{B}$, then $d(x) = n - 2$ and if $x \in \mathcal{R}$, then $d(x) = 2$ (see Figure 3 for an example). Since $n \geq 5$, then Γ has exactly $n - 1$ vertices of degree $n - 2$. Let $B = \{\{1,2\}\}$ and $O = \{\{1,i\}\colon 3 \leq i \leq n\}$. Clearly $\{B, O\}$ is a partition of $\mathcal{B}$. Let $R = \{\{2,i\}\colon 3 \leq i \leq n\}$ and $G = \{\{i,j\}\colon 3 \leq i < j \leq n\}$. Clearly $R = N(\{1,2\})$ and $\{R, G\}$ is a partition of $\mathcal{R}$. In Figure 3 we show the partition $\{B, O, R, G\}$ of $V(\Gamma)$.

Now, by Theorem 2.3, $\mathrm{Aut}(K_{1,n-1}) \leq \mathrm{Aut}(\Gamma)$. It is well-known that $\mathrm{Aut}(K_{1,n-1}) = S_{n-1}$ and hence it is enough to show that $|\mathrm{Aut}(\Gamma)| \leq (n-1)!$

Let x be the vertex $\{1,2\}$ in Γ and let $\Gamma_1 = \Gamma\langle N(x)\rangle$. In this case $N(x) = R$ and $\Gamma_1 = \overline{K_{n-2}}$. Therefore $\mathrm{Aut}(\Gamma_1) = S_{n-2}$. We have that $|O(x)| \leq n-1$ because x has degree $n-2$ and there are exactly $n-1$ vertices in Γ of degree $n-2$. If an automorphism f of Γ belongs to $\mathrm{Stab}(x)$, then $f(N(x)) = N(x)$ and hence $f|_{N(x)} \in \mathrm{Aut}(\Gamma_1)$. Let $\varphi\colon \mathrm{Stab}(x) \to \mathrm{Aut}(\Gamma_1)$ be the function defined by $\varphi(f) = f|_{N(x)}$. We will prove that $\mathrm{Ker}\,\varphi = \{id\}$.

Let $f \in \mathrm{Ker}\,\varphi$. We will prove that $f(Y) \subset \mathrm{Fix}(f)$, for every $Y \in \{B, O, R, G\}$ in the following order: B, R, O, G. Since $f \in \mathrm{Stab}(x)$ and $f \in \mathrm{Ker}\,\varphi$, then $f(\{1,2\}) = \{1,2\}$ and $f(y) = y$, for every $y \in N(x)$, respectively. That is, $f(B) \cup f(R) \subset \mathrm{Fix}(f)$. Now we prove that $f(O) \subset \mathrm{Fix}(f)$. Let $w \in O$, that is $w = \{1,i\}$, for some $i \in \{3,\ldots,n\}$. Note that $N(\{2,i\}) = \{\{1,2\}, \{1,i\}\}$ and hence $\{1,i\} \in N(\{2,i\})$. Since $f(\{2,i\}) = \{2,i\}$, then $f(N(\{2,i\})) = N(\{2,i\})$. But $f(\{1,2\}) = \{1,2\}$ and then $f(\{1,i\}) = \{1,i\}$, for every $i \in \{3,\ldots,n\}$. Then $f(O) \subset \mathrm{Fix}(f)$. Finally, let $w \in G$, that is $w = \{r,s\}$, with

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$r, s \in \{3, \ldots, n\}$, $r \neq s$. We have that $N(\{r, s\}) = \{\{1, r\}, \{1, s\}\}$, that is a subset of O. Then $N(\{r, s\}) \subset \text{Fix}(f)$. Since $|N(y) \cap N(z)| < 2$, for every $y, z \in \mathcal{R}$, with $y \neq z$, then $f(\{r, s\}) = \{r, s\}$, for $r, s \in \{3, \ldots, n\}$, $r \neq s$. Therefore $f(G) \subset \text{Fix}(f)$. We conclude that $f = id$ because $\{B, O, R, G\}$ is a partition of $V(\Gamma)$.

In this way $\text{Ker } \varphi = \{id\}$ and then $|\text{Stab}(x)| \leq (n-2)!$ Now we use that $|\text{Aut}(\Gamma)| = |O(x)||\text{Stab}(x)|$ to obtain that $|\text{Aut}(\Gamma)| \leq (n-1)(n-2)! = (n-1)!$ as desired. $\qquad \square$

The following conjecture is based on experimental results.

Conjecture 4.2. *Let n be an integer and $3 \leq k \leq n/2$. If $k \neq n/2$ then $\text{Aut}(F_k(K_{1,n})) = \text{Aut}(K_{1,n})$, and if $k = n/2$, then $\text{Aut}(F_k(K_{1,n})) = S_2 \times \text{Aut}(K_{1,n})$.*

Fan graphs

The fan graph $A_{1,n}$ is the joint graph $K_1 + P_n$. The vertices of $A_{1,n}$ are the disjoint union $V(K_1) \cup V(P_n)$, where $V(K_1) = \{v\}$ and $V(P_n) = \{u_1, \ldots, u_n\}$, where $u_i \sim u_{i+1}$ in P_n, for every $1 \leq i \leq n-1$. In Figure 4 we show $F_2(A_{1,7})$.

It is well-known that $\text{Aut}(A_{1,n}) \simeq S_2$, for $n \neq 3$. For $n \geq 5$, it can be shown (by Observation 2.1) that the degrees of vertices in $F_2(A_{1,n-1})$ are as follows:

- $d(\{u_1, u_2\}) = 3, d(\{u_1, u_i\}) = 5$, for $3 \leq i \leq n-1$, $d(\{u_1, v\}) = n$, $d(\{u_1, u_n\}) = 4$;

- $d(\{u_i, u_j\}) \in \{4, 6\}$, for $i, j \in \{2, \ldots, n-1\}$, $i \neq j$;

- $d(\{u_i, u_n\}) = 5$, for $2 \leq i \leq n-2$, $d(\{u_{n-1}, u_n\}) = 3, d(\{u_n, v\}) = n$;

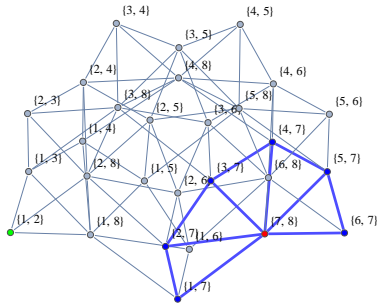- $d(u_i, v) = n+1$, for every $2 \leq i \leq n-1$.

Figure 4: The 2-token graph of $A_{1,7}$, where $V(A_{1,7}) = \{1, \ldots, 8\}$ and $d(8) = 7$. In the proof of Theorem 4.3, $x = \{1, 2\}$, and R is equal to the set of red and blue vertices.
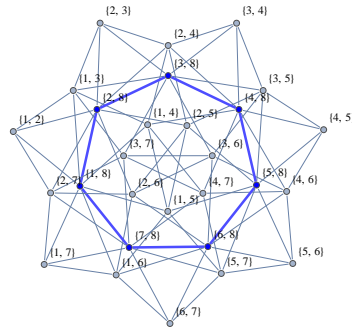

Figure 5: The 2-token graph of $W_{1,7}$, where $V(W_{1,7}) = \{1, \ldots, 8\}$ and $d(8) = 7$. In the proof of Theorem 4.5, Γ_C is the blue subgraph.

Therefore, if $w \in V(F_2(A_{1,n}))$, then $d(w) \in \{3, 4, 5, 6, n, n+1\}$. Note that for $n \geq 8$, there exists exactly two vertices of degree 3 and exactly two vertices of degree n in $F_2(A_{1,n})$. By computer we obtain that $\text{Aut}(A_{1,3}) \simeq S_2 \times S_2$.

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Theorem 4.3. *Let $n \geq 4$ be an integer. Then* $\operatorname{Aut}(F_2(A_{1,n})) = \operatorname{Aut}(A_{1,n})$.

Proof. The proof is by induction on n. The cases for $n \in \{4, \ldots, 8\}$ were obtained by computer. In the rest of the proof $n \geq 9$. Assume as induction hypothesis that the result is true for every $4 \leq m < n$.

Let $\Gamma = F_2(A_{1,n})$. By Theorem 2.3, $\operatorname{Aut}(A_{1,n}) \leq \operatorname{Aut}(\Gamma)$ and hence it is enough to prove that $|\operatorname{Aut}(\Gamma)| \leq 2$. Let

$$R = \{\{w, u_n\} \in V(\Gamma) \colon w \in V(A_{1,n}), w \neq u_n\}$$

and $\Gamma_R = \Gamma\langle R \rangle$. We have that

$$R = N\left(\{v, u_n\}\right) \cup \{\{v, u_n\}\} \setminus \{\{v, u_{n-1}\}\}.$$

Note that Γ_R is isomorphic to $A_{1,n-1}$, where $\{v, u_n\}$ is the vertex of degree $n - 1$ in Γ_R. Let $\Gamma_{\overline{R}} = \Gamma - R$. Note that $\Gamma_{\overline{R}} = F_2(A_{1,n-1})$, where the vertices of $A_{1,n-1}$ are given by $V(K_1) = \{v\}$ and $V(P_{n-1}) = \{u_1, \ldots, u_{n-1}\}$.

Let $x \in V(\Gamma)$ be the vertex $\{u_1, u_2\}$. We have that $|\operatorname{Orb}(x)| \leq 2$ because $d(x) = 3$ and there are exactly two vertices of degree 3 in Γ.

Let $\Gamma_1 = \Gamma\langle N(x)\rangle$, where $N(x) = \{\{u_1, u_3\}, \{v, u_1\}, \{v, u_2\}\}$. Since the graph Γ_1 is isomorphic to P_3, then $\operatorname{Aut}(\Gamma_1) = S_2$. Let $\varphi \colon \operatorname{Stab}(x) \to \operatorname{Aut}(\Gamma_1)$ be the homomorphism given by $f \mapsto f|_{N(x)}$. Since the vertices $\{u_1, u_3\}$ and $\{v, u_2\}$ have different degree in Γ, then $f|_{N(x)} = id$, for every $f \in \operatorname{Stab}(x)$. Therefore $\operatorname{Im}(\varphi) = \{id\}$ and hence $|\operatorname{Stab}(x)| = |\operatorname{Ker} \varphi|$. We will prove that $\operatorname{Ker} \varphi = \{id\}$.

Let $f \in \operatorname{Ker} \varphi$. As $f \in \operatorname{Stab}(x)$ and $f|_{N(x)} = id$, we have that

$$\{\{u_1, u_2\}, \{u_1, u_3\}, \{v, u_1\}, \{v, u_2\}\} \subseteq \operatorname{Fix}(f).$$

The unique vertices in Γ of degree n are $\{v, u_1\}$ and $\{v, u_n\}$. Therefore $f(\{v, u_n\}) = \{v, u_n\}$ and this implies that $f(N(\{v, u_n\})) = N(\{v, u_n\})$. Note that $\{v, u_{n-1}\}$ is the unique vertex of degree $n + 1$ in $N(\{v, u_n\})$ and hence $f(\{v, u_{n-1}\}) = \{v, u_{n-1}\}$. Then $f(R) = R$ and hence $f|_R \in \operatorname{Aut}(\Gamma_R)$ (remember that $R = N\left(\{v, u_n\}\right) \cup \{\{v, u_n\}\} \setminus \{\{v, u_{n-1}\}\}$). This implies that $f|_{V(\Gamma_{\overline{R}})} \in \operatorname{Aut}(\Gamma_{\overline{R}})$.

Since $\operatorname{Aut}(\Gamma_R)$ is isomorphic to $\operatorname{Aut}(A_{1,n-1})$, the image of $\{u_1, u_n\}$ under $f|_R$ has only two possibilities: $\{u_1, u_n\}$ or $\{u_{n-1}, u_n\}$. But $f(\{u_1, u_n\}) = \{u_{n-1}, u_n\}$ is imposible because $\{u_1, u_n\}$ and $\{u_{n-1}, u_n\}$ have different degrees in Γ. Thus $f|_R = id$. Now $\operatorname{Aut}(\Gamma_{\overline{R}}) = \operatorname{Aut}(F_2(A_{1,n-1}))$ and by induction we have that either $f|_{V(\Gamma_{\overline{R}})} = id$ or $f|_{V(\Gamma_{\overline{R}})} = g$, where g is the automorphism in $\operatorname{Aut}(F_2(A_{1,n-1}))$ that moves the vertex $\{u_1, u_2\}$. But $f(\{u_1, u_2\}) = \{u_1, u_2\}$ and hence $f|_{V(\Gamma_{\overline{R}})} = id$. Therefore $f = id$. In this way $|\operatorname{Stab}(x)| = 1$ and hence $|\operatorname{Aut}(\Gamma)| \leq 2$ as desired. $\square$

Conjecture 4.4. *Let n be an integer and $3 \leq k \leq n/2$. If $k \neq n/2$ then $\operatorname{Aut}(F_k(A_{1,n})) = \operatorname{Aut}(A_{1,n})$ and if $k = n/2$, then $\operatorname{Aut}(F_k(A_{1,n})) = S_2 \times \operatorname{Aut}(A_{1,n})$.*

Wheel graphs

The wheel graph $W_{1,n}$, $n \geq 3$, is defined as the join graph $K_1 + C_n$. In Figure 5 we show $F_2(W_{1,7})$. It is well-known that $\operatorname{Aut}(W_{1,n}) \simeq D_{2n}$. By computer software we obtain that $\operatorname{Aut}(W_{1,3}) \simeq S_2 \times D_6$.

Theorem 4.5. *Let $n \geq 4$ be integer. Then $\text{Aut}(F_2(W_{1,n})) = \text{Aut}(W_{1,n})$.*

Proof. In this proof, the vertex set of $W_{1,n}$ is $\{v, u_1, \ldots, u_n\}$, where $V(K_1) = \{v\}$ and $V(C_n) = \{u_1, \ldots, u_n\}$, with $u_1 \sim u_n$ and $u_i \sim u_{i+1}$, $1 \leq i \leq n$ in C_n. The cases $n \in \{4, 5\}$ were verified by computer and hence we suppose that $n \geq 6$. Let $\Gamma = F_2(W_{1,n})$. Let $T = \{\{u_i, u_j\} \in V(\Gamma) : 1 \leq i < j \leq n\}$ and $C = \{\{u_i, v\} \in V(\Gamma) : 1 \leq i \leq n\}$. Let $\Gamma_T = \Gamma\langle T \rangle$ and $\Gamma_C = \Gamma\langle C \rangle$. We have that $\Gamma_T = F_2(C_n)$ and $\Gamma_C \simeq C_n$ (see Figure 5 for an example), where an isomorphism between C_n and Γ_C is given by $u_i \mapsto (u_i, v)$. In this proof we use that every automorphism in $\text{Aut}(\Gamma_T)$ (that is equal to $\text{Aut}(F_2(C_n))$) is induced by some automorphism in $\text{Aut}(C_n)$. For the case of $\text{Aut}(\Gamma_C)$ we have that every automorphism $\theta \in \text{Aut}(C_n)$ induces an automorphism g in $\text{Aut}(\Gamma_C)$ given by $g(\{u_i, v\}) = (\{\theta(u_i), v\})$. We know that $\text{Aut}(W_{1,n}) \leq \text{Aut}(F_2(W_{1,n}))$ and we will prove that $\text{Aut}(F_2(W_{1,n})) = \text{Aut}(W_{1,n})$ by showing that every automorphism f in $\text{Aut}(F_2(W_{1,n}))$ is induced by some automorphism θ in $\text{Aut}(W_{1,n})$, i.e., $f(\{a, b\}) = \{\theta(a), \theta(b)\}$, for every $\{a, b\} \in V(F_2(W_{1,n}))$.

Let $y \in V(F_2(W_{1,n}))$. Note that if $y \in T$, then $d(y) \in \{4, 6\}$, and if $y \in C$, then $d(y) = n + 1$. As $n \geq 6$, then $d(y) \notin \{4, 6\}$ when $y \in C$. Therefore, we have that $f|_T \in \text{Aut}(\Gamma_T)$ and $f|_C \in \text{Aut}(\Gamma_C)$, for every $f \in \text{Aut}(F_2(W_{1,n}))$. As $f|_T \in \text{Aut}(F_2(C_n))$, then there exists $\alpha \in \text{Aut}(C_n)$ such that $f|_T(\{a, b\}) = \{\alpha(a), \alpha(b)\}$, for any $\{a, b\} \in V(\Gamma_T)$. For the case of $f|_C$ we have that $f|_C(\{a, v\}) = \{\beta(a), v\}$, for some $\beta \in \text{Aut}(C_n)$.

We will prove by contradiction that $\alpha = \beta$. Suppose that $\alpha \neq \beta$. Without loss of generality, we suppose that $\alpha(u_1) \neq \beta(u_1)$.

Claim 4.6. *If $\alpha(u_1) \neq \beta(u_1)$, then $\alpha(u_1) = \beta(u_2), \alpha(u_2) = \beta(u_1)$ and $\alpha(u_j) = \beta(u_j)$, for every $j \in \{3, \ldots, n\}$.*

Proof. As $\{u_1, u_2\} \sim \{u_1, v\}$, then $f(\{u_1, u_2\}) \sim f(\{u_1, v\})$, that is $\{\alpha(u_1), \alpha(u_2)\} \sim \{\beta(u_1), v\}$. By the definition of 2-token graph $|\{\alpha(u_1), \alpha(u_2)\} \cap \{\beta(u_1), v\}| = 1$. But $v \notin \{\alpha(u_1), \alpha(u_2)\}$ and $\alpha(u_1) \neq \beta(u_1)$, an then we have that $\alpha(u_2) = \beta(u_1)$. On the other hand $\{u_1, u_2\} \sim \{u_2, v\}$ and then $f(\{u_1, u_2\}) \sim f(\{u_2, v\})$. That is $\{\alpha(u_1), \alpha(u_2)\} \sim \{\beta(u_2), v\}$. But we have proved that $\alpha(u_2) = \beta(u_1)$ and hence $\{\alpha(u_1), \beta(u_1)\} \sim \{\beta(u_2), v\}$. Similarly as in previous case the equality $|\{\alpha(u_1), \beta(u_1)\} \cap \{\beta(u_2), v\}| = 1$ implies that $\alpha(u_1) = \beta(u_2)$.

Now, for $j \in \{3, \ldots, n\}$, we have that $\{u_1, u_j\} \sim \{u_j, v\}$. Then $f(\{u_1, u_j\}) \sim f(\{u_j, v\})$, that is $\{\alpha(u_1), \alpha(u_j)\} \sim \{\beta(u_j), v\}$. But $\alpha(u_1) = \beta(u_2)$ and hence $\{\beta(u_2), \alpha(u_j)\} \sim \{\beta(u_j), v\}$. The unique option is that $\alpha(u_j) = \beta(u_j)$ and the proof of the claim is completed. $\square$

Let $a = \beta(u_1)$ and $b = \beta(u_2)$. Using previous claim, it is easy to check that $\alpha\beta^{-1}(a) = b$, $\alpha\beta^{-1}(b) = a$, and $\alpha\beta^{-1}(c) = c$, for every $c \in V(C_n) \setminus \{a, b\}$. That is, $\alpha\beta^{-1}$ is equal to the transposition (a, b) (written in cyclic notation). But $\alpha\beta^{-1} \in \text{Aut}(C_n) = D_{2n}$ and the dihedral group D_{2n} has not transpositions when $n > 3$. This contradiction shows that $\alpha = \beta$. Therefore, f is the automorphism induced by $\theta \in \text{Aut}(W_{1,n})$, where $\theta(v) = v$ and $\theta(u_i) = \alpha(u_i)$, for every $i \in \{1, \ldots, n\}$. $\square$

Conjecture 4.7. *Let n be an integer and $3 \leq k \leq n/2$. If $k \neq n/2$ then $\text{Aut}(F_k(W_{1,n})) = \text{Aut}(W_{1,n})$ and if $k = n/2$, then $\text{Aut}(F_k(W_{1,n})) = S_2 \times \text{Aut}(W_{1,n})$.*

5   Proof of Theorem 1.2

In this section we use P_n to denote the path graph with $V(P_n) = \{1, \dots, n\}$ and $E(P_n) = \{\{i, i+1\} \colon 1 \le i \le n-1\}$. It is well-known that $\mathrm{Aut}(P_1) \simeq S_1$ and $\mathrm{Aut}(P_n) \simeq S_2$, for $n \ge 2$. Explicitly, if $n \ge 2$, then $\mathrm{Aut}(P_n) = \{id, \theta\}$, where

$$\theta = (1,\ n)(2,\ n-1)\dots(n/2,\ n/2+1), \text{ when } n \text{ is even}$$

and

$$\theta = (1,\ n)(2,\ n-1)\dots(\lceil n/2 \rceil - 1,\ \lceil n/2 \rceil + 1)(\lceil n/2 \rceil), \text{ when } n \text{ is odd}.$$

(We are writing permutation θ in its cycle notation).

Our main result in this section is Theorem 1.2 about the automorphism group of $F_k(P_n)$, for $n \ne 2k$. In Figure 6 and 7 we show $F_2(P_6)$ and $F_3(P_7)$, respectively.
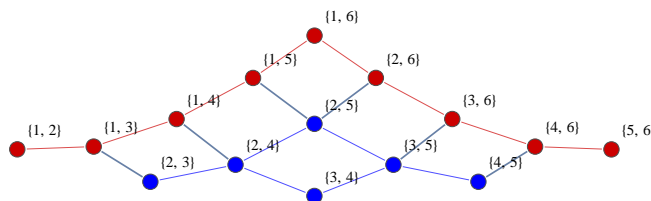

Figure 6: The 2-token graph of P_6. The red (resp. blue) subgraph is induced by the set L_1 (resp. L_2) defined in the proof of Theorem 1.2.

First, we present some auxiliary results. Without loss of generality, we write the elements of a vertex $\{a_1, \dots, a_k\} \in V(F_k(P_n))$ in ascending order, that is $a_1 < \cdots < a_k$. Using Observation 2.1 we obtain the following facts about the vertices in $F_k(P_n)$.

Observation 5.1. *Let P_n be a path graph of order $n \ge 3$ and let $2 \le k \le n-2$ be an integer.*

1. *Let $v = \{a_1, a_2, \dots, a_k\}$ be a vertex in $F_k(P_n)$. The degree of v is even if and only if $\{a_1, a_k\} = \{1, n\}$ or $\{a_1, a_k\} \cap \{1, n\} = \emptyset$.*

2. *The vertices of degree 2 in $F_k(P_n)$ are either of the form $\{a, a+1, \dots, a+(k-1)\}$, for $2 \le a \le n-k$, or $\{1, \dots, m, n-(k-m-1), n-(k-m), \dots, n\}$, for $1 \le m \le k-1$.*

3. *The graph $F_k(P_n)$ has exactly two vertices of degree one, say $\{1, \dots, k\}$ and $\{n-(k-1), \dots, n\}$.*

The formula for the graph distance $d(u, v)$ between vertices u and v in $F_k(P_n)$ is given in the following result.

Lemma 5.2. *Let* $u = \{u_1, \ldots, u_k\}$ *and* $v = \{v_1, \ldots, v_k\}$ *be vertices in* $F_k(P_n)$, *where* $u_1 < \cdots < u_k$ *and* $v_1 < \cdots < v_k$. *Then*

$$d(u, v) = \sum_{i=1}^{k} |v_i - u_i|.$$

Proof. The L_1-distance on $\mathbb{Z}^k$ is defined as

$$d_{L_1}(a, b) = \sum_{i=1}^{k} |b_i - a_i|,$$

for every $a = \{a_1, \ldots, a_k\}, b = \{b_1, \ldots, b_k\} \in \mathbb{Z}^k$. The grid graph $\mathbb{Z}^k$ is constructed as follows: two points in $\mathbb{Z}^k$ are adjacent if their L_1-distance is equal to one. It is well-known (see, e.g., [13, p. 333]) that the L_1-distance is equal to the path distance on the grid graph $\mathbb{Z}^k$. The result follows by observing that $F_k(P_n)$ is a subgraph of the grid graph $\mathbb{Z}^k$, where vertex $\{a_{i_1}, \ldots, a_{i_k}\}$ in $F_k(P_n)$ correspond to the unique vertex $(a_1, \ldots, a_k)$ in $\mathbb{Z}^k$ such that $a_1 < \cdots < a_k$ and $\{a_{i_1}, \ldots, a_{i_k}\} = \{a_1, \ldots, a_k\}$. $\square$

The proof of previous lemma also follows by using the formula of distance in [15]. The case for $F_2(P_n)$ was also proved in [9]. In the proof of the following theorem, we use several times the following fact: if H is a graph, then

$$d(a, b) = d(g(a), g(b)),$$

for every $g \in \text{Aut}(H)$ and every $a, b \in V(H)$.

Proof of Theorem 1.2. If $k = 1$, then $F_1(P_n) \simeq P_n$ and the result is trivially true. The cases $n \leq 7$, $1 \leq k \leq n/2$, were verified by computer and hence we suppose that $n \geq 8$. Let $\Gamma = F_k(P_n)$. By Theorem 2.3, it follows that $\text{Aut}(P_n) \leq \text{Aut}(\Gamma)$ and hence it is enough to prove that $|\text{Aut}(\Gamma)| \leq 2$.

Let $x \in \Gamma$ be the vertex $\{1, \ldots, k\}$. Note that $d(v) = 1$ and hence $|O(x)| \leq 2$. In fact, $|O(x)| = 2$ because the non-identity automorphism, say ϕ, in $\text{Aut}(P_n)$ induces a non-identity automorphism g_ϕ in $\text{Aut}(\Gamma)$ such that $g_\phi(\{1, \ldots, k\}) = \{n - (k-1), \ldots, n\}$. Let $\Gamma_1 = \Gamma\langle N(x)\rangle$. Let $\varphi : \text{Stab}(x) \to \text{Aut}(\Gamma_1)$ be the function defined by $\varphi(f) = f|_{N(x)}$. As $\text{Aut}(\Gamma_1) = \text{Id}$, we have that $\text{Stab}(x) = \text{Ker}\, \varphi$. We will prove that $\text{Ker}\, \varphi = \{id\}$, which implies that $|\text{Aut}(\Gamma)| \leq 2$.

First we prove the case $k = 2$. Suppose by induction that $F_2(P_m) = \text{Aut}(P_m)$, for every $5 \leq m < n$. Let $L_1 = \{\{a_1, a_2\} \in V(\Gamma): a_1 = 1 \text{ or } a_2 = n\}$ and $L_2 = V(\Gamma) \setminus L_1$. Let $\Gamma_{L_1} = \Gamma\langle L_1\rangle$ and let $\Gamma_{L_2} = \Gamma\langle L_2\rangle$. Note that $\Gamma_{L_1} \simeq P_{2n-3}$ and, by Proposition 2.2, we have that $\Gamma_{L_2} \simeq F_2(P_{n-2})$ (see Figure 6 for an example). It is easy to see that, with the exception of $\{1, n\}$, all the vertices in L_1 have either degree 1 or degree 3. In fact L_1 has all the vertices of Γ that have degree 1 or 3. Note that $\{1, n\}$ is the unique vertex in Γ of degree 2 in with its two neighbors of degree 3. Consequently, if $f \in \text{Aut}(\Gamma)$, then $f(L_1) = L_1$ and hence $f|_{L_1} \in \text{Aut}(\Gamma_{L_1})$. Now, let $f \in \text{Ker}\, \varphi$. Since $\text{Aut}(L_1) \simeq \text{Aut}(P_{2n-3})$ and $f(\{1, 2\}) = \{1, 2\}$, we have that $f|_{L_1} = id$ which shows that $f(L_1) \subset \text{Fix}(f)$. Using this, the proof that $f(L_2) \subset \text{Fix}(f)$ follows immediately by induction because $\Gamma_{L_2} \simeq F_2(P_{n-2})$ and $n - 2 \geq 6$ (see Figure 6 for an example).

Now we prove the case $3 \leq k < n/2$. The proof is by induction on n. As bases cases, we have proved the result for $F_2(P_n)$, for every $n \geq 5$, and for $n = 8$, we have verified every

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$k \in \{2,3\}$ by computer. We suppose that the result is true for any graph $P_{n'}$ with $n' < n$. That is, $\mathrm{Aut}(F'_k(P_{n'})) = \mathrm{Aut}(P_{n'})$, for $2 \le k' < n'/2 < n/2$.

Let $f \in \mathrm{Ker}\, \varphi$. Since $f \in \mathrm{Stab}(x)$, we have that $f(\{1, \ldots, k-1, k+1\}) = (\{1, \ldots, k-1, k+1\}$ and $f(\{n-(k-1), \ldots, n-1, n\}) = (\{n-(k-1), \ldots, n-1, n\})$ (the grey vertices in the example in Figure 7).

We define the following subsets of $V(\Gamma)$

$$
\begin{aligned}
A &= \{\{a_1, \ldots, a_k\} \in V(\Gamma) \colon a_k = n\} \\
B &= \{\{b_1, \ldots, b_k\} \in V(\Gamma) \colon b_1 = 1\} \\
C &= \{\{c_1, \ldots, c_k\} \in V(\Gamma) \colon c_1 \neq 1, c_k \neq n\}
\end{aligned}
$$

Note that $C = V(\Gamma) \setminus (A \cup B)$. Let $\Gamma_A = \Gamma\langle A\rangle$, $\Gamma_B = \Gamma\langle B\rangle$ and $\Gamma_C = \Gamma\langle C\rangle$. We will prove that if f in $\mathrm{Ker}\, \varphi$, then f is the identity permutation by showing that $X \subset \mathrm{Fix}(f)$, for every $X \in \{A, B, C\}$. We do this in four steps: first we prove that if $f \in \mathrm{Ker}\, \varphi$, then $f(\{1, \ldots, k-1, n\}) = (\{1, \ldots, k-1, n\}$ (Claim 5.4), second we work the case of A, third the case of B and finally the case of C. In Figure 7, we use colors to illustrate the four steps.
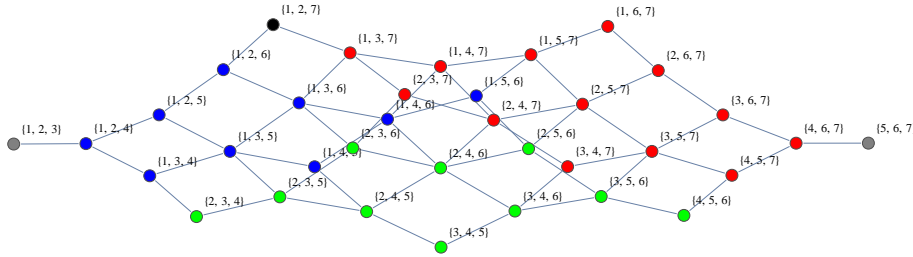

Figure 7: The 3-token graph of P_7. In the proof of Theorem 1.2, for $f \in \mathrm{Ker}\, \varphi$, we prove that $f(v) = v$, for every $v \in V(\Gamma)$, in the following order: grey, black, red, blue and green vertices.

We need the following:

Claim 5.3. *Let y be the vertex $\{1, \ldots, k-1, n\}$.*

1. *If $3 \le a \le n-k-1$, then $\{a, a+1, \ldots, a+(k-1)\} \notin O(y)$,*

2. *If $2 \le m \le k-2$, then $\{1, \ldots, m, n-(k-m-1), n-(k-m), \ldots, n\} \notin O(y)$*

Proof. The elements in $O(y)$ should be vertices of degree 2. We will use the fact that if $g \in \mathrm{Aut}(\Gamma)$, then $N(g(v)) = g(N(v))$, for every $v \in V(\Gamma)$. Note that $N(y) = \{y_1, y_2\}$, where $y_1 = \{1, \ldots, k-2, k, n\}$ and $y_2 = \{1, \ldots, k-1, n-1\}$. The condition $k \ge 3$ implies that $d(y_1) = 4$ and $d(y_2) = 3$.

Proof of (1). Let $w = \{a, a+1, \ldots, a+(k-1)\}$, with $3 \le a \le n-k-1$, then $N(w) = \{w_1, w_2\}$, where $w_1 = \{a-1, a+1, \ldots, a+(k-1)\}$, $w_2 = \{a, a+1, \ldots, a+(k-2), a+k\}$, $d(w_1) = d(w_2) = 4$. Therefore $f(y) \neq w$.

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Proof of (2). Let $z = \{1, \ldots, m, n - (k - m - 1), n - (k - m), \ldots, n\}$, with $2 \leq m \leq k - 2$. Then $N(z) = \{z_1, z_2\}$, where $z_1 = \{\{1, \ldots, m - 1, m + 1, n - (k - m - 1), n - (k - m), \ldots, n\}$, $z_2 = \{1, \ldots, m, n - (k - m - 1) - 1, n - (k - m), \ldots, n\}$ and $d(z_1) = d(z_2) = 4$ (here we are using that $k < n/2$). Therefore $f(y) \neq z$. $\square$

Step 1. We prove the following fact.

Claim 5.4. *Let y be the vertex $\{1, \ldots, k - 1, n\}$. If $f \in \operatorname{Ker} \varphi$, then $f(y) = y$.*

Proof. By Observation 5.1(2) and Claim 5.3 we have that $f(y)$ has only the following options:

1. $\{2, 3, \ldots, k + 1\}$,

2. $\{n - k, n - (k - 1), \ldots, n - 1\}$,

3. $\{1, n - (k - 2), \ldots, n\}$

4. $\{1, \ldots, k - 1, n\}$,

Now we use several times the fact that $d(a, b) = d(g(a), g(b))$, for every $g \in \operatorname{Aut}(\Gamma)$ and every $a, b \in V(\Gamma)$.

Case 1. Suppose that $f(\{1, \ldots, k - 1, n\}) = \{2, 3, \ldots, k + 1\}$. Then

$$
\begin{aligned}
d(\{1, 2, \ldots, k\}, \{1, \ldots, k - 1, n\}) &= d(\{1, 2, \ldots, k\}, \{2, 3, \ldots, k + 1\}) \\
n - k &= k
\end{aligned}
$$

which implies that $n = 2k$, a contradiction.

Case 2. Suppose that $f(\{1, \ldots, k - 1, n\}) = \{n - k, n - (k - 1), \ldots, n - 1\}$. Then

$$
\begin{aligned}
d(\{1, 2, \ldots, k\}, \{1, \ldots, k - 1, n\}) &= d(\{1, 2, \ldots, k\}, \{n - k, n - (k - 1), \ldots, n - 1\}) \\
n - k &= \sum_{i=1}^{k}(n - i) - \sum_{i=1}^{k} i
\end{aligned}
$$

From which we obtain $k^2 - nk + n = 0$. The discriminant of equation $x^2 - nx + n = 0$ is $D = n^2 - 4n$. As $n \geq 7$, D is positive and not a square number. Therefore $x^2 - nx + n = 0$ has not integer solutions, which is a contradiction.

Case 3. Suppose that $f(\{1, \ldots, k - 1, n\}) = \{1, n - (k - 2), \ldots, n\}\}$. Then

$$
\begin{aligned}
d(\{1, 2, \ldots, k\}, \{1, \ldots, k - 1, n\}) &= d(\{1, 2, \ldots, k\}, \{1, n - (k - 2), \ldots, n\}) \\
n - k &= \sum_{i=0}^{k-2}(n - i) - \sum_{i=2}^{k} i \\
n - k &= (n - k)(k - 1)
\end{aligned}
$$

As $n \neq k$, then $k = 2$, a contradiction that $k \geq 3$.

Therefore, the unique option is that $f(y) = y$ as desired. $\square$

In Figure 7 we show an example where vertex $\{1,\ldots,k-1,n\}$ is colored black.

Step 2. We will prove that $f(A) \subset \text{Fix}(f)$ (the red vertices in the example in Figure 7).

First we prove that $f(A) = A$. Let $v = \{a_1,\ldots,a_{k-1},n\}$ be a vertex in A. Suppose that $f(v) = \{c_1,\ldots,c_k\}$. Then

$$d(\{1,2,\ldots,k\},\{a_1,\ldots,a_{k-1},n\}) = d(\{1,2,\ldots,k\},\{c_1,\ldots,c_k\}),$$

from which follows that

$$n + a_1 + \cdots + a_{k-1} = c_1 + c_2 + \cdots + c_k \tag{1}$$

Now, by Claim 5.4, $f(\{1,\ldots,k-1,n\}) = \{1,\ldots,k-1,n\}$. Then

$$d(\{1,\ldots,k-1,n\},\{a_1,\ldots,a_{k-1},n\}) = d(\{1,\ldots,k-1,n\},\{c_1,\ldots,c_k\}),$$

from which follows that

$$a_1 + \cdots + a_{k-1} = n - c_k + (c_1 + c_2 + \cdots + c_{k-1}) \tag{2}$$

Combining equations (1) and (2) we obtain that $c_k = n$ and hence $f(v) \in A$. Then $f(A) \subseteq A$. In fact, $f(A) = A$ because f is a bijection. Therefore $f|_A \in \text{Aut}(\Gamma_A)$.

Notice that Γ_A is isomorphic to $F_{k-1}(P_{n-1})$ (see Figure 8 for the case of $F_3(P_7)$), with an isomorphism given by $\{a_1,\ldots,a_{k-1},n\} \mapsto \{a_1,\ldots,a_{k-1}\}$. The inequality $3 \le k < n/2$, implies that $2 \le k-1 < (n-1)/2$. By the induction hypothesis, $\text{Aut}(F_{k-1}(P_{n-1})) = S_2$. The unique option is that $f|_A = id$ because $\{n-(k-1),\ldots,n-1,n\}$ is the unique vertex in A with degree 1 simultaneously in Γ_A and $F_k(P_n)$ (see Figure 8 for an example).
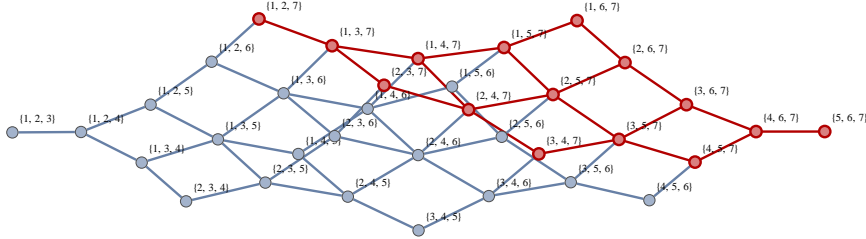

Figure 8: The 3-token graph of P_7. The red subgraph H is isomorphic to $F_2(P_6)$. Note that the vertex $\{1,2,7\}$ has degree one in H but degree two in $F_3(P_7)$.

Step 3. We will prove that $f(B) \subset \text{Fix}(f)$ (the blue vertices in the example in Figure 7).

First, we prove that $f(B) = B$. Let $v = \{1,v_2,\ldots,v_k\} \in B$. We have two cases: $v_k = n$ or $v_k \ne n$. If $v_k = n$, then $v \in A$ and hence $f(v) \in B$, because we have proved that $f(v) = v$, for every $v \in A$. Suppose now that $v_k \ne n$, that is $v \notin A$. By Observation 5.1(1) it follows that $d(v)$ is odd and hence $d(f(v))$ should be odd. Also by Observation 5.1(1), $f(v)$ has only two options. The first one is $f(v) = \{1,\ldots,d_k\}$, with $d_k \ne n$, in which case $f(v) \in B$ as desired. The second option is $f(v) = \{a_1,\ldots,n\}$, with $a_1 \ne 1$. In this case $f(v) \in A$ and hence $f(v) = v \in A$, a contradiction.

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Therefore, $f(B) \subseteq B$ and hence $f(B) = B$ because f is a bijection. As $B \simeq F_{k-1}(P_{n-1})$, with an isomorphism given by $\{1, b_2, \ldots, b_k\} \mapsto \{b_2, \ldots, b_k\}$, then $\mathrm{Aut}(\Gamma_B) \simeq S_2$ and hence $f|_B = id$ as in the case of Γ_A. Therefore $f(A \cup B) \subset \mathrm{Fix}(f)$.

Step 4. We will prove that $f(C) \subset \mathrm{Fix}(f)$ (the green vertices in the example in Figure 7).

It is easy to see that $\Gamma \langle C \rangle \simeq F_k(P_{n-2})$. However, we can not apply the induction hypothesis on C because it is possible that $k \geq (n-2)/2$ (for example, if $n = 10$ and $k = 4$). We solve this inconvenient by using the graph distance in Γ.

Suppose that $f(\{c_1, \ldots, c_k\}) = \{d_1, \ldots, d_k\}$.

$$d(\{1, c_2, \ldots, c_k\}, \{c_1, c_2, \ldots, c_k\}) = d(\{1, c_2, \ldots, c_k\}, \{d_1, \ldots, d_k\})$$
$$c_1 - 1 = d_1 - 1 + \sum_{i=2}^{k} |d_i - c_i|$$

and this implies that $c_1 \geq d_1$. Now

$$d(\{1, d_2, \ldots, d_k\}, \{c_1, c_2, \ldots, c_k\}) = d(\{1, d_2, \ldots, d_k\}, \{d_1, \ldots, d_k\})$$
$$c_1 - 1 + \sum_{i=2}^{k} |d_i - c_i| = d_1 - 1$$

and this implies that $d_1 \geq c_1$. Therefore, $d_1 = c_1$. Now we will prove that $d_k = c_k$.

$$d(\{c_1, c_2, \ldots, c_{k-1}, n\}, \{c_1, \ldots, c_k\}) = d(\{c_1, c_2, \ldots, c_{k-1}, n\}, \{d_1, \ldots, d_k\})$$
$$n - c_k = n - d_k + \sum_{i=1}^{k-1} |d_i - c_i|$$

and this implies that $d_k \geq c_k$. Now

$$d(\{d_1, d_2, \ldots, d_{k-1}, n\}, \{c_1, c_2, \ldots, c_k\}) = d(\{d_1, d_2, \ldots, d_{k-1}, n\}, \{d_1, \ldots, d_k\})$$
$$n - c_k + \sum_{i=1}^{k-1} |d_i - c_i| = n - d_k$$

and this implies that $d_k \leq c_k$. Therefore, $d_k = c_k$.

Finally we prove that $c_i = d_i$, for every $2 \leq i \leq k - 1$. We have proved that if $c = \{c_1, c_2, \ldots, c_{k-1}, c_k\}$, then $f(c) = \{c_1, d_2, \ldots, d_{k-1}, c_k\}$. Therefore

$$d(\{1, d_2, \ldots, d_{k-1}, n\}, \{c_1, c_2, \ldots, c_{k-1}, c_k\}) = d(\{1, d_2, \ldots, d_{k-1}, n\}, \{c_1, d_2, \ldots, d_{k-1}, c_k\})$$
$$n - c_k + c_1 - 1 + \sum_{i=2}^{k-1} |d_i - c_i| = n - c_k + c_1 - 1$$

From which we obtain that $\sum_{i=2}^{k-1} |d_i - c_i| = 0$. But as every $|d_i - c_i|$ is non negative, then $|d_i - c_i| = 0$, for every $i \in \{2, \ldots, k-1\}$ and hence $d_i = c_i$, for every $i \in \{2, \ldots, k-1\}$. So we have that $f(c) = c$, for every $c \in C$.

Therefore $f = id$ and the proof of the theorem is completed. $\qquad \square$

We believe that the following is true.

Conjecture 5.5. *Let n be an even integer. Then $\mathrm{Aut}(F_{n/2}(P_n)) \simeq S_2 \times S_2$.*

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