The Landau function and the Riemann hypothesis

Marc Deléglise, Jean-Louis Nicolas*

July 18, 2019

Abstract

The Landau function $g(n)$ is the maximal order of an element of the symmetric group $\mathfrak{S}_n$; it is also the largest product of powers of primes whose sum is $\le n$. The main result of this article is that the property "For all $n \ge 1$, $\log g(n) < \sqrt{\mathrm{li}^{-1}(n)}$ " (where li^{-1} denotes the inverse function of the logarithmic integral) is equivalent to the Riemann hypothesis.

2010 *Mathematics Subject Classification*: Primary 11A25; Secondary 11N37, 11N05, 11-04.
Keywords: distribution of primes, champion number, highly composite number, Landau function, Riemann hypothesis.

1 Introduction

Let n be a positive integer. In [13], Landau introduced the function $g(n)$ as the maximal order of an element in the symmetric group $\mathfrak{S}_n$; he showed that

$$g(n) = \max_{\ell(M)\le n} M \tag{1.1}$$

where ℓ is the additive function such that $\ell(p^\alpha) = p^\alpha$ for p prime and $\alpha \ge 1$. In other words, if the standard factorization of M is $M = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_j^{\alpha_j}$ we have $\ell(M) = q_1^{\alpha_1} + q_2^{\alpha_2} + \cdots + q_j^{\alpha_j}$ and $\ell(1) = 0$. He also proved that

$$\log g(n) \sim \sqrt{n \log n}, \quad n \to \infty.$$

*Research partially supported by CNRS, Institut Camille Jordan, UMR 5208.

A function close to the Landau function is the function $h(n)$ defined for $n \geq 2$ as the greatest product of a family of primes $q_1 < q_2 < \cdots < q_j$ the sum of which does not exceed n. If μ denotes the Möbius function, $h(n)$ can also be defined by

$$h(n) = \max_{\substack{\ell(M)\leq n\\ \mu(M)\neq 0}} M. \tag{1.2}$$

The above equality implies $h(1) = 1$. Note that

$$\ell(g(n)) \leq n \quad \text{and} \quad \ell(h(n)) \leq n. \tag{1.3}$$

From (1.2) and (1.1), it follows that

$$h(n) \leq g(n), \quad (n \geq 1). \tag{1.4}$$

Sequences $(g(n))_{n\geq 1}$ and $(h(n))_{n\geq 1}$ are sequences A000793 and A159685 in the OEIS (*On-line Encyclopedia of Integer Sequences*). One can find results about $g(n)$ in [15, 17, 7, 11, 20], see also [18] and [4, §10.10]. In the introductions of [7, 11], other references are given. The three papers [8, 9, 10] are devoted to $h(n)$. A fast algorithm to compute $g(n)$ (resp. $h(n)$) is described in [11] (resp. [8, §8]). In [9, (4.13)], it is shown that

$$\log h(n) \leq \log g(n) \leq \log h(n) + 5.68\,(n \log n)^{1/4}, \quad n \geq 1. \tag{1.5}$$

Let li denote the logarithmic integral and li^{-1} its inverse function (cf. below §2.2). In [15, Theorem 1 (i)], it is proved that

$$\log g(n) = \sqrt{\mathrm{li}^{-1}(n)} + \mathcal{O}\left(\sqrt{n}\exp(-a\sqrt{\log n})\right) \tag{1.6}$$

holds for some positive a. The asymptotic expansion of $\log g(n)$ does coincide with the one of $\sqrt{\mathrm{li}^{-1}(n)}$ (cf. [15, Corollaire, p. 225]) and also, from (1.5), with the one of $\log h(n)$:

$$\left.\begin{array}{r}\log h(n)\\ \log g(n)\\ \sqrt{\mathrm{li}^{-1}(n)}\end{array}\right\} = \sqrt{n\log n}\left(1 + \frac{\log\log n - 1}{2\log n} - \frac{(\log\log n)^2 - 6\log\log n + 9 + o(1)}{8\log^2 n}\right) \tag{1.7}$$

In [15, Théorème 1 (iv)], it is proved that under the Riemann hypothesis the inequality

$$\log g(n) < \sqrt{\mathrm{li}^{-1}(n)} \tag{1.8}$$

holds for n large enough. In August 2009, the second author received an e-mail of Richard Brent asking whether it was possible to replace "n large enough" by "$n \geq n_0$" with a precise value of n_0. The aim of this paper is to anwer this question positively. For $n \geq 2$, let us introduce the sequences

$$\log g(n) = \sqrt{\mathrm{li}^{-1}(n)} - a_n (n \log n)^{1/4} \quad \text{i.e.} \quad a_n = \frac{\sqrt{\mathrm{li}^{-1}(n)} - \log g(n)}{(n \log n)^{1/4}}, \tag{1.9}$$

$$\log h(n) = \sqrt{\mathrm{li}^{-1}(n)} - b_n (n \log n)^{1/4} \quad \text{i.e.} \quad b_n = \frac{\sqrt{\mathrm{li}^{-1}(n)} - \log h(n)}{(n \log n)^{1/4}}, \tag{1.10}$$

and the constant

$$c = \sum_\rho \frac{1}{|\rho(\rho+1)|} = 0.046\,117\,644\,421\,509\ldots \tag{1.11}$$

where ρ runs over the non trivial zeros of the Riemann ζ function. The computation of the above numerical value is explained in [10, Section 2.4.2]. We prove

Theorem 1.1. *Under the Riemann hypothesis,*

(i) $\log g(n) < \sqrt{\mathrm{li}^{-1}(n)}$ *for* $n \geq 1$,

(ii) $a_n \geq \dfrac{2-\sqrt{2}}{3} - c - \dfrac{0.43\ \log\log n}{\log n} > 0 \quad$ *for* $n \geq 2$,

(iii) $a_n \leq \dfrac{2-\sqrt{2}}{3} + c + \dfrac{1.02\ \log\log n}{\log n} \quad$ *for* $n \geq 19425$,

(iv) $0.11104 < a_n \leq a_2 = 0.9102\ldots \quad$ *for* $n \geq 2$,

(v) $0.149\ldots = \dfrac{2-\sqrt{2}}{3} - c \leq \liminf a_n \leq \limsup a_n \leq \dfrac{2-\sqrt{2}}{3} + c = 0.241\ldots$

(vi) When $n \to \infty$,

$$\Big(\frac{2-\sqrt{2}}{3} - c\Big)\Big(1 + \frac{\log\log n + \mathcal{O}(1)}{4\log n}\Big) \leq a_n \leq \Big(\frac{2-\sqrt{2}}{3} + c\Big)\Big(1 + \frac{\log\log n + \mathcal{O}(1)}{4\log n}\Big).$$

Remark 1.2. *It does not seem easy to calculate* $\inf_{n\geq 2} a_n$, *and to decide whether it is a minimum or not.*

Corollary 1.3. *Each of the six points of Theorem 1.1 is equivalent to the Riemann hypothesis.*

Proof. If the Riemann hypothesis fails, it is proved in [15, Theorem 1 (ii)] that there exists $b > 1/4$ such that

$$\log g(n) = \sqrt{\mathrm{li}^{-1}(n)} + \Omega_{\pm}((n \log n)^b) \tag{1.12}$$

which contradicts (i), (ii), …, (vi) of Theorem 1.1. □

In the paper [10], the following theorem is proved:

Theorem 1.4. *Under the Riemann hypothesis,*

(i) $\log h(n) < \sqrt{\mathrm{li}^{-1}(n)}$ *for* $n \geq 1$,

(ii) $b_{17} = 0.49795\ldots \leq b_n \leq b_{1137} = 1.04414\ldots$ *for* $n \geq 2$,

(iii) $b_n \geq \dfrac{2}{3} - c - \dfrac{0.23\ \log\log n}{\log n}$ *for* $n \geq 18$,

(iv) $b_n \leq \dfrac{2}{3} + c + \dfrac{0.77\ \log\log n}{\log n}$ *for* $n \geq 4\,422\,212\,326$,

(v) $2/3 - c = 0.620\ldots \leq \liminf b_n \leq \limsup b_n \leq 2/3 + c = 0.712\ldots$

(vi) and, when $n \to \infty$,

$$\left(\frac{2}{3} - c\right)\left(1 + \frac{\log\log n + \mathcal{O}(1)}{4\log n}\right) \leq b_n \leq \left(\frac{2}{3} + c\right)\left(1 + \frac{\log\log n + \mathcal{O}(1)}{4\log n}\right).$$

The main tools in the proof of Theorem 1.4 in [10] are the explicit formulas for $\sum_{p^m \leq x} p$ and $\sum_{p^m \leq x} \log p$.

We deduce Theorem 1.1 about $g(n)$ from Theorem 1.4 about $h(n)$ by studying the difference $\log g(n) - \log h(n)$ in view of improving inequalities (1.5). More precisely, we prove

Theorem 1.5. *Without any hypothesis,* *(i) For n tending to infinity,*

$$\log\frac{g(n)}{h(n)} = \frac{\sqrt{2}}{3}(n\log n)^{1/4}\Bigg(1+\frac{\log\log n - 4\log 2 - 11/3}{4\log n}$$
$$-\frac{\frac{3}{32}(\log\log n)^2-\left(\frac{3\log 2}{4}+\frac{15}{16}\right)\log\log n+\frac{(\log 2)^2}{2}+\frac{29\log 2}{12}+\frac{635}{288}}{(\log n)^2}\Bigg)$$
$$+\mathcal{O}\left(\frac{(\log\log n)^3}{(\log n)^3}\right)$$

(ii)

$$\log\frac{g(n)}{h(n)} \le \frac{\sqrt{2}}{3}(n\log n)^{1/4}\left(1+\frac{\log\log n+2.43}{4\log n}\right)$$

for $n \ge 3\,997\,022\,083\,663$,

(iii) $\log\dfrac{g(n)}{h(n)} \ge \dfrac{\sqrt{2}}{3}(n\log n)^{1/4}\left(1+\dfrac{\log\log n-11.6}{4\log n}\right)$ *for $n \ge 4\,230$,*

(iv) For $n \ge 1$ we have $\dfrac{g(n)}{h(n)} \ge 1$ with equality for $n = 1, 2, 3, 5, 6, 8, 10, 11, 15, 17, 18, 28, 41, 58, 77.$

(v) For $n \ge 1$, we have $\log\dfrac{g(n)}{h(n)} \le 0.62066\ldots(n\log n)^{1/4}$ with equality for $n = 2243$.

Remark 1.6. *From the asymptotic expansion (i), it follows that, for n very large, the inequality* $\log(g(n)/h(n)) > (\sqrt{2}/3)(n\log n)^{1/4}$ *holds. But finding the largest n for which*
$(\log(g(n)/h(n)))/(n\log n)^{1/4}$ *does not exceed* $\sqrt{2}/3 = 0.471\ldots$ *seems difficult.*

Theorem 1.7. *For $n \ge 373\,623\,863$,*

$$\sqrt{n\log n}\left(1+\frac{\log\log n-1}{2\log n}-\frac{(\log\log n)^2}{8\log^2 n}\right) \le \log h(n) \le \log g(n) \qquad (1.13)$$

and, for $n \ge 4$,

$$\log h(n) \le \log g(n) \le \sqrt{n\log n}\left(1+\frac{\log\log n-1}{2\log n}\right) \qquad (1.14)$$

The lower bound (1.13) of $\log h(n)$ improves on [9, Theorem 4] where it was shown that, for $n \geq 77\,615\,268$, we have $\log h(n) \geq \sqrt{n\log n}\left(1 + \frac{\log\log n - 1.16}{2\log n}\right)$. Inequality (1.14) improves on the result of [15, Corollaire, p. 225], where -1 was replaced by -0.975. From the common asymptotic expansion (1.7) of $\log h(n)$ and $\log g(n)$, one can see that the constants 8 in (1.13) and -1 in (1.14) are optimal.

1.1 Notation

- $\pi_r(x) = \sum_{p\leq x} p^r$. For $r = 0$, $\pi(x) = \pi_0(x) = \sum_{p\leq x} 1$ is the prime counting function. $\pi_r^-(x) = \sum_{p<x} p^r$.
- $\theta(x) = \sum_{p\leq x} \log p$ is the Chebichev function. $\theta^-(x) = \sum_{p<x} \log p$.
- $\mathcal{P} = \{2, 3, 5, \dots\}$ denotes the set of primes. $p_1 = 2,\ p_2 = 3, \dots, p_j$ is the j-th prime. For $p \in \mathcal{P}$ and $n \in \mathbb{N}$, $v_p(n)$ denotes the largest exponent such that $p^{v_p(n)}$ divides n.
- $\mathrm{P}^+(n)$ denotes the largest prime factor of n.
- $\mathrm{li}(x)$ denotes the logarithmic integral of x (cf. below Section 2.2). The inverse function is denoted by li^{-1}.
- If $\lim_{n\to\infty} u_n = +\infty$, $v_n = \Omega_\pm(u_n)$ is equivalent to $\limsup_{n\to\infty} v_n/u_n > 0$ and $\liminf_{n\to\infty} v_n/u_n < 0$.
- We use the following constants:
  - x_1 takes three values (cf. (2.3)),
  - $x_0 = 10^{10}+19$ is the smallest prime exceeding 10^{10}, $\log(x_0) = 23.025\,850\dots$,
  - $v_0 = 2\ 220\ 832\ 950\ 051\ 364\ 840 = 2.22\dots 10^{18}$ is defined below in (3.17),
  - $\log v_0 = 42.244414\dots, \log\log v_0 = 3.743472\dots$
  - The numbers $(\lambda_j)_{j\geq 2}$ described in Lemma 3.8 and $(x_j^{(0)})_{2\leq j\leq 29}$ defined in (3.18).
  - For convenience, we sometimes write L for $\log n$, λ for $\log\log n$, L_0 for $\log v_0$ and λ_0 for $\log\log v_0$.

We often implicitly use the following result : for u and v positive and w real, the function

$$t \mapsto \frac{(\log t - w)^u}{t^v} \quad \text{is decreasing for} \quad t > \exp(w + u/v). \tag{1.15}$$

Also, if ϵ and ϵ_0 are real numbers satisfying $0 \le \epsilon \le \epsilon_0 < 1$ we shall use the following upper bound

$$\frac{1}{1-\epsilon} = 1 + \frac{\epsilon}{1-\epsilon} \le 1 + \frac{\epsilon}{1-\epsilon_0}. \tag{1.16}$$

Let us write $\sigma_0 = 0$, $N_0 = 1$, and, for $j \ge 1$,

$$N_j = p_1 p_2 \cdots p_j \quad \text{and} \quad \sigma_j = p_1 + p_2 + \cdots + p_j = \ell(N_j). \tag{1.17}$$

For $n \ge 0$, let $k = k(n)$ denote the integer $k \ge 0$ such that

$$\sigma_k = p_1 + p_2 + \cdots + p_k \le n < p_1 + p_2 + \cdots + p_{k+1} = \sigma_{k+1}. \tag{1.18}$$

In [8, Proposition 3.1], for $j \ge 1$, it is proved that

$$h(\sigma_j) = N_j. \tag{1.19}$$

In the general case, one writes $n = \sigma_k + m$ with $0 < m < p_{k+1}$ and, from [8, Section 8], we have

$$h(n) = N_k G(p_k, m) \tag{1.20}$$

where $G(p_k, m)$ can be calculated by the algorithm described in [11, Section 9].

1.2 Plan of the article

- In Section 2, we recall some effective bounds for the Chebichev function $\theta(x)$ and for $\pi_r(x) = \sum_{p \le x} p^r$. We give also some properties of the logarithmic integral $\mathrm{li}(x)$ and its inverse li^{-1}.
- Section 3 is devoted to the definition and properties of ℓ-superchampion numbers. These numbers, defined on the model of the *superior highly composite numbers* introduced by Ramanujan in [22], are crucial for the study of the Landau function. They allow the construction of an infinite number of integers n for which $g(n)$ is easy to calculate. To reduce the running time of computation, an argument of convexity is given in Section 3.3 and used in Section 5.5 and in Lemma 8.1 in conjunction with the tools presented in Section 4.

- In Section 4 we present some methods used to compute efficiently: how to quickly enumerate the superchampion numbers, how to find the largest integer n in a finite interval $[a, b]$ which does not satisfy a boolean property $ok(n)$, by computing only a small number of values $ok(n)$.
- In Section 5 we prove Theorem 1.7.
- In Section 6, in preparation to the proof of Theorem 1.5, we study the function $\log g(n) - \log h(n)$ for which we give an effective estimate for $n \geq \nu_0$ (defined in (3.17)) and also an asymptotic estimate.
- In Section 7, we prove Theorem 1.5, first for $n \geq \nu_0$, by using the results of Section 6, and further, for $n < \nu_0$, by explaining the required computation.
- In Section 8, we prove Theorem 1.1. For $n \geq \nu_0$, it follows from the reunion of the proofs of Theorem 1.4 (given in [10]) and of Theorem 1.5. For $n < \nu_0$, some more computation is needed.

All computer calculations have been implemented in Maple and C^{++}. Maple programs are slow but can be executed by anyone disposing of Maple. C^{++} programs are much faster. They use real double precision, except for the demonstration of the Lemma 8.1 where we used the GNU-MPFR Library to compute with real numbers with a mantissa of 80 bits. The most expensive computations are the proof of theorem 1.5.(ii) and the proof of Lemma 8.1 which took respectively 40 hours and 10 hours of CPU (with the C^{++} programs). The Maple programs can be loaded on [27].

2 Useful results

2.1 Effective estimates

In [5], Büthe has proved

$$\theta(x) = \sum_{p \leq x} \log p < x \quad \text{for } x \leq 10^{19} \tag{2.1}$$

while Platt and Trudgian in [21] have shown that

$$\theta(x) < (1 + 7.5 \cdot 10^{-7})\, x \text{ for } x \geq 2 \tag{2.2}$$

so improving on results of Schoenfeld [26]. Without any hypothesis, we know that

$$|\theta(x) - x] < \frac{\alpha\, x}{\log^3 x} \text{ for } x \geq x_1 = x_1(\alpha) \tag{2.3}$$

with

$$\alpha = \begin{cases} 1 & \text{and} \quad x_1 = \quad 89\,967\,803 \quad \text{(cf. [12, Theorem 4.2])} \\ 0.5 & \text{and} \quad x_1 = \quad 767\,135\,587 \quad \text{(cf. [12, Theorem 4.2])} \\ 0.15 & \text{and} \quad x_1 = \quad 19\,035\,709\,163 \quad \text{(cf. [3, Theorem 1.1])} . \end{cases}$$

Lemma 2.1. *Let us denote* $\theta^-(x) = \sum_{p<x} \log p$. *Then*

$$\theta(x) \geq \theta^-(x) \ \geq x - 0.0746 \, \frac{x}{\log x} \qquad (x > 48757) \tag{2.4}$$

$$\theta(x) \ \leq x\Big(1 + \frac{0.000079}{\log x}\Big) \qquad (x > 1) \tag{2.5}$$

$$\pi(x) \ < 1.26 \frac{x}{\log x} \qquad (x > 1). \tag{2.6}$$

Proof. – From [26, Corollary 2*, p. 359], for $x \geq 70877$, we have $\theta(x) > F(x)$ with $F(t) = t(1 - 1/(15 \log t))$. As $F(t)$ is increasing for $t > 0$, this implies that if $x > 70877$ then $\theta^-(x) \geq F(x)$ holds. Indeed, if x is not prime, we have $\theta^-(x) = \theta(x)$ while if $x > 70877$ is prime then $x - 1 > 70877$ holds and we have $\theta^-(x) = \theta(y) > F(y)$ for y satisfying $x - 1 < y < x$. When y tends to x, we get $\theta^-(x) \geq F(x)$ and, as $1/15 < 0.0746$ holds, this proves (2.4) for $x > 70877$. Now, let us assume that $48757 < x \leq 70877$ holds. For all primes p satisfying $48757 \leq p < 70877$ and p^+ the prime following p we consider the function $f(t) = t(1 - 0.0746/\log t)$ for $t \in (p, p^+]$. As f is increasing, the maximum of f is $f(p^+)$ and $\theta^-(t)$ is constant and equal to $\theta(p)$. So, to complete the proof of (2.4), we check that $\theta(p) \geq f(p^+)$ holds for all these p's.

– (2.5) follows from (2.1) for $x \leq 10^{19}$, while, for $x > 10^{19}$, from (2.3), we have

$$\theta(x) \leq x \left(1 + \frac{0.15}{\log^3 x}\right) \leq x \left(1 + \frac{0.15}{(\log^2 10^{19})(\log x)}\right) = x \left(1 + \frac{0.0000783\ldots}{\log x}\right).$$

– (2.6) is stated in [24, (3.6)].

□

Lemma 2.2. *Let us set*

$$W(x) = \sum_{p \leq x} \frac{\log p}{1 - 1/p}. \tag{2.7}$$

Then, for $x > 0$,

$$\frac{W(x)}{x} \le \omega = \begin{cases} W(7)/7 = 1.045\,176\ldots & \text{if } x \le 7.32 \\ 1.000014 & \text{if } x > 7.32. \end{cases} \tag{2.8}$$

Proof. First, we calculate $W(p)$ for all primes $p < 10^6$. For $11 \le p < 10^6$, $W(p) < p$ holds while, for $p \in \{2, 3, 5, 7\}$, the maximum of $W(p)/p$ is attained for $p = 7$. If p and p^+ are consecutive primes, $W(x)$ is constant and $W(x)/x$ is decreasing on $[p, p^+)$. As $W(7) = 7.316\ldots$ this proves (2.8) for $x \le 7.32$ and $W(x) < x$ for $7.32 < x \le 10^6$.

Let us assume now that $x > y = 10^6$ holds. We have

$$\begin{aligned} W(x) = W(y) + \sum_{y<p\le x} \frac{\log p}{1-1/p} &\le W(y) + \frac{y}{y-1} \sum_{y<p\le x} \log p \\ &= W(y) - \frac{y}{y-1}\theta(y) + \frac{y}{y-1}\theta(x) = 12.240\,465\ldots + \frac{10^6}{10^6-1}\theta(x) \end{aligned}$$

and, from (2.2),

$$W(x) \le 12.241\frac{x}{10^6} + \frac{10^6}{10^6-1}(1 + 7.5 \times 10^{-7})x < 1.00001399\ldots\, x,$$

which completes the proof of Lemma 2.2. □

Lemma 2.3. *Let $K \ge 0$ and $\alpha > 0$ be two real numbers. Let us assume that there exists $X_0 > 1$ such that, for $x \ge X_0$,*

$$x - \frac{\alpha x}{\log^{K+1} x} \le \theta(x) \le x + \frac{\alpha x}{\log^{K+1} x}. \tag{2.9}$$

If a is a positive real number satisfying $a < \log^{K+1} X_0$, for $x \ge X_0$, we have

$$\pi\left(x + \frac{a\,x}{\log^K x}\right) - \pi(x) \ge b\,\frac{x}{\log^{K+1} x} \tag{2.10}$$

with

$$b = \left(1 - \frac{a}{\log^{K+1} X_0}\right)\left(a - \frac{2\alpha}{\log X_0} - \frac{\alpha\, a}{\log^{K+1} X_0}\right).$$

Proof. Let us set $y = x(1 + a/\log^K x)$. For $x \ge X_0$, we have

$$1 < X_0 \le x < y = x\left(1 + \frac{a}{\log^K x}\right) \le x\left(1 + \frac{a}{\log^K X_0}\right)$$

and

$$0 < \log x < \log y < \log x + \frac{a}{\log^K x} = (\log x)\left(1 + \frac{a}{\log^{K+1} x}\right)$$
$$\le (\log x)\left(1 + \frac{a}{\log^{K+1} X_0}\right) < \frac{\log x}{1 - a/\log^{K+1} X_0}. \quad (2.11)$$

Further,

$$\pi(y) - \pi(x) = \sum_{x<p\le y} 1 \ge \sum_{x<p\le y} \frac{\log p}{\log y} = \frac{1}{\log y}(\theta(y) - \theta(x))$$

and, from (2.9),

$$\begin{aligned}
\pi(y) - \pi(x) &\ge \frac{1}{\log y}\left(y - x - \frac{\alpha\, y}{\log^{K+1} y} - \frac{\alpha\, x}{\log^{K+1} x}\right)\\
&= \frac{1}{\log y}\left(\frac{a\, x}{\log^K x} - \frac{\alpha\, y}{\log^{K+1} x} - \frac{\alpha\, x}{\log^{K+1} x}\right)\\
&= \frac{x}{(\log y)\log^K x}\left(a - \frac{2\alpha}{\log x} - \frac{\alpha\, a}{\log^{K+1} x}\right)\\
&\ge \frac{x}{(\log y)\log^K x}\left[a - \frac{2\alpha}{\log X_0} - \frac{\alpha\, a}{\log^{K+1} X_0}\right].
\end{aligned}$$

If the above bracket is ≤ 0 then b is also ≤ 0 and (2.10) trivially holds. If the bracket is positive then (2.10) follows from (2.11), which ends the proof of Lemma 2.3. □

Corollary 2.4. *For $x \ge x_0 = 10^{10} + 19$,*

$$\pi(x(1 + 0.045/\log^2 x)) - \pi(x) \ge 0.012\sqrt{x}. \quad (2.12)$$

Proof. Since, for $x \ge x_0$, (2.3) implies (2.9) with $\alpha = 1/2$, $K = 2$ and $X_0 = x_0$, we may apply Lemma 2.3 that yields $\pi(x(1+0.045/\log^2 x)) - \pi(x) \ge b\,x/\log^3 x$ with $b = 0.001568\ldots$

From (1.15), for $x \ge x_0$, $\sqrt{x}/\log^3 x$ is increasing and $\sqrt{x_0}/\log^3 x_0 = 8.19\ldots$, so that $\pi(x(1+0.045/\log^2 x)) - \pi(x) \ge b\,x/\log^3 x \ge 8.19\,b\,\sqrt{x} \ge 0.012\,\sqrt{x}$. □

2.2 The logarithmic integral

For x real > 1, we define $\mathrm{li}(x)$ as (cf. [1, p. 228])

$$\mathrm{li}(x) = \fint_0^x \frac{dt}{\log t} = \lim_{\varepsilon\to 0^+}\left(\int_0^{1-\varepsilon} + \int_{1+\varepsilon}^x \frac{dt}{\log t}\right) = \int_2^x \frac{dt}{\log t} + \mathrm{li}(2).$$

From the definition of li(x), it follows that

$$\frac{d}{dx}\,\mathrm{li}(x) = \frac{1}{\log x} \qquad \text{and} \qquad \frac{d^2}{dx^2}\,\mathrm{li}(x) = -\frac{1}{x\log^2 x}.$$

For $x \to \infty$, the logarithmic integral has the asymptotic expansion

$$\mathrm{li}(x) = \sum_{k=1}^{N} \frac{(k-1)!x}{(\log x)^k} + \mathcal{O}\left(\frac{x}{(\log x)^{N+1}}\right). \tag{2.13}$$

The function $t \mapsto \mathrm{li}(t)$ is an increasing bijection from $(1, +\infty)$ onto $(-\infty, +\infty)$. We denote by $\mathrm{li}^{-1}(y)$ its inverse function that is defined for all $y \in \mathbb{R}$. Note that $\mathrm{li}^{-1}(y) > 1$ always holds.

To compute numerical values of li(x), we used the formula, due to Ramanujan (cf. [6, p. 126-131]),

$$\mathrm{li}(x) = \gamma_0 + \log\log x + \sqrt{x}\sum_{n=1}^{\infty} a_n(\log x)^n \quad \text{with} \quad a_n = \frac{(-1)^{n-1}}{n!\,2^{n-1}} \sum_{m=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{1}{2m+1}.$$

The computation of $\mathrm{li}^{-1} y$ is carried out by solving the equation $\mathrm{li}(x) = y$ by the Newton method.

2.3 Study of $\pi_r(x) = \sum_{p\le x} p^r$

In the article [10], we have deduced from (2.3) the following proposition:

Proposition 2.5. *Let α, $x_1 = x_1(\alpha)$ be two real numbers such that $0 < \alpha \le 1$, $x_1 \ge 89\,967\,803$ and $|\theta(x) - x| < \alpha\, x/\log^3 x$ for $x \ge x_1$. Then, for $r \ge 0.6$ and $x \ge x_1$,*

$$\pi_r(x) \le C_0 + \frac{x^{r+1}}{(r+1)\log x} + \frac{x^{r+1}}{(r+1)^2\log^2 x} + \frac{2x^{r+1}}{(r+1)^3\log^3 x} + \frac{(51\alpha r^4 + 176\alpha r^3 + 222\alpha r^2 + 120\alpha r + 23\alpha + 168)x^{r+1}}{24(r+1)^4\log^4 x} \tag{2.14}$$

with

$$C_0 = \pi_r(x_1) - \frac{x_1^r\theta(x_1)}{\log x_1} - \frac{3\alpha\, r^4 + 8\alpha\, r^3 + 6\alpha\, r^2 + 24 - \alpha}{24}\,\mathrm{li}(x_1^{r+1}) + \frac{(3\alpha\, r^3 + 5\alpha\, r^2 + \alpha\, r + 24 - \alpha\,)x_1^{r+1}}{24\log x_1} + \frac{\alpha\,(3r^2 + 2r - 1)x_1^{r+1}}{24\log^2 x_1} + \frac{\alpha\,(3r-1)x_1^{r+1}}{12\log^3 x_1} - \frac{\alpha\, x_1^{r+1}}{4\log^4 x_1}. \tag{2.15}$$

The unique positive root $r_0(\alpha)$ of the equation $3r^4 + 8r^3 + 6r^2 - 24/\alpha - 1 = 0$ is decreasing on α and satisfies $r_0(1) = 1.1445\ldots$, $r_0(0.5) = 1.4377\ldots$ and $r_0(0.15) = 2.1086\ldots$ For $0.06 \le r \le r_0(\alpha)$ and $x \ge x_1(\alpha)$, we have

$$\pi_r(x) \ge \widehat{C_0} + \frac{x^{r+1}}{(r+1)\log x} + \frac{x^{r+1}}{(r+1)^2\log^2 x} + \frac{2x^{r+1}}{(r+1)^3\log^3 x} - \frac{(2\alpha r^4 + 7\alpha r^3 + 9\alpha r^2 + 5\alpha r + \alpha - 6)x^{r+1}}{(r+1)^4\log^4 x} \tag{2.16}$$

while, if $r > r_0(\alpha)$ and $x \ge x_1(\alpha)$, we have

$$\pi_r(x) \ge \widehat{C_0} + \frac{x^{r+1}}{(r+1)\log x} + \frac{x^{r+1}}{(r+1)^2\log^2 x} + \frac{2x^{r+1}}{(r+1)^3\log^3 x} - \frac{(51\alpha r^4 + 176\alpha r^3 + 222\alpha r^2 + 120\alpha r + 23\alpha - 168)x^{r+1}}{24(r+1)^4\log^4 x}, \tag{2.17}$$

with

$$\widehat{C_0} = \pi_r(x_1) - \frac{x_1^r\theta(x_1)}{\log x_1} + \frac{3\alpha r^4 + 8\alpha r^3 + 6\alpha r^2 - \alpha - 24}{24}\,\mathrm{li}(x_1^{r+1}) - \frac{(3\alpha r^3 + 5\alpha r^2 + \alpha r - \alpha - 24)x_1^{r+1}}{24\log x_1} - \frac{\alpha(3r^2 + 2r - 1)x_1^{r+1}}{24\log^2 x_1} - \frac{\alpha(3r-1)x_1^{r+1}}{12\log^3 x_1} + \frac{\alpha x_1^{r+1}}{4\log^4 x_1}. \tag{2.18}$$

Corollary 2.6. *For $x \ge 110\,117\,910$, we have*

$$\pi_1(x) \le \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{107\,x^2}{160\log^4 x} \tag{2.19}$$

and, for $x \ge 905\,238\,547$,

$$\pi_1(x) \ge \frac{x^2}{2\log x} + \frac{x^2}{4\log^2 x} + \frac{x^2}{4\log^3 x} + \frac{3\,x^2}{20\log^4 x}. \tag{2.20}$$

Proof. It is Corollary 2.7 of [10], cf. also [2, Theorem 6.7 and Proposition 6.9]. □

Corollary 2.7. *For $x \ge 60\,173$,*

$$\pi_2(x) \le \frac{x^3}{3\log x} + \frac{x^3}{9\log^2 x} + \frac{2x^3}{27\log^3 x} + \frac{1181x^3}{648\log^4 x} \tag{2.21}$$

and for $x \geq 60\,297$,

$$\pi_2(x) \leq \frac{x^3}{3\log x}\left(1+\frac{0.385}{\log x}\right) \tag{2.22}$$

while, for $x \geq 1\,091\,239$, *we have*

$$\pi_2(x) \geq \frac{x^3}{3\log x}+\frac{x^3}{9\log^2 x}+\frac{2x^3}{27\log^3 x}-\frac{1069x^3}{648\log^4 x}. \tag{2.23}$$

and for $x > 32\,321$, *with* $\pi_2^-(x) = \sum_{p<x} p^2$,

$$\pi_2(x) \geq \pi_2^-(x) \geq \frac{x^3}{3\log x}\left(1+\frac{0.248}{\log x}\right). \tag{2.24}$$

Proof. From (2.3), the hypothesis $|\theta(x)-x| \leq \alpha x/\log^3 x$ is satisfied with $\alpha = 1$ and $x_1 = 89\,967\,803$. By computation, we find $\pi_2(x_1) = 13\,501\,147\,086\,873\,627\,946\,348$, $\theta(x_1) = 89\,953\,175.416\,013\,726\ldots$ and C_0, defined by (2.15) with $r = 2$ and $\alpha = 1$ is equal to $-1.040\ldots \times 10^{18} < 0$ so that (2.21) follows from (2.14) for $x \geq x_1$. From (1.15), the right-hand side of (2.21) is increasing on x for $x \geq e^{4/3} = 3.79\ldots$ We check that (2.21) holds when x runs over the primes p satisfying $60209 \leq p \leq x_1$ but not for $p = 60169$. For $x = 60172.903\ldots$, the right-hand side of (2.21) is equal to $\pi_2(60169)$, which completes the proof of (2.21).

Let us set $x_2 = 315\,011$. From (2.21), for $x \geq x_2$, we have

$$\begin{aligned}\pi_2(x) &\leq \frac{x^3}{3\log x}\left(1+\frac{1}{\log x}\left(\frac{1}{3}+\frac{2}{9\log x_2}+\frac{1181}{216\log^2 x_2}\right)\right)\\ &\leq \frac{x^3}{3\log x}\left(1+\frac{0.385}{\log x}\right)\end{aligned}$$

which proves (2.22) for $x \geq x_2$. Further, we check that (2.22) holds when x runs over the primes p such that $60317 \leq p \leq x_2$ but does not hold for $p = 60293$. Solving the equation $\pi_2(60293) = t^3/(3\log t)(1+0.385/\log t)$ yields $t = 60296.565\ldots$ which completes the proof of (2.22).

Similarly, $\widehat{C_0}$ defined by (2.18) is equal to $8.022\ldots \times 10^{18} > 0$ which implies (2.23) from (2.17) for $x \geq x_1$. Let us define $F(t) = \frac{t^3}{3\log t}+\frac{t^3}{9\log^2 t}+\frac{2t^3}{27\log^3 t}-\frac{1069t^3}{648\log^4 t}$. We have $F'(t) = \frac{t^2}{648\log^5 t}(648\log^4 t - 3351\log t + 4276)$ which is positive for $t > 1$ and thus, $F(t)$ is increasing for $t > 1$. For all primes p satisfying $1\,091\,239 \leq p \leq x_1$, we denote by p^+ the prime following p and we check that $\pi_2(p) \geq F(p^+)$, which proves (2.23).

Let us set $f(t) = \frac{t^3}{3\log t}(1 + \frac{0.248}{\log t})$, so that $F(t) - f(t) = \frac{t^3}{81000\log^4 t}(2304\log^2 t + 6000\log t - 133625)$. The largest root of the trinomial on $\log t$ is $6.424\ldots$ so that $F(t) > f(t)$ holds for $t \geq 618 > \exp(6.425)$, which, as (2.23) holds for $x \geq 1\,091\,239$, proves (2.24) for $x \geq 1\,091\,239$.

After that, we check that $\pi_2(p) \geq f(p^+)$ for all pairs (p, p^+) of consecutive primes satisfying $32321 \leq p < p^+ \leq 1\,091\,239$, which, for $x \geq 32321$, proves that $\pi_2(x) \geq f(x)$ and (2.24) if x is not prime. For x prime and $x > 32321$, we have $x - 1 > 32321$ and we consider y satisfying $x - 1 < y < x$. We have $\pi_2^-(x) = \pi_2(y) \geq f(y)$, which proves (2.24) when y tends to x. □

Corollary 2.8. *We have*

$$\pi_3(x) \leq 0.271\,\frac{x^4}{\log x} \quad \textit{for} \quad x \geq 664, \tag{2.25}$$

$$\pi_4(x) \leq 0.237\,\frac{x^5}{\log x} \quad \textit{for} \quad x \geq 200, \tag{2.26}$$

$$\pi_5(x) \leq 0.226\,\frac{x^6}{\log x} \quad \textit{for} \quad x \geq 44 \tag{2.27}$$

$$\pi_r(x) \leq \frac{\log 3}{3}\left(1 + \left(\frac{2}{3}\right)^r\right)\frac{x^{r+1}}{\log x} \quad \textit{for} \quad x > 1 \textit{ and } r \geq 5. \tag{2.28}$$

Proof. First, from (2.15), with $r \in \{3, 4, 5\}$, $\alpha = 1$ and $x_1 = 89\,967\,803$, we calculate

$$C_0(3) = -1.165\ldots 10^{26}, \quad C_0(4) = -1.171\ldots 10^{34}, \quad C_0(5) = -1.123\ldots 10^{42}.$$

As these three numbers are negative, from (2.14), for $r \in \{3, 4, 5\}$ and $x \geq x_1$, we have

$$\pi_r(x) \leq \frac{x^{r+1}}{(r+1)\log x}\bigg(1 + \frac{1}{(r+1)\log x_1} + \frac{2}{(r+1)^2\log^2 x_1} + \frac{51r^4 + 176r^3 + 222r^2 + 120r + 191}{24(r+1)^3\log^3 x_1}\bigg) \tag{2.29}$$

and

$$\pi_3(x) \leq 0.254\,\frac{x^4}{\log x}, \quad \pi_4(x) \leq 0.203\,\frac{x^5}{\log x} \quad \text{and} \quad \pi_5(x) \leq 0.169\,\frac{x^6}{\log x}.$$

If p and p^+ are two consecutive primes, for $r \geq 3$, it follows from (1.15) that the function $t \mapsto \frac{\pi_r(t)\log t}{t^{r+1}}$ is decreasing on t for $p \leq t < p^+$. Therefore, to complete the proof of (2.25), one checks that $\frac{\pi_3(p)\log p}{p^4} \le 0.271$ holds for $673 \le p \le x_1$ and we solve the equation $\pi_3(t)\log t = 0.271\ t^4$ on the interval $[661, 673)$ whose root is $663.35\ldots$ The proof of (2.26) and (2.27) are similar.

Since the mapping $t \mapsto (\log t)/t^6$ is decreasing for $t \ge 2$, the maximum of $\pi_5(x)(\log x)/x^6$ is attained on a prime p. In view of (2.27), calculating $\pi_5(p)(\log p)/p^6$ for $p = 5, 7, \ldots, 43$ shows that the maximum for $x \ge 5$ is attained for $x = 5$ and is equal to $0.350\ldots < (\log 3)/3$. Let r be a number ≥ 5. By applying the trivial inequality

$$\pi_r(x) \le x^{r-5}\pi_5(x), \quad r \ge 5,$$

we deduce that $\pi_r(x) < (\log 3)\, x^{r+1}/(3\log x)$ for $x \ge 5$, and, by calculating $\pi_r(2)(\log 2)/2^{r+1} = (\log 2)/2$ and $\pi_r(3)(\log 3)/3^{r+1} = \left(1 + \left(\frac{2}{3}\right)^r\right)\frac{\log 3}{3}$, we obtain (2.28). □

3 ℓ-superchampion numbers

3.1 Definition of ℓ-superchampion numbers

Definition 3.1. *An integer N is said ℓ-superchampion (or more simply superchampion) if there exists $\rho > 0$ such that, for all integer $M \ge 1$*

$$\ell(M) - \rho\log M \ge \ell(N) - \rho\log N. \tag{3.1}$$

When this is the case, we say that N is a ℓ-superchampion associated *to ρ.*

Geometrically, if we represent $\log M$ in abscissa and $\ell(M)$ in ordinate for all $M \ge 1$, the vertices of the convex envelop of all these points represent the ℓ-superchampion numbers (cf. [11, Fig. 1, p. 633]). If N is an ℓ-superchampion, the following property holds (cf. [11, Lemma 3]):

$$N = g(\ell(N)). \tag{3.2}$$

Similar numbers, the so-called *superior highly composite numbers* were first introduced by S. Ramanujan (cf. [22]). The ℓ-superchampion numbers were also used in [19, **?**, 16, 15, 17, 20, 7]. Let us recall the properties we will need. For more details, cf. [11, Section 4].

Lemma 3.2. *Let ρ satisfy $\rho \ge 5/\log 5 = 3.11\ldots$. Then, depending on ρ, there exists an unique decreasing sequence $(\xi_j)_{j\ge1}$ such that $\xi_1 > \exp(1)$ and, for all $j \ge 2$,*

$$\xi_j > 1 \quad \textit{and} \quad \frac{\xi_j^j - \xi_j^{j-1}}{\log \xi_j} = \frac{\xi_1}{\log \xi_1} = \rho. \tag{3.3}$$

We have also $\xi = \xi_1 \ge 5$ and $\xi_2 \ge 2$.

Definition 3.3. *For each prime $p \in \mathcal{P}$, let us define the sets*

$$\mathcal{E}_p = \left\{ \frac{p}{\log p}, \frac{p^2 - p}{\log p}, \ldots, \frac{p^{i+1} - p^i}{\log p}, \ldots \right\}, \quad \mathcal{E} = \bigcup_{p \in \mathcal{P}} \mathcal{E}_p. \tag{3.4}$$

Remark 3.4. *Note that all the elements of $\mathcal{E}_p$ are distinct at the exception, for $p = 2$, of $\frac{2}{\log 2} = \frac{2^2 - 2}{\log 2}$ and that, for $p \neq q$, $\mathcal{E}_p \cap \mathcal{E}_q = \emptyset$ holds.*

Furthermore if $\rho \in \mathcal{E}_p$, there is an unique $\widehat{j} = \widehat{j}(\rho) \geq 1$ such that $\xi_{\widehat{j}}$ is an integer; this integer is p and $\widehat{j}$ is given by

$$\widehat{j} = \begin{cases} 1 & \text{if } \rho = p/\log p \\ j & \text{if } \rho = (p^j - p^{j-1})/\log p \quad (j \geq 2). \end{cases} \tag{3.5}$$

Proposition 3.5. *Le $\rho > 5/\log 5$, $\xi_j = \xi_j(\rho)$ defined by* (3.3) *and N_ρ, N_ρ^+ defined by*

$$N_\rho = \prod_{j \geq 1} \Big(\prod_{p < \xi_j} p \Big) = \prod_{j \geq 1} \Big(\prod_{\xi_{j+1} \leq p < \xi_j} p^j \Big) \tag{3.6}$$

and

$$N_\rho^+ = \prod_{j \geq 1} \Big(\prod_{p \leq \xi_j} p \Big) = \prod_{j \geq 1} \Big(\prod_{\xi_{j+1} < p \leq \xi_j} p^j \Big) \tag{3.7}$$

Then,

1. *If $\rho \notin \mathcal{E}$, $N_\rho = N_\rho^+$ is the unique superchampion associated to ρ.*
2. *If $\rho \in \mathcal{E}_p$, N_ρ and N_ρ^+ are two consecutive superchampions, they are the only superchampions associated to ρ, and*

$$N_\rho^+ = pN_\rho = \xi_{\widehat{j}}(\rho) N_\rho. \tag{3.8}$$

From (3.3) we deduce that the upper bound for j in (3.6) and (3.7) is $\lfloor J \rfloor$ with J defined by

$$\frac{2^J - 2^{J-1}}{\log 2} = \rho = \frac{\xi}{\log \xi} \quad \text{i.e.} \quad J = \frac{\log \xi + \log(2\log 2) - \log\log \xi}{\log 2} < \frac{\log \xi}{\log 2}, \tag{3.9}$$

as $\xi \geq 5$ is assumed.

Definition 3.6. *Let us suppose $n \geq 7$. Depending on n, we define ρ, N', N'', n', n'', ξ, and $(\xi_j)_{j \geq 1}$.*

1. *ρ is the unique element of $\mathcal{E}$ such that*

$$\ell(N_\rho) \leq n < \ell(N_\rho^+). \tag{3.10}$$

2. N', N'', n', n'' *are defined by*

$$N' = N_\rho, \quad N'' = N_\rho^+ \quad n' = \ell(N'), \text{ and } n'' = \ell(N''). \tag{3.11}$$

3. *For* $j \geq 1$, ξ_j *is defined by* (3.3) *and* ξ *is defined by* $\xi = \xi_1$ *i.e.* $\log \xi / \xi = \rho$.

Proposition 3.7. *Let us suppose* $n \geq 7$, ρ, N', N'', n', n'' *and* ξ *defined by Definiton 3.6. Then*

$$n' \leq n < n'' \quad \text{and} \quad N' = g(n') \leq g(n) < N'' = g(n'') = pN'. \tag{3.12}$$

$$\ell(N') - \rho \log N' = \ell(N'') - \rho \log N''. \tag{3.13}$$

$$N'' \leq \xi N'. \tag{3.14}$$

$$\ell(N'') - \ell(N') \leq \xi. \tag{3.15}$$

Proof. – (3.12) results of (3.10), (3.11) and (3.8).

– By applying (3.1) first to $M = N'$, $N = N''$ and further to $M = N''$, $N = N'$ we get (3.13).

– Equality (3.8) gives $N'' = \xi_{\hat{j}(\rho)} N'$; with the decreasingness of (ξ_j) and the definiton $\xi = \xi_1$ we get (3.14).

– By using (3.13) and (3.14) we have $\ell(N'') - \ell(N') = \rho \log(N\varepsilon/N')) \leq \rho \log \xi = \xi$.

□

In the array of Fig. (1), for a small n, one can read the value of N', N'', ρ and ξ as given in Definition 3.6. For instance, for $n = 45$, we have $N' = 60\,060$, $N'' = 180\,180$, $\rho = 6/\log 3$ and $\xi = 14.667\ldots$ We also can see the values of the parameter associated to a superchampion number N. For instance, $N = 360\,360$ is associated to all values of ρ satisfying $4/\log 2 \leq \rho \leq 17/\log 17$.

As another example, let us consider $x_0 = 10^{10} + 19$, the smallest prime exceeding 10^{10}, and the two ℓ-superchampion numbers N_0' and N_0'' associated to $\rho = x_0/\log x_0 \in \mathcal{E}_{x_0}$. We have

$$N_0' = 2^{29}3^{18}5^{12}7^{10}11^{8}13^{8}17^{7}19^{7}23^{6}\ldots 31^{6}37^{5}\ldots 71^{5}73^{4}\ldots 211^{4}223^{3}\ldots 1459^{3}$$
$$1471^{2}\ldots 69557^{2}69593\ldots 9\,999\,999\,967 \quad \text{and} \quad N_0'' = x_0\, N_0', \tag{3.16}$$

$$v_0 = \ell(N_0') = 2\,220\,832\,950\,051\,364\,840 = 2.22\ldots 10^{18}, \qquad J = 29.165\ldots \tag{3.17}$$

n	N	$\ell(N)$	ρ	ξ
	$12 = 2^2 \cdot 3$	7		
7..11			$5/\log 5 = 3.11$	5
	$60 = 2^2 \cdot 3 \cdot 5$	12		
12..18			$7/\log 7 = 3.60$	7
	$420 = 2^2 \cdot 3 \cdot 5 \cdot 7$	19		
19..29			$11/\log 11 = 4.59$	11
	$4\,620 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$	30		
30..42			$13/\log 13 = 5.07$	13
	$60\,060 = 2^2 \cdot 3 \dots 13$	43		
43..48			$(9-3)/\log 3 = 5.46$	14.66
	$180\,180 = 2^2 \cdot 3^2 \cdot 5 \dots 13$	49		
49..52			$(8-4)/\log 2 = 5.77$	16
	$360\,360 = 2^3 \cdot 3^2 \cdot 5 \dots 13$	53		
53..69			$17/\log 17 = 6.00$	17
	$6\,126\,120 = 2^3 \cdot 3^2 \cdot 5 \dots 17$	70		
70..88			$19/\log 19 = 6.45$	19
	$116\,396\,280 = 2^3 \cdot 3^2 \cdot 5 \dots 19$	89		

Figure 1: The first superchampion numbers

and, for $2 \le j \le 29$, $\xi_j = x_j^{(0)}$ with

$$x_2^{(0)} = 69588.8\dots, x_3^{(0)} = 1468.8\dots, x_4^{(0)} = 220.2\dots, x_5^{(0)} = 71.5\dots$$
$$\text{and } \; x_{29}^{(0)} = 2.0\dots \tag{3.18}$$

A complete table of values of $x_j^{(0)}$ is given in [27].

Let n be an integer, and $\xi = \xi(n)$ (Definition 3.6). Let us suppose $n \ge \nu_0 = \ell(N_0')$; then, by (3.10), $\rho \ge \rho_0$, ie $\xi/\log\xi \ge x_0/\log x_0$. So that

$$\xi \ge x_0. \tag{3.19}$$

3.2 Estimates of ξ_j defined by (3.3)

Lemma 3.8. *(i) For $\xi \ge 5$ and $j \ge 2$, we have*

$$\xi_j \le \xi^{1/j}. \tag{3.20}$$

(ii) For $2 \le j \le 8$ and $\xi \ge \lambda_j$, we have

$$\xi_j \le \left(\frac{\xi}{j}\right)^{1/j}, \tag{3.21}$$

with $\lambda_2 = 80$, $\lambda_3 = 586$, $\lambda_4 = 6381$, $\lambda_5 = 89017$, $\lambda_6 = 1\,499\,750$, $\lambda_7 = 29\,511\,244$, $\lambda_8 = 663\,184\,075$.

(iii) For $\xi \geq 5$ *and* j *such that* $\xi_j \geq x_j^{(0)} \geq 2$ *(where* $x_j^{(0)}$ *is defined in* (3.18)*), we have*

$$\xi_j \leq \left(\frac{\xi}{j(1 - 1/x_j^{(0)})}\right)^{1/j} \leq \left(\frac{2\xi}{j}\right)^{1/j}. \tag{3.22}$$

Proof. (i) As the function $t \mapsto \frac{t^j - t^{j-1}}{\log t}$ is increasing, it suffices to show that

$$\frac{(\xi^{1/j})^j - (\xi^{1/j})^{j-1}}{\log(\xi^{1/j})} = \frac{\xi - \xi^{(j-1)/j}}{(1/j)\log\xi} \geq \frac{\xi}{\log\xi}$$

which is equivalent to

$$\xi \geq \left(1 + \frac{1}{j-1}\right)^j. \tag{3.23}$$

But the sequence $(1+1/(j-1))^j$ decreases from 4 to $\exp(1)$ when j increases from 2 to ∞, and $\xi \geq 5$ is assumed, which implies (3.23) and (3.20).

(ii) Here, we have to prove

$$\frac{\xi/j - (\xi/j)^{(j-1)/j}}{(1/j)\log(\xi/j)} \geq \frac{\xi}{\log\xi}$$

which is equivalent to

$$\frac{\xi^{1/j}}{\log\xi} \geq \frac{j^{1/j}}{\log j}. \tag{3.24}$$

For $2 \leq j \leq 8$, we have $\lambda_j > e^j$ and, from (1.15), the function $\xi \mapsto \xi^{1/j}/\log\xi$ is increasing for $\xi \geq \lambda_j$ and its value for $\xi = \lambda_j$ exceeds $j^{1/j}/\log j$ so that inequality (3.24) is satisfied.

(iii) Let us suppose that $\xi \geq 5$ and $\xi_j \geq x_j^{(0)} \geq 2$ hold. From (3.3) and (3.20), we have

$$\xi_j^j = \frac{\xi\log\xi_j}{\log\xi(1 - 1/\xi_j)} \leq \frac{\xi}{j(1 - 1/\xi_j)} \leq \frac{\xi}{j(1 - 1/x_j^{(0)})} \leq \frac{2\xi}{j}$$

which proves (3.22). □

Corollary 3.9. *For* $n \geq 7$, $\rho = \rho(n)$, $\xi = \xi(n)$ *defined in Definition 3.6, the powers* p^j *of primes dividing* $N' = N_\rho$ *or* $N'' = N_\rho^+$ *in* (3.6) *or* (3.7) *do not exceed* ξ.

Proof. This follows from (3.20), (3.6) and (3.7). □

3.3 Convexity

In this paragraph we prove Lemma 3.13 which is used to speed-up the computations done in Section 5.5 to prove Inequality (1.14) of Theorem 1.7, and and in Section 8.4 to prove Theorem 1.1 (iv).

Lemma 3.10. *The function $t \mapsto \sqrt{\mathrm{li}^{-1}(t)}$ is concave for $t > \mathrm{li}(e^2) = 4.954\ldots$.*

Let $a \le 1$ be a real number. Then, for $t \ge 31$, the function $t \mapsto \sqrt{\mathrm{li}^{-1}(t)} - a(t(\log t))^{1/4}$ is concave.

Proof. The proof is a good exercise of calculus: cf. [10, Lemma 2.5]. □

Lemma 3.11. *Let u be a real number, $0 \le u \le e$. The function Φ_u defined by*

$$\Phi_u(t) = \sqrt{t\log t}\left(1 + \frac{\log\log t - 1}{2\log t} - u\frac{(\log\log t)^2}{\log^2 t}\right) \tag{3.25}$$

is increasing and concave for $t \ge e^e = 15.15\ldots$

Proof. Let us write L for $\log t$ and λ for $\log\log t$. One calculates (cf. [27])

$$\frac{d\,\Phi_u}{dt} = \frac{1}{4L^2\sqrt{tL}}\big(2L(L^2 - u\lambda^2) + L(L - \lambda + 3) + \lambda(L^2 - 2u) + 6u\lambda(\lambda - 1)\big)$$

$$\frac{d^2\Phi_u}{dt^2} = \frac{-1}{8L^2(tL)^{3/2}}(L^2(L^2 - 2u\lambda^2) + L^3(\lambda - 1) + L(2L - 3\lambda) + (L^4 + 11L - 22u\lambda) + 2u(15\lambda^2 - 21\lambda + 8)).$$

For $t \ge e^e$, we have $L \ge e$, $\lambda \ge 1$, $L = e^\lambda > e\lambda \ge u\lambda$, so that Φ_u' is positive.

In Φ_u'', $L^4 + 11L \ge (e^3 + 11)L \ge (e^3 + 11)u\lambda > 22u\lambda$, the trinomial $15\lambda^2 - 21\lambda + 8$ is always positive and the three first terms of the parenthesis are also positive, so that Φ_u'' is negative. □

Lemma 3.12. *Let $n' = \ell(N')$, $n'' = \ell(N'')$ where N' and N'' are two consecutive ℓ-superchampion numbers associated to the same parameter ρ. Let Φ be a concave function on $[n', n'']$ such that $\log N' \le \Phi(n')$ and $\log N'' \le \Phi(n'')$. Then,*

$$\text{For } n \in [n', n''], \quad \log g(n) \le \Phi(n).$$

Proof. From (3.2), it follows that $N' = g(n')$ and $N'' = g(n'')$. Let us set $N = g(n)$ so that, from (1.3), we have $n \ge \ell(N)$. From the definition (3.1) of superchampion numbers and (3.13), we have

$$n - \rho\log N \ge \ell(N) - \rho\log N \ge n' - \rho\log N' = n'' - \rho\log N''. \tag{3.26}$$

Now, we may write

$$n = \alpha n' + \beta n'' \quad \text{with} \quad 0 \le \alpha \le 1 \quad \text{and} \quad \beta = 1 - \alpha \tag{3.27}$$

and (3.26) implies

$$\begin{aligned}\log N &\le \frac{1}{\rho}(n - (n' - \rho \log N')) \\ &= \frac{1}{\rho}(\alpha n' + \beta n'' - \alpha(n' - \rho \log N') - \beta(n'' - \rho \log n'')) \\ &= \alpha \log N' + \beta \log N''. \end{aligned} \tag{3.28}$$

From the concavity of Φ, $\log N' \le \Phi(n')$ and $\log N'' \le \Phi(n'')$, (3.28) and (3.27) imply

$$\log g(n) = \log N \le \alpha\, \Phi(n') + \beta\, \Phi(n'') \le \Phi(\alpha n' + \beta n'') = \Phi(n),$$

which completes the proof of Lemma 3.12. □

For $n \ge 2$, let us define z_n by

$$\log g(n) = \sqrt{n \log n}\left(1 + \frac{\log\log n - 1}{2 \log n} - z_n \frac{(\log\log n)^2}{\log^2 n}\right). \tag{3.29}$$

Lemma 3.13. *Let N' and N'' be two consecutive ℓ-superchampion numbers and $\ell(N') \le n \le \ell(N'')$.*

(i) If $n' \ge 43$ and if $a_{n'}$ and $a_{n''}$ (defined by (1.9)*) both belong to* $[0, 1]$ *then*

$$a_n \ge \min(a_{n'}, a_{n''}).$$

(ii) If $n' \ge 19$ and if $z_{n'}$ and $z_{n''}$ (defined by (3.29)*) both belong to* $[0, e]$ *then*

$$z_n \ge \min(z_{n'}, z_{n''}).$$

Proof. From (3.2), it follows that $N' = g(n')$ and $N'' = g(n'')$. Let us set $N = g(n)$ and

$$\Phi(t) = \sqrt{\mathrm{li}^{-1}(t)} - \min(a_{n'}, a_{n''})(t \log t)^{1/4}.$$

From Lemma 3.10, Φ is concave on $[n', n'']$. Moreover, from the definition (1.9) of $a_{n'}$ and $a_{n''}$, we have $\log N' \le \Phi(n')$ and $\log N'' \le \Phi(n'')$ which, from Lemma 3.12, implies $\log g(n) \le \Phi(n)$ and, from (1.9), $a_n \ge \min(a_{n'}, a_{n''})$ holds, which proves (i).

The proof of (ii) is similar. We set $u = \min(z_{n'}, z_{n''})$. From Lemma 3.11, Φ_u is concave on $[n', n'']$, $\log g(n') \le \Phi_u(n')$ and $\log g(n'') \le \Phi_u(n'')$ so that, from Lemma 3.12, we have $\log g(n) \le \Phi_u(n)$ and, from (3.29), $z_n \ge u$ holds, which completes the proof of Lemma 3.13. □

3.4 Estimates of ξ_2 defined by (3.3)

By iterating the formula $\xi_2 = \sqrt{\frac{\xi \log \xi_2}{\log \xi} + \xi_2}$ (cf. (3.3)), for any positive integer K, we get

$$\xi_2 = \sqrt{\frac{\xi}{2}}\left(1 + \sum_{k=1}^{K-1} \frac{\alpha_k}{\log^k \xi} + \mathcal{O}\left(\frac{1}{\log^K \xi}\right)\right), \quad \xi \to \infty, \tag{3.30}$$

with

$$\alpha_1 = -\frac{\log 2}{2}, \quad \alpha_2 = -\frac{(\log 2)(\log 2 + 4)}{8}, \quad \alpha_3 = -\frac{(\log 2)(\log^2 2 + 8\log 2 + 8)}{16}$$

Proposition 3.14. *We have the following bounds for ξ_2 :*

$$\xi_2 < \sqrt{\frac{\xi}{2}}\left(1 - \frac{\log 2}{2\log \xi}\right) < \sqrt{\frac{\xi}{2}}\left(1 - \frac{0.346}{\log \xi}\right) \quad \textit{for } \xi \geq 31643 \tag{3.31}$$

$$\xi_2 > \sqrt{\frac{\xi}{2}}\left(1 - \frac{0.366}{\log \xi}\right) \quad \textit{for } \xi \geq 4.28 \times 10^9. \tag{3.32}$$

Proof. Let us suppose $\xi \geq 4$ and $a \leq 0.4$. We set

$$\Phi = \Phi_a(\xi) = \sqrt{\frac{\xi}{2}}\left(1 - \frac{a}{\log \xi}\right) \geq \sqrt{\frac{4}{2}}\left(1 - \frac{0.4}{\log 4}\right) = 1.006\ldots > 1$$

and

$$W = W_a(\xi) = \frac{(\Phi^2 - \Phi)\log \xi}{\xi} - \log \Phi = \frac{(\log \Phi)(\log \xi)}{\xi}\left(\frac{\Phi^2 - \Phi}{\log \Phi} - \frac{\xi}{\log \xi}\right)$$
$$= \frac{(\log \Phi)(\log \xi)}{\xi}\left(\frac{\Phi^2 - \Phi}{\log \Phi} - \frac{\xi_2^2 - \xi_2}{\log \xi_2}\right).$$

As $\Phi > 1$ and $t \mapsto (t^2 - t)/\log t$ is increasing, we have

$$W_a(\xi) > 0 \quad \Longleftrightarrow \quad \xi_2 < \Phi(\xi) = \sqrt{\frac{\xi}{2}}\left(1 - \frac{a}{\log \xi}\right). \tag{3.33}$$

By the change of variable

$$\xi = \exp(2t), \quad t = \frac{1}{2}\log \xi \geq \log 2,$$

with the help of Maple (cf. [27]), we get

$$
\begin{aligned}
W &= -a + \frac{a^2}{4t} + \frac{3}{2}\log 2 + \log t - \log(2t-a) - e^{-t}\frac{\sqrt{2}}{2}(2t-a),\\
W' &= \frac{dW}{dt} = \frac{U}{4t^2(2t-a)} \quad \text{with}\\
U &= -2a(a+2)t + a^3 + 2\sqrt{2}e^{-t}V(4t^4 - 4(a+1)t^3 + a(a+2)t^2),\\
U' &= \frac{dU}{dt}\\
&= -2a(a+2) + 2\sqrt{2}e^{-t}(-4t^4 + 4(a+5)t^3 - (a^2+14a+12)t^2 + 2a(a+2)t)\\
U'' &= \frac{d^2U}{dt^2} = 2\sqrt{2}e^{-t}V\\
&\text{with } V = 4t^4 - 4(a+9)t^3 + (a^2+26a+72)t^2 - 4(a^2+8a+6)t + 2a(a+2).
\end{aligned}
$$

The sign of U'' is the same than the sign of the polynomial V that is easy to find. For a fixed and $t \geq \log 2$ (i.e. $\xi \geq 4$), one can successively determine, the variation and the sign of U', U and W.

For $a = (\log 2)/2 = 0.346\ldots$

t	$\log 2$		0.96		2.39		4.23		6.38		∞
U''		$+$		$+$	0	$-$		$-$	0	$+$	
					16.09						-1.62
U'		↗	0	↗		↘	0	↘		↗	
	-2.79								-10.03		

t	$\log 2$		0.96		1.49		4.23		8.0		∞
U'		$-$	0	$+$		$+$	0	$-$		$-$	
	-1.76						29.6				
U		↘		↗	0	↗		↘	0	↘	
			-2.18								$-\infty$

t	$\log 2$		1.49		5.18		8.0		∞
U or W'		$-$	0	$+$		$+$	0	$-$	
	-0.036						0.021		
W		↘		↗	0	↗		↘	
			-0.27						0

(3.34)

The root of W is $w_0 = 5.1811243\ldots$ and $\exp(2w_0) = 31642.25\ldots$. Therefore, (3.31) follows from array (3.34).

For $a = 0.366$

t	$\log 2$		0.97		2.39		4.23		6.39		∞
U''		$+$		$+$	0	$-$		$-$	0	$+$	
U'	-2.957	$\nearrow$	0	$\nearrow$	15.91	$\searrow$	0	$\searrow$	-10.09	$\nearrow$	-1.731

t	$\log 2$		0.9797		1.5208		4.231		7.892		∞
U'		$-$	0	$+$		$+$	0	$-$		$-$	
U	-1.830	$\searrow$	-2.308	$\nearrow$	0	$\nearrow$	29.04	$\searrow$	0	$\searrow$	$-\infty$

t	$\log 2$		1.52		6.37		7.89		11.08		∞
W'		$-$	0	$+$		$+$	0	$-$		$-$	
W	-0.025	$\searrow$	-0.28	$\nearrow$	0	$\nearrow$	0.004	$\searrow$	0	$\searrow$	-0.019

(3.35)

The root w_0 of W is equal to $11.08803\ldots$ and $\exp(2w_0) = 4.27505\ldots \times 10^9$ so that (3.32) follows from (3.33) and from array (3.35).

□

Remark 3.15. *By solving the system $W = 0$, $U = 0$ on the two variables t and a, one finds*

$$a = a_0 = 0.370612465\ldots, \qquad t = t_0 = 7.86682407\ldots$$

and for $a = a_0$, $t = t_0$ is a double root of W_{a_0}. By studying the variation of W_{a_0}, we find an array close to (3.35), *but $W_{a_0}(t_0) = W'_{a_0}(t_0) = 0$, so that W_{a_0} is nonpositive for $t \geq \log 2$, which proves*

$$\xi_2 \geq \sqrt{\frac{\xi}{2}}\left(1 - \frac{a_0}{\log \xi}\right) > \sqrt{\frac{\xi}{2}}\left(1 - \frac{0.371}{\log \xi}\right) \quad \textit{for} \quad \xi \geq 4.$$

Corollary 3.16. *If $\xi \geq x_0 = 10^{10} + 19$ holds and ξ_2 is defined by* (3.3), *then*

$$\frac{2}{\log \xi} \leq \frac{2}{\log \xi}\left(1 + \frac{\log 2}{\log \xi}\right) \leq \frac{1}{\log \xi_2} \leq \frac{2}{\log \xi}\left(1 + \frac{0.75}{\log \xi}\right) \leq \frac{2.07}{\log \xi}. \tag{3.36}$$

Proof. Since $\xi \geq x_0$ holds and ξ_2 is increasing on ξ, we have $\xi_2 \geq x_2^{(0)} = 69588.859\ldots$ (cf. (3.18)) and (3.21) implies $\xi_2 \leq \sqrt{\xi/2}$, i.e.

$$\log \xi_2 \leq \frac{1}{2}\log \xi - \frac{1}{2}\log 2 = \frac{\log \xi}{2}\left(1 - \frac{\log 2}{\log \xi}\right)$$

and

$$\frac{1}{\log \xi_2} \geq \frac{2}{(\log \xi)(1 - (\log 2)/\log \xi)} \geq \frac{2}{\log \xi}\left(1 + \frac{\log 2}{\log \xi}\right) \geq \frac{2}{\log \xi}.$$

On the other hand, (3.32) implies

$$\begin{aligned}\log \xi_2 &\geq \frac{1}{2}\log \xi - \frac{1}{2}\log 2 + \log\left(1 - \frac{0.366}{\log \xi}\right)\\ &\geq \frac{1}{2}\log \xi - \frac{1}{2}\log 2 + \log\left(1 - \frac{0.366}{\log x_0}\right)\\ &\geq \frac{1}{2}\log \xi - 0.363 = \frac{\log \xi}{2}\left(1 - \frac{0.726}{\log \xi}\right) \geq \frac{\log \xi}{2}\left(1 - \frac{0.726}{\log x_0}\right) \geq \frac{\log \xi}{2.07}\end{aligned}$$

and by (1.16),

$$\begin{aligned}\frac{1}{\log \xi_2} &\leq \frac{2}{(\log \xi)(1 - 0.726/\log \xi)} \leq \frac{2}{\log \xi}\left(1 + \frac{0.726}{(\log \xi)(1 - 0.726/\log x_0)}\right)\\ &\leq \frac{2}{\log \xi}\left(1 + \frac{0.75}{\log \xi}\right) \leq \frac{2}{\log \xi}\left(1 + \frac{0.75}{\log x_0}\right) \leq \frac{2.07}{\log \xi},\end{aligned}$$

which completes the proof of (3.36). □

Corollary 3.17. *If* $\xi \geq x_0 = 10^{10} + 19$ *holds and* ξ_2 *is defined by* (3.3), *then*

$$\sqrt{\frac{\xi}{2}}\left(1 - \frac{0.521}{\log \xi}\right) \leq \theta^-(\xi_2) = \sum_{p < \xi_2} \log p \leq \theta(\xi_2) \leq \sqrt{\frac{\xi}{2}}\left(1 - \frac{0.346}{\log \xi}\right). \quad (3.37)$$

Proof. We have $\xi_2 > x_2^{(0)} > 69588$ and (2.4), (3.36) and (3.32) imply

$$\begin{aligned}\theta^-(\xi_2) &\geq \xi_2\left(1 - \frac{0.0746}{\log \xi_2}\right) \geq \xi_2\left(1 - \frac{0.0746 \times 2.07}{\log \xi}\right) \geq \xi_2\left(1 - \frac{0.155}{\log \xi}\right)\\ &\geq \sqrt{\frac{\xi}{2}}\left(1 - \frac{0.366}{\log \xi}\right)\left(1 - \frac{0.155}{\log \xi}\right) \geq \sqrt{\frac{\xi}{2}}\left(1 - \frac{0.521}{\log \xi}\right).\end{aligned}$$

Similarly, for the upper bound, we use (2.5), (3.36) and (3.31) to get

$$\begin{aligned}\theta(\xi_2) &\leq \xi_2\left(1 + \frac{0.000079}{\log \xi_2}\right) \leq \xi_2\left(1 + \frac{0.000079 \times 2.07}{\log \xi}\right)\\ &\leq \xi_2\left(1 + \frac{0.000164}{\log \xi}\right) \leq \sqrt{\frac{\xi}{2}}\left(1 - \frac{\log 2}{2\log \xi}\right)\left(1 + \frac{0.000164}{\log \xi}\right)\\ &\leq \sqrt{\frac{\xi}{2}}\left(1 - \frac{0.346}{\log \xi}\right)\end{aligned}$$

which proves the upper bound of (3.37) □

Corollary 3.18. *Let* $\xi \geq x_0 = 10^{10} + 19$ *be a real number and* ξ_2 *be defined by* (3.3)*. Then*

$$\frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{0.122}{\log\xi}\right) \leq \pi_2^-(\xi_2) = \sum_{p<\xi_2} p^2 \leq \pi_2(\xi_2)$$
$$\leq \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{0.458}{\log\xi}\right). \tag{3.38}$$

Proof. First, from (3.3), we observe that

$$\frac{\xi_2^3}{\log\xi_2} = \xi_2\left(\frac{\xi_2^2 - \xi_2}{\log\xi_2}\right) + \frac{\xi_2^2}{\log\xi_2} = \xi_2\frac{\xi}{\log\xi} + \frac{\xi_2^2}{\log\xi_2} \geq \xi_2\frac{\xi}{\log\xi}, \tag{3.39}$$

whence, from (2.24) and (3.36), since $\xi \geq x_0$ and $\xi_2 \geq x_0^{(2)}$ are assumed,

$$\pi_2^-(\xi_2) \geq \frac{\xi_2^3}{3\log\xi_2}\left(1 + \frac{0.248}{\log\xi_2}\right) \geq \xi_2\frac{\xi}{3\log\xi}\left(1 + \frac{0.496}{\log\xi}\right)$$

and, from (3.32),

$$\begin{aligned}\pi_2^-(\xi_2) &\geq \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{0.496}{\log\xi}\right)\left(1 - \frac{0.366}{\log\xi}\right)\\ &= \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{0.13}{\log\xi} - \frac{0.496 \times 0.366}{\log^2\xi}\right)\\ &\geq \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{0.13}{\log\xi} - \frac{0.496 \times 0.366}{(\log x_0)\log\xi}\right) \geq \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{0.122}{\log\xi}\right),\end{aligned}$$

which proves the lower bound of (3.38).

To prove the upper bound, as (3.20) implies $\xi_2 \leq \sqrt{\xi}$ and $\xi_2/\log\xi_2 \leq 2\sqrt{\xi}/\log\xi$, from (3.39), we observe that

$$\begin{aligned}\frac{\xi_2^3}{\log\xi_2} &= \xi_2\frac{\xi}{\log\xi} + \frac{\xi_2^2}{\log\xi_2} = \xi_2\frac{\xi}{\log\xi} + \frac{\xi_2^2 - \xi_2}{\log\xi_2} + \frac{\xi_2}{\log\xi_2}\\ &= \frac{\xi}{\log\xi}(\xi_2 + 1) + \frac{\xi_2}{\log\xi_2} \leq \frac{\xi}{\log\xi}\left(\xi_2 + 1 + \frac{2}{\sqrt{\xi}}\right)\end{aligned}$$

and, from (3.31) and (3.36),

$$\begin{aligned}\frac{\xi_2^3}{\log \xi_2} &\le \frac{\xi}{\log \xi}\left(1+\frac{2}{\sqrt{\xi}}+\sqrt{\frac{\xi}{2}}\left(1-\frac{\log 2}{2\log \xi}\right)\right)\\ &= \frac{\xi^{3/2}}{\sqrt{2}\log \xi}\left(1-\frac{1}{\log \xi}\left(\frac{\log 2}{2}-\frac{\sqrt{2}(1+2/\sqrt{\xi})\log^2 \xi}{\sqrt{\xi}}\right)\right)\\ &\le \frac{\xi^{3/2}}{\sqrt{2}\log \xi}\left(1-\frac{1}{\log \xi}\left(\frac{\log 2}{2}-\frac{\sqrt{2}(1+2/\sqrt{x_0})\log^2 x_0}{\sqrt{x_0}}\right)\right)\\ &\le \frac{\xi^{3/2}}{\sqrt{2}\log \xi}\left(1-\frac{0.339}{\log \xi}\right).\end{aligned}$$

Further, from (3.36), we have $0.385/\log \xi_2 \le 2.07 \times 0.385/\log \xi \le 0.797/\log \xi$, whence from (2.22),

$$\begin{aligned}\pi_2(\xi_2) \le \frac{\xi_2^3}{3\log \xi_2}\left(1+\frac{0.385}{\log \xi_2}\right) &\le \frac{\xi^{3/2}}{3\sqrt{2}\log \xi}\left(1-\frac{0.339}{\log \xi}\right)\left(1+\frac{0.797}{\log \xi}\right)\\ &\le \frac{\xi^{3/2}}{3\sqrt{2}\log \xi}\left(1+\frac{0.458}{\log \xi}\right),\end{aligned}$$

which completes the proof of Corollary 3.18. □

3.5 The additive excess and the multiplicative excess

3.5.1 The additive excess

Let N be a positive integer and $Q(N) = \prod_{p|N} p$ be the squarefree part of N. The additive excess $E(N)$ of N is defined by

$$E(N) = \ell(N) - \ell(Q(N)) = \ell(N) - \sum_{p|N} p = \sum_{p|N} (p^{v_p(N)} - p). \tag{3.40}$$

If N' and N'' are two consecutive superchampion numbers of common parameter $\rho = \xi/\log \xi$ (cf. Proposition 3.5), then, from (3.40) and (3.6),

$$\ell(N') = \sum_{p|N'} p + E(N') = \sum_{p<\xi} p + E(N'). \tag{3.41}$$

Proposition 3.19. *Let n be an integer satisfying $n \ge v_0$ (defined by (3.17)) and N' and ξ defined by Defintion 3.6 so that $\xi \ge x_0 = 10^{10} + 19$ holds, then the additive excess $E(N')$ satisfies*

$$\frac{\xi^{3/2}}{3\sqrt{2}\log \xi}\left(1+\frac{0.12}{\log \xi}\right) \le E(N') \le \frac{\xi^{3/2}}{3\sqrt{2}\log \xi}\left(1+\frac{0.98}{\log \xi}\right). \tag{3.42}$$

Proof. With J defined by (3.9), from (3.6), we have

$$E(N') = \sum_{p|N'} \left(p^{v_p(N')} - p\right) = \sum_{j=2}^{J} \sum_{\xi_{j+1} \le p < \xi_j} (p^j - p). \tag{3.43}$$

For an asymptotic estimates of $E(N')$ see below (6.13).

The lower bound. From (3.43), we deduce

$$E(N') \ge \sum_{p<\xi_2} p^2 - \sum_{p\le\xi_2} p = \pi_2^-(\xi_2) - \pi_1(\xi_2). \tag{3.44}$$

As $\xi_2 \ge x_2^{(0)} > 69588$ holds, from (3.21) we have $\xi_2^2 \le \xi/2$ and from (2.6), $\pi_1(\xi_2) \le \xi_2\pi(\xi_2) \le 1.26\xi_2^2/\log\xi_2 \le 0.63\xi/\log 69588 \le 0.057\xi$. From (3.44) and (3.38), it follows that

$$\begin{aligned}
E(N') &\ge \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{0.122}{\log\xi}\right) - 0.057\xi \\
&= \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{1}{\log\xi}\left(0.122 - \frac{0.057 \times 3\sqrt{2}\log^2\xi}{\sqrt{\xi}}\right)\right) \\
&\ge \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{1}{\log\xi}\left(0.122 - \frac{0.057 \times 3\sqrt{2}\log^2 x_0}{\sqrt{x_0}}\right)\right) \\
&\ge \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{0.12}{\log\xi}\right)
\end{aligned}$$

which proves the lower bound of (3.42).

The upper bound. Let us consider an integer j_0, $3 \le j_0 \le 29$, that will be fixed later; (3.43) implies

$$E(N') = \sum_{p<\xi_{j_0}} \left(p^{v_p(N')} - p\right) + \sum_{j=2}^{j_0-1} \sum_{\xi_{j+1}\le p<\xi_j} (p^j - p) \le S_1 + S_2. \tag{3.45}$$

with

$$S_1 = \sum_{p<\xi_{j_0}} p^{v_p(N')} \quad \text{and} \quad S_2 = \sum_{j=2}^{j_0-1} \pi_j(\xi_j).$$

Let $\rho = \xi/\log\xi$ be the common parameter of N' and N''. From (3.6), for $p < \xi_{j_0}$, we have $p^{v_p(N')} \le \rho(\log p)/(1 - 1/p)$ so that Lemma 2.2 leads to $S_1 \le \rho W(\xi_{j_0}) \le$

$\omega\rho\xi_{j_0}$ with $\omega = 1.000014$ if $j_0 \le 10$ and $\omega = 1.346$ if $j_0 \ge 11$, since $x_{11}^{(0)} = 6.55 < 7.32 < x_{10}^{(0)} = 7.96$. In view of applying (3.21) and (3.22), we set $\beta_j = j$ for $2 \le j \le 8$ and $\beta_j = j(1 - 1/x_j^{(0)})$ (with $x_j^{(0)}$ defined by (3.18)) for $9 \le j \le 29$. Therefore, from Lemma 2.2, (3.21) and (3.22), we get

$$S_1 \le \omega \frac{\xi\xi_{j_0}}{\log\xi} \le \frac{\omega\xi^{1+1/j_0}}{\beta_{j_0}^{1/j_0}\log\xi} = \frac{\xi^{3/2}}{3\sqrt{2}\log^2\xi}\left(\frac{3\sqrt{2}\,\omega\log\xi}{\beta_{j_0}^{1/j_0}\xi^{1/2-1/j_0}}\right)$$
$$\le \frac{\xi^{3/2}}{3\sqrt{2}\log^2\xi}\left(\frac{3\sqrt{2}\,\omega\log x_0}{\beta_{j_0}^{1/j_0}x_0^{1/2-1/j_0}}\right). \quad (3.46)$$

In view of applying (2.25)–(2.28), we set $\alpha_3 = 0.271$, $\alpha_4 = 0.237$, $\alpha_5 = 0.226$ and, for $j \ge 6$, $\alpha_j = (1 + (2/3)^j)(\log 3)/3$. Therefore, for $3 \le j \le 29$, it follows from (2.25)–(2.28), (3.21) and (3.22) that $\pi_j(\xi_j) \le \alpha_j\xi_j^{j+1}/\log\xi_j$, $\xi_j \le (\xi/\beta_j)^{1/j}$ and

$$\pi_j(\xi_j) \le \alpha_j\frac{\xi_j^{j+1}}{\log\xi_j} \le \frac{\alpha_j(\xi/\beta_j)^{1+1/j}}{(1/j)\log(\xi/\beta_j)} = \frac{j\alpha_j(\xi/\beta_j)^{1+1/j}}{(\log\xi)(1-(\log\beta_j)/\log\xi)} \le \gamma_j\frac{\xi^{1+1/j}}{\log\xi} \quad (3.47)$$

with

$$\gamma_j = \frac{j\alpha_j}{\beta_j^{1+1/j}(1-(\log\beta_j)/\log x_0)}.$$

Further, for $3 \le j \le j_0 - 1$, we have

$$\gamma_j\frac{\xi^{1+1/j}}{\log\xi}\frac{3\sqrt{2}\log^2\xi}{\xi^{3/2}} = \frac{3\sqrt{2}\gamma_j\log\xi}{\xi^{1/2-1/j}} \le \delta_j \quad \text{with} \quad \delta_j = \frac{3\sqrt{2}\gamma_j\log x_0}{x_0^{1/2-1/j}}$$

which implies from (3.47)

$$\pi_j(\xi_j) \le \frac{\delta_j\xi^{3/2}}{3\sqrt{2}\log^2\xi}. \quad (3.48)$$

From the definition of S_2, from (3.38) and from (3.48), one gets

$$S_2 \le \sum_{j=2}^{j_0-1}\sum_{\xi_{j+1}\le p<\xi_j} p^j \le \sum_{j=2}^{j_0-1}\pi_j(\xi_j)$$
$$\le \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1+\frac{1}{\log\xi}\left(0.458+\sum_{j=3}^{j_0-1}\delta_j\right)\right).$$

Finally, from (3.45) and (3.46), we conclude

$$E(N') \le \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1+\frac{1}{\log\xi}\left(\frac{3\sqrt{2}\,\omega\log x_0}{\beta_{j_0}^{1/j_0}x_0^{1/2-1/j_0}}+0.458+\sum_{j=3}^{j_0-1}\delta_j\right)\right)$$

which, by choosing $j_0 = 6$, completes the proof of (3.42) (cf. [27]). □

Remark 3.20. *When* $n = \nu_0$, $N' = N'_0$ *is given by* (3.16) *and* $E(N'_0) = 10\,517\,469\,635\,602$. *Observing that* $E(N'_0)$ *is equal to* $\dfrac{x_0^{3/2}}{3\sqrt{2}\log x_0}\left(1+\dfrac{0.632\ldots}{\log x_0}\right)$ *shows that the constant* 0.98 *in* (3.42) *cannot be shortened below* 0.632.

3.5.2 The multiplicative excess

Let N be a positive integer and $Q(N) = \prod_{p|N} p$ be the squarefree part of N. The multiplicative excess $E^*(N)$ of N is defined by

$$E^*(N) = \log\left(\frac{N}{Q(N)}\right) = \log N - \sum_{p|N}\log p = \sum_{p|N}(\upsilon_p(N)-1)\log p. \quad (3.49)$$

If N' and N'' are two consecutive superchampion numbers of common parameter $\rho = \xi/\log\xi$ (cf. Proposition 3.5), then from (3.6),

$$\log N' = \sum_{p|N'}\log p + E^*(N') = \sum_{p<\xi}\log p + E^*(N'). \quad (3.50)$$

Proposition 3.21. *Let n be an integer satisfying $n \ge \nu_0$ (defined by* (3.17)*) and N' and ξ defined by Definition 3.6 so that $\xi \ge x_0 = 10^{10} + 19$ holds. Then the multiplicative excess $E^*(N')$ satisfies*

$$\sqrt{\frac{\xi}{2}}\left(1-\frac{0.521}{\log\xi}\right) \le E^*(N') \le \sqrt{\frac{\xi}{2}}\left(1+\frac{0.305}{\log\xi}\right) \le 0.72\sqrt{\xi}. \quad (3.51)$$

Remark 3.22. *When* $n = \nu_0$, $N' = N'_0$ *is given by* (3.16) *and* $E^*(N'_0) = 70954.46\ldots = \sqrt{x_0/2}(1 + (0.079385\ldots)/\log x_0)$*, so that the constant* 0.305 *in* (3.51) *cannot be shortened below* 0.079.

Proof. **The lower bound**. From (3.49), (3.6) and (3.9), we may write

$$E^*(N') = \sum_{j\ge 2}\ \sum_{\xi_{j+1}\le p<\xi_j}(j-1)\log p = \sum_{j=2}^{J}\theta^-(\xi_j) \quad (3.52)$$

with $\theta^-(x) = \sum_{p<x} \log p$. From (3.52), we deduce $E^*(N') \geq \theta^-(\xi_2)$ which, from (3.37), proves the lower bound of (3.51).

The upper bound. From (3.52) and (3.9), it follows that

$$E^*(N') \leq \sum_{j=2}^{J} \theta(\xi_j). \tag{3.53}$$

Let us fix $j_0 = 26$. From (2.2) with $\epsilon = 7.5 \times 10^{-7}$, we write

$$\sum_{j=3}^{J} \theta(\xi_j) \leq (1+\epsilon)\left(\sum_{j=3}^{j_0-1} \xi_j + \sum_{j=j_0}^{J} \xi_j\right) = (1+\epsilon)(S_1 + S_2). \tag{3.54}$$

From (3.21) and (3.22) with $\beta_j = j$ for $3 \leq j \leq 8$ and $\beta_j = j(1 - 1/x_j^{(0)})$ for $9 \leq j \leq j_0$, we get

$$\begin{aligned} S_1 = \sum_{j=3}^{j_0-1} \xi_j \leq \sum_{j=3}^{j_0-1} \left(\frac{\xi}{\beta_j}\right)^{1/j} &= \sqrt{\frac{\xi}{2}}\left(\frac{1}{\log \xi}\right) \sum_{j=3}^{j_0-1} \frac{\sqrt{2}\log \xi}{\beta_j^{1/j} \xi^{1/2-1/j}} \\ &\leq \sqrt{\frac{\xi}{2}}\left(\frac{1}{\log \xi}\right) \sum_{j=3}^{j_0-1} \frac{\sqrt{2}\log x_0}{\beta_j^{1/j} x_0^{1/2-1/j}} = 0.627703\ldots \sqrt{\frac{\xi}{2}}\left(\frac{1}{\log \xi}\right). \end{aligned} \tag{3.55}$$

Further, from (3.9) and (3.22), since ξ_j is decreasing on j, we have

$$\begin{aligned} S_2 = \sum_{j=j_0}^{J} \xi_j \leq J\xi_{j_0} \leq \frac{(\log \xi)\xi^{1/j_0}}{(\log 2)\beta_{j_0}^{1/j_0}} &= \sqrt{\frac{\xi}{2}}\left(\frac{1}{\log \xi}\right)\left(\frac{\sqrt{2}\log^2 \xi}{(\log 2)\beta_{j_0}^{1/j_0} \xi^{1/2-1/j_0}}\right) \\ &\leq \sqrt{\frac{\xi}{2}}\left(\frac{1}{\log \xi}\right)\left(\frac{\sqrt{2}\log^2 x_0}{(\log 2)\beta_{j_0}^{1/j_0} x_0^{1/2-1/j_0}}\right) = 0.022597\ldots \sqrt{\frac{\xi}{2}}\left(\frac{1}{\log \xi}\right). \end{aligned} \tag{3.56}$$

Finally, from (3.53), (3.37), (3.54), (3.55) and (3.56), we conclude

$$\begin{aligned} E^*(N') &\leq \sqrt{\frac{\xi}{2}}\left(1 + \frac{1}{\log \xi}(-0.346 + (1+\epsilon)(0.6278 + 0.0226))\right) \\ &< \sqrt{\frac{\xi}{2}}\left(1 + \frac{0.305}{\log \xi}\right) \end{aligned}$$

which ends the proof of Proposition 3.21. □

3.5.3 The number $s(n)$ of primes dividing $h(n)$ but not N'

Let $n \geq 7$ and N', N'' and ξ defined by Definition 3.6. Let us denote by p_{i_0} the largest prime factor of N'. Note, from (3.6), that p_{i_0} is the largest prime $< \xi$. If ξ is prime, we have $\xi = p_{i_0+1}$ while, if ξ is not prime $p_{i_0+1} > \xi$. In both cases, we have

$$p_{i_0} < \xi \leq p_{i_0+1} \tag{3.57}$$

From the definition of the additive excess (3.40), we define $s = s(n) \geq 0$ by

$$p_{i_0+1} + \ldots + p_{i_0+s} \leq n - \ell(N') + E(N') < p_{i_0+1} + \ldots + p_{i_0+s+1}. \tag{3.58}$$

Proposition 3.23. *If $n \geq \nu_0$ (defined by (3.17)) and ξ defined in Definition 3.6, we have*

$$\frac{\sqrt{\xi}}{3\sqrt{2}\log \xi}\left(1 + \frac{0.095}{\log \xi}\right) \leq s \leq \frac{\sqrt{\xi}}{3\sqrt{2}\log \xi}\left(1 + \frac{1.01}{\log \xi}\right). \tag{3.59}$$

Proof. **The upper bound.** Since $p_{i_0+1} \geq \xi$ and $n - \ell(N') < \ell(N'') - \ell(N') \leq \xi$ hold (cf. (3.57) and (3.13)), (3.58) and (3.42) imply

$$\begin{aligned} s\xi \leq n - \ell(N') + E(N') &\leq \xi + \frac{\xi^{3/2}}{3\sqrt{2}\log \xi}\left(1 + \frac{0.98}{\log \xi}\right) \\ &\leq \frac{\xi^{3/2}}{3\sqrt{2}\log \xi}\left(1 + \frac{1}{\log \xi}\left(0.98 + \frac{3\sqrt{2}\log^2 \xi}{\sqrt{\xi}}\right)\right) \\ &\leq \frac{\xi^{3/2}}{3\sqrt{2}\log \xi}\left(1 + \frac{1}{\log \xi}\left(0.98 + \frac{3\sqrt{2}\log^2 x_0}{\sqrt{x_0}}\right)\right) \\ &\leq \frac{\xi^{3/2}}{3\sqrt{2}\log \xi}\left(1 + \frac{1.01}{\log \xi}\right) \end{aligned} \tag{3.60}$$

which yields the upper bound of (3.59).

The lower bound. First, from (3.60), we observe that

$$\begin{aligned}
s+1 &\le \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\left(1+\frac{1.01}{\log\xi}\right)+1\\
&= \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\left(1+\frac{1}{\log\xi}\left(1.01+\frac{3\sqrt{2}\log^2\xi}{\sqrt{\xi}}\right)\right)\\
&\le \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\left(1+\frac{1}{\log\xi}\left(1.01+\frac{3\sqrt{2}\log^2 x_0}{\sqrt{x_0}}\right)\right)\\
&\le \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\left(1+\frac{1.033}{\log\xi}\right),
\end{aligned} \tag{3.61}$$

which implies

$$s+1 \le \frac{\xi^{1/2}}{3\sqrt{2}\log x_0}\left(1+\frac{1.033}{\log x_0}\right) < 0.011\sqrt{\xi}. \tag{3.62}$$

From Corollary 2.4, the number of primes between ξ and $\xi(1+0.045/\log^2\xi)$ is $\ge 0.011\sqrt{\xi} > s+1$ so that, in (3.58), we have

$$p_{i_0+s+1} \le \xi\left(1+\frac{0.045}{\log^2\xi}\right) \le \xi\left(1+\frac{0.045}{(\log x_0)(\log\xi)}\right) \le \xi\left(1+\frac{0.002}{\log\xi}\right). \tag{3.63}$$

From (3.12), we get $n-\ell(N')\ge 0$. Therefore, from (3.42), (3.58) and (3.63), we have

$$\frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1+\frac{0.12}{\log\xi}\right) \le E(N') \le n-\ell(N')+E(N') \le (s+1)\xi\left(1+\frac{0.002}{\log\xi}\right)$$

which yields

$$\begin{aligned}
s+1 &\ge \frac{\xi^{1/2}(1+0.12/\log\xi)}{3\sqrt{2}(\log\xi)(1+0.002/\log\xi)} \ge \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\left(1+\frac{0.12}{\log\xi}\right)\left(1-\frac{0.002}{\log\xi}\right)\\
&\ge \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\left(1+\frac{0.118}{\log\xi}-\frac{0.002\times 0.12}{(\log x_0)\log\xi}\right) \ge \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\left(1+\frac{0.1179}{\log\xi}\right)
\end{aligned}$$

and

$$s \geq \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\Big(1 + \frac{0.1179}{\log\xi}\Big) - 1$$
$$= \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{1}{\log\xi}\left(0.1179 - \frac{3\sqrt{2}\log^2\xi}{\sqrt{\xi}}\right)\right)$$
$$\geq \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{1}{\log\xi}\left(0.1179 - \frac{3\sqrt{2}\log^2 x_0}{\sqrt{x_0}}\right)\right)$$
$$\geq \frac{\xi^{1/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{0.095}{\log\xi}\right) \tag{3.64}$$

and the proof of Proposition 3.23 is completed. □

From (3.63), we deduce for $\xi \geq x_0$

$$\frac{\log p_{i_0+s+1}}{\log\xi} \leq 1 + \frac{0.002}{\log^2\xi} \leq 1 + \frac{0.002}{(\log x_0)(\log\xi)} \leq 1 + \frac{0.0001}{\log\xi} \tag{3.65}$$

and, from (3.61),

$$(s+1)\log p_{i_0+s+1} \leq \frac{\sqrt{\xi}}{3\sqrt{2}}\left(1 + \frac{1.033}{\log\xi}\right)\left(1 + \frac{0.0001}{\log\xi}\right)$$
$$= \frac{\sqrt{\xi}}{3\sqrt{2}}\left(1 + \frac{1.0331}{\log\xi} + \frac{0.0001033}{\log^2\xi}\right)$$
$$\leq \frac{\sqrt{\xi}}{3\sqrt{2}}\left(1 + \frac{1.0331}{\log\xi} + \frac{0.0001033}{(\log x_0)(\log\xi)}\right)$$
$$\leq \frac{\sqrt{\xi}}{\sqrt{2}}\left(\frac{1}{3} + \frac{0.345}{\log\xi}\right). \tag{3.66}$$

We shall also deduce from (3.64) the following inequality valid for $\xi \geq x_0$:

$$(s-1)\log\xi \geq \frac{\sqrt{\xi}}{3\sqrt{2}}\left(1 + \frac{0.095}{\log\xi}\right) - \log\xi = \frac{\sqrt{\xi}}{3\sqrt{2}}\left(1 + \frac{0.095}{\log\xi} - \frac{3\sqrt{2}\log^2\xi}{(\log\xi)\sqrt{\xi}}\right)$$
$$\geq \frac{\sqrt{\xi}}{3\sqrt{2}}\left(1 + \frac{0.095}{\log\xi} - \frac{3\sqrt{2}\log^2 x_0}{(\log\xi)\sqrt{x_0}}\right) \geq \frac{\sqrt{\xi}}{3\sqrt{2}}\left(1 + \frac{0.0724}{\log\xi}\right). \tag{3.67}$$

4 Some computational points

4.1 Enumeration of superchampion numbers

Let us recall that m is said a *squarefull integer* if, for every prime factor p of m, p^2 divides m. Each $n \geq 1$ may be writen in a unique way $n = ab$, with a squarefull, b squarefree and a, b coprime. In the case where n is a superchampion number N, we will say that a is the prefix[1] of N.

Let $N \leq N'$ be two consecutive superchampions, and A, A' their prefixes. In most of the cases $A' = A$ and $N' = p'N$ where p' is the prime following $\mathrm{P}^+(N)$.

When this is not the case, $N' = qN$ and $A' = qA$ where q is a prime facteur of A, or the prime following $\mathrm{P}^+(A)$. In this case, we say that N' is a superchampion of type 2.

For example, let us consider the figure 1. In this table two superchampions are of type 2, 180 180 which is equal to 3 times its predecessor, and 360 360 which is equal to 2 times its predecessor.

The superchampions of type 2 are not very numerous. There are 455059774 superchampions N satisfying $12 \leq N \leq N'_0$ (cf. (3.17)) whose 7265 are of type 2. We have precomputed the table TabT2, which, for each of these 7265 numbers N, keeps the triplet $(\ell(N), q, \log N)$, where q is the quotient of N by its predecessor (which, generally, is not of type 2). For example, entries associated to the superchampions 180 180 and 360 360 are the triplets $(49, 3, 12.101\ldots)$ and $(53, 2, 12.794\ldots)$. With this table it is very fast to enumerate the increasing sequence of $(\ell(N), \log N)$ for all the superchampion numbers. Let us associate to each superchampion N the quadruple $(\ell(N), \log N, \mathrm{P}^+(N), j)$, where j is the smallest integer such that $\mathrm{TabT2}[j][1] > \ell(N)$.

The following function, written in Python's programming language, computes the quadruple associated to the successor of N.

```
def next_super_ch(n, logN, pplusN, j):
   p =  next_prime(pplusN)
   if  n + p <= TabT2[j][1]:
       return (n+p, logN + log(p), p, j)
   else
      return(TabT2[j][1], logN + log(TabT2[j][2]), pplusN, j+1)
```

Figure 2: Enumeration of super-champion numbers

Using a prime generator function, which computes the sequence of successive primes up to n in time $O(n \log\log n)$, we wrote a C^{++} function which computes

[1] In [11] the term prefix is used with a different meaning.

the pairs $(\ell(N), \log N)$, for all superchampion numbers up to N_0', in time about 22 seconds.

4.2 Computing and bounding $g(n)$ and $h(n)$ on finite intervals

4.2.1 Computating an isolate value of $\log h(n)$ or $\log g(n)$

The computation of an isolate value $\log h(n)$ by (1.20) (cf. [8, Section 8]) or $\log g(n)$ (by the algorithm described in [11]) is relatively slow. The table below shows the time of these computations in ms. for n randomly choosen in intervals $[1, 10^j]$ for $j = 9, 12, 15, 16, 17, 18$ (on a MacBook 2016 computer).

n	10^9	10^{12}	10^{15}	10^{16}	10^{17}	10^{18}
log h(n)	5.55	5.78	6.74	7.38	8.39	9.80
log g(n)	10.1	38.2	221.	589.	2 467.	7 980.

For $n > 10^{16}$ the computation of an isolate value $\log g(n)$ takes a few seconds, and it is impossible to compute more than some thousands of these values.

4.2.2 Bounding by slices

We will need effectives bounds of $\log g(n)$ or $\log h(n)$ on intervals up to $v_0 = 2.22\cdot 10^{18}$ (cf. (3.17)). It is impossible to compute a lot of these values for ordinary large integers n. Nethertheless, by using `next_super_ch`, we can enumerate quickly the seqence $(N, \log N)$ of superchampions and of their logarithms. If, in the same time, we enumerate the values $k(\ell(N))$ (cf. (1.18)), by using lemma 4.1 we get good estimates of $\log h(n)$ and $\log g(n)$ on the intervals $[\ell(N),\ \ell(N')]$, for values of $\ell(N)$ up to v_0.

Lemma 4.1. *Let N_1, N_2 be two consecutives superchampion numbers. Let us define $n_1 = \ell(N_1)$, $k_1 = k(n_1)$, $m_1 = n_1 - \sigma_{k_1}$, $n_2 = \ell(N_2)$, $k_2 = k(n_2)$, $m_2 = n_2 - \sigma_{k_2}$ and q as the smalllest prime not smaller than $p_{k+1} - m_1$. Then*

$$\log h(n_1) \geq \theta(p_{k_1+1}) - \log q \tag{4.1}$$

$$\log h(n_2) \leq \theta(p_{k_2+1}) - \log(p_{k_2} - m_2) \tag{4.2}$$

Proof. The lower bound for $h(n_1) = N_{k_1} G(p_{k_1}, m_1)$ (cf. (1.20)) comes from [11, Proposition 8]), applied to $G(p_{k_1}, m_1)$, and the upper bound for $h(n_2)$ from the same proposition applied to $G(p_{k_2}, m_2)$. □

4.2.3 A dichotomic algorithm

We recall the algorithm `ok_rec(n1, n2)`, presented in ([10, Section 4.9]), which we will use several times. Let us suppose that `ok(n)` is a boolean function with the following side effect: when it returns false, before returning, it prints *n does not satisfy property ok*. We suppose that we also have at our disposal a boolean function `good_interval(n1, n2)` such that, when it returns true, the property $ok(n)$ is satisfied by all $n \in [n_1, n_2]$; in other words $good\_intervall(n_1, n_2)$ is a sufficient condition (most often not necessary) ensuring that $ok(n)$ is true on $[n_1, n_2]$.

Then the procedure `ok_rec(n1, n2)` returns true if and only $ok(n)$ is true for every $n \in [n_1, n_2]$, and, when it return false, before returning, it prints the value of the largest n in $[n_1, n_2]$ which does not satisfy `ok(n)`.

This procedure is used in Sections 5.3 (Theorem 1.7), 7.2 (Theorem 1.5 (ii) and (iii)), 7.4 (Theorem 1.5 (v)) and 8.3 (Theorem 1.1 (iii)).

5 Proof of Theorem 1.7

5.1 Estimates of ξ in terms of n

Lemma 5.1. *Let n and δ be two numbers satisfying $n > e^6 = 403.42\ldots$ and $1 \le \delta \le 2$. Let us define $f : [\sqrt{n}, n] \longrightarrow \mathbb{R}$ by*

$$f(t) = f_{n,\delta}(t) = \sqrt{n(2\log t - \delta)}.$$

(i) $f([\sqrt{n}, n])$ is included in $(\sqrt{n}, n)$.

(ii) f is increasing and $f'(t) \le 1/2$ holds.

(iii) The equation $t = f(t)$ has a unique root $R = R(n,\delta)$ in $(\sqrt{n}, n)$. If $R < t \le n$, then we have $R < f(t) < t$ while, if $\sqrt{n} \le t < R$, $R > f(t) > t$ holds.

Proof. we have $f(\sqrt{n}) = \sqrt{n(\log n - \delta)} \ge \sqrt{n(\log n - 2)} \ge \sqrt{4n} > \sqrt{n}$. On the other hand, we have $f(n) = \sqrt{n(2\log n - \delta)} < \sqrt{2n\log n}$ and by using the inequality $\log n \le n/e$, $f(n) \le (\sqrt{2/e})n$, which completes the proof of (i).

The derivative $f'(t) = \frac{\sqrt{n}}{t\sqrt{2\log t - \delta}}$ is clearly positive and we have

$$f'(t) \le \frac{\sqrt{n}}{\sqrt{n}\sqrt{\log n - 2}} \le \frac{1}{\sqrt{\log(e^6) - 2}} = \frac{1}{2},$$

which proves (ii).

From (ii), the derivative of $t \mapsto f(t) - t$ is negative while, from (i), $f(t) - t$ is positive for $t = \sqrt{n}$ and negative for $t = n$, whence the existence of the root R and for $\sqrt{n} \le t \le n$, the equivalences

$$f(t) < t \iff t > f(t) > R \quad \text{and} \quad f(t) > t \iff t < f(t) < R,$$

which proves (iii). □

Let us recall that $k = k(n)$ is defined by (1.18) and let us set

$$x = x(n) = p_{k+1} \tag{5.1}$$

so that

$$\pi_1(x) - x \le n < \pi_1(x) \tag{5.2}$$

holds. Further, $n \ge 7$ being given, one defines N' and N'' by Definition 3.6, $\rho = \rho(n)$ is the common parameter of N' and N'' (cf. Proposition 3.5) and $\xi = \xi(n)$ is defined by $\rho = \xi / \log \xi$. If p_{i_0} denotes the largest prime factor of N', from (3.57), we have $p_{i_0} < \xi \le p_{i_0+1}$ and, from (3.6), $\ell(N') \ge \pi_1(p_{i_0})$. Therefore, from (3.12) and (5.2), we get

$$\pi_1(p_{i_0}) \le \ell(N') \le n < \pi_1(x) = \pi_1(p_{k+1}),$$

which implies $p_{i_0} < p_{k+1}$, $p_{i_0+1} \le p_{k+1}$ and from (3.57),

$$\xi = \xi(n) \le p_{i_0+1} \le p_{k+1} = x = x(n). \tag{5.3}$$

Note that in N'' (cf. (3.7)), from Corollary 3.9, all prime powers dividing N'' do not exceed ξ, so that $n < \ell(N'') \le \xi\pi(\xi) \le \xi^2$ and, with (5.3),

$$\log n \le 2 \log \xi \le 2 \log x. \tag{5.4}$$

Proposition 5.2 improves on Lemma 2.8 of [10].

Proposition 5.2. *For $n \ge \nu_0$ (defined by (3.17)),*

$$\sqrt{n \log n}\left(1 + \frac{\log\log n - 1}{2\log n} - \frac{(\log\log n)^2}{8\log^2 n} + 0.38\,\frac{\log\log n}{\log^2 n}\right) \le \xi \le x \tag{5.5}$$

while, for $n \ge \pi_1(x_0) = 2\,220\,822\,442\,581\,729\,257 = 2.22\ldots 10^{18}$,

$$\xi \le x \le \sqrt{n \log n}\left(1 + \frac{\log\log n - 1}{2\log n} - \frac{13\,(\log\log n)^2}{10000\,\log^2 n}\right). \tag{5.6}$$

Proof. **The lower bound** (5.5)**.** First, from the definition of v_0 (cf. (3.17)), ξ, N' and N'' (cf. Definition 3.6), it follows that $n \geq v_0$ implies $\xi \geq x_0$ and that, from (3.13) and (3.14),

$$n < \ell(N'') = \ell(N') + \ell(N'') - \ell(N') = \ell(N') + \rho \log(N''/N') \leq \ell(N') + \xi$$

which, from (3.41) and (3.42), yields

$$\begin{aligned}
n &\leq \ell(N') + \xi = \sum_{p<\xi} p + E(N') + \xi \leq \pi_1(\xi) + \xi + \frac{\xi^{3/2}}{3\sqrt{2}\log\xi}\left(1 + \frac{0.98}{\log\xi}\right) \\
&= \pi_1(\xi) + \frac{\xi^2}{\log^4\xi}\left(\frac{\log^4\xi}{\xi} + \frac{\log^3\xi}{3\sqrt{2}\sqrt{\xi}}\left(1 + \frac{0.98}{\log\xi}\right)\right) \\
&\leq \pi_1(\xi) + \frac{\xi^2}{\log^4\xi}\left(\frac{\log^4 x_0}{x_0} + \frac{\log^3 x_0}{3\sqrt{2}\sqrt{x_0}}\left(1 + \frac{0.98}{\log x_0}\right)\right) \\
&\leq \pi_1(\xi) + 0.0301\frac{\xi^2}{\log^4\xi}. \qquad (5.7)
\end{aligned}$$

Further, as $107/160 + 0.0301 < 7/10$ holds, it follows from (2.19) and (5.7), that

$$n \leq \frac{\xi^2}{2\log\xi} + \frac{\xi 2}{4\log^2\xi} + \frac{\xi^2}{4\log^3\xi} + \frac{7\,\xi^2}{10\log^4\xi}. \qquad (5.8)$$

Let us consider the polynomial

$$\begin{aligned}
P &= \left(\frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{4} + \frac{7\,t^4}{10}\right)\left(\frac{2}{t} - 1 - 0.584\,t\right) \\
&= 1 - 0.042\,t^2 + 1.004\,t^3 - 0.846\,t^4 - 0.4088\,t^5.
\end{aligned}$$

The polynomial $P - 1$ has a double root in 0 and three other roots $-2.92\ldots$, $0.0434574\ldots$ and $0.809\ldots$. Therefore, $P \leq 1$ holds for $0 \leq t \leq 1/\log x_0 = 0.0434294\ldots$ and (5.8) implies

$$n \leq \frac{\xi^2}{2\log\xi - 1 - 0.584/\log\xi} \quad \text{for} \quad \xi \geq x_0. \qquad (5.9)$$

Therefore, from (5.4), (5.9) yields

$$n \leq \frac{\xi^2}{2\log\xi - 1 - 1.168/\log n},$$

which implies

$$\xi \geq f(\xi) \quad \text{with} \quad f(t) = f_{n,\delta}(t) = \sqrt{n(2\log t - \delta)} \qquad (5.10)$$

with

$$1 < \delta = 1 + \frac{1.168}{\log n} \le 1 + \frac{1.168}{\log \nu_0} \le 1.03. \tag{5.11}$$

From Lemma 5.1 (ii), the equation $t = f(t)$ has one root $R \in (\sqrt{n}, n)$. Let us set

$$t_1 = \sqrt{n \log n \left(1 + \frac{\log\log n - \delta}{\log n}\right)}. \tag{5.12}$$

We have $t_1 \in (\sqrt{n}, n)$ and

$$f(t_1) = \sqrt{n(\log n + \log\log n + \log(1+u) - \delta)} \quad \text{with} \quad u = \frac{\log\log n - \delta}{\log n}. \tag{5.13}$$

As $n \ge \nu_0$ holds, u is positive and $f(t_1)^2 - t_1^2 = n\log(1+u) > 0$ so that $f(t_1) > t_1$ and, from Lemma 5.1 (iii), the root R satisfies $R > f(t_1)$. But, from (5.10), $\xi \ge f(\xi)$, which implies

$$\xi \ge R \ge f(t_1). \tag{5.14}$$

By Taylor formula, since the third derivative of $t \mapsto \log(1+t)$ is positive, we have $\log(1+u) \ge u - u^2/2$. For convenience, from now on, we write L for $\log n$, λ for $\log n$, L_0 for $\log \nu_0 = 42.244414\ldots$, and λ_0 for $\log\log \nu_0 = 3.743472\ldots$ With (1.15),

$$\frac{u^2}{2} = \frac{(\lambda-\delta)^2}{2L^2} \le \frac{\lambda - 1}{2L}\left(\frac{\lambda-\delta}{L}\right) \le \frac{\lambda_0 - 1}{2L_0}\left(\frac{\lambda-\delta}{L}\right) \le 0.04\,\frac{\lambda-\delta}{L}$$

and

$$\log(1+u) \ge u - \frac{u^2}{2} \ge \frac{\lambda-\delta}{L} - 0.04\,\frac{\lambda-\delta}{L} = 0.96\,\frac{\lambda-\delta}{L},$$

which, from (5.14) and (5.13), yields

$$\xi \ge f(t_1) = \sqrt{n(\log n)(1+v)} \tag{5.15}$$

with

$$\frac{\lambda-\delta}{L}\left(1 + \frac{0.96}{L}\right) \le v = \frac{\lambda-\delta}{L} + \frac{\log(1+u)}{L} \le \frac{\lambda-\delta}{L} + \frac{u}{L} = \frac{\lambda-\delta}{L}\left(1+\frac{1}{L}\right). \tag{5.16}$$

For $v > 0$, by Taylor formula, since the third derivative of $t \mapsto \sqrt{1+t}$ is positive, we have $\sqrt{1+v} \ge 1 + v/2 - v^2/8$ and we need an upper bound for $v^2/8$. From

(5.11), we get

$$\frac{v^2}{8} = \frac{(\lambda-\delta)^2}{8L^2}\left(1+\frac{1}{L}\right)^2 = \frac{(\lambda-\delta)^2}{8L^2}\left(1+\frac{2}{L}\left(1+\frac{1}{2L}\right)\right)$$
$$\leq \frac{(\lambda-\delta)^2}{8L^2}\left(1+\frac{2}{L}\left(1+\frac{1}{2L_0}\right)\right)$$
$$\leq \frac{(\lambda-\delta)^2}{8L^2}\left(1+\frac{2.03}{L}\right) \leq \frac{(\lambda-1)^2}{8L^2}+\frac{2.03\,\lambda^2}{8\,L^3}$$
$$= \frac{\lambda^2-2\lambda+1}{8L^2}+\frac{\lambda}{8L^2}\left(\frac{2.03\,\lambda}{L}\right) \leq \frac{\lambda^2}{8L^2}-\frac{2\lambda}{8L^2}+\frac{\lambda}{8\lambda_0 L^2}+\frac{\lambda}{8L^2}\left(\frac{2.03\,\lambda_0}{L_0}\right)$$
$$\leq \frac{\lambda^2}{8L^2}+\frac{\lambda}{8L^2}(-2+0.27+0.18) \leq \frac{\lambda^2}{8L^2}-0.19\frac{\lambda}{L^2}. \tag{5.17}$$

Finally, from (5.15), (5.16), (5.17) and (5.11),

$$\begin{aligned}
\frac{\xi}{\sqrt{n\log n}} &\geq \sqrt{1+v} \geq 1+\frac{v}{2}-\frac{v^2}{8}\\
&\geq 1+\frac{\lambda-\delta}{2L}+0.48\frac{\lambda-\delta}{L^2}-\frac{\lambda^2}{8L^2}+0.19\frac{\lambda}{L^2}\\
&\geq 1+\frac{\lambda-1}{2L}-\frac{1.168}{2L^2}+0.48\,\frac{\lambda-1.03}{L^2}-\frac{\lambda^2}{8L^2}+0.19\frac{\lambda}{L^2}\\
&\geq 1+\frac{\lambda-1}{2L}-\frac{\lambda^2}{8L^2}+0.67\,\frac{\lambda}{L^2}-\frac{(0.584+0.48\times 1.03)\lambda}{L^2\lambda_0}\\
&\geq 1+\frac{\lambda-1}{2L}-\frac{\lambda^2}{8L^2}+\frac{\lambda}{L^2}(0.67-0.29)\\
&= 1+\frac{\lambda-1}{2L}-\frac{\lambda^2}{8L^2}+0.38\,\frac{\lambda}{L^2},
\end{aligned} \tag{5.18}$$

which proves (5.5).

The upper bound (5.6)**.** We assume $x \geq x_0 = 10^{10}+19$. As

$$\frac{3x^2}{20\log^4 x}-x = \frac{x^2}{\log^4 x}\left(\frac{3}{20}-\frac{\log^4 x}{x}\right) \geq \frac{x^2}{\log^4 x}\left(\frac{3}{20}-\frac{\log^4 x_0}{x_0}\right) \geq 0.149\,\frac{x^2}{\log^4 x},$$

(2.20) and (5.2) imply

$$n \geq \pi_1(x)-x \geq \frac{x^2}{2\log x}+\frac{x^2}{4\log^2 x}+\frac{x^2}{4\log^3 x}+0.149\,\frac{x^2}{\log^4 x}. \tag{5.19}$$

Let us set

$$Q = \left(\frac{t}{2}+\frac{t^2}{4}+\frac{t^3}{4}+0.149\,t^4\right)\left(\frac{2}{t}-1-0.492\,t\right)$$
$$= 1+0.004\,t^2-0.075\,t^3-0.272\,t^4-0.073308\,t^5.$$

The polynomial $Q-1$ has a double root in 0 and three other roots

$$-3.4052\ldots,\ -0.3508\ldots,\ 0.04567\ldots,$$

and $Q \geq 1$ holds for $0 \leq t \leq 1/\log x_0 = 0.0434294\ldots$,which, from (5.19), proves

$$n \geq \frac{x^2}{2\log x - 1 - 0.492/\log x} \quad \text{for} \quad x \geq x_0.$$

Further, (5.19) implies $n \geq x^2/(2\log x)$, whence

$$\begin{aligned}\log n \geq 2\log x - \log(2\log x) &= (\log x)\left(2 - \frac{\log(2\log x)}{\log x}\right)\\ &\geq (\log x)\left(2 - \frac{\log(2\log x_0)}{\log x_0}\right) = 1.8336\ldots\log x\end{aligned}$$

and, as $1.8336 \times 0.492 \geq 0.902$,

$$n \geq \frac{x^2}{2\log x - 1 - 0.902/\log n} \quad \text{for} \quad n \geq \nu_0 \tag{5.20}$$

and

$$x \leq f(x) \text{ with } f(t) = f_{n,\delta}(t) = \sqrt{n(2\log t - \delta)},\ b = 0.902,\ \delta = 1 + \frac{b}{\log n}. \tag{5.21}$$

This time, one chooses

$$t_2 = A\sqrt{n\log n} \quad \text{with} \quad A = 1 + \frac{\lambda - 1}{2L} \leq 1 + \frac{\lambda_0 - 1}{2L_0} \leq 1.033$$

and one calculates

$$f(t_2) = \sqrt{B\, n\log n}$$

with

$$B = 1 + \frac{\lambda - \delta}{L} + \frac{2}{L}\log\left(1 + \frac{\lambda - 1}{2L}\right) \leq B' = 1 + \frac{\lambda - \delta}{L} + \frac{\lambda - 1}{L^2}.$$

We have

$$A^2 - B \geq A^2 - B' = \frac{1}{4L^2}(\lambda^2 - 6\lambda + 5 + 4b) = \frac{1}{4L^2}(\lambda^2 - 6\lambda + 8.608)$$

and

$$\lambda^2 - 6\lambda + 8.608 = 0.011\lambda^2 + (0.989\lambda^2 - 6\lambda + 8.608).$$

The roots of the above trinomial are $2.327\ldots$ and $3.738\ldots < \lambda_0$ so that it is positive for $\lambda \geq \lambda_0$ and one gets $A^2 - B \geq 0.011\lambda^2/(4L^2)$, $A > \sqrt{B}$ and

$$A - \sqrt{B} = \frac{A^2 - B}{A + \sqrt{B}} \geq \frac{A^2 - B}{2A} \geq \frac{0.011\lambda^2}{4 \times 2.066\, L^2} \geq 0.0013\frac{\lambda^2}{L^2}, \tag{5.22}$$

so that $t_2 > f(t_2)$ holds. By Lemma 5.1, the root R of the equation $t = f(t)$ satisfies $R < f(t_2)$ and (5.21) implies

$$\begin{aligned} x \leq f(t_2) = \sqrt{B\, n \log n} = \sqrt{n \log n}(A - (A - \sqrt{B})) \\ \leq \sqrt{n \log n}(A - 0.0013\lambda^2/L^2) \end{aligned}$$

which proves (5.6). □

Corollary 5.3. *For* $n \geq \nu_0$,

$$\sqrt{n \log n}\left(1 + \frac{\log\log n - 1.019}{2\log n}\right) \leq \xi \leq x \leq \sqrt{n \log n}\left(1 + \frac{\log\log n - 1}{2\log n}\right). \tag{5.23}$$

Proof. The upper bound follows from (5.6). From (5.5),

$$\xi \geq \sqrt{n \log n}\left(1 + \frac{\log\log n - y(\log\log n)}{2\log n}\right)$$
$$\text{with} \quad y(t) = 1 + (t^2/4 - 0.76\, t)\exp(-t).$$

The derivative $y'(t) = (-0.25\, t^2 + 1.26\, t - 0.76)\exp(-t)$ vanishes for $t = 0.7005\ldots$ and $t = 4.339\ldots$ so that, for $t \geq \lambda_0$, $y(t)$ is maximal for $t = 4.339\ldots$ and its value is $1.01838\ldots$. □

5.2 Proof of the lower bound (1.13) for $n \geq \pi_1(x_0)$.

Let us recall that $x_0 = 10^{10} + 19$, and let us suppose first that $n \geq \pi_1(x_0) = 2\,220\,822\,442\,581\,729\,257$, so that $x = x(n) = p_{k+1}$ defined by (5.1) is $\geq x_0$. As the function h is nondecreasing, from (2.3) with $\alpha = 1/2$,

$$\log h(n) \geq \log N_k = \theta(p_k) = \theta(x) - \log x \geq x - \frac{x}{2\log^3 x} - \log x. \tag{5.24}$$

Inequality (5.24) together with (1.15) and (5.4) yield, by noting L for $\log n$, λ for $\log\log n$, L_0' for $\log \pi_1(x_0)$, λ_0' for $\log\log \pi_1(x_0)$,

$$\begin{aligned}
\frac{\log h(n)}{x} &\geq 1 - \frac{1}{2\log^3 x} - \frac{\log x}{x} = 1 - \left(\frac{1}{2} + \frac{\log^4 x}{x}\right)\frac{1}{\log^3 x}\\
&\geq 1 - \left(\frac{1}{2} + \frac{\log^4 x_0}{x_0}\right)\frac{1}{\log^3 x} \geq 1 - \frac{0.500029}{\log^3 x}\\
&\geq 1 - \frac{4.0003}{\log^3 n} \geq 1 - \frac{4.0003\,\lambda}{(\log^2 n)\lambda_0' L_0'} \geq 1 - \frac{0.026\,\lambda}{L^2}
\end{aligned}$$

and, from (5.5),

$$\begin{aligned}
\frac{\log h(n)}{\sqrt{n\log n}} &\geq \left(1 + \frac{\lambda - 1}{2L} - \frac{\lambda^2}{8L^2} + \frac{0.38\,\lambda}{L^2}\right)\left(1 - \frac{0.026\,\lambda}{L^2}\right)\\
&\geq 1 + \frac{\lambda - 1}{2L} - \frac{\lambda^2}{8L^2} + \frac{0.38\,\lambda}{L^2} - \frac{0.026\,\lambda}{L^2}\left(1 + \frac{\lambda}{2L} + \frac{0.38\,\lambda}{L^2}\right)\\
&\geq 1 + \frac{\lambda - 1}{2L} - \frac{\lambda^2}{8L^2} + \frac{0.38\,\lambda}{L^2} - \frac{0.026\,\lambda}{L^2}\left(1 + \frac{\lambda_0'}{2L_0'} + \frac{0.38\,\lambda_0'}{L_0'^2}\right)\\
&\geq 1 + \frac{\lambda - 1}{2L} - \frac{\lambda^2}{8L^2} + \frac{0.35\,\lambda}{L^2}, \qquad (5.25)
\end{aligned}$$

which proves (1.13) for $x \geq \pi_1(x_0)$. □

5.3 Proof of the lower bound (1.13) for $n < \pi_1(x_0)$.

Let Φ_u defined by (3.25) and $n_1 \leq n_2$ such that the following inequality

$$\log h(n_1) \geq \Phi_{1/8}(n_2) \qquad (5.26)$$

is true. Then, by the non decreasingness of h and $\Phi_{1/8}$, $\log h(n) \geq \Phi_{1/8}(n)$ is true on the whole interval $[n_1,\ n_2]$. In particular, (1.13) is satisfyed on $[\sigma_k, \sigma_{k+1}]$ if the following inequality is true

$$\theta(p_k) = \log h(\sigma_k) > \Phi_{1/8}(\sigma_{k+1}), \qquad (5.27)$$

By enumerating p_k, σ_k and θ_k until $p_{k+1} = x_0 = 10^{10} + 19$ we remark that (5.27) is satisfyied for $k \geq k_1 = 9\,018$. This proves that inequality (1.13) is true for $n \geq \sigma_{k_1} = 398\,898\,277$.

It remains to compute the largest n in $[2, \sigma_{k_1}]$ such that $\Phi_{1/8}(n) \leq \log h(n)$ fails. This is done by dichotomy (cf. Section 4.2.3), calling `ok_rec(2, 398 898`

277) with $ok(n)$ which returns true if and and only if $\log h(n) \geq \Phi_{1/8}(n)$ and $good\_interval(n_1, n_2)$ which returns true if and only if (5.26) is true. This gives the largest n in $[2, 398\,898\,277]$, which does not satisfy (1.13), $n = 373\,623\,862$, and this call of `ok_rec` computes 3577 values of $good\_interval$ and 2 values of $ok(n)$. $\square$

5.4 Proof of the upper bound (1.14) for $n \geq \nu_0$.

For $n \geq \nu_0$ (defined by (3.17)), one defines N', N'' and ξ by Definition 3.6. The inequalities $\xi \geq x_0$ and $N'' \leq \xi N'$ hold (cf. (3.14)). From (3.12), (3.50) and (3.51),

$$\log g(n) \leq \log N'' = \log N' + \log \frac{N''}{N'} = \sum_{p<\xi} \log p + E^*(N') + \log \frac{N''}{N'}$$
$$\leq \theta(\xi) + 0.72\sqrt{\xi} + \log \xi.$$

Further, from (5.23), with our notation $L = \log n$, $\lambda = \log L$, $L_0 = \log \nu_0$, $\lambda_0 = \log L_0$,

$$\log g(n) \leq \theta(\xi) + \sqrt{\xi}\left(0.72 + \frac{\log \xi}{\sqrt{\xi}}\right) \leq \theta(\xi) + \sqrt{\xi}\left(0.72 + \frac{\log x_0}{\sqrt{x_0}}\right)$$
$$\leq \theta(\xi) + 0.73\sqrt{\xi} \leq \theta(\xi) + 0.73(n\log n)^{1/4}\left(1 + \frac{\lambda - 1}{4L}\right)$$
$$\leq \theta(\xi) + 0.73(n\log n)^{1/4}\left(1 + \frac{\lambda_0 - 1}{4L_0}\right) \leq \theta(\xi) + 0.75(n\log n)^{1/4}. \quad (5.28)$$

Now, we consider two cases, according to $\xi \leq 10^{19}$ or not.

– If $x_0 \leq \xi \leq 10^{19}$, then (5.28), (2.1) and (5.6) imply

$$\log g(n) \leq \xi + 0.75(n\log n)^{1/4} = \xi + \frac{\lambda^2}{L^2}\sqrt{n\log n}\left(\frac{0.75L^{7/4}}{n^{1/4}\lambda^2}\right)$$
$$\leq \xi + \frac{\lambda^2}{L^2}\sqrt{n\log n}\left(\frac{0.75L_0^{7/4}}{\nu_0^{1/4}\lambda_0^2}\right)$$
$$\leq \sqrt{n\log n}\left(1 + \frac{\lambda - 1}{2L} - \frac{\lambda^2}{L^2}\left(\frac{13}{10^4} - 10^{-3}\right)\right)$$
$$= \sqrt{n\log n}\left(1 + \frac{\lambda - 1}{2L} - \frac{3\lambda^2}{10^4 L^2}\right)$$

which proves (1.14) for $x_0 \leq \xi \leq 10^{19}$.

– If $\xi > x_6 = 10^{19}$, then from (3.6), (3.7) and (2.20),

$$n \geq \ell(N') \geq \pi_1(\xi) - \xi \geq \xi^2/(2\log\xi) - \xi \geq x_6^2/(2\log x_6) - x_6$$
$$\geq \nu_1 \stackrel{def}{=\!=} 10^{36}.$$

From (5.6), by setting $L_1 = \log \nu_1 = 82.89\ldots$, $\lambda_1 = \log L_1 = 4.41\ldots$,

$$\xi \leq \sqrt{n\log n}\left(1 + \frac{\lambda - 1}{2L}\right) \leq \sqrt{n\log n}\left(1 + \frac{\lambda_1 - 1}{2L_1}\right) \leq 1.021\sqrt{n\log n},$$

and, from (2.3) with $\alpha = 0.15$ and (5.4),

$$\theta(\xi) - \xi \leq \frac{0.15\,\xi}{\log^3\xi} \leq \frac{1.2\,\xi}{\log^3 n} \leq \frac{1.2 \times 1.021\,\lambda^2}{L^2}\sqrt{n\log n}\frac{1}{L\lambda^2}$$
$$\leq \frac{1.23\,\lambda^2}{L^2}\sqrt{n\log n}\frac{1}{L_1\lambda_1^2} \leq \frac{8\,\lambda^2}{10^4\,L^2}\sqrt{n\log n}. \quad (5.29)$$

We also have

$$0.75(n\log n)^{1/4} = \frac{\lambda^2}{L^2}\sqrt{n\log n}\left(\frac{0.75\,L^{7/4}}{n^{1/4}\lambda^2}\right)$$
$$\leq \frac{\lambda^2}{L^2}\sqrt{n\log n}\left(\frac{0.75\,L_1^{7/4}}{\nu_1^{1/4}\lambda_1^2}\right) \leq \frac{9\,\lambda^2}{10^8\,L^2}\sqrt{n\log n}. \quad (5.30)$$

Finally, from (5.6), (5.28), (5.29) and (5.30),

$$\log g(n) \leq \theta(\xi) + 0.75(n\log n)^{1/4} \leq \xi + \sqrt{n\log n}\,\frac{\lambda^2}{L^2}\left(\frac{8}{10^4} + \frac{9}{10^8}\right)$$
$$\leq \sqrt{n\log n}\left(1 + \frac{\lambda - 1}{2L} - \frac{\lambda^2}{L^2}\left(\frac{13 - 8 - 0.0009}{10^4}\right)\right)$$
$$\leq \sqrt{n\log n}\left(1 + \frac{\lambda - 1}{2L} - \frac{4\,\lambda^2}{10^4\,L^2}\right), \quad (5.31)$$

which completes the proof of (1.14) for $n \geq \nu_0$.

□

5.5 Proof of the upper bound (1.14) for $n < \nu_0$.

The inequality (1.14) for $4 \leq n < \nu_0$ will follow from the lemma:

Lemma 5.4. *For* $4 \le n \le \nu_0$*,* z_n *defined by* (3.29) *satisfies*

$$z_6 = 3.18\ldots \ge z_n \ge z_{\nu_2} = 0.005\,455\,048\,036\ldots > 0 \qquad \textit{with} \quad \nu_2 = 6\,473\,549\,497\,145\,122. \tag{5.32}$$

Proof. For $4 \le n \le 18$, we calculate z_n and obtain $z_{12} = 1.73\ldots \le z_n \le z_6 = 3.18\ldots$ For $n \ge 19$, we compute $z_{\ell(N)}$ for all superchampion numbers N satisfying $19 \le \ell(N) \le \nu_0$. The minimum is attained in ν_2 and the maximum is $z_{19} = 1.53\ldots$, which, by applying Lemma 3.13 (ii), completes the proof of (5.32). We have $z_2 = -2.05\ldots$ and $z_3 = -2.38\ldots$ It is possible that $z_n \ge z_{\nu_2}$ holds for all $n \ge 4$ but we have not been able to prove it. We have only proved, from (5.31) and (5.32), that $z_n \ge 0.0004$ holds for $n \ge 4$. □

6 Study of $g(n)/h(n)$ for large n's

6.1 Effective estimates of $\log g(n) - \log h(n)$

Proposition 6.1. *If* $n \ge \nu_0$ *(defined by* (3.17)*), we have*

$$\frac{\sqrt{2}}{3}(n\log n)^{1/4}\left(1+\frac{\log\log n - 11.6}{4\log n}\right) \le \log\frac{g(n)}{h(n)} \le \frac{\sqrt{2}}{3}(n\log n)^{1/4}\left(1+\frac{\log\log n + 2.43}{4\log n}\right). \tag{6.1}$$

Proof. For $n \ge \nu_0$, we consider the two superchampion numbers N' and N'' and ξ defined in Definition 3.6. From (3.14), we have $N'' \le \xi N'$ and from (3.12), $N' \le g(n) < N''$.

In view of estimating $h(n)$, we need the value of $k = k(n)$ defined by (1.18). For that, we have to convert the additive excess $E(N')$ (cf. (3.40)) in large primes. More precisely, if p_{i_0} denotes the largest prime factor of N' and $\sigma_{i_0} = \sum_{p \le p_{i_0}} p$ (cf. (1.17), from (3.7) and (3.57), we have $\sum_{p \mid N'} p = \sum_{p<\xi} p = \sigma_{i_0}$ and from (3.41), $\ell(N') - E(N') = \sigma_{i_0}$ so that, from the definition (3.58) of $s = s(n)$,

$$\sigma_{i_0+s} \le n < \sigma_{i_0+s+1} \tag{6.2}$$

and, from (1.18), $k = k(n) = i_0 + s$. As h is nondecreasing on n, from (1.19), one deduces

$$h(\sigma_{i_0+s}) = N_{i_0+s} \le h(n) \le N_{i_0+s+1} = h(\sigma_{i_0+s+1})$$

and

$$\frac{N'}{N_{i_0+s+1}} \le \frac{g(n)}{h(n)} \le \frac{N''}{N_{i_0+s}} \le \frac{\xi N'}{N_{i_0+s}}. \tag{6.3}$$

The lower bound. Observing from (3.50) that $\log N' = \sum_{p|N'} \log p + E^*(N') = \log N_{i_0} + E^*(N')$, from (6.3), (3.51) and (3.66), one gets

$$\log \frac{g(n)}{h(n)} \geq \log \frac{N'}{N_{i_0+s+1}} = E^*(N') - \sum_{i=i_0+1}^{i_0+s+1} \log p_i \geq E^*(N') - (s+1) \log p_{i_0+s+1}$$
$$\geq \sqrt{\frac{\xi}{2}} \left(1 - \frac{0.521}{\log \xi} - \frac{1}{3} - \frac{0.345}{\log \xi}\right) = \frac{\sqrt{2\xi}}{3} \left(1 - \frac{1.299}{\log \xi}\right). \quad (6.4)$$

Now, as the third derivative of $u \mapsto \sqrt{1+u}$ is positive, from Taylor's formula and Corollary 5.3,

$$\sqrt{\xi} \geq (n \log n)^{1/4} \left(1 + \frac{u}{2} - \frac{u^2}{8}\right) \quad \text{with} \quad u = \frac{\log \log n - 1.019}{2 \log n}. \quad (6.5)$$

By writing L for $\log n$, λ for $\log \log n$, L_0 for $\log \nu_0$ and λ_0 for $\log \log \nu_0$, from (1.15),

$$\frac{u^2}{8} = \frac{(\lambda - 1.019)^2}{4 \times 8\, L^2} \leq \frac{(\lambda_0 - 1.019)^2}{4 \times 8\, L_0 L} \leq \frac{0.022}{4L},$$
$$1 + \frac{u}{2} - \frac{u^2}{8} \geq 1 + \frac{\lambda - 1.019}{4L} - \frac{0.022}{4L} = 1 + \frac{\lambda - 1.041}{4L}$$

so that, from (6.5), one gets $\sqrt{\xi} \geq (n \log n)^{1/4}(1 + (\lambda - 1.041)/(4L))$. Further, from (5.4), $1.299/\log \xi < 10.392/(4L)$ holds and (6.4) implies

$$\log \frac{g(n)}{h(n)} \geq \frac{\sqrt{2}}{3}(n \log n)^{1/4} \left(1 + \frac{\lambda - 1.041}{4L}\right)\left(1 - \frac{10.392}{4L}\right)$$
$$\geq \frac{\sqrt{2}}{3}(n \log n)^{1/4} \left(1 + \frac{\lambda - 11.433}{4L} - \frac{10.392(\lambda_0 - 1.041)}{(4L_0)(4L)}\right)$$
$$\geq \frac{\sqrt{2}}{3}(n \log n)^{1/4} \left(1 + \frac{\lambda - 11.6}{4L}\right)$$

which proves the lower bound of (6.1).

The upper bound. Similarly, from (6.3), (3.57), (3.51) and (3.67), we have

$$\log \frac{g(n)}{h(n)} \leq \log \frac{\xi N'}{N_{i_0+s}} = \log \xi + E^*(N') - \sum_{i=i_0+1}^{i_0+s} \log p_i \leq E^*(N') - (s-1) \log \xi$$
$$\leq \sqrt{\frac{\xi}{2}} \left(1 + \frac{0.305}{\log \xi} - \frac{1}{3} - \frac{0.0724}{3 \log \xi}\right) = \frac{\sqrt{2\xi}}{3} \left(1 + \frac{0.4213}{\log \xi}\right). \quad (6.6)$$

Further, by using the inequality $\sqrt{1+t} \le 1 + t/2$, it follows from (5.23) and (5.4) that

$$\begin{aligned}\log \frac{g(n)}{h(n)} &\le \frac{\sqrt{2}}{3}(n \log n)^{1/4}\left(1+\frac{\lambda-1}{4L}\right)\left(1+\frac{3.3704}{4L}\right)\\ &\le \frac{\sqrt{2}}{3}(n \log n)^{1/4}\left(1+\frac{\lambda+2.3704}{4L}+\frac{3.3704(\lambda_0-1)}{(4L_0)(4L)}\right)\\ &\le \frac{\sqrt{2}}{3}(n \log n)^{1/4}\left(1+\frac{\lambda+2.43}{4L}\right)\end{aligned}$$

which ends the proof of Proposition 6.1. □

6.2 Asymptotic expansion of $\log g(n) - \log h(n)$

Proposition 6.2. *Let n be an integer tending to infinity. N', N'', ξ and ξ_2 are defined by Definition 3.6. Then, for any real number K, when n and ξ tend to infinity, we have*

$$\log \frac{g(n)}{h(n)} = \left(\xi_2 - \mathrm{li}(\xi_2^3)\frac{\log \xi}{\xi}\right)\left(1+\mathcal{O}_K\left(\frac{1}{\log^K \xi}\right)\right). \qquad (6.7)$$

Proof. The proof follows the lines of the proof of Proposition 6.1. Let K be a real number as large as we wish. First, from the Prime Number Theorem, for $r \ge 0$, it is easy to deduce

$$\pi_r(x) = \sum_{p \le x} p^r = (\mathrm{li}(x^{r+1}))(1+\mathcal{O}(1/\log^K x)), \qquad x \to \infty. \qquad (6.8)$$

From Proposition 3.14, when $\xi \to \infty$, we know that

$$\xi_2 \sim \sqrt{\xi/2} \qquad (6.9)$$

and, from Proposition 3.23, that

$$s \sim \frac{\sqrt{\xi}}{3\sqrt{2}\log \xi}. \qquad (6.10)$$

Note that (6.10) implies

$$s \pm 1 = s(1+\mathcal{O}(1/\log^K \xi)). \qquad (6.11)$$

By using the crude estimate $\pi_j(\xi_j) \le \xi_j^{j+1}$, from (3.43), (3.20) and (3.9), it follows that

$$E(N') \le \sum_{j=2}^{J} \pi_j(\xi_j) \le \pi_2(\xi_2) + \sum_{j=3}^{J} \xi_j^{j+1} \le \pi_2(\xi_2) + \sum_{j=3}^{J} \xi^{1+1/j}$$
$$\le \pi_2(\xi_2) + J\,\xi^{4/3} = \pi_2(\xi_2) + \mathcal{O}(\xi^{4/3} \log \xi). \quad (6.12)$$

In the same way, from (3.44), one gets $E(N') \ge \pi_2(\xi_2) + \mathcal{O}(\xi)$ which, together with (6.12), (6.8) and (6.9) yields

$$E(N') = (\mathrm{li}(\xi_2^3))(1 + \mathcal{O}(1/\log^K \xi_2)) = (\mathrm{li}(\xi_2^3))(1 + \mathcal{O}(1/\log^K \xi)). \quad (6.13)$$

From (3.52), similarly, we have

$$E^*(N') = \theta(\xi_2) + \mathcal{O}(J\,\xi_3) = \xi_2(1 + \mathcal{O}(1/\log^K \xi)). \quad (6.14)$$

It follows from Lemma 2.3 and (6.10) that the number of primes between ξ and $\xi(1 + 1/\log^K \xi)$ satisfies, for n and ξ large enough, $\pi(\xi(1 + 1/\log^K \xi)) - \pi(\xi) > \xi/(2\log^{K+1} \xi) > \sqrt{\xi} > s + 1$, which, via (3.57), (3.58) and (3.13), implies

$$\xi \le p_{i_0+1} \le p_{i_0+s+1} \le \xi(1 + \mathcal{O}(1/\log^K \xi)) \quad (6.15)$$

and

$$\begin{aligned}(s-1)\xi &\le p_{i_0+1} + \ldots + p_{i_0+s} - \xi \le n - \ell(N') - \xi + E(N') \\ &\le \ell(N'') - \ell(N') - \xi + E(N' \le E(N')) \\ &\le n - \ell(N') + E(N') \le p_{i_0+1} + \ldots + p_{i_0+s+1} \\ &\le (s+1)p_{i_0+s+1} \le (s+1)\,\xi\,(1 + \mathcal{O}(1/\log^K \xi)).\end{aligned}$$

From (6.11), it follows that $E(N') = s\xi(1 + \mathcal{O}(1/\log^K \xi))$ and, from (6.13) ,

$$s = \frac{\mathrm{li}(\xi_2^3)}{\xi}\left(1 + \mathcal{O}\left(\frac{1}{\log^K \xi}\right)\right). \quad (6.16)$$

From (6.4) and (6.6), we have

$$E^*(N') - (s+1)\log p_{i_0+s+1} \le \log \frac{g(n)}{h(n)} \le E^*(N') - (s-1)\log \xi, \quad (6.17)$$

which, from (6.11), (6.15), (6.16) and (6.14), proves (6.7). □

7 Proof of Theorem 1.5

7.1 Proof of Theorem 1.5 (i).

We assume that n tends to infinity. N', N'' and ξ are defined by (3.12). From Proposition 6.2 one deduces

$$\log \frac{g(n)}{h(n)} \asymp \xi_2 - \mathrm{li}(\xi_2^3)\frac{\log\xi}{\xi} = \frac{\sqrt{2\xi}}{3}F \quad \text{with} \quad F = \frac{3}{\sqrt{2\xi}}\left(\xi_2 - \mathrm{li}(\xi_2^3)\frac{\log\xi}{\xi}\right). \tag{7.1}$$

By using (3.30) and (2.13), we get the asymptotic expansion of F in terms of $t = 1/\log\xi$ (cf. [27])

$$F = F(t) = 1 - \frac{2+3\log 2}{6}t - \frac{32+48\log 2 + 9\log^2 2}{72}t^2 + \dots \tag{7.2}$$

From (3.6), (3.12), (3.14), (3.50) an d (3.51), we have

$$\theta^-(\xi) \le \log N' \le \log g(n) \le \log N''$$
$$\le \log N' + \log\xi = \theta^-(\xi) + E^*(N') + \log\xi = \theta(\xi) + \mathcal{O}(\sqrt{\xi})$$

so that, from the Prime Number Theorem and (1.6), for any real number K, we have

$$\xi = \sqrt{\mathrm{li}^{-1} n}\,(1 + \mathcal{O}_K(1/\log^K n)) \tag{7.3}$$

that we write $\xi \asymp \sqrt{\mathrm{li}^{-1} n}$. Therefore, from (7.1) and (1.7), we can get the asymptotic expansion of $\log(g(n)/h(n))$. More precisely, we may use Theorem 2 of [25] to get

$$\log \frac{g(n)}{h(n)} \asymp \frac{\sqrt{2}}{3}(n\log n)^{1/4}\left(1 + \sum_{j\ge 1}\frac{P_j(\log\log n)}{\log^j n}\right) \tag{7.4}$$

where P_j is a polynomial of degree j satisfying the induction relation

$$\frac{d}{dt}(P_{j+1}(t) - P_j(t)) = \left(\frac{1}{4} - j\right)P_j(t). \tag{7.5}$$

For that, one sets $y = \log(\mathrm{li}^{-1}(n))$, $n = \mathrm{li}(e^y)$, $\xi \asymp e^{y/2}$ and, from (2.13), $n \asymp \frac{e^y}{y}\sum_{k\ge 0}\frac{k!}{y^k}$ so that, from (7.1),

$$\log \frac{g(n)}{h(n)} \asymp \frac{\sqrt{2}}{3}e^{y/4}F(2/y)$$

holds. Finally, we apply the procedure *theorem2_part2* of [25, p. 234] with $\alpha = 1, \beta = 1/4, \gamma = 0, G(t) = F(2t)$ (with F defined by (7.2)), $d(t) = \sum_{k\geq 0} k!t^k$ and $x = n$. □

The values of the polynomials P_j can be found on the website [27] (cf. [23] and [14] for similar results).

7.2 Proof of Theorem 1.5 (ii) and (iii).

Let us define β_n as the unique number such that

$$\log \frac{g(n)}{h(n)} = \frac{\sqrt{2}}{3}(n\log n)^{1/4}\left(1 + \frac{\log\log n + \beta_n}{4\log n}\right) \tag{7.6}$$

An easy computation gives

$$\beta_n = 6\sqrt{2}(\log n)^{3/4}\frac{\log g(n) - \log h(n)}{n^{1/4}} - 4\log n - \log\log n. \tag{7.7}$$

From the non decreasingness of the functions log, log log, g and h we deduce from (7.7) the following

Lemma 7.1. *Let $n_1 \leq n_2$ be two integers. Then, for every $n \in [n_1,\ n_2]$,*

$$\beta_n \leq 6\sqrt{2}(\log n_2)^{3/4}\ \frac{\log g(n_2) - \log h(n_1)}{n_1^{1/4}} - 4\log n_1 - \log\log n_1 \tag{7.8}$$

and, if $g(n_1) \geq h(n_2)$ is satisfied

$$\beta_n \geq 6\sqrt{2}(\log n_1)^{3/4}\ \frac{\log g(n_1) - \log h(n_2)}{n_2^{1/4}} - 4\log n_2 - \log\log n_2 \tag{7.9}$$

The expensive operations in the computation of the bounds given in (7.8) and (7.9) are the computations of $g(n_1), g(n_2), h(n_1), h(n_2)$. In the particular case where $n_1, n_2 = \ell(N_1), \ell(N_2)$ for two consecutive superchampions N_1, N_2, we will use the following lemma to quickly bound β_n on the slice $[n_1,\ n_2]$.

Lemma 7.2. *Let $n_1 = \ell(N_1)$, $n_2 = \ell(N_2)$ where N_1, N_2 are two consecutive superchampions, $k_1 = k(n_1)$ (resp. $k_2 = k(n_2)$), $m_1 = n_1 - \sigma_{k_1}$ (resp. $m_2 = n_2 - \sigma_{k_2}$) and q the first prime not smaller than $p_{k+1} - m_1$. Then, for every n in $[n_1,\ n_2]$,*

$$\beta_n \leq 6\sqrt{2}(\log n_2)^{3/4}\ \frac{\log N_2 - \theta(p_{k_1+1}) + \log q}{n_1^{1/4}} - 4\log n_1 - \log\log n_1 \tag{7.10}$$

and, if $\log N_1 > \theta(p_{k_2+1}) - \log(p_{k_2+1} - m_2)$

$$\beta_n \geq 6\sqrt{2}(\log n_1)^{3/4} \frac{\log N_1 - \theta(p_{k_2+1}) + \log(p_{k_2+1} - m_2)}{n_2^{1/4}} - 4\log n_2 - \log\log n_2. \tag{7.11}$$

Proof. – We get (7.10) from (7.8) by noticing that $\log g(n_2) = \log N_2$ and by using (4.1) to minimize $h(n_1)$.

– We get (7.11) from (7.9) by noticing that $\log g(n_1) = \log N_1$ and by using (4.2) to maximize $h(n_2)$.

□

Proof of Theorem 1.5.(ii). Considering (7.6) we have to prove that $\beta_n \leq 2.43$ for $n > \nu_5 = 3\,997\,022\,083\,662$. For $n > \nu_0$, it results of Proposition 6.1.

We first enumerate all the pairs of consecutive superchampions $\leq N_0'$, and for each of these pairs we compute the upper bound given by (7.10). It appears that for $n \geq \nu_3 = 23\,542\,052\,569\,006$, $\beta_n < 2.43$. To get the largest n which does not satisfy Theorem 1.5.ii we use the dichotomic procedure `ok_rec` described in Section 4.2.3 on the interval $[2, \nu_3]$, choosing the functions $ok(n)$ which returns true if and only if $\beta_n \leq 2.43$, and the function $good\_interval(n_1, n_2)$ which returns true if and only if the right term of (7.10) is not greater than 2.43. The call `ok_rec(2,` ν_3`)` gives ν_5 as the largest number n such that $\beta_n = 2.430\,001\,869\ldots > 2.43$. This computation generated 2 calls of `ok(n)`. and 5 017 255 calls of `good_interval`, and it took 40h. □

Proof of Theorem 1.5.(iii). For $n > \nu_0$, it results of Proposition 6.1.

As in the previous paragraph, we first enumerate all the pairs of consecutive superchampions $N_1, N_2 \leq N_0'$. We have checked that, for $\ell(N_1) \geq 1487$, $\log N_1 > \theta(p_{k_2+1}) - \log(p_{k_2+1} - m_2)$, so that (7.11) holds, and then, we verify that for $n \geq \nu_4 = 1\,017\,810$, $\beta_n > -11.6$ holds.

Now the call `ok_rec(1487,` ν_4`)` with the function $good\_interval(n_1, n_2)$ which returns true if and only if the right term of (7.11) is smaller than -11.6 and the function $ok(n)$ which returns true if and only if $\beta_n > -11.6$, we get $n = 4\,229$ as the largest n sucht that $\beta_n < -11.6$.

□

7.3 Proof of Theorem 1.5 (iv).

The inequality $g(n) \geq h(n)$ follows from (1.1) and (1.2). For $n \geq 4\,230$, inequality $g(n) > h(n)$ is an easy consequence of point (iii). We end the proof by computing $g(n)$ and $h(n)$ for $1 \leq n < 4\,230$. □

7.4 Proof of Theorem1.5 (v).

Let d_n defined by

$$d_n = b_n - a_n = \frac{\log g(n) - \log h(n)}{(n \log n)^{1/4}} = \frac{\sqrt{2}}{3}\left(1 + \frac{\log\log n + \beta_n}{4 \log n}\right). \tag{7.12}$$

For $n > v_5$, from point (ii), we have

$$\log \frac{g(n)}{h(n)} \leq \frac{\sqrt{2}}{3}(n \log n)^{1/4}\left(1 + \frac{\log\log v_5 + 2.43}{4\log v_5}\right) \leq 0.5\,(n \log n)^{1/4}.$$

By the non-decreasingness of g and h, if $n_1 \leq n_2$, d_n is bounded above on $[n_1,\ n_2]$ by $M(n_1, n_2)$, with

$$M(n_1, n_2) = (\log g(n_2) - \log h(n_1)(n_1 \log n_1)^{1/4}. \tag{7.13}$$

Thus, the inequality

$$M(n_1, n_2) < 0.62 \tag{7.14}$$

is a sufficient condition ensuring that $d_n < 0.62$ on the whole interval $[n_1,\ n_2]$.

As in paragraph 7.2, for all the pairs $n_1 = \ell(N_1)$, $n_2 = \ell(N_2)$ where N_1, N_2 are two consecutive superchampions with $\ell(N_2) \leq v_5$ we quickly get an upper bound of $M(n_1, n_2)$ by using $g(n_2) = \log(N_2)$ and bounding below $\log h(n_1)$ by (4.1). It appears that this bound is smaller than 0.62 for $n \geq 49\,467\,083$.

Now the call `ok_rec(2, 49467083)` using $ok(n)$ which returns true if and only if $d_n < 0.62059$ and $good\_interval(n_1, n_2)$ which returns true if and only $M(n_1, n_2) < 0.62059$, gives us the last value of d_n which is greater than 0.62059, this value is $d_{2243} = 0.620\,665\,265\,68...$ Note that $g(2243)$ is a superchampion number associated to $\rho = 139/\log 139$ and $149/\log 149$. Finally, by computing $g(n)$ and $h(n)$, we checked that $d_n < 0.62$ holds for $2 \leq n < 2243$. □

8 Proof of Theorem 1.1

8.1 Proof of Theorem 1.1 (i).

For $n \geq 2$, the point (i) of Theorem 1.1 follows from the definition (1.9) of a_n and from the point (iv), below. For $n = 1$, $\mathrm{li}^{-1}(1) = 1.96\ldots$ and $g(1) = 1$ so that $\log g(1) < \sqrt{\mathrm{li}^{-1}(1)}$ holds. □

8.2 Proof of Theorem 1.1 (ii).

From now on, the following notation is used : $L = \log n$, $\lambda = \log\log n = \log L$ and $\lambda_0 = \log\log \nu_0$.

From (1.10), for $n \geq \nu_0$ (defined by (3.17)), we have

$$\log g(n) = \log h(n) + \log \frac{g(n)}{h(n)} = -b_n (n\log n)^{1/4} + \sqrt{\mathrm{li}^{-1} n} + \log \frac{g(n)}{h(n)}$$

and, from (1.9), Theorem 1.4 (iii) and Proposition 6.1, one gets

$$\begin{aligned} a_n = \frac{\sqrt{\mathrm{li}^{-1} n} - \log g(n)}{(n\log n)^{1/4}} &= b_n - \frac{\log(g(n)/h(n))}{(n\log n)^{1/4}} \\ &\geq \frac{2}{3} - c - \frac{0.23\,\lambda}{L} - \frac{\sqrt{2}}{3}\left(1 + \frac{\lambda + 2.43}{4L}\right) \\ &= \frac{2-\sqrt{2}}{3} - c - \frac{\lambda}{L}\left(0.23 + \frac{\sqrt{2}}{12} + \frac{2.43\sqrt{2}/\lambda}{12}\right) \\ &\geq \frac{2-\sqrt{2}}{3} - c - \frac{\lambda}{L}\left(0.23 + \frac{\sqrt{2}}{12} + \frac{2.43\sqrt{2}/\lambda_0}{12}\right) \geq \frac{2-\sqrt{2}}{3} - c - \frac{0.43\,\lambda}{L}, \end{aligned}$$

which proves point (ii) for $n > \nu_0$.

Lemma 8.1. *For* $2 \leq n \leq \nu_0$, a_n *defined by* (1.9) *satisfies*

$$a_n \geq a_{6\,473\,580\,667\,603\,736} = 0.193938608602\ldots. \tag{8.1}$$

Proof. For $2 \leq n \leq 42$, one checks that $a_n \geq 0.4$. For $43 \leq n < \nu_0$, by Lemma 3.13 (i), the minimum is attained in $\ell(N)$ with N being a superchampion number satisfying $43 \leq \ell(N) \leq \nu_0 = \ell(N_0')$. So, by enumerating $(N, \log N) = (N, \log g(\ell(n)))$ for $N \leq N_0'$ we check that the minimum is $0.193938602\ldots$, attained for $\ell(N) = 6\,473\,580\,667\,603\,736$. □

For $3 \leq n < \nu_0$, Lemma 8.1 shows that $a_n \geq (2-\sqrt{2})/3 - c = 0.149\ldots$ holds, which, as $\log\log n$ is positive, proves point (ii). For $n = 2$, $a_2 = 0.91\ldots$ and point (ii) is still satisfied. □

8.3 Proof of Theorem 1.1 (iii).

For n $> \nu_0$**.** From Theorem 1.4 (iv) and Proposition 6.1, we have

$$\begin{aligned} a_n &= b_n - \frac{\log(g(n)/h(n))}{(n\log n)^{1/4}} \leq \frac{2}{3} + c + \frac{0.77\,\lambda}{L} - \frac{\sqrt{2}}{3}\left(1 + \frac{\lambda - 11.6}{4L}\right) \\ &\leq \frac{2-\sqrt{2}}{3} + c + \frac{\lambda}{L}\left(0.77 - \frac{\sqrt{2}}{12} + \frac{11.6\sqrt{2}/\lambda_0}{12}\right) \leq \frac{2-\sqrt{2}}{3} + c + \frac{1.02\,\lambda}{L}. \end{aligned}$$

For n $\le v_0$. Let us suppose $16 \le n_1 \le n_2$. The non decreasingness of li, g, the positivity of a_n (cf. (8.1)) and therefore of $\sqrt{\mathrm{li}^{-1}(n)} - \log g(n)$ imply that, for $n \in [n_1, n_2]$,

$$a_n = \frac{\sqrt{\mathrm{li}^{-1}(n)} - \log g(n)}{(n \log n)^{1/4}} \le R(n_1, n_2) = \frac{\sqrt{\mathrm{li}^{-1}(n_2)} - \log g(n_1)}{(n_1 \log n_1)^{1/4}} \tag{8.2}$$

Since $n_1 \ge 16$, the function $\log\log n / \log n$ is decreasing on $[n_1, n_2]$, and, in view of (8.2),

$$R(n_1, n_2) \le \frac{2 - \sqrt{2}}{3} + c + m \frac{\log\log n_2}{\log n_2} \tag{8.3}$$

is a sufficient condition ensuring that, for all $n \in [n_1, n_2]$,

$$a_n < \frac{2 - \sqrt{2}}{3} + c + m \frac{\log\log n}{\log n}. \tag{8.4}$$

In the case $n_1 = \ell(N_1)$, $n_2 = \ell(N_2)$, where N_1, N_2 are consecutive superchampions, $g(n_1) = \log(N_1)$, and, by enumerating all the pairs of consecutive superchampions $\le N_0'$, we check that, if $m = 1.02$, inequality (8.2) is satisfied for $n \ge 5\,432\,420$. To compute the largest n which does not satisfy this inequality we call `ok_rec(2, 5432420)` with the boolean fonction $ok(n)$ which returns true if and only if (8.4) is true, and the procedure `good_interval(n1, n2)` which returns true if and only (8.3) is satisfied. This gives us 19424 as the largest integer which does not satisfy point (iii). □

8.4 Proof of Theorem 1.1 (iv).

For $n \ge v_0$, from point (ii) it follows that

$$a_n \ge \frac{2 - \sqrt{2}}{3} - c - \frac{0.43 \log\log v_0}{\log v_0} = 0.11104\ldots$$

while, by Lemma 8.1, $a_n \ge 0.1939$ for $n \le v_0$.

By computing a_n for $2 \le n \le 19424$, it appears that $a_n < a_2 = 0.9102\ldots$, while, for $n \ge 19425$, by point (iii) and the decreasingness of $\log\log n / \log n$,

$$a_n < \frac{2 - \sqrt{2}}{3} + c + \frac{1.02 \log\log 19425}{\log 19425} = 0.477\ldots$$ □

8.5 Proof of Theorem 1.1 (v).

The point (v) of Theorem 1.1 follows from the points (ii) and point (iii). □

8.6 Proof of Theorem 1.1 (vi).

From (7.12) and Theorem 1.5 (i), we have

$$b_n - a_n = d_n = \frac{\log(g(n)/h(n))}{(n\log n)^{1/4}} = \frac{\sqrt{2}}{3}\left(1 + \frac{\log\log n + \mathcal{O}(1)}{4\log n}\right)$$

whence, from Theorem 1.4 (vi),

$$\begin{aligned} a_n &= b_n - d_n \\ &\le \left(\frac{2}{3} + c\right)\left(1 + \frac{\log\log n + \mathcal{O}(1)}{4\log n}\right) - \frac{\sqrt{2}}{3}\left(1 + \frac{\log\log n + \mathcal{O}(1)}{4\log n}\right) \\ &= \left(\frac{2-\sqrt{2}}{3} + c\right)\left(1 + \frac{\log\log n + \mathcal{O}(1)}{4\log n}\right), \end{aligned}$$

which proves the upper bound of (vi). The proof of the lower bound is similar. □

Acknowledgements. We thank very much Richard Brent for attracting our attention on the problem studied in this article.

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Marc Deléglise, Jean-Louis Nicolas,
Univ. Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208
Institut Camille Jordan, Mathématiques, Bât. Doyen Jean Braconnier,
43 Bd du 11 Novembre 1918, F-69622 Villeurbanne cedex, France.

m.h.deleglise@gmail.com,
http://math.univ-lyon1.fr/homes-www/deleglis/

nicolas@math.univ-lyon1.fr,
http://math.univ-lyon1.fr/homes-www/nicolas/