

ON SPECIAL MATRICES RELATED TO CAUCHY AND TOEPLITZ MATRICES

SAJAD SALAMI

ABSTRACT. In this paper, we are going to calculate the determinant of a certain type of square matrices, which are related to the well-known Cauchy and Toeplitz matrices. Then, we will use the results to determine the rank of special non square matrices.

1. INTRODUCTION AND MAIN RESULT

Throughout this paper, we fix a field F of characteristic zero, $2 \leq r < n$ arbitrary integers, and $\{a_\ell\}_{\ell=1}^\infty \subset F^*$ an infinite sequence of distinct elements. For any pair of indexes (ℓ, e) , we define $d_{(\ell, e)} := a_\ell - a_e$ and $d_\ell := d_{(\ell+1, \ell)}$. Then, we consider the $(n-r) \times (r+1)$ matrix $C := [C_{(i-r, j)}]$, such that for $j = 0, \dots, r$ and $r+1 \leq i \leq n$ we have

$$C_{(i-r, 0)} = (-1)^r \prod_{e < \ell \in I} d_{(i, \ell)} d_{(\ell, e)}, \quad C_{(i-r, j)} = (-1)^{r+j} a_i \prod_{e < \ell \in I_j} a_\ell d_{(i, \ell)} d_{(\ell, e)},$$

$$D_r = (-1)^r \prod_{e < \ell \in I} a_\ell (a_\ell - a_e) = (-1)^r \prod_{e < \ell \in I} a_\ell d_{(\ell, e)} \neq 0,$$

where $I = \{1, \dots, r\}$ and $I_j := I \setminus \{j\}$ for $j \in I$. Define $\mathbf{C}_r^n := [C|D]$ as a $(n-r) \times (n+1)$ blocked matrix, where D is a $(n-r) \times (n-r)$ diagonal matrix with entries D_r . A special case of \mathbf{C}_r^n is related to the Hilbert and Toeplitz's matrices [1, 2, 3]. For more details, see Section 2.

The main result of this paper concerns with calculating the rank of the matrix \mathbf{C}_r^n ,

Theorem 1.1. *Let F be a field of characteristic zero and $\{a_\ell\}_{\ell=1}^\infty \subset F^*$ be an infinite sequence of distinct elements of F . Then the matrix \mathbf{C}_r^n has full rank $n-r$ for integers $2 \leq r < n$.*

In order to prove the above theorem, we will calculate the determinant of certain square matrices which are related to the well-known Cauchy's matrices [4, 5]. We notice that it is used the author's forthcoming paper [6] to show the non-singularity of a certain family of complete intersection varieties satisfying the Bombieri-Lang conjecture in the Diophantine geometry [7].

2010 *Mathematics Subject Classification.* Primary 15B05; Secondary 15A15 .

Key words and phrases. Determinant and Rank of Matrices; Cauchy, Hilbert, and Toeplitz matrices.

The organization of the present paper is as follows. In the section 2, we recall the definition and determinant of the square Cauchy's and Hilbert's matrix over a field of characteristic zero. In section 3, we calculate the determinant of a certain square matrix that are related to the Cauchy's matrix. Finally, in Section 4, we use the result of Section 3 to prove Theorem 1.1.

2. CAUCHY'S AND TOEPLITZ MATRICES

In 1841, Augustin Louis Cauchy introduced a certain type of matrices with certain properties, see [4, 5]. We are going to recall the definition and determinant of these matrices in this section.

An $n \times n$ square *Cauchy's matrix* defined by disjoint subsets of distinct nonzero elements $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ in a field of characteristic zero F , is the square matrix $X_n := [x_{ij}]$ with

$$x_{ij} = \frac{1}{x_i - y_j}, \quad 1 \leq i, j \leq n.$$

Note that any submatrix of a Cauchy's matrix is itself a Cauchy's matrix. The determinant of a Cauchy's matrix is known as *Cauchy's determinant* in the literature, which is always nonzero because $x_i \neq y_j$. Following proposition shows that how one can calculate the determinant of Cauchy's matrices.

Proposition 2.1. *Let $n \geq 1$ be an integer and X_n a $n \times n$ Cauchy's matrix defined as above over a field F of characteristic zero. Then*

$$|X_n| = \frac{\prod_{i < j \in I} (x_i - x_j)(y_i - y_j)}{\prod_{i \in I} \prod_{j \in I} (x_i - y_j)}, \quad I = \{1, 2, \dots, n\}.$$

Proof. See the lemma (11.3) in [9] for an analytic proof, when $F = \mathbb{C}$. For an arbitrary field F , we will use the elementary column and row operations to get the desired result. Subtracting the first column of U from others gives that

$$x_{ij} = \frac{(y_1 - y_j)}{(x_i - y_1)} \cdot \frac{1}{(x_i - y_j)} \quad (1 \leq i, j \leq n).$$

Extracting the factor $1/(x_i - y_1)$ from i -th row for $i = 1, \dots, n$, and $y_1 - y_j$ from j -th column for $j = 2, \dots, n$ leads to

$$|U| = \frac{1}{(x_1 - y_1)} \cdot \frac{\prod_{j=2}^n (y_1 - y_j)}{\prod_{i=2}^n (x_i - y_1)} \cdot \begin{vmatrix} 1 & \frac{1}{(x_1 - y_2)} & \cdots & \frac{1}{(x_1 - y_n)} \\ 1 & \frac{1}{(x_2 - y_2)} & \cdots & \frac{1}{(x_2 - y_n)} \\ \vdots & \cdots & \ddots & \vdots \\ 1 & \frac{1}{(x_n - y_2)} & \cdots & \frac{1}{(x_n - y_n)} \end{vmatrix}.$$

Now, denoting the last determinant by $|x'_{ij}|$ and subtracting its first row from others, we get

$$x'_{i1} = 0, \quad x'_{ij} = \frac{(x_1 - x_i)}{(x_1 - y_j)} \cdot \frac{1}{(x_i - y_j)} \quad 2 \leq i, j \leq n.$$

Extracting the factor $(x_1 - x_i)$ from each rows, and $1/(x_1 - y_j)$ from each column, for $2 \leq i, j \leq n$, gives that

$$|X_n| = \frac{1}{(x_1 - y_1)} \prod_{i,j=2}^n \frac{(y_1 - y_j)(x_1 - x_i)}{(x_i - y_1)(x_1 - y_i)} \begin{vmatrix} \frac{1}{(x_2 - y_2)} & \frac{1}{(x_2 - y_3)} & \cdots & \frac{1}{(x_2 - y_n)} \\ \frac{1}{(x_3 - y_2)} & \frac{1}{(x_3 - y_3)} & \cdots & \frac{1}{(x_3 - y_n)} \\ \vdots & \cdots & \ddots & \vdots \\ \frac{1}{(x_n - y_2)} & \frac{1}{(x_n - y_3)} & \cdots & \frac{1}{(x_n - y_n)} \end{vmatrix}.$$

Repeating this procedure, we obtain that

$$|X_n| = \frac{1}{\prod_{i \in I} (x_i - y_i)} \cdot \frac{\prod_{i < j \in I} (y_i - y_j)(x_j - x_i)}{\prod_{i < j \in I} (x_i - y_j)(x_j - y_i)} = \frac{\prod_{i < j \in I} (x_i - x_j)(y_i - y_j)}{\prod_{i \in I} \prod_{j \in I} (x_i - y_j)}.$$

□

In [1], Hilbert introduced a certain square matrix which is a special case of the Cauchy square matrix. The *Hilbert's matrix* is an $n \times n$ matrix $\mathbf{H}_n = [h_{ij}]$ with entries $h_{ij} = 1/(i + j - 1)$, where $1 \leq i, j \leq n$. Using the proposition 2.1, one can calculate the determinant of a Hilbert's matrix as

$$|\mathbf{H}_n| = \frac{c_n^A}{c_{2n}}, \quad c_n = \prod_{i=1}^{n-1} i!.$$

He also mentioned that the determinant of \mathbf{H}_n is the reciprocal of a well known integer which follows from the following identity

$$\frac{1}{|\mathbf{H}_n|} = \frac{c_{2n}}{c_n^A} = n! \cdot \prod_i^{2n-1} \binom{i}{[i/2]}.$$

For more information see the sequence A005249 in OEIS [8]. For a recent work related to the Cauchy's and Hilbert's matrices one can see [10].

The other type of matrices, which we are going to recall here, are the Toeplitz matrices. An $n \times n$ *Toeplitz matrix* with entries in a field F is the square matrix

$$V_n := \begin{bmatrix} v_0 & v_1 & v_2 & \cdots & v_{n-1} \\ v_{-1} & v_0 & v_1 & \cdots & v_{n-2} \\ v_{-2} & v_{-1} & v_0 & \cdots & v_{n-3} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ v_{1-n} & v_{2-n} & v_{3-n} & \cdots & v_0 \end{bmatrix}.$$

These are one of the most well studied and understood classes of matrices that arise in most areas of the mathematics: algebra [11], algebraic geometry [12], and graph theory [13]. In [3], the author obtained a unique LU factorizations and an explicit formula for the determinant and also the inversion of Toeplitz matrices. And, the inverse, determinants, eigenvalues, and eigenvectors of symmetric Toeplitz matrices over real number field with linearly increasing entries have been studied in [14]. In [15], the author

showed that every $n \times n$ square matrix is generically a product of $\lfloor n/2 \rfloor + 1$ and always a product of at most $2n + 5$ Toeplitz matrices.

3. DETERMINANT OF CERTAIN SQUARE MATRIX

In this section, we calculate the determinant of certain square matrices with entries in a field F of characteristic zero, which are related to the determinant of Cauchy's matrix. In special case, the determinant of our matrix is related to the determinant of a certain Toeplitz matrix. First, let us to give the following elementary result for a given infinite sequence $\{a_\ell\}_{\ell=1}^\infty$ of distinct nonzero elements in a field F of characteristic zero.

Lemma 3.1. *For indexes e, ℓ, s , and t , we have*

$$a_s d_{(\ell, e)} - a_\ell d_{(s, e)} = -a_e d_{(s, \ell)}, \quad \frac{d_{(t, e)}}{d_{(t, \ell)}} - \frac{d_{(s, e)}}{d_{(s, \ell)}} = \frac{d_{(t, s)} d_{(\ell, e)}}{d_{(t, \ell)} d_{(s, \ell)}}.$$

Proof. For indexes e, ℓ , and s , by definition $d_{(s, e)} = d_{(s, \ell)} + d_{(\ell, e)}$, so

$$\begin{aligned} a_s d_{(\ell, e)} - a_\ell d_{(s, e)} &= a_s d_{(\ell, e)} - a_\ell (d_{(s, \ell)} + d_{(\ell, e)}) \\ &= (a_s - a_\ell) d_{(\ell, e)} - a_\ell d_{(s, \ell)} \\ &= d_{(s, \ell)} (d_{(\ell, e)} - a_\ell) = -a_e d_{(s, \ell)} \end{aligned}$$

For indexes e, ℓ, s , and t , one has

$$\begin{aligned} \frac{d_{(t, e)}}{d_{(t, \ell)}} - \frac{d_{(s, e)}}{d_{(s, \ell)}} &= \frac{d_{(t, e)} d_{(s, \ell)} - d_{(s, e)} d_{(t, \ell)}}{d_{(t, \ell)} d_{(s, \ell)}} \\ &= \frac{1}{d_{(t, \ell)} d_{(s, \ell)}} \cdot \begin{vmatrix} d_{(t, t)} & d_{(t, \ell)} \\ d_{(s, e)} & d_{(s, \ell)} \end{vmatrix} \\ &= \frac{1}{d_{(t, \ell)} d_{(s, \ell)}} \cdot \begin{vmatrix} d_{(t, e)} - d_{(s, e)} & d_{(t, \ell)} - d_{(s, \ell)} \\ d_{(s, e)} & d_{(s, \ell)} \end{vmatrix} \\ &= \frac{1}{d_{(t, \ell)} d_{(s, \ell)}} \cdot \begin{vmatrix} d_{(t, s)} & d_{(t, s)} \\ d_{(s, e)} & d_{(s, \ell)} \end{vmatrix} \\ &= \frac{d_{(t, s)}}{d_{(t, \ell)} d_{(s, \ell)}} \cdot \begin{vmatrix} 1 & 1 \\ d_{(s, e)} & d_{(s, \ell)} \end{vmatrix} \\ &= \frac{d_{(t, s)} (d_{(s, \ell)} - d_{(s, e)})}{d_{(t, \ell)} d_{(s, \ell)}} = \frac{d_{(t, s)} d_{(\ell, e)}}{d_{(t, \ell)} d_{(s, \ell)}}. \end{aligned}$$

□

For any integer $n \geq 1$, define $(n+1) \times (n+1)$ matrix A_n as:

$$A_n := \begin{bmatrix} 1 & \frac{a_{i_1}}{d(i_1, e_1)} & \frac{a_{i_1}}{d(i_1, e_2)} & \cdots & \frac{a_{i_1}}{d(i_1, e_n)} \\ 1 & \frac{a_{i_2}}{d(i_2, e_1)} & \frac{a_{i_2}}{d(i_2, e_2)} & \cdots & \frac{a_{i_2}}{d(i_2, e_n)} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \frac{a_{i_n}}{d(i_n, e_1)} & \frac{a_{i_n}}{d(i_n, e_2)} & \cdots & \frac{a_{i_n}}{d(i_n, e_n)} \\ 1 & \frac{a_{i_{n+1}}}{d(i_{n+1}, e_1)} & \frac{a_{i_{n+1}}}{d(i_{n+1}, e_2)} & \cdots & \frac{a_{i_{n+1}}}{d(i_{n+1}, e_n)} \end{bmatrix},$$

where $\{a_{i_1}, \dots, a_{i_{n+1}}\}$ and $\{a_{e_1}, \dots, a_{e_n}\}$ are disjoint subsets of the infinite sequence $\{a_\ell\}_{\ell=1}^\infty$. The following proposition gives the determinant of A_n . We will use Lemma 3.1 in its proof.

Proposition 3.2. *Let $I = \{1, 2, \dots, n\}$ and $J = \{1, 2, \dots, n+1\}$. Then, one has*

$$|A_n| = \frac{D_r \cdot \prod_{s' < s \in J} d(i_s, i_{s'})}{\prod_{s \in J} \prod_{j \in I} d(i_s, e_j)}.$$

Proof. Subtracting first row from others and using lemma (3.2), gives that

$$\begin{aligned} |A_n| &= \begin{vmatrix} 1 & \frac{a_{i_1}}{d(i_1, e_1)} & \cdots & \frac{a_{i_1}}{d(i_1, e_n)} \\ 0 & \frac{a_{i_2} d(i_1, e_1) - a_{i_1} d(i_2, e_1)}{d(i_1, e_1) d(i_2, e_1)} & \cdots & \frac{a_{i_2} d(i_1, e_n) - a_{i_1} d(i_2, e_n)}{d(i_1, e_n) d(i_2, e_n)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{a_{i_{n+1}} d(i_1, e_1) - a_{i_1} d(i_{n+1}, e_1)}{d(i_1, e_1) d(i_{n+1}, e_1)} & \cdots & \frac{a_{i_{n+1}} d(i_1, e_n) - a_{i_1} d(i_{n+1}, e_n)}{d(i_1, e_n) d(i_{n+1}, e_n)} \end{vmatrix} \\ &= \begin{vmatrix} 1 & \frac{-a_{i_1}}{d(i_1, e_1)} & \cdots & \frac{-a_{i_1}}{d(i_1, e_n)} \\ 0 & \frac{-a_{e_1} d(i_2, i_1)}{d(i_1, e_1) d(i_2, e_1)} & \cdots & \frac{-a_{e_n} d(i_2, i_n)}{d(i_1, e_n) d(i_2, e_n)} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{-a_{e_1} d(i_{n+1}, i_1)}{d(i_1, e_1) d(i_{n+1}, e_1)} & \cdots & \frac{-a_{e_n} d(i_{n+1}, i_n)}{d(i_1, e_n) d(i_{n+1}, e_n)} \end{vmatrix} \end{aligned}$$

By extracting the factor $-a_{e_j}/d(i_1, e_j)$ from each columns ($1 \leq j \leq n$) and $d(i_s, i_1)$ from each rows ($2 \leq s \leq n+1$), one gets that

$$|A_n| = (-1)^n \prod_{j=1}^n \frac{a_{e_j}}{d(i_1, e_j)} \cdot \prod_{s=2}^{n+1} d(i_s, i_1) \cdot |B_n|$$

where

$$B_n := \begin{bmatrix} \frac{1}{d(i_2, e_1)} & \frac{1}{d(i_2, e_2)} & \cdots & \frac{1}{d(i_2, e_n)} \\ \frac{1}{d(i_3, e_1)} & \frac{1}{d(i_3, e_2)} & \cdots & \frac{1}{d(i_3, e_n)} \\ \vdots & \cdots & \vdots & \vdots \\ \frac{1}{d(i_{n+1}, e_1)} & \frac{1}{d(i_{n+1}, e_2)} & \cdots & \frac{1}{d(i_{n+1}, e_n)} \end{bmatrix}.$$

Since the matrix B_n is a Cauchy's matrix defined by

$$x_1 = a_{i_2}, \dots, x_n = a_{i_{n+1}}, \quad y_1 = a_{e_1}, \dots, y_n = a_{e_n},$$

so using Proposition 2.1 we have

$$|B_n| = \frac{\prod_{s' < s \in \{2, \dots, n+1\}} d(i_s, i_{s'}) \cdot \prod_{i < j \in J} d(e_j, e_i)}{\prod_{s=2}^{n+1} \prod_{j=1}^n d(i_s, e_j)},$$

and hence,

$$|A_n| = \frac{(-1)^n \prod_{j=1}^n \frac{a_{e_j}}{d(i_1, e_j)} \cdot \prod_{s' < s \in J} d(i_s, i_{s'})}{\prod_{s=1}^{n+1} \prod_{j=1}^n d(i_s, e_j)} = \frac{D_r \cdot \prod_{s' < s \in J} d(i_s, i_{s'})}{\prod_{s=1}^{n+1} \prod_{j=1}^n d(i_s, e_j)}.$$

□

We note that the matrix B_n in the proof of the above proposition is related to a certain $n \times n$ Toeplitz matrix. Indeed, if we consider the sequence $a_\ell = 1/\ell$ for $\ell = 1, 2, \dots$ and indexes $e_j = j$ and $i_s = n + s - 1$ for $j = 1, \dots, n$ and $s = 1, \dots, n + 1$, then a simple calculation shows that $B_n = (-1)^n (2n)! V_n$, where V_n is the following $n \times n$ Toeplitz matrix

$$V_n = \begin{bmatrix} \frac{1}{n} & \frac{1}{n-1} & \frac{1}{n-2} & \cdots & \frac{1}{2} & 1 \\ \frac{1}{n+1} & \frac{1}{n} & \frac{1}{n-1} & \cdots & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{n+2} & \frac{1}{n+1} & \frac{1}{n} & \cdots & \frac{1}{4} & \frac{1}{3} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{1}{2n-2} & \frac{1}{2n-3} & \frac{1}{2n-4} & \cdots & \frac{1}{n} & \frac{1}{n-1} \\ \frac{1}{2n-1} & \frac{1}{2n-2} & \frac{1}{2n-3} & \cdots & \frac{1}{n+1} & \frac{1}{n} \end{bmatrix} = (-1)^k \mathbf{H}_n,$$

where $k = n/2$ if n is even and $k = (n-1)/2$ if n is odd; and the last equality comes by changing j -th column with $(n-j+1)$ -th column of V_n .

4. PROOF OF THEOREM 1.1

In order to prove Theorem 1.1, we need the following result.

Proposition 4.1. *Given integers $2 \leq r < n$ satisfying $r+1 \leq n-r$, consider the indexes $r+1 \leq i_1, \dots, i_{r+1} \leq n$ and let $I = \{1, 2, \dots, r\}$ and $J = \{1, \dots, r+1\}$. Then, the matrix $C' := [C_{i_s-r, j}]$, where $s \in J$ and $0 \leq j \leq r$, has nonzero determinant as*

$$|C'| = (-1)^{r^2+3r} D_r^{r+1} \prod_{s' < s \in J} d(i_s, i_{s'}).$$

Proof. Extracting the factor $(-1)^r \prod_{e < \ell \in I} d(\ell, e)$ and $(-1)^{r+j} \prod_{e < \ell \in I_j} a_\ell d(\ell, e)$, respectively, from first column and the j -th column for $s \in J$ and $2 \leq j \leq r$, where $I_j = I \setminus \{j\}$, gives that

$$\begin{aligned} |C'| &= (-1)^{3r(r+1)/2} \prod_{e < \ell \in I} d(\ell, e) \prod_{e < \ell \in I_1} a_\ell d(\ell, e) \prod_{e < \ell \in I_r} a_\ell d(\ell, e) \cdot |C''| \\ &= (-1)^{r(r+3)/2} D_r^r \cdot |C''|, \end{aligned}$$

where C'' is the following $(r+1) \times (r+1)$ matrix

$$C'' := \begin{bmatrix} \prod_{\ell \in I} d_{(i_1, \ell)} & a_{i_1} \prod_{\ell \in I_1} d_{(i_1, \ell)} & \cdots & a_{i_1} \prod_{\ell \in I_r} d_{(i_1, \ell)} \\ \prod_{\ell \in I} d_{(i_2, \ell)} & a_{i_2} \prod_{\ell \in I_1} d_{(i_2, \ell)} & \cdots & a_{i_2} \prod_{\ell \in I_r} d_{(i_2, \ell)} \\ \vdots & \cdots & \vdots & \vdots \\ \prod_{\ell \in I} d_{(i_{r+1}, \ell)} & a_{i_{r+1}} \prod_{\ell \in I_1} d_{(i_{r+1}, \ell)} & \cdots & a_{i_{r+1}} \prod_{\ell \in I_r} d_{(i_{r+1}, \ell)} \end{bmatrix}.$$

By extracting the factor $\prod_{\ell \in I} d_{(i_s, \ell)}$ from s -th row $1 \leq s \leq r+1$, we obtain

$$|C''| = \prod_{s \in J} \prod_{j \in I} d_{(i_s, j)} \cdot \begin{vmatrix} 1 & \frac{a_{i_1}}{d_{(i_1, 1)}} & \cdots & \frac{a_{i_1}}{d_{(i_1, r)}} \\ 1 & \frac{a_{i_2}}{d_{(i_2, 1)}} & \cdots & \frac{a_{i_2}}{d_{(i_2, r)}} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \frac{a_{i_{r+1}}}{d_{(i_{r+1}, 1)}} & \cdots & \frac{a_{i_{r+1}}}{d_{(i_{r+1}, r)}} \end{vmatrix}$$

Considering $t = r$ and $e_j = j$ for $j = 1, \dots, r$, and using Proposition 3.2 for calculating the last determinant, one can conclude that

$$\begin{aligned} |C'| &= (-1)^{r(r+3)/2} D_r^{r-1} \prod_{s \in J} \prod_{j \in I} d_{(i_s, \ell)} \cdot \frac{D_r \cdot \prod_{s' < s \in J} d_{(i_s, i_{s'})}}{\prod_{s \in J} \prod_{j \in I} d_{(i_s, j)}} \\ &= (-1)^{r^2+3r} D_r^{r+1} \prod_{s' < s \in J} d_{(i_s, i_{s'})}. \end{aligned}$$

□

We notice that above proposition is a special case of the next general one.

Proposition 4.2. *Given integers $2 \leq r < n$, and $m \leq \min\{r+1, n-r\}$, any $m \times m$ sub-matrix of the matrix $C = [C_{(i-r, j)}]$ has non-zero determinant, where $r+1 \leq i \leq n$ and $0 \leq j \leq r$; therefore C has maximal rank equal to $\min\{n-r, r+1\}$.*

Proof. We may assume that $r+1 \leq n-r$, the other case is similar. For $m \leq r+1$, we denote by C_m any $m \times m$ sub-matrix of C . By proposition 4.1, the determinant of C_m is nonzero for $m = r+1$. Thus, we may suppose that $m < r+1$ and $C_m = [C_{(i_s-r, j_t)}]$, where $r+1 \leq i_s \leq n-r$, $0 \leq j_{s'} \leq r$ for $1 \leq s, s' \leq m$. If we suppose that $0 = j_1 < j_2, \dots, j_m$, then

$$C_m = \begin{bmatrix} C_{(i_1-r, 0)} & C_{(i_1-r, j_2)} & \cdots & C_{(i_1-r, j_m)} \\ C_{(i_2-r, 0)} & C_{(i_2-r, j_2)} & \cdots & C_{(i_2-r, j_m)} \\ \vdots & \vdots & \cdots & \vdots \\ C_{(i_m-r, 0)} & C_{(i_m-r, j_2)} & \cdots & C_{(i_m-r, j_m)}. \end{bmatrix},$$

such that

$$\begin{aligned} C_{(i_s-r, 0)} &= (-1)^r \prod_{e < \ell \in I} d_{(i_s, \ell)} d_{(\ell, e)}, \\ C_{(i_s-r, j_{s'})} &= (-1)^{r+j_{s'}} a_{i_s} \prod_{e < \ell \in I_{j_{s'}}} a_{\ell} d_{(i_s, \ell)} d_{(\ell, e)}, \end{aligned}$$

where $I = \{1, 2, \dots, r\}$ and $I_{j_{s'}} = I \setminus \{j_{s'}\}$. Extracting $(-1)^r \prod_{e < \ell \in I} d_{(\ell, e)}$ and $(-1)^{r+j_{s'}} \prod_{e < \ell \in I_{j_{s'}}} a_{\ell} d_{(\ell, e)}$ from first and s' -th columns, respectively, and then $\prod_{\ell \in I} d_{(i_s, \ell)}$ from s -th row for $1 \leq s < s' \leq m$, gives that

$$|C_m| = (-1)^{r'} \prod_{e < \ell \in I} d_{(\ell, e)} \cdot \prod_{t=2}^m \prod_{e < \ell \in I_{j_t}} a_{\ell} d_{(\ell, e)} \cdot \prod_{s=1}^m \prod_{\ell \in I} d_{(i_s, \ell)}$$

$$\times \begin{vmatrix} 1 & \frac{a_{i_1}}{d_{(i_1, j_2)}} & \cdots & \frac{a_{i_1}}{d_{(i_1, j_m)}} \\ 1 & \frac{a_{i_2}}{d_{(i_2, j_2)}} & \cdots & \frac{a_{i_2}}{d_{(i_2, j_m)}} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \frac{a_{i_m}}{d_{(i_m, j_2)}} & \cdots & \frac{a_{i_m}}{d_{(i_m, j_m)}} \end{vmatrix},$$

where $r' = mr + j_2 + \dots + j_m$ and the above is nonzer by Propositions 3.2. Otherwise, if suppose that $1 \leq j_1 < j_2 < \dots < j_m$, then extracting the factor $(-1)^{r+j_{s'}} a_{i_s} \prod_{e < \ell \in I_{j_{s'}}} a_{\ell} d_{(\ell, e)}$ from s' -th column, and then $\prod_{\ell \in I} d_{(i_s, \ell)}$ from s -th row of the matrix $C_m = [C_{i_s - r, j_{s'}}]$, where $1 \leq s, s' \leq m$, gives that

$$|C_m| = (-1)^{r''} \prod_{s, t=1}^m \prod_{e < \ell \in I_{j_t}} a_{\ell} d_{(\ell, e)} d_{(i_s, \ell)}$$

$$\times \begin{vmatrix} \frac{1}{d_{(i_1, j_1)}} & \frac{1}{d_{(i_1, j_2)}} & \cdots & \frac{1}{d_{(i_1, j_m)}} \\ \frac{1}{d_{(i_2, j_1)}} & \frac{1}{d_{(i_2, j_2)}} & \cdots & \frac{1}{d_{(i_2, j_m)}} \\ \vdots & \cdots & \vdots & \vdots \\ \frac{1}{d_{(i_m, j_1)}} & \frac{1}{d_{(i_m, j_2)}} & \cdots & \frac{1}{d_{(i_m, j_m)}} \end{vmatrix},$$

where $r'' = mr + j_1 + \dots + j_m$ and the last determinant is nonzero by Propositions 2.1. This completes the proof of the proposition. \square

Now we are ready to prove the main theorem 1.1, using the above results.

Proof. For integers $2 \leq r < n$, recall that $\mathbf{C}_r^n := [C|D]$ is a $(n-r) \times (n+1)$ blocked matrix, where $C = [C_{(i-r, j)}]$ is $(n-r) \times (r+1)$ matrix defined as in the first section and D is a $(n-r) \times (n-r)$ diagonal matrix with entries D_r . By Proposition 4.2, any $m \times m$ sub-matrix of the matrix C has non-zero determinant and C has maximal rank equal to $\min\{n-r, r+1\}$. It is clear that the matrix D has full rank equal to $n-r$. By exchanging the columns, if it is necessary, one can see that any $(n-r) \times (n-r)$ submatrix of \mathbf{C}_r^n is a diagonal blocked matrix with blocks equal to D_r or $m \times m$ submatrices of C with $1 \leq m \leq \min\{n-r, r+1\}$, which have non-zero determinant. Therefore, any $(n-r) \times (n-r)$ submatrix of \mathbf{C}_r^n has nonzero determinant, and hence it has maximal rank $n-r$, as desired. \square

REFERENCES

- [1] Hilbert D. Ein betrag zur theorie des Legendre'schen polynoms. Acta Mathematica, Vol. 18, 155-159, (1894).

- [2] Choi M-D. Tricks or Treats with the Hilbert Matrix. Amer. Math. Month., Vol. 90, No. 5, 301-312, 1983.
- [3] Li HSUAN-CHU On Calculating the Determinants of Toeplitz Matrices. Journal of Applied Mathematics and Bioinformatics, Vol. 1, No. 1, 55-64 (2011).
- [4] Cauchy AL. Mémoire sur les fonctions alternées et sur les somme alternées. Exercices d Analyse et de Phys. Math., Vol. II, 151-159, (1841).
- [5] Pólya G, Szego G. Zweiter Band. Springer, Berlin, Vol., (1925).
- [6] Salami S. Rational points on a certain family of complete intersection varieties. Under Preparation (2019).
- [7] Lang S. Number Theory III: Survey of Diophantine Geometry. Encyclopaedia of Mathematical Sciences, Springer, Berlin, Vol. 60, (1991).
- [8] Sloane N.J.A. The On-Line Encyclopedia of Integer Sequences. <http://oeis.org>. Sequence A005249.
- [9] Davis PH.J. Interpolation and approximation. Dover Publication Inc., New-Yourk (NY) (1975).
- [10] Fiedle M. Notes on Hilbert and Cauchy matrices, Linear Algebra and its Applications, Vol. 432, 351-356, (2010).
- [11] Rietsch K. Totally positive Toeplitz matrices and quantum cohomology of partial flag varieties. J. Amer. Math. Soc., Vol. 16, no. 2. 2003. p. 363-392.
- [12] Englis M. Toeplitz operators and group representations. J. Fourier Anal. Appl., Vol. 13, no. 3, 243-265, (2007).
- [13] Euler R. Characterizing bipartite Toeplitz graphs. Theoret. Comput. Sci., Vol. 263, no. 1-2, 47-58, (2001).
- [14] Bunger F. Inverse, determinants, eigenvalues, and eigenvectors of real symmetric Toeplitz matrices with linearly increasing entrie. Linear Algebra and its Applications, Vol. 459, 595-619, (2014).
- [15] Ye KE, Lim LH. Every Matrix is a Product of Toeplitz Matrices. Found. Comput. Math., Vol. 16, no. 1-2, 577-598, (2016).

(Sajad Salami) INSTITUTO DA MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE ESTADUAL DO RIO DO JANEIRO, BRAZIL

E-mail address, Sajad Salami: sajad.salami@ime.uerj.br

URL: <https://sites.google.com/a/ime.uerj.br/sajadsalami/>