Sects, rooks, pyramids, partitions and paths for type $DIII$ clans

Aram Bingham[1] and Özlem Uğurlu[2]

[1]Tulane University, New Orleans; abingham@tulane.edu

[2]Palm Beach State College, Boca Raton; ugurluo@palmbeachstate.edu

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Abstract

Borel subgroup orbits of the classical symmetric space SO_{2n}/GL_n are parametrized by (n, n)-clans of type $DIII$. We describe explicit bijections between such clans, diagonally symmetric rook placements, certain pairs of minimally intersecting set partitions, and a class of weighted Delannoy paths. Then we group the clans into "sects" corresponding to Schubert cells of the orthogonal Grassmannian and use this to conjecture on the closure order of the largest sect. Clans of the largest sect are in bijection with fixed-point-free partial involutions.

Keywords: Bruhat order, lattice paths, rook placements, fixed-point-free partial involutions.

MSC: 05A15, 05A19, 14M15

1 Introduction

This paper is a continuation of the program described in [2], to understand the Borel subgroup orbits of classical symmetric spaces in which the fixed-point subgroup is a Levi factor of a parabolic subgroup. This is the third of three cases in which this occurs; similar analysis was performed for type $AIII$ ($SL_{p+q}/S(GL_p \times GL_q)$) in [3] and [6], and for type CI (Sp_{2n}/GL_n) in [2]. The symmetric space of type $DIII$ refers to the quotient SO_{2n}/GL_n, where GL_n is realized as the fixed point subgroup of an order two automorphism of SO_{2n}. Note that all groups are taken to be over the complex numbers.

For B, a Borel subgroup of SO_{2n}, the B-orbits in SO_{2n}/GL_n are parameterized by combinatorial objects we call *type $DIII$ (n, n)-clans*. After setting down some notation and terminology in Section 2, our first result is Proposition 3.3, which gives an explicit formula for the number of type $DIII$ (n, n)-clans.

Type $DIII$ clans can be thought of as subsets of clans which parameterize Borel orbits in the type CI and $AIII$ cases, and it turns out that the additional restrictiveness gives combinatorial coincidences with other well-studied families of objects. The first of these involves the number of inequivalent placements of $2n$ non-attacking rooks on a $2n \times 2n$ board with symmetry across each of the main diagonals, which was written about in the classic text [12]. A bijection between type $DIII$ (n, n)-clans and such rook placements is given in Section 3.2, by extracting a triangular portion of the square board and analyzing this resulting "pyramid."

These pyramids also make it easy to describe a (near) bijection with objects studied by Pittel in [16]. Precisely, these are ordered pairs (p, p') of partitions of an n-element set such that p consists of exactly two blocks. This map is described in Section 3.3.

The symmetric subgroup GL_n can be realized as the Levi factor of a maximal parabolic subgroup P such that SO_{2n}/P is $\mathrm{OGr}(n, \mathbb{C}^{2n})$, the *orthogonal Grassmannian* of maximal (n-dimensional) isotropic subspaces of $\mathbb{C}^{2n}$. This gives us a canonical projection map $\pi : SO_{2n}/GL_n \to \mathrm{OGr}(n, \mathbb{C}^{2n})$. Borel orbits in Grassmann varieties can be parameterized by lattice paths which are also a tool for understanding their geometry. As a step towards extending these ideas to the symmetric space above, we present another bijection between (n, n)-clans and certain *weighted Delannoy paths* in Section 3.4, completing the initial combinatory analysis of type $DIII$ clans.

Section 4 supplies background on the weak order on type $DIII$ clans and works out an example in detail. This order gives the (n, n)-clans the structure of a graded poset, whose rank function (which we call its *length function*, referencing the length of an underlying involution) is presented in the following section. Also in Section 5, we give a recurrence relation for the length generating polynomials of this poset.

In Section 6, we give background on isotropic flags and parabolic subgroups of SO_{2n} so that we can understand the projection map π, and decompose the pre-image of a Schubert cell as a collection of clans called a *sect*. From results of [3], the sects provide a cell decomposition and basis for (co)homology of SO_{2n}/GL_n. To describe the sects, we take the equivalent view of type $DIII$ clans as parameterizing GL_n-orbits in the component of the isotropic flag variety identified with SO_{2n}/B, and construct representative flags for each orbit.

Finally, we look at the pre-image of the dense Schubert cell of SO_{2n}/GL_n, which we call the *big sect*. We prove that the clans of the big sect are in bijection with the set of partial fixed-point-free involutions of an n-element set, denoted $\mathcal{PF}_n$. The elements of $\mathcal{PF}_n$ parametrize congruence orbits of the upper triangular invertible matrices on the set of skew-symmetric matrices, as described in [8]. Equipped with the closure order of the orbits of that action, they form a poset which has been studied in [4]. We conjecture that the closure order on Borel orbits corresponding to type $DIII$ clans of the largest sect is isomorphic to this order.

2 Notation and Preliminaries

Let n be a positive integer. First, we describe our realization of SO_{2n}; all linear algebraic groups that follow are taken to be over the field of complex numbers. In this manuscript, we follow most of the notations of [19].

Let J_{2n} denote a $2n \times 2n$ matrix with 1's along the anti-diagonal and 0's elsewhere. Then, we set

$$SO_{2n} := \{g \in GL_{2n} \mid g^t J_{2n} g = J_{2n}\}.$$

Let $\text{int}(g) : GL_{2n} \to GL_{2n}$ denote the map defined by

$$\text{int}(g)(h) = ghg^{-1}.$$

Now define the matrix

$$I_{n,n} := \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Then we check that we have an involution θ on GL_{2n} defined by

$$\theta := \text{int}(iI_{n,n}).$$

Since $(iI_{n,n})^{-1} = -iI_{n,n}$, then if

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is the $n \times n$ block form of g, we have

$$\theta(g) = \begin{pmatrix} iI_n & 0 \\ 0 & -iI_n \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} -iI_n & 0 \\ 0 & iI_n \end{pmatrix} = \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}.$$

Observe that the restriction of θ to SO_{2n} induces an involution on that group as well. The fixed points of this involution must be block diagonal, that is

$$\theta(g) = g \iff g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}.$$

Furthermore, membership in the special orthogonal group forces $D = J_n(A^{-1})^t J_n$. Thus, A can be any invertible $n \times n$ matrix and this completely determines g, so the fixed point subgroup is isomorphic to GL_n.

The symmetric group of permutations of $[n] := \{1, \ldots, n\}$ is denoted by $\mathcal{S}_n$. If $\pi \in \mathcal{S}_n$, then its one-line notation is the string $\pi_1\pi_2\ldots\pi_n$, where $\pi_i = \pi(i)$ for $1 \leq i \leq n$. For instance, $\pi = 164578329$ is the one-line notation for the permutation $\pi \in \mathcal{S}_9$ with cycle decomposition $1\ (2, 6, 8)(3, 4, 5, 7)\ 9$.

An involution is an element of $\mathcal{S}_n$ of order at most two, and the set of involutions in $\mathcal{S}_n$ is denoted by $\mathcal{I}_n$. We write involutions $\pi \in \mathcal{I}_n$ in cycle notation as

$$\pi = (a_1, b_1)(a_2, b_2)\ldots(a_k, b_k)d_1 \ldots d_{n-2k},$$

where $a_i < b_i$ for all $1 \leq i \leq k$, $a_1 < a_2 < \cdots < a_k$, and $d_1 < \cdots < d_{n-2k}$.

Next, we define clans.

Definition 2.1. Let p and q be two positive integers such that $q \leq p$. A (p, q)*-clan* γ is a string of symbols from $\mathbb{N} \cup \{+, -\}$ such that

1. there are $p - q$ more $+$'s than $-$'s;

2. if a natural number appears in γ, then it appears exactly twice.

For example, $12+21$ is a $(3, 2)$ clan and $+1+1$ is a $(3, 1)$ clan. Clans γ and γ' are considered to be equivalent if the positions of the matching number pairs are the same in both of them. For example, $\gamma = 1122$ and $\gamma' = 2211$ are the same $(2, 2)$-clan, since both of γ and γ' have matching numbers in positions $(1, 2)$ and in positions $(3, 4)$.

Let γ be a clan of the form $\gamma = c_1 \cdots c_n$. The *reverse* of γ, denoted by $rev(\gamma)$, is the clan

$$rev(\gamma) := c_n c_{n-1} \cdots c_1.$$

We obtain the *negative* of γ, denoted by $-\gamma$, by changing all $+$'s in γ to $-$'s, and vice versa, leaving the natural numbers unchanged. Now, we define symmetric and skew-symmetric clans.

Definition 2.2. A (p, q)-clan γ is called *symmetric* if

$$\gamma = rev(\gamma),$$

and is called *skew-symmetric* if

$$\gamma = -rev(\gamma).$$

Let us illustrate these definitions with the following example.

Example 2.3. Consider the clan $\gamma = +-123312+-$. Its negative reverse is $-+213321-+$. Since $\gamma = -rev(\gamma)$, it is a skew-symmetric $(5, 5)$-clan.

The clan $\gamma_1 = 1234545321$ is a skew-symmetric $(5, 5)$-clan which is also symmetric, as it contains no $\pm$ symbols.

Clans originated in [14] to parametrize symmetric subgroup orbits in complex flag manifolds of classical type. The notation for clans has morphed with their development through subsequent works, notably [21], [19], and [6]. Let us finish this section by defining signed involutions, which provide an alternative manner of presenting clans.

Definition 2.4. A *signed* (p, q)*-involution* is an involution $\pi \in \mathcal{I}_{p+q}$ together with an assignment of $+$ and $-$ signs to the fixed points of π such that there are $p - q$ more $+$'s than $-$'s, where $q \leq p$.

For example, $\pi = (1\ 8)(2\ 4)(3^+)(5^-)(6^-)(7^+)$ is a signed $(4, 4)$-involution which can be regarded as the $(4, 4)$-clan $12+2--+1$. This is accomplished by placing matching natural numbers at the positions that appear in each transposition, and placing the signature $+$ or $-$ at the position of each signed fixed point. Observe here that p is equal to the number of fixed points in π with a $+$ sign attached plus the number of two-cycles in π, while q is equal to the number of fixed points in π with a $-$ sign attached plus the number of two-cycles in π.

3 Counting Type $DIII$ Clans

3.1 A Formula

Here, we record a formula for the number of (n, n)-clans of type $DIII$, that is those which parametrize the Borel subgroup orbits in the classical symmetric space SO_{2n}/GL_n. These are (n, n)-clans $\gamma = c_1 \cdots c_{2n}$ satisfying the additional conditions (see [19] for details):

1. γ is skew-symmetric, that is $\gamma = -rev(\gamma)$;
2. if $c_i \in \mathbb{N}$, then $c_i \neq c_{2n+1-i}$;
3. the total number of $-$'s and pairs of matching natural numbers among $c_1 \cdots c_n$ is even.

Let $\Delta(n)$ denote the set of (n, n)-clans of type $DIII$ and Δ_n its cardinality. Let $\delta_{k,n}$ denote the number of such clans which contain k pairs of natural numbers.

Lemma 3.1. *If $\gamma = c_1 \cdots c_{2n}$ is an (n, n)-clan with k pairs of natural numbers among its symbols, then k is even.*

Proof. Suppose $c_i = c_j = a \in \mathbb{N}$. By property 2 of type $DIII$ clans, $j \neq 2n+1-i$. Since γ is skew-symmetric, upon taking the negative transpose we have that $a \neq c_{2n+1-i} = c_{2n+1-j} = b \in \mathbb{N}$. Thus, each natural number pair in γ has an "opposing" natural number pair with which it exchanges place upon reversal. Since natural number pairs then come in twos (so accounting for four symbols c_i in total), k is even. ☐

From now on, we are free to write $k = 2r$ for some $r \in \mathbb{N}$, when referring to this statistic. For a given type $DIII$ clan $\gamma = c_1 \cdots c_{2n}$, we will partition the natural number pairs into two sets. Let

$$\Pi_0 = \{(c_i, c_j) \mid c_i = c_j \in \mathbb{N} \text{ and } 1 \leq i \leq n < j \leq 2n\},$$

and

$$\Pi_1 = \{(c_i, c_j) \mid c_i = c_j \in \mathbb{N} \text{ and } 1 \leq i < j \leq n \text{ or } n+1 \leq i < j \leq 2n\}.$$

It is easy to see that if a natural number pair (c_i, c_j) is in either one of these sets, then its opposing pair is in the same set. Then we can write $|\Pi_0| = 2p$ and $|\Pi_1| = 2q$, and $2p + 2q = 2r$. Next, observe that if γ contains k pairs of natural numbers, then exactly k of the symbols among $c_1 \cdots c_n$ will be natural numbers, by the skew-symmetry condition.

In order to count Δ_n, first we will count $\delta_{k,n}$. To do this, we will sum over the combinations of p and q that give a fixed k.

Lemma 3.2. *With the above notation,*

$$\delta_{k,n} = 2^{n-k-1} \binom{n}{k} \frac{k!}{\frac{k}{2}!}. \tag{3.1}$$

Proof. There are $\binom{n}{k}$ choices for where to place natural numbers among $c_1 \cdots c_n$, and by skew-symmetry, each choice determines placement among $c_{n+1} \cdots c_{2n}$, so one needs only to specify which symbols are natural numbers for the "first half" of the clan. This accounts for the binomial coefficient in the equation (3.1).

Next, notice that for fixed p, there are $2p$ distinct natural numbers without pairs among $c_1 \cdots c_n$, and $r - p = q$ (type Π_1) pairs of natural numbers among $c_1 \cdots c_n$. Hence, condition 3 for type $DIII$ clans says that q plus the number of minus signs among $c_1 \cdots c_n$ is even. When, q is even, the number of minus signs in the first half is even, and when q is odd, the number of minus signs odd.

Once the positions of the k natural numbers are fixed, one can view the possible placements of plus and minus signs among the remaining symbols of $c_1 \cdots c_n$ as determining subsets of a size $n-k$ set, where $-$ denotes membership in the subset. Then, when q is even or odd, the possible arrangements for $+$ and $-$ symbols enumerate subsets of a size $n - k$ set which have even or odd cardinality, respectively. It is well known that the subsets of even and odd cardinality are equinumerous in the power set of any finite set. Thus, for any p (and so any q), there are 2^{n-k-1} possible placements of $+$ and $-$ among the $n-k$ symbols of $c_1 \cdots c_n$ which are not natural numbers. Since this placement determines the $+$ and $-$ pattern among $c_{n+1} \cdots c_{2n}$, we have accounted for the term 2^{n-k-1}.

Recall the following basic fact from combinatorics: there are $\binom{2m}{m}\frac{m!}{2^m} = \frac{(2m)!}{m!2^m}$ ways to form m pairs from $2m$ objects. Indeed, there are $\binom{2m}{m}$ ways to choose the first element in every pair, then $m!$ ways to choose mates for each of these. Then one divides by 2^m to account for the fact that it doesn't matter which element in each pair is chosen first.

The next step is to form $\frac{k}{2} = r$ pairs from the k positions of natural number symbols among $c_1 \cdots c_n$. For each of these r pairs (i, j), one then decides whether c_i and c_j are distinct natural numbers which are the first entries of opposing Π_0 pairs, or whether $(c_i, c_j) \in \Pi_1$. Thus, we multiply $\frac{2r!}{r!2^r}$ by another factor of 2^r, yielding the last term $\frac{k!}{\frac{k}{2}!}$ of equation (3.1). $\square$

Using $k = 2r$, we have the following count for Δ_n.

Proposition 3.3. *The number of (n, n)-clans of type $DIII$ is*

$$\Delta_n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2r-1} \frac{n!}{r!(n-2r)!}. \tag{3.2}$$

Proof. The formula comes from rewriting the equation (3.1) in terms of r, expanding the binomial coefficient, and then summing over possible values for r. $\square$

The first values of Δ_n, beginning with $n = 1$, are 1, 3, 10, 38, 156, 692, 3256,... This is in fact the number of inequivalent placements of $2n$ rooks on a $2n \times 2n$ board having symmetry across each diagonal ([15], A000902), as we will show next.

3.2 Rooks and Pyramids

Given an $n \times n$ grid of squares (modeling an $n \times n$ chess board) the rook problem asks how many ways n rooks can be placed on the board so that none can attack any other. Necessarily, each placement will exhibit exactly one rook in each row and each column. In [12], Lucas refines this question to ask how many placements possess symmetry with respect to a given subgroup of the dihedral group D_8, which acts as the symmetry group of the board. We are interested in rook placements which are invariant under reflection across each main diagonal d and d' as depicted in the figure below.

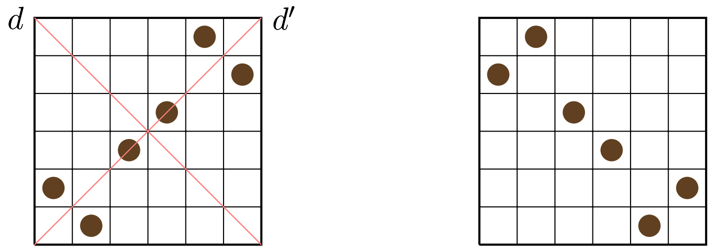


Figure 3.1: A diagonally invariant 6×6 rook placement and its rotational equivalent.

Furthermore, we are only interested in these placements up to equivalence, where two placements are said to be equivalent if there is an element of D_8 that transforms one to the other. The next claims will help us with this count; they are stated without proof in [12] and [18], but a thorough proof is hopefully welcome.

Proposition 3.4. *When $n \geq 2$, there is no placement of n rooks on an $n \times n$ board with symmetry group D_8.*

Proof. Recall that D_8 is generated by two elements F and R, satisfying the relations $F^2 = R^4 = (FR)^2 = e$, where e is the identity element of the group. Geometrically, we can consider F to be the reflection of the square board across any of the four lines which bisect opposite sides or opposite angles, and we can take R to be a rotation by $\pi/2$. Here, we will take F to be the reflection across the diagonal d (see Figure 3.1), and R to be a counter-clockwise rotation. Under this assignment, reflection across d' has the formula $R^{-1}FR$.

Let's introduce some notation for the board spaces to keep track of the effect of D_8 on rook placements. We label each space where a rook can be placed on the board as $b_{i,j}$, with i and j ranging from 1 to n from lower left to upper right. It may be helpful to think of the underlying board as stationary and of D_8 as acting on the labels of this initial configuration. One sees easily from the Figure 3.2 that

$$F \cdot b_{i,j} = b_{n+1-j,n+1-i}, \qquad R \cdot b_{i,j} = b_{j,n+1-i}, \qquad \text{and} \qquad R^{-1}FR \cdot b_{i,j} = b_{j,i}.$$

A rook placement ρ is given by a set of n indexed spaces, $\rho = \{b_{i_1,j_1}, \ldots, b_{i_n,j_n}\}$ such that

$$(i_k = i_l \text{ or } j_k = j_l) \implies k = l. \tag{3.3}$$

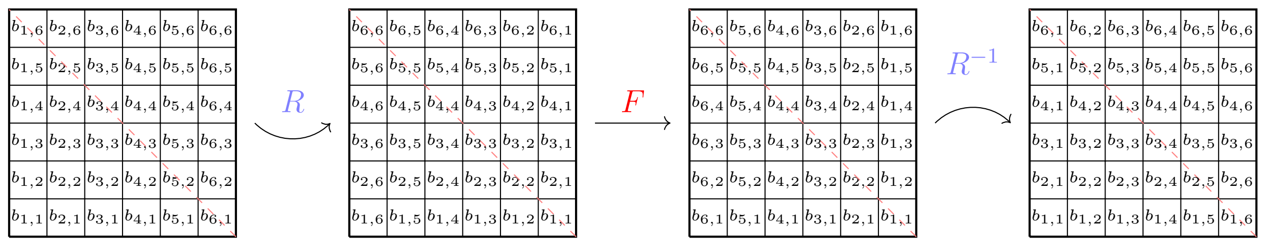


Figure 3.2: Reflection across the anti-diagonal d' as an element of D_8.

If a rook placement ρ is symmetric with respect to $g \in D_8$, this means that if $b_{i,j} \in \rho$, then $g \cdot b_{i,j} \in \rho$ as well. Since $D_8 = \langle F, R \rangle$, it suffices to show that a rook placement cannot be symmetric with respect to both of these elements unless $n = 1$.

Suppose that ρ is symmetric with respect to both F and R, with a rook at $b_{i,j}$. Applying F, R, and R^{-1} we see that we must also have

$$b_{n+1-j,n+1-i}, \qquad b_{j,n+1-i}, \qquad \text{and} \qquad b_{n+1-j,i}$$

in ρ, respectively. Since $F \cdot b_{i,j}$ and $R \cdot b_{i,j}$ share the same second index $n+1-i$, the condition (3.3) forces that their first indices must match as well. Together with the same reasoning for $F \cdot b_{i,j}$ and $R^{-1} \cdot b_{i,j}$, we have the equations

$$j = n+1-j \qquad \text{and} \qquad i = n+1-i,$$

so $i = j = \frac{n+1}{2}$. This means the rook at $b_{i,j}$ is in the center of the board for some n odd. However, these equations must hold for every rook in the arrangement ρ, so must be that the central rook is the *only* rook in ρ, in which case $n = 1$. $\square$

Corollary 3.5. *When $n \geq 2$, every placement of n rooks on an $n \times n$ board that is symmetric with respect to reflection across both diagonals is equivalent to exactly one other arrangement.*

Proof. In the notation of the previous proposition, the diagonal reflections F and $R^{-1}FR$ together generate a subgroup V of D_8 isomorphic to the Klein four-group $\mathbb{Z}_2 \times \mathbb{Z}_2$. V has index 2 in D_8, so it is a maximal proper subgroup. Then, by the previous proposition, any rook placement stabilized by V has exactly V as its symmetry group, so by the orbit-stabilizer theorem, the size of the orbit of a diagonally symmetric rook placement under the action of D_8 is just

$$[D_8 : V] = \frac{|D_8|}{|V|} = \frac{8}{4} = 2.$$

$\square$

The following fact is also discussed in [12]. From now on, let d_n denote the number of diagonally symmetric inequivalent arrangements of n non-attacking rooks on an $n \times n$ board.

Lemma 3.6. *For all $n \geq 1$, $d_{2n} = d_{2n+1}$.*

Proof. Suppose we have a diagonally symmetric arrangement ρ on a $(2n+1) \times (2n+1)$ board. Then for some $b_{i,j} \in \rho$, we have $i = n+1$. By symmetry (applying F and $R^{-1}FR$), $b_{2n+2-j,n+1}$ and $b_{j,n+1}$ are also in ρ. Since the second indices of these two rooks match, it must be that $j = 2n+2-j$ so $j = n+1$ as well. We then see that the central square is necessarily occupied in a rook placement on an odd sidelength board. Thus, the data of ρ is really determined by the placement of rooks in the remaining $2n$ rows and columns[1]. One obtains a bijection between placements on a $(2n+1) \times (2n+1)$ board and placements on a $2n \times 2n$ board by deleting or adding back this central row and column. Moreover, this bijection is equivariant with respect to the action of D_8.

□

Figure 3.3 illustrates an instance of this correspondence, where $n = 2$.

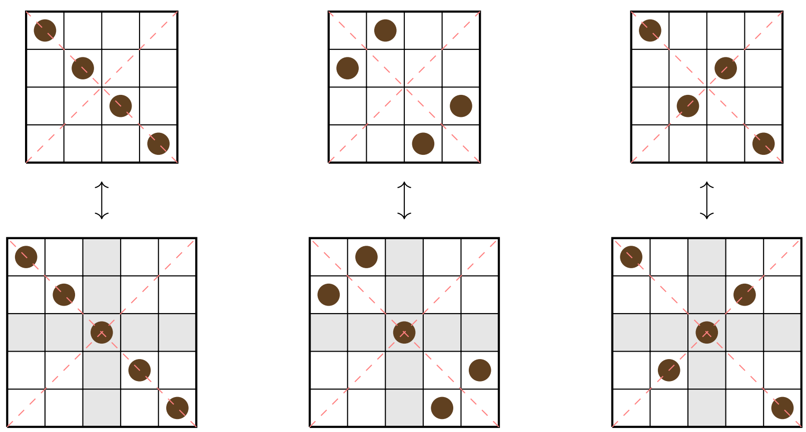


Figure 3.3: The three 4×4 and 5×5 inequivalent diagonally symmetric rook placements.

We see now that one only needs to count diagonally symmetric placements on even-sided boards to understand the whole picture. We will prove that $\Delta_n = d_{2n}$ by exhibiting an explicit bijection between type $DIII$ clans and diagonally symmetric rook placements. This will allow us to make use of a recurrence relation which was known for the rook placements to give a generating function for the number of type $DIII$ clans and the orbits they parametrize. At present, a direct proof of the recursion using the formula of Proposition 3.3 eludes us.

First, we state the recurrence relation, also given in [12], pp. 217.

Proposition 3.7. *Taking $d_0 = d_2 = 1$, the numbers d_k satisfy the recurrence relation*

$$d_{2n} = 2d_{2n-2} + (2n-2)d_{2n-4} \tag{3.4}$$

for all $n \geq 2$.

[1]The rows and columns are often called "rank" and "file" elsewhere.

Proof. A diagonally symmetric $2n \times 2n$ rook placement must have a rook on the bottom row of the board. This breaks down into two cases: either a rook is in one of the corners $b_{1,1}$ or $b_{1,2n}$, or the rook is in between at $b_{1,j}$ for $1 < j < 2n$.

In either of the first cases, by symmetry, there will be a rook in the opposite corner. Then, we may eliminate the border squares from consideration for placing any of the remaining $2n-2$ rooks, and they can be placed in any allowable $(2n-2) \times (2n-2)$ arrangement on the interior of the board. This accounts for the first term in the recurrence.

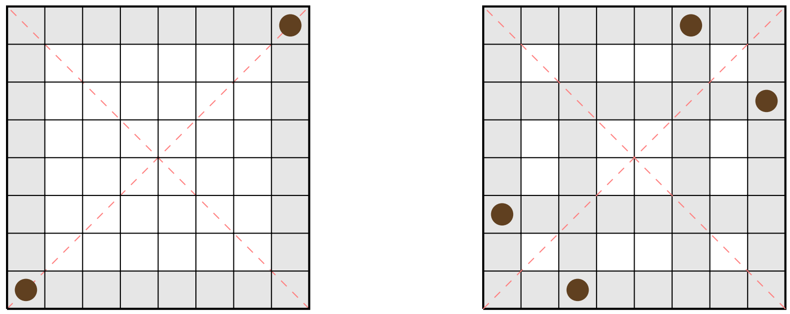


Figure 3.4: Deletion of rows and columns based on the position of border rooks.

Suppose now that there is a rook at $b_{1,j}$, with $1 < j < 2n$. By symmetry, there are rooks at $b_{2n+1-j,2n}$, $b_{j,1}$, and $b_{2n,2n+1-j}$ as well. Then, we remove the rows and columns with index 1, j, $2n+1-j$, and $2n$ from consideration for placing the remaining $(2n-4)$ rooks on the board obtained by deletion of these rows and columns. Since there were $(2n-2)$ choices for j, we have accounted for the second term in the recurrence. $\square$

The diagonals of a $2n \times 2n$ board divide it into four triangles, and one sees that the information of a diagonally symmetric rook placement is captured within any of these triangles. In identifying rook placements, we will then extract one of the triangles of the board, including the squares along each diagonal, to obtain a *pyramid*.

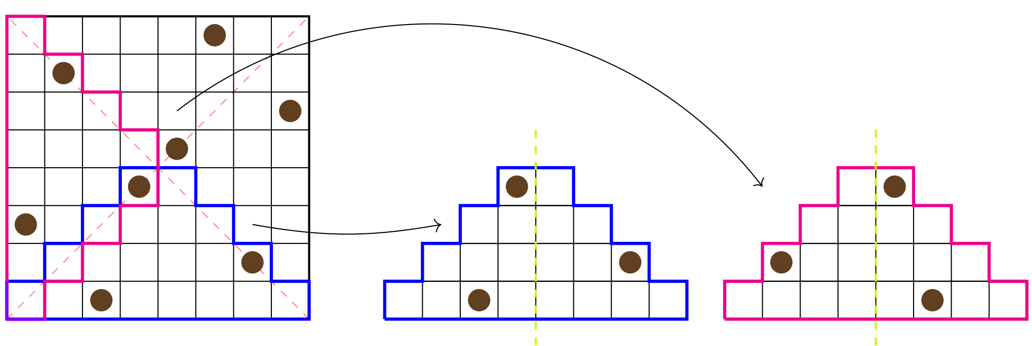


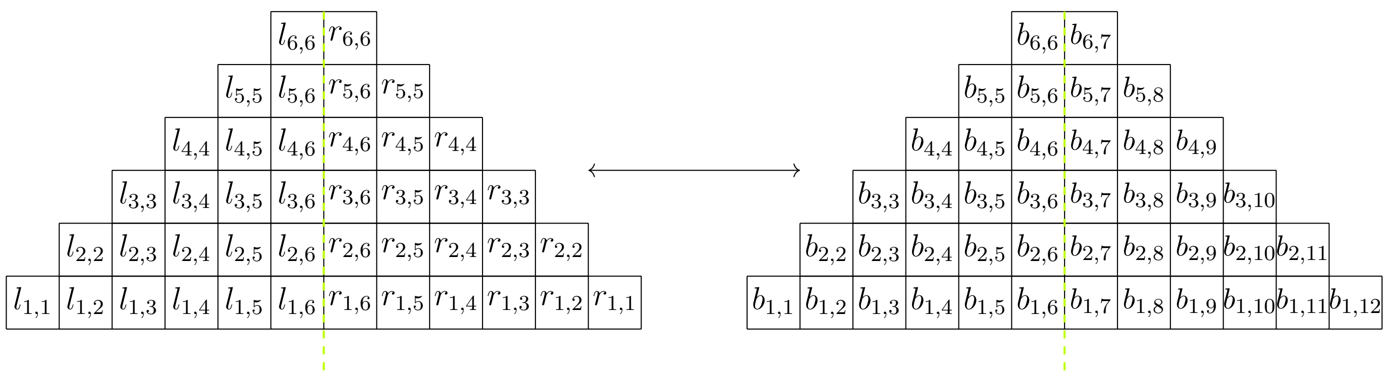
Figure 3.5: The two possible pyramids of a diagonally symmetric rook placement.

If you lay pyramids from different triangles on top of one another, the rook placements in pyramids which appear opposite in the board are identical, while they differ from those of adjacent triangles by a reflection across the line which perpendicularly bisects the base of the pyramid. We describe pyramids more precisely below.

In order to describe a bijection between type $DIII$ clans and diagonally symmetric rook placements, it will be convenient to introduce coordinates on the blocks of the pyramids by dividing them into *left* and *right* halves on either side of the axis of symmetry. The coordinates on the left $l_{i,j}$ increase moving up and to the right, while those on the right $r_{i,j}$ increase as we move up and to the left.

We can convert these pyramid coordinates to those inherited from the board, where, for instance, the pyramid appears at the bottom of the board. An example illustrating both of these coordinate systems is drawn below.

Figure 3.6: Coordinates on pyramids.



Lemma 3.8. *A pyramid corresponds to a diagonally symmetric rook placement if and only if for each $1 \leq k \leq n$, there is a unique block of the pyramid, $l_{i,j}$ or $r_{i,j}$, on which a rook is placed and for which either $i = k$, or $j = k$, or both $i = j = k$.*

Proof. Clearly there can be at most one rook in each row and column of the pyramid. Starting with a rook placement with rook at $b_{i,j}$, reflect it around the board and further rule out the rows and columns of the resulting rooks. Tracing these prohibited rows and columns into the pyramid gives the statement in terms left and right pyramid coordinates. $\square$

The following describes an algorithm for obtaining a pyramid from a clan. Basically, we will read the symbols c_1 through c_n in reverse order, and place rooks as we descend rows of the pyramid. We will introduce an auxiliary variable X, which acts as a "switch" that can take values l or r, telling us on which side of the pyramid to place a rook. Every time we encounter a $-$ or the first number in a Π_1 pair, the switch gets "flipped" from l to r or from r to l.

Algorithm 3.9. Given a type $DIII$ (n, n)-clan γ, we construct a pyramid corresponding to a diagonally symmetric $2n \times 2n$ rook placement as follows.

set $i = n$, $X = l$.
while $i \geq 1$:
    if $c_i = +$:
        place rook at $X_{i,i}$
    if $c_i = -$:
        flip switch
        place rook at $X_{i,i}$
    if $c_i \in \mathbb{N}$:
        find $c_j = c_i$ in γ
        if $j > n$ and $2n + 1 - j > i$: [second condition prevents redundancy]
            place a rook at $X_{i,2n+1-j}$ [indices corr. to opposing Π_0 pairs]
        if $i < j \leq n$: [positions of a Π_1 pair]
            flip switch
            place rook at $X_{i,j}$
        else:
            pass
    subtract 1 from i

One sees that this algorithm produces a pyramid that satisfies the condition of Lemma 3.8, yielding a diagonally symmetric rook placement. Indeed, in each placement step of the algorithm, the index i, j, or $2n + 1 - j$ corresponds to the index of a unique symbol from $c_1 \cdots c_n$ which is used exactly once. As an example, the blue pyramid in Figure 3.5 is obtained from the $(4, 4)$-clan, $1 - 1 + -2 + 2$.

Without trouble, this algorithm can be reversed to give a map from pyramids to clans. However, not all pyramids yield type $DIII$ clans. In fact, for every diagonally symmetric rook placement, just one of the pyramids will be the one corresponding to a clan. If one attempts to apply the reverse algorithm to the wrong pyramid, one obtains a clan which violates the third condition for type $DIII$. For example, the pink pyramid in Figure 3.5 would yield the clan $1 - 1 - +2 + 2$, which has three total minus signs and pairs of matching numbers among the first four symbols. In fact, the clans obtained from the two pyramids of a diagonally symmetric rook placement differ exactly by a swap of the symbols c_n and c_{n+1} of $\gamma = c_1 \cdots c_{2n}$. In the notation of [19], pp. 115, reflection of the pyramid across its center line corresponds to the "flip" operation $\mathrm{Flip}(\gamma)$ on the corresponding clan γ. It is easy to see that whether c_n is a sign or a natural number, Flip will always change the total number of $-$ signs and pairs of matching natural numbers by one. Hence, exactly one of the pyramids gives a $DIII$ clan.

Algorithm 3.9 and it's reverse are clearly injective, so the fact that each diagonally symmetric rook placement contains a unique pyramid which gives a type $DIII$ clan completes the bijection.

Theorem 3.10. *Diagonally symmetric rook placements on a $2n \times 2n$ board and (n, n)-clans of type $DIII$ are in bijection, whence $\Delta_n = d_{2n}$.*

Corollary 3.11. *Taking $\Delta_0 = \Delta_1 = 1$, the number of type DIII clans satisfies the recurrence relation*

$$\Delta_n = 2\Delta_{n-1} + (2n-2)\Delta_{n-2}, \tag{3.5}$$

and has exponential generating function

$$\sum_{n=0}^{\infty} \Delta_n \frac{x^n}{n!} = \frac{1}{2}(e^{2x+x^2} + 1). \tag{3.6}$$

Proof. The recurrence follows from Theorem 3.10 and Proposition 3.7. The exponential generating function is found in [18], pp. 203.[2] □

3.3 Minimally Intersecting Set Partitions

Let $[n]$ denote the set $\{1, \ldots, n\}$ and consider partitions of $[n]$ ordered by refinement. Two partitions p and p' are said to be *minimally intersecting* if the partition whose blocks are the pairwise intersections of blocks from p and p' is the minimal partition

$$p_{\min} = \{\{1\}, \{2\}, \ldots, \{n\}\}.$$

Lemma 2 of [16] says that the right hand side of the equation (3.6) is equal to e^x plus the exponential generating function for the number of ordered pairs of minimally intersecting partitions (p, p') of an n-element set such that p consists of exactly two blocks. In other words, the number of (n, n)-clans of type $DIII$ is one more than the number of ordered pairs (p, p') of minimally intersecting set partitions of $[n]$ where p has two blocks ([15], A000902). In this subsection, we present maps between pyramids and such pairs of set partitions that reflect this equality.

Remark 3.12. There is a well known bijection between $n \times n$ staircase rook placements and partitions of $[n+1]$ (see [11], pp. 77-78). Observing that each pyramid consists of two staircase shapes with "complementary" rook placements, one could also define a correspondence with pairs of partitions of $[n+1]$ with certain properties. We leave this description to the motivated reader.

Consider a pyramid that corresponds to an (n, n)-clan which is not $+\cdots+-\cdots-$, along with the coordinates $l_{i,j}$, $r_{i,j}$ of Figure 3.6. First we describe how to obtain the corresponding two-block partition, which we will write as $p = \{L, R\}$.

1. If there is a rook at $l_{i,i}$, then $i \in L$; if there is a rook at $r_{i,i}$, then $i \in R$.

2. If there is a rook at $l_{i,j}$ for $i \neq j$, then $j \in L$ and $i \in R$. Similarly, if there is a rook at $r_{i,j}$, then $j \in R$ and $i \in L$.

[2] What actually appears in this source is the exponential generating function for $2\Delta_n$, without taking $\Delta_0 = 1$; the addition of one and multiplication by one half above account for this discrepancy.

Then we construct p' by taking $\{i,j\}$ as a block for each rook at $l_{i,j}$ or $r_{i,j}$. Thus, the blocks of p' have maximum size two, and rooks at $l_{i,i}$ or $r_{i,i}$ give blocks that are singletons. For a block of size two $\{i,j\}$, our construction guarantees that i and j are in opposite blocks in p, so the pair (p,p') is in fact minimally intersecting.

Notice that either pyramid from a diagonally symmetric rook placement produces the same ordered pair (p,p'); the only difference is that the blocks L and R will be switched from one to the other. Given a pyramid, to guarantee that neither of L or R is empty, we only need one rook placed at some $l_{i,j}$ or $r_{i,j}$ where $i \neq j$, or if all of the rooks are at $l_{i,i}$'s or $r_{i,i}$'s (which occurs when the underlying clan consists only of $+$'s and $-$'s), then at least one rook needs to be on each half. This only fails for the clan $+\cdots+-\cdots-$, which gives a pyramid with rooks only at $l_{i,i}$ for all $1 \leq i \leq n$. In all other cases, the rook placements on the pyramids clearly determine unique pairs of minimally intersecting partitions.

Example 3.13. The blue pyramid of Figure 3.5 gives the pair (p,p') with $p = \{\{3,4\},\{1,2\}\}$ and $p' = \{\{1,3\},\{2\},\{4\}\}$.

Now we describe how to obtain a pyramid from a pair (p,p'), where $p = \{L,R\}$. Notice that p' cannot have any blocks of size greater than two; suppose otherwise that b was a block in p' of size three or greater. Then, by the pigeonhole principle we would have at least two of the elements of b in either L or R. The pairwise intersections of these blocks would leave us with a block of size at least two, and so p and p' could not be minimally intersecting. With this in mind, we construct a pyramid from a pair (p,p') as follows.

1. If $i \in L$ and i is a singleton in p', then place a rook at $l_{i,i}$; if $i \in R$ and i is a singleton in p', then place a rook at $r_{i,i}$.
2. If $j \in L$ and $\{i,j\}$ is a block of p' with $i < j$, then place a rook at $l_{i,j}$; if $j \in R$ and $\{i,j\}$ is a block of p' with $i < j$ then place a rook at $r_{i,j}$.

It is clear that the pyramid constructed will satisfy the criterion of Lemma 3.8, and so it corresponds to a unique diagonally symmetric rook placement and type $DIII$ clan. In fact, this recipe inverts the (partial) map from pyramids to partition pairs described above. Again, the pyramid corresponding to the clan $+\cdots+-\cdots-$ can not be constructed through this process. This discussion establishes the following.

Theorem 3.14. *The set of type $DIII$ (n,n)-clans without the clan $+\cdots+-\cdots-$ is in bijection with the set of ordered pairs (p,p') of minimally intersecting pairs of partitions of $[n]$, where p has exactly two blocks.*

3.4 Lattice Paths

In this section, we describe another sequence of combinatorial objects whose cardinalities are given by Δ_n. Similar constructions were made in [6] and [2] for type $AIII$ and type CI, respectively. While $DIII$ clans can be viewed as subsets of the clans of either of those types, the weighted lattice paths presented here are only a subset of those presented in the latter work.

Recall that an (n, n) *Delannoy path* is an integer lattice path from $(0, 0)$ to (n, n) in the plane $\mathbb{R}^2$ consisting only of single north, east, or diagonally northeast steps. Alternatively, one can consider strings from the alphabet $\{N, E, D\}$ such that the number of N's plus the number of D's is equal to the sum of the numbers of E's and D's (which is equal to n). We will demonstrate a bijection between the set of (n, n)-clans of type $DIII$ and the set of (n, n) Delannoy paths with certain labels which are defined as follows.

Definition 3.15. By a *labelled step* we mean a pair (L, l), where $L \in \{N, E, D\}$ and l is a positive integer such that $l = 1$ if $L = N$ or $L = E$. A *weighted* (n, n) *Delannoy path* is a word of the form $W := W_1 \dots W_r$, where the W_i's $(i = 1, \dots, r)$ are labeled steps $W_i = (L_i, l_i)$ such that

- $L_1 \dots L_r$ is an (n, n) Delannoy path;
- $L_i = N$ if and only if $L_{r+1-i} = E$;
- letting $k_i = \#\{j < i \mid l_j \neq 1\}$, if $L_i = D$ then $2 \leq l_i \leq 2n + 1 - 2(i + 2k_i)$ for $1 \leq i \leq \lfloor \frac{r}{2} \rfloor$, and $W_{r+1-i} = (D, 2n + 3 - 2(i + 2k_i) - l_i)$;
- either $L_{\frac{r}{2}} = E$ (so that $L_{\frac{r}{2}+1} = N$) or $W_{\frac{r}{2}} = (D, 3)$ (so that $W_{\frac{r}{2}+1} = (D, 2)$).[3]

The set of all weighted (n, n) Delannoy paths is denoted by $\mathcal{D}^\omega(n)$.

Theorem 3.16. *There is a bijection between the set of weighted (n, n) Delannoy paths and the set of Type DIII (n, n) clans. In particular, we have*

$$\Delta_n = \sum_{W \in \mathcal{D}^\omega(n)} 1.$$

Proof. Let a_n denote the cardinality of $\mathcal{D}^\omega(n)$. We will prove that the sequence of a_n's obeys the same recurrence as Δ_n, and that it satisfies the same initial conditions. Let W be an arbitrary element of $\mathcal{D}^\omega(n)$. Suppose W ends with an N-step from $(n, n-1)$ to (n, n), so that it begins with a E-step between $(0, 0)$ and $(1, 0)$. Then there are a_{n-1} possible ways of completing these steps between $(1, 0)$ and $(n, n-1)$ to a weighted (n, n) Delannoy path.

Similarly, if W ends with E-step from $(n-1, n)$ to (n, n), then it begins with a N-step between $(0, 0)$ and $(0, 1)$. Again, there are a_{n-1} possible ways of completing these steps between $(0, 1)$ and $(n-1, n)$ to a weighted (n, n) Delannoy path. Together with the previous case, we see that $2a_{n-1}$ appears in the recurrence relation for a_n.

Next, consider the case where W ends in a D-step between $(n-1, n-1)$ and (n, n) so that it also begins with a D-step between $(0, 0)$ and $(1, 1)$. There are $2n - 2$ possible labelings for l_1, which determines the last step's label. Then, there are a_{n-2} possible ways of completing these steps between $(1, 1)$ and $(n-1, n-1)$ to a weighted (n, n) Delannoy path.

Combining the above observations, we see that $a_n = 2a_{n-1} + (2n-2)a_{n-2}$ as desired. By inspection, one can verify that $a_1 = 1$ and $a_2 = 3$, so $a_n = \Delta_n$ for all n. $\square$

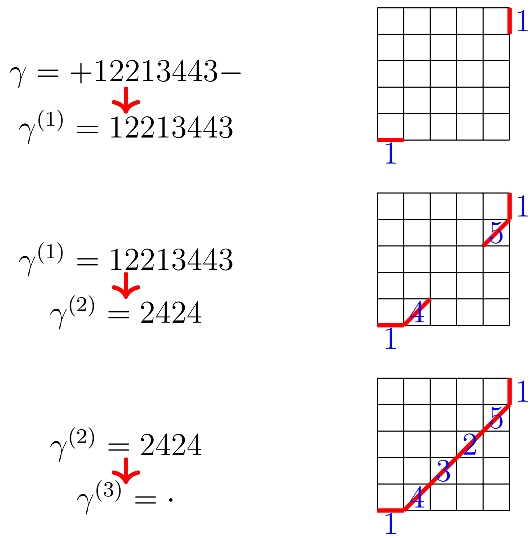


Figure 3.7: Algorithmic construction of the bijection onto weighted Delannoy paths.

Now, we indicate how to obtain a weighted (n,n) Delannoy path from a type $DIII$ clan $\gamma = c_1 \cdots c_{2n}$. If c_{2n} is a $-$ sign, then c_1 is a $+$ sign, and so we draw an N-step from $(n, n-1)$ to (n,n) and an E-step between $(0,0)$ and $(1,0)$ (each step labelled with 1). Next, we remove c_1 and c_{2n} from γ to obtain $\gamma^{(1)} = c_2 \cdots c_{2n-1}$.

In a similar manner, if $c_{2n} = +$ so that $c_1 = -$, we draw an E-step from $(n-1, n)$ to (n,n) and an N-step between $(0,0)$ and $(0,1)$ (both labelled 1). Again we remove c_1 and c_{2n} from γ, but in this case we then swap c_n and c_{n+1} to obtain $\gamma^{(1)} = c_2 \cdots c_{n+1} c_n \cdots c_{2n-1}$.

If c_{2n} is a natural number from Π_0 pair (c_i, c_{2n}) $(i \leq n)$, there there is an opposing pair Π_0 pair (c_1, c_{2n+1-i}). In this case, we draw a D-step between $(n-1, n-1)$ and (n,n) and label this step i, and we draw another D-step between $(0,0)$ and $(1,1)$ and label this step $2n+1-i$. Then we remove all four symbols c_1, c_i, c_{2n+1-i}, and c_{2n} from γ and call the resulting $(n-2, n-2)$-clan $\gamma^{(1)}$.

In case c_{2n} is a natural number from Π_1 pair (c_j, c_{2n}) $(j > n)$ with opposing Π_1 pair (c_1, c_{2n+1-j}), then we draw a D-step between $(n-1, n-1)$ and (n,n) and label this step j, and we draw another D-step between $(0,0)$ and $(1,1)$ and label this step with $2n+1-j$. We remove all four symbols c_1, c_j, c_{2n+1-j}, and c_{2n} from γ, then swap c_n and c_{n+1} and call the resulting $(n-2, n-2)$-clan $\gamma^{(1)}$.

After performing this first step, we apply the same procedure to $\gamma^{(1)}$ accordingly by examining its last symbol, thereby obtaining $\gamma^{(2)}$ and so on, building the path from the corners inwards.

Let us illustrate our construction by an example.

Example 3.17. Let γ denote the $(5,5)$-clan of type $DIII$, $\gamma = +12213443-$. The steps of our construction are shown in Figure 3.7.

[3]As a consequence of the preceding properties, r is guaranteed to be even.

To supply further examples, we depict the weighted Delannoy paths corresponding to type $DIII$ (3,3)-clans in Figure 4.1 as a poset with the weak order, which is described in the next section.

4 The Weak Order on Type $DIII$ Clans

Let $B \subset SO_{2n}$ be the Borel subgroup of upper triangular matrices, and let $GL_n \subset SO_{2n}$ be realized as

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \;\middle|\; A \in GL_n \right\},$$

as in Section 2. Here we recall a description, given in [19], of the weak order on GL_n-orbits in SO_{2n}/B[4] in terms of clans. See that source for further details.

Let $T \subset SO_{2n}$ be the maximal torus of diagonal matrices with Lie algebra $\mathfrak{t}$. By the condition defining the special orthogonal group, we have that

$$T = \{\mathrm{diag}(t_1, \ldots t_n, t_n^{-1}, \ldots, t_1^{-1} \mid t_i \in \mathbb{C}^*)\},$$

so that

$$\mathfrak{t} = \{\mathrm{diag}(a_1, \ldots, a_n, -a_n, \ldots, -a_1) \mid a_i \in \mathbb{C}\}.$$

We declare simple roots $\alpha_i := Y_i - Y_{i+1}$ for $1 \leq i \leq n-1$ and $\alpha_n := Y_{n-1} + Y_n$ where $Y_i \in \mathfrak{t}^*$ is given by $Y_i(\mathrm{diag}(a_1, \ldots, a_n, -a_n, \ldots, -a_1)) = a_i$. The simple roots act on the GL_n-orbits corresponding to type $DIII$ clans, and this is denoted by

$$s_{\alpha_i} \cdot Q_\gamma = Q_{\gamma'},$$

though we may also abuse notation and write $s_{\alpha_i} \cdot \gamma = \gamma'$ to mean the same. When $\gamma' \neq \gamma$, then $\dim(Q_{\gamma'}) = \dim(Q_\gamma) + 1$ and these orbits are related in the weak order, denoted by $Q_\gamma \prec Q_{\gamma'}$[5]. The weak order is the transitive closure of the relations generated by the action of the simple roots.

We also define the *flip* of $\gamma = c_1 \cdots c_n c_{n+1} \cdots c_{2n}$ by

$$\mathrm{Flip}(\gamma) = c_1 \cdots c_{n+1} c_n \cdots c_{2n}.$$

Note that if γ is a type $DIII$ clan then $\mathrm{Flip}(\gamma)$ is not, though we will still act on it combinatorially by the simple roots as if it were.

Now, suppose that $\gamma = c_1 \cdots c_{2n}$ is a type $DIII$ clan.

(1) For $1 \leq i \leq n-1$, α_i is called *complex* for the orbit Q_γ (and $s_{\alpha_i} \cdot Q_\gamma \neq Q_\gamma$) if and only if one of the following holds:

   i) c_i is a sign, c_{i+1} is a natural number, and the mate for c_{i+1} occurs to the right of c_{i+1}.

[4] Or equivalently, B-orbits in SO_{2n}/GL_n.

[5] Or abusively, again, $\gamma \prec \gamma'$.

ii) c_i is a number, c_{i+1} is a sign, and the mate for c_i occurs to the left of c_i.

iii) c_i and c_{i+1} are unequal natural numbers, the mate of c_i occurs to the left of the mate of c_{i+1}, and $(c_i, c_{i+1}) \neq (c_{2n-i}, c_{2n-i+1})$.

In the above cases, $s_{\alpha_i} \cdot Q_\gamma = Q_{\gamma'}$, where γ' is the clan obtained from γ by interchanging c_i and c_{i+1}, and also c_{2n-i} and c_{2n-i+1}.

(2) For $1 \leq i \leq n-1$, α_i is called *non-compact imaginary* for the associated orbit Q_γ (and $s_{\alpha_i} \cdot Q_\gamma \neq Q_\gamma$) if and only if c_i and c_{i+1} (and, by skew-symmetry, c_{2n+1-i} and c_{2n-i}) are opposite signs.

In this case, $s_{\alpha_i} \cdot Q_\gamma = Q_{\gamma''}$, where γ'' is the clan obtained from γ by replacing the signs in positions i and $i+1$ by a pair of matching natural numbers, and the signs in positions $2n+1-i$ and $2n-i$ by a second pair of matching natural numbers.

(3) The root α_n is complex for Q_γ if and only if the string $c_{n-1}c_nc_{n+1}c_{n+2}$ satisfies one of the following:

i) c_{n-1} is a number, c_{n+2} is a different number, c_n and c_{n+1} are opposite signs, and the mate for c_{n-1} lies to the left of c_{n-1} (implying, by skew-symmetry, that the mate for c_{n+2} lies to the right of c_{n+2}).

ii) c_{n-1} and c_{n+2} are opposite signs, c_n is a number, c_{n+1} is a different number, and the mate for c_n lies to the left of c_n (implying, by skew-symmetry, that the mate for c_{n+1} lies to the right of c_{n+1}).

iii) c_{n-1}, c_n, c_{n+1}, and c_{n+2} are 4 distinct numbers, with the mate of c_{n-1} lying to the left of the mate of c_{n+1} (implying, by skew-symmetry, that the mate of c_n lies to the left of the mate of c_{n+2}).

In the above cases, $s_{\alpha_n} \cdot Q_\gamma = Q_{\gamma'''}$, where

$$\gamma''' = \mathrm{Flip}(s_{\alpha_{n-1}} \cdot \mathrm{Flip}(\gamma)).$$

(4) The root α_n is non-compact imaginary for Q_γ if and only if $c_{n-1}c_nc_{n+1}c_{n+2}$ is either $++--$ or $--++$. Again in this case, $s_{\alpha_n} \cdot Q_\gamma = Q_{\gamma'''}$, where

$$\gamma''' = \mathrm{Flip}(s_{\alpha_{n-1}} \cdot \mathrm{Flip}(\gamma)).$$

In order to have a better understanding of this construction, let us work out the following example.

Example 4.1. Consider the set of type $DIII$ $(3,3)$-clans. We know that there are 10 elements which are

$$12+-12,\ 1+21-2,\ 1-12+2,\ +1212-,\ 11-+22,\ -1122+,$$
$$+++---,\ +--++-,\ -+-+-+,\ --+-++\ .$$

- Let $\gamma = +++---$. In this case, α_3 is non-compact imaginary for γ because $c_2c_3c_4c_5$ is $++--$. So, to obtain $\gamma^{'''}$, first we flip γ to obtain $\text{Flip}(\gamma) = ++-+--$ for which s_2 is non-compact imaginary, and its action yields $+1122-$. Flipping one more time, we have
$$\gamma^{'''} = \text{Flip}(s_{\alpha_2} \cdot \text{Flip}(\gamma)) = s_3 \cdot +++--- = +1212-.$$

- Let $\gamma = +--++-$ for which α_1 is non-compact imaginary since c_1 and c_2 are opposite signs. Then we will have $\gamma^{''} = s_1 \cdot +--++- = 11-+22$ by (2) above.

  Since $c_2c_3c_4c_5$ is $--++$, α_3 is also non-compact imaginary. We flip γ to obtain $\text{Flip}(\gamma) = +-+-+-$, and acting by s_2 on the result yields us $+1122-$ by (2) above. Flipping one more time gives us
$$\gamma^{'''} = \text{Flip}(s_2 \cdot \text{Flip}(\gamma)) = s_3 \cdot +--++- = +1212-.$$

- Let $\gamma = -+-+-+$. Start with α_1 which is non-compact imaginary for γ since c_1 and c_2 are opposite signs. So, we will have $\gamma^{''} = s_{\alpha_2} \cdot \gamma = 11-+22$ by (2) above.

  It is easy to see that α_2 is also non-compact imaginary for γ because c_2 and c_3 are opposite signs. So in this case, we will have $\gamma^{''} = s_{\alpha_2} \cdot \gamma = -1122+$.

- If $\gamma = --+-++$, then α_2 is non-compact imaginary for γ, and $s_{\alpha_2} \cdot \gamma = -1122+$.

- α_1 is complex for the clan $\gamma = +1212-$ since c_1 is $+$ and c_2 is a natural number whose mate occurs to its right. Thus by (1.i), we have
$$s_1 \cdot +1212- = 1+21-2.$$

- Let $\gamma = 11-+22$. Since c_2 is a natural number whose mate occurs to its left and c_3 is $-$, α_2 is complex for γ. Thus by (1.ii), we have $s_2 \cdot 11-+22 = 1-12+2$.

  Now, since c_2 and c_5 are different natural numbers where the mate of c_2 occurs to its left and c_3 and c_4 are opposite signs, α_3 is also complex for γ. Thus, by (3.i), we have
$$s_3 \cdot 11-+22 = \text{Flip}(s_2 \cdot \text{Flip}(\gamma)) = 1+21-2.$$

- Let $\gamma = -1122+$. Since c_1 is $-$ and c_2 is a natural number whose mate occurs to its right, α_1 is complex for γ. Thus by (1.i), we have
$$s_1 \cdot -1122+ = 1-12+2.$$

- Let $\gamma = 1+21-2$. Since c_2 is $+$ and c_3 is a natural number whose mate occurs to its right, α_2 is complex for γ. Thus, by (1.i), we have
$$s_2 \cdot 1+21-2 = 12+-12.$$

- Finally, let $\gamma = 1 - 12 + 2$. Since c_2 and c_5 are opposite signs and c_3 and c_4 are different numbers where the mate of c_2 occurs to its left, α_3 is complex for γ. Thus by (3.ii), we have
$$s_3 \cdot 1 - 12 + 2 = \mathrm{Flip}(s_2 \cdot \mathrm{Flip}(\gamma)) = 12 + -12.$$

This list exhausts the weak order relations given by the action of simple roots. Using the bijection between type $DIII$ clans and weighted Delannoy paths, we finish this section by drawing the weak order poset of $(\Delta(3), \prec)$, substituting the corresponding lattice paths; see Figure 4.1 below.

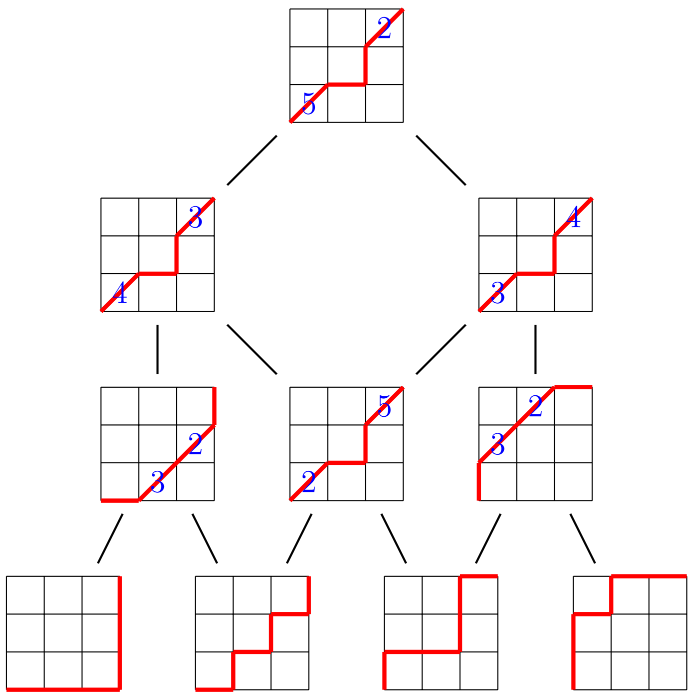


Figure 4.1: Weak order on $\mathcal{D}^\omega(3)$

5 Length Generating Function

In this section, we will consider a t-analog of Δ_n: the length generating function $A_n(t)$ of the weak order on (n,n)-clans of type $DIII$. This will be defined precisely below, but readers may recognize it as the rank polynomial for the weak order poset.

Definition 5.1. We define the length $L(\gamma)$ of a type $DIII$ clan $\gamma = c_1c_2 \ldots c_{2n}$ as follows:

$$L(\gamma) = \frac{1}{2}\Big(l(\gamma) - \#\{a \in \mathbb{N} \mid a = c_u = c_t \text{ and } u \le n < t \le 2n+1-u\}\Big)$$

where $l(\gamma) = \sum\limits_{\substack{c_i=c_j\in\mathbb{N}\\ i<j}} \big(j - i - \#\{a \in \mathbb{N} \mid a = c_u = c_t \text{ and } u < i < t < j\}\big)$.

Note that l is the length function for (p, q)-clans (type $AIII$) which appears in [21]. Moreover, in the notation of Section 3,

$$L(\gamma) = \frac{1}{2}(l(\gamma) - p) \tag{5.1}$$

where $2p = |\Pi_0|$. Let's introduce some terminology that will aid in the discussion that follows. For a natural number $a = c_i = c_j$ which appears in $\gamma = c_1 \cdots c_{2n}$, the *spread* of a is the quantity $s(a) := j - i$. The *weave* of a will be the quantity

$$w(a) := \#\{b \in \mathbb{N} \mid b = c_u = c_t \text{ and } u < i < t < j\}.$$

Now, we can write

$$L(\gamma) = \frac{1}{2}\left(\left(\sum_{a=c_i=c_j} s(a) - w(a)\right) - p\right). \tag{5.2}$$

Next, we verify that L is indeed a length function in the sense of [17]. It is straightforward from the definition that the length of a clan containing only $\pm$ signs is zero. Now, we will show that $L(s_{\alpha_i}\cdot\gamma) = L(\gamma)+1$ for each case in the list of weak order relations of the previous section. Throughout, we let $\gamma = c_1 \cdots c_{2n}$ with $L(\gamma) = r$, and we write $s_{\alpha_i}\cdot\gamma = \gamma' = c'_1 \cdots c'_{2n}$.

Case (1.i) Assume that γ is a clan to which we can apply (1.i), that is $c_i = \pm$ and $c_{i+1} = c_j = a \in \mathbb{N}$ for some $i + 1 < j$ with $i < n$. It follows that $c_{2n+1-i} = \mp$ and c_{2n-i} is another natural number, say b. After applying s_{α_i}, we will have $c'_i = a$, $c'_{i+1} = \pm$, $c'_{2n+1-i} = b$, and $c'_{2n-i} = \mp$ in the new clan γ'. In this case, since the mate of a occurs to its right and the mate of b to its left, the only thing that changes under s_{α_i} is that the spread of each natural number a and b has increased by 1. We can conclude that the length of γ' is $r + 1$.

Case (1.ii) Assume that γ is a clan where $c_j = c_i = a \in \mathbb{N}$ for $j < i$ and $c_{i+1} = \pm$ so that we can apply (1.ii). It follows that $c_{2n-i} = \mp$ and c_{2n+1-i} is an another natural number, say b. After applying s_{α_i}, we will have $c'_i = \pm$, $c'_{i+1} = a$, $c'_{2n+1-i} = \mp$, and $c'_{2n+1-i} = b$ in the new clan γ'. Again, the only change is that the spreads of a and b have each increased by one, so we can conclude that $L(\gamma') = r + 1$.

Case (1.iii) In this case $c_i = c_j = a \in \mathbb{N}$, $c_{i+1} = c_k = b \in \mathbb{N}$ for $j < k$, and $(c_i, c_{i+1}) \neq (c_{2n-i,2n+1-i})$. It follows from the last condition that $c_{2n-i} = c_l = c \in \mathbb{N}$ and $c_{2n+1-i} = c_m = d \in \mathbb{N}$ with $l < m$. Now there are a few possibilities for what happens after applying s_{α_i}.

$\mathbf{j} < \mathbf{k} < \mathbf{i}$: The changes in the spreads of a and b cancel ($s(a)$ goes up one, $s(b)$ goes down one), but the weave of b decreases by one, and by skew-symmetry the same is true for the spreads of d and c, and the weave of d, respectively.

$\mathbf{j < i < k}$: The spreads of a and b both increase by one, but the weave of b increases by one, and by skew-symmetry, the same is true for the spreads of c and d and the weave of d, respectively.

$\mathbf{i < j < k}$: The changes in the spreads of a and b cancel ($s(a)$ goes down one, $s(b)$ goes up one), but the weave of b increases by one, and by skew-symmetry, the same is true for the spreads of d and c and the weave of d, respectively.

Consulting the equation (5.2), we see that the net effect in each of these situations is that $L(\gamma') = r + 1$.

Case (2) Here, γ is a clan where c_i and c_{i+1} are opposite signs. After applying s_{α_i}, we will have $c_i = c_{i+1} = a \in \mathbb{N}$ and $c_{2n-i} = c_{2n+1-i} = b \in \mathbb{N}$. It is clear that both a and b contribute a spread of one to the length formula, and hence the length of γ' is $r + 1$.

Case (3.i) Now $c_{n-1} = c_j = a \in \mathbb{N}$ with $j < n-1$, and $c_{n+2} = c_k = b \in \mathbb{N}$ with $k > n+2$. Also $c_n = \pm$ and $c_{n+1} = \mp$. After flipping c_n and c_{n+1}, we apply (1.ii) with α_{n-1} and obtain $c_j = c_n = a$, $c_{n-1} = \pm$, $c_{n+1} = c_k = b$ and $c_{n+2} = \mp$. After flipping one more time, we will have $c'_j = c'_{n+1} = a$, $c'_{n-1} = \pm$, $c'_n = c'_k = b$ and $c'_{n+2} = \mp$ in γ', that is $\gamma' = \cdots a \cdots \pm ba \mp \cdots b \cdots$. In all, $s(a)$ and $s(b)$ have each increased by two, but also $w(b)$ and p have each increased by one, so $L(\gamma') = r + 1$.

Case (3.ii) In this case, $c_{n-1} = \pm$, $c_n = c_j = a \in \mathbb{N}$ with $j < n$, $c_{n+1} = c_k = b \in \mathbb{N}$ with $k > n+1$, and $c_{n+2} = \mp$. After applying s_{α_n}, we get $\gamma' = \cdots a \cdots b \pm \mp a \cdots b \cdots$. The statistics change in the same way as the previous case, again giving $L(\gamma') = r + 1$.

Case (3.iii) We have $c_{n-1} = c_i = a$, $c_{n+1} = c_k = c$ with $i < k$, and $c_n = c_j = b$, $c_{n+2} = c_l = d$ for $j < l$. After applying s_{α_n}, we have $c_{n+1} = c_j = a$, $c_{n+2} = c_{j'} = b$, $c_{n-1} = c_i = c$ and $c_n = c_{i'} = d$ in γ'. Similar to (1.iii), there are three cases we indicate briefly.

$\mathbf{j < k < n}$: $\gamma = \cdots a \cdots c \cdots abcd \cdots b \cdots d \cdots$. Net spread doesn't change, net weave decreases by two, and p is constant.

$\mathbf{j < n < k}$: $\gamma = \cdots a \cdots c \cdots abcd \cdots c \cdots d \cdots$. In all, spread increases by eight, weave by four, and p by two.

$\mathbf{n < j < k}$: $\gamma = \cdots b \cdots d \cdots abcd \cdots a \cdots c \cdots$. Net spread doesn't change, net weave decreases by two, and p is constant.

In every case, $L(\gamma') = r + 1$.

Case (4) Now $c_{n-1}c_nc_{n+1}c_{n+2}$ is either $++--$ or $--++$. After applying s_{α_n}, we have $c_{n-1}c_nc_{n+1}c_{n+2} = aabb$ and hence total spread increases by two, increasing the length by one.

Note that the inclusion poset of Borel orbit closures in SO_{2n}/GL_n contains all of the weak order relations on type $DIII$ (n, n)-clans, possibly with additional relations. This is a graded poset and its rank is equal to the length of its maximal element. It follows from the description of the weak order that the unique maximal clan in this poset is of the form

$$\begin{aligned} \gamma_0 &= 12\ldots(n-1)n(n-1)n\ldots12, \quad \text{if } n \text{ is even,} \\ \gamma_0 &= 12\ldots(n-1)n+-(n-1)n\ldots12, \quad \text{if } n \text{ is odd.} \end{aligned} \tag{5.3}$$

Remark 5.2. If γ is the type $DIII$ clan corresponding to the Borel orbit R_γ, then the dimension of R_γ is equal to $L(\gamma)+c$, where c is the dimension of any closed Borel orbit in SO_{2n}/GL_n. Thus, studying $L(\gamma)$ is equivalent to studying dimensions of Borel orbits in the type $DIII$ symmetric space (or GL_n-orbits in SO_{2n}/B). Moreover, the dimension of the closed orbits is equal to the dimension of the flag variety of GL_n, which is $\frac{n(n-1)}{2}$.

Proposition 5.3. *If γ_0 is the maximal element in the weak order poset, then its length is $L(\gamma_0)=\frac{n(n-1)}{2}$.*

Proof. SO_{2n} has the dimension of its Lie algebra, which consists of skew-symmetric $2n\times 2n$ matrices. This is $\frac{2n(2n-1)}{2}$ dimensional. GL_n has dimension n^2, and so

$$\dim SO_{2n}/GL_n = n(2n-1)-n^2 = n(n-1).$$

Since the maximal element corresponds to a dense orbit $R_\gamma \subseteq SO_{2n}/GL_n$, then $\dim R_\gamma = n(n-1)$ as well. By the preceding remark, this is also equal to $L(\gamma_0)+\frac{n(n-1)}{2}$, finishing the proof. ☐

As before, $\Delta(n)$ denotes the set of type $DIII$ (n,n)-clans.

Definition 5.4. The length generating function of $\Delta(n)$ is defined by

$$A_n(t) = \sum_{\gamma\in\Delta(n)} t^{L(\gamma)}.$$

For example, it is easy to see from Figure 4.1 that $A_3(t)=t^3+2t^2+3t+4$. Now, we provide a recurrence relation for $A_n(t)$, which collapses to equation (3.5) when $t=1$.

Proposition 5.5. *The length generating function $A_n(t)$ satisfies the following recurrence:*

$$A_n(t) = 2A_{n-1}(t) + (t+t^2+\cdots t^{n-2}+2t^{n-1}+t^n+\cdots+t^{2n-3})A_{n-2}(t). \tag{5.4}$$

Proof. We break this into two parts, one for each of the recursive terms.

Coefficient of $\mathbf{A_{n-1}(t)}$: Let γ be an arbitrary clan from $\Delta(n-1)$. Then, we can always create a new clan $+\gamma- \in \Delta(n)$ simply by inserting a $+$ at the beginning of the string and appending a $-$ at the end of the string. It is clear that this procedure does not affect the value of the length function.

We can create a different clan $\mathrm{Flip}(-\gamma+)\in\Delta(n)$ similarly, where the flip is required to ensure that there are an even number of $-$'s and Π_1 pairs among the first n symbols. In this situation, there are a few possibilities. Let $\gamma = c_2\cdots c_n c_{n+1}\cdots c_{2n-1}$ for convenience.

$[\mathbf{c_n c_{n+1}} = \pm\mp]$: Attaching the new symbols and flipping has no consequence for any portion of the length function computation.

$[\gamma = \cdots \mathbf{a} \cdots \mathbf{ab} \cdots \mathbf{b} \cdots]$: Attaching $-$ and $+$ has no effect. Upon flipping, $s(a)$ and $s(b)$ both increase by one, but so does $w(b)$ and p, so there is no net effect on the length function.

$[\gamma = \cdots \mathbf{a} \cdots \mathbf{ba} \cdots \mathbf{b} \cdots]$: Identical to the previous case, but change "increase" to "decrease."

We see that given an arbitrary $(n-1, n-1)$-clan, we can create two different (n, n)-clans for which the length function evaluates to the same number, which accounts for the first term in the equation (5.4). These comprise all of the clans in $\Delta(n)$ which start and end with $+$ or $-$.

Coefficient of $\mathbf{A_{n-2}(t)}$: Now let γ be an arbitrary clan from $\Delta(n-2)$. We obtain a new clan $\gamma' = c'_1 \cdots c'_{2n} \in \Delta(n)$ by inserting $a \in \mathbb{N}$ as c'_1 and c'_i and $b \in \mathbb{N}$ as c'_{2n+1-i} and c'_{2n}. Observe that a and b each contribute a spread of $i-1$, and $w(a) = 0$ for every i.

If $i \leq n$, then both a and b go in as Π_1 pairs, so p is unchanged. If $w(b)$ results positive due to any natural number pair (c'_u, c'_t), this contribution will cancel in the length formula by the fact that the first b at c'_{2n+1-i} is increasing the spread of that pair by one, as compared to its placement in γ. The insertion of b cannot affect the weave of any other natural number because the last symbol is b. If a contributes to the weave of any other natural number r, this is likewise cancelled out by an increase of one in $s(r)$. Thus, the length is only affected by the spreads of a and b, and increases by $i-1$ on balance. Since this holds any choice of $2 \leq i \leq n$, we see that $(t + t^2 + \ldots t^{n-1})A_{n-2}(t)$ appears in the recurrence.

Now suppose $i > n$. Both a and b go in as Π_0 pairs, so p increases by one. As with the previous case, weave contributions of a and b cancel with spread contributions to other numbers with one exception: the pair of a's (c'_1, c'_i) contributes one to the weave of b which is not compensated in any other manner. Thus, compared to the length of γ, $L(\gamma')$ is augmented by $i-1$ from the spreads of a and b, and but diminished by one from the change in p and $w(b)$. In all, each choice of $n < i \leq 2n-1$ gives a different clan whose length is $i-2$ greater than $L(\gamma)$, accounting for an additional term of $(t^{n-1} + t^n + \ldots t^{2n-3})A_{n-2}(t)$ in the recurrence formula.

Adding these cases together gives the claim. $\square$

For consideration, we also mention that $A_1(t) = 1$, $A_2(t) = t + 2$, and, in view of Figure 5.1, $A_4(t) = t^6 + 3t^5 + 4t^4 + 7t^3 + 7t^2 + 8t + 8$. In that figure, the black edges between $\gamma \prec \gamma'$ are labelled with the index i of the root α_i such that $s_{\alpha_i} \cdot \gamma = \gamma'$, while the red edges are those from the full closure order on corresponding orbits. The closed orbits in blue are represented by just their first four symbols due to space considerations.

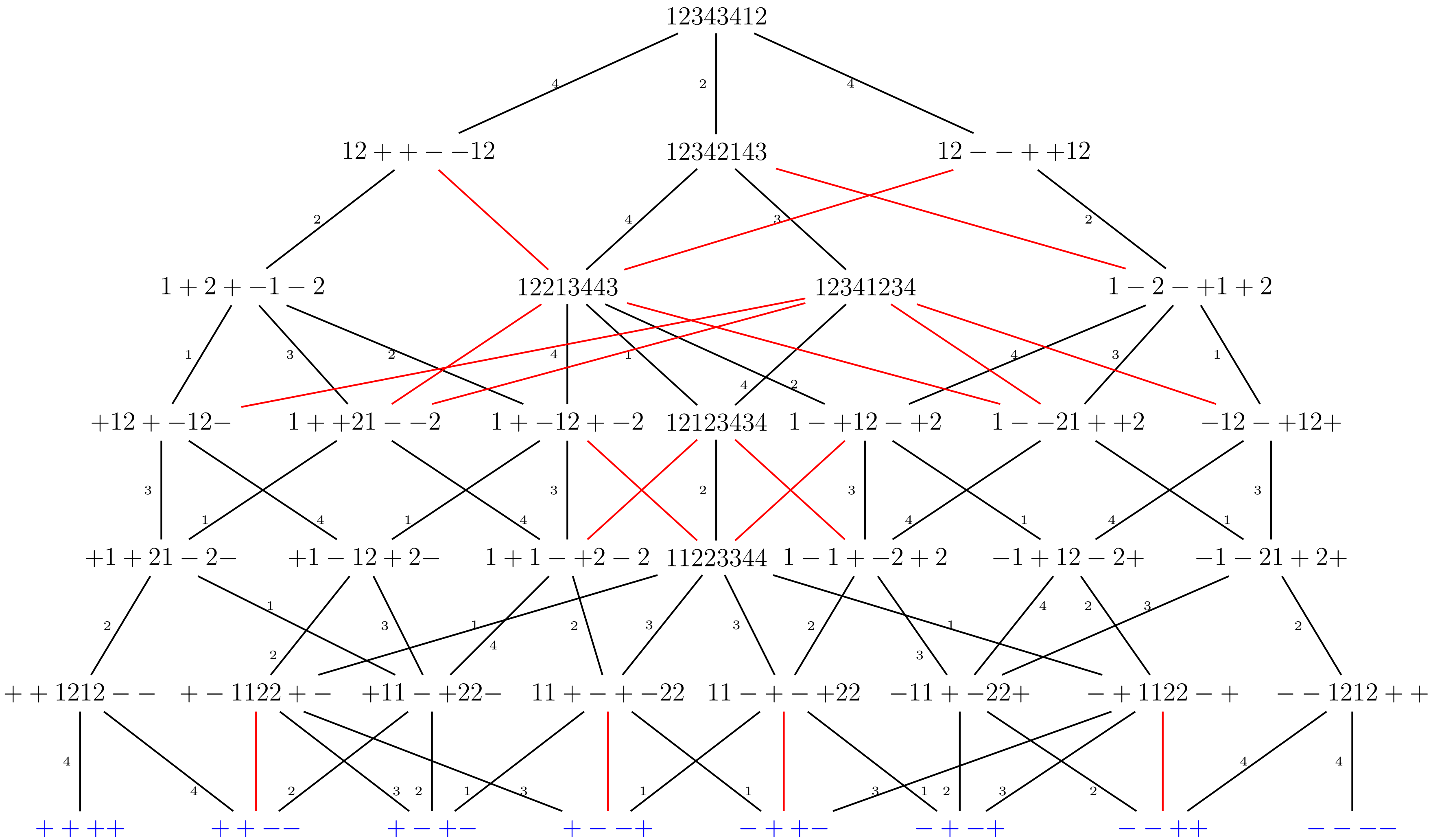


Figure 5.1: The weak and full closure orders on $\Delta(4)$.

6 Sects

6.1 Background

In order to present sects for type $DIII$ clans, we must visit the theory of parabolic subgroups of special orthogonal groups. We refer to [13] for more detail.

Given a vector space V with bilinear form ω, recall that an *isotropic subspace* W is one such that $\omega(\mathbf{u}, \mathbf{v}) = 0$ for all vectors $\mathbf{u}, \mathbf{v} \in W$. If we also use ω to stand for the matrix which represents this bilinear form in a particular choice of basis, this condition becomes $\mathbf{u}^t \omega \mathbf{v} = \mathbf{0}$. A *polarization* of V is then an orthogonal decomposition (with respect to the usual dot product, in this case) of V into subspaces which are each isotropic (with respect to ω), that is $V = V_- \oplus V_+$. We also define an *isotropic flag* as a sequence of vector spaces

$$\{\mathbf{0}\} \subset V_1 \subset \dots V_r \subset V$$

such that V_i is an isotropic subspace of V for all $1 \leq i \leq r$.

Taking $V = \mathbb{C}^{2n}$, we have a bilinear form given by J_{2n} which defines the special orthogonal group SO_{2n}. From [13], Proposition 12.13, the parabolic subgroups of $G := SO_{2n}$ are precisely the stabilizers of flags which are isotropic with respect to J_{2n}.[6]

Let E_n be the subspace generated by standard basis vectors $\{\mathbf{e}_i \mid 1 \leq i \leq n\}$. It is easy to check that this is an isotropic subspace of $\mathbb{C}^{2n}$ with respect to J_{2n}, and in fact this space is maximally isotropic. There is a polarization of $\mathbb{C}^{2n}$ as

$$V = E_n \bigoplus \tilde{E}_n,$$

where $\tilde{E}_n$ is the subspace spanned by $\{\mathbf{e}_{n+1}, \dots, \mathbf{e}_{2n}\}$. This is a feature that the type $DIII$ symmetric space has in common with those of type $AIII$ and CI (see [10], pp. 511).

The stabilizer of the the flag $\{\mathbf{0}\} \subset E_n \subset \mathbb{C}^{2n}$ is the parabolic subgroup P consisting of matrices with $n \times n$ block form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \text{ and which has Levi factor } L = \left\{ \begin{pmatrix} A & 0 \\ 0 & J_n(A^{-1})^t J_n \end{pmatrix} \,\middle|\, A \in GL_n \right\}. \tag{6.1}$$

See [13], pp. 144 or Section 8.1 of[9] for related discussion. Thus, we see that the Levi subgroup of this parabolic subgroup coincides with the symmetric subgroup in type $DIII$, that is $G^\theta = L$ where θ is the involution of Section 2.

The upshot of this coincidence is that we have a G-equivariant projection map

$$\pi : G/L \longrightarrow G/P$$

which we can analyze. Letting B be the Borel subgroup of upper triangular matrices in G ([13], pp. 39) we can relate the B-orbits in G/P to the B-orbits in G/L. The B-orbits in

[6] Malle and Testerman define their special orthogonal group by a form which is a scalar multiple (by one-half) of the one presented here. The resulting group that leaves the form invariant is the same, by the commutativity of scalar multiplication.

G/P are called *Schubert cells*. The equivariance of π allows us to ask precisely which clans constitute the pre-image of a particular Schubert cell. We call such a collection of clans the *sect* associated to the Schubert cell.

In the literature, clans usually parametrize L orbits in G/B (which can be identified with one component of the isotropic flag variety) by encoding the information of how flags in that orbit relate to the polarization. For J_{2n}, a *full* isotropic flag $F_\bullet$ in $\mathbb{C}^{2n}$ is a sequence of vector subspaces $\{V_i\}_{i=0}^n$ such that

$$\{\mathbf{0}\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_n, \tag{6.2}$$

where $\dim V_i = i$ for all i and V_n is a maximal isotropic subspace. We find it convenient to write

$$F_\bullet = \langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \rangle$$

to indicate that $F_\bullet$ is the flag with $V_i = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ for all $1 \le i \le n$. Any full isotropic flag is canonically extended to a full flag in $\mathbb{C}^{2n}$

$$\{\mathbf{0}\} \subset V_1 \subset \ldots \subset V_{2n-1} \subset V_{2n} = \mathbb{C}^{2n}$$

by assigning

$$V_{2n-i} = V_i^\perp := \{\mathbf{v} \in \mathbb{C}^{2n} \mid \omega(\mathbf{v}, \mathbf{w}) = 0,\ \forall \mathbf{w} \in V_i\},$$

so we may abuse notation slightly by using $F_\bullet$ to refer to either one. For instance, the standard isotropic full flag is extended as

$$E_\bullet := \langle \mathbf{e}_1, \ldots, \mathbf{e}_n, \mathbf{e}_{n+1}, \ldots, \mathbf{e}_{2n} \rangle.$$

If $g \in G$ is a matrix whose i^{th} column is a vector $\mathbf{v}_i$, then the isotropic flag corresponding to the coset gB will be given by $F_\bullet = \langle \mathbf{v}_1, \ldots, \mathbf{v}_{2n} \rangle$, and vice versa. For example, the identity matrix I_{2n} corresponds to the standard isotropic flag $E_\bullet$.

The space of full flags isotropic with respect to J_{2n} is a disconnected double cover of G/B; it consists of two isomorphic SO_{2n} orbits. Since we will represent flags by matrices g that identify cosets $gB \in G/B$, and we want the standard flag $E_\bullet$ to identify with the identity coset, to guarantee that a J_{2n}-isotropic flag $F_\bullet$ is in the same SO_{2n} orbit as $E_\bullet$ we must add the additional condition that $\dim(F_n \cap E_n) \equiv n \bmod 2$ ([19], pp. 106). This set of such flags is then an honest homogeneous space for SO_{2n}.

We must also present a few definitions before describing the process of obtaining orbit-representative flags; see also [3].

Definition 6.1. Given an (n,n)-clan $\gamma = c_1 \cdots c_{2n}$, one obtains the *default signed clan* associated to γ by assigning to c_i a "signature" of $-$ and to c_j a "signature" of $+$ whenever $c_i = c_j \in \mathbb{N}$ and $i < j$. We denote this default signed clan as $\tilde{\gamma} = \tilde{c}_1 \cdots \tilde{c}_{2n}$.

For instance, $\tilde{\gamma} = +1_-2_-1_+2_+-$ is the default signed clan of $\gamma = +1212-$. Note that the signature of $\tilde{c}_i$ is just the symbol itself in case c_i is $+$ or $-$.

Definition 6.2. Given a default signed clan $\tilde{\gamma}$, define a permutation $\sigma \in \mathcal{S}_{2n}$ which, for $i \leq n$:

- assigns $\sigma(i) = i$ and $\sigma(2n+1-i) = 2n+1-i$ if c_i is a symbol with signature $+$.
- assigns $\sigma(i) = 2n+1-i$ and $\sigma(2n+1-i) = i$ if c_i is a symbol with signature $-$.

We call σ the *default permutation* associated to γ.

Returning to our example, $+1212-$ has default permutation 154326 (in one-line notation). Note that σ is an involution, and it is the σ' which results from choosing $\sigma'' = id$ in the context of [21], Theorem 3.2.11.

6.2 Sects for $DIII$ Clans

Fix, as before, $G = SO_{2n}$, B its Borel subgroup of upper triangular matrices, and P and L as defined by (6.1). Next, we show how to obtain representative flags for L-orbits in G/B from type $DIII$ (n,n)-clans using a variant of the methods in [21]. To ensure that we obtain a complete set of representative flags from $\Delta(n)$, we apply the following instance of [19], Theorem 1.5.8.

Theorem 6.3. *For the symmetric pair $(G, L) = (SO_{2n}, GL_n)$ of type $DIII$, each L-orbit of G/B is equal to the intersection of an $S(GL_n \times GL_n)$-orbit in the flag variety X' of SL_{2n} with the isotropic flag variety, viewed as a subvariety $X \subseteq X'$.*

This theorem allows one to consider type $DIII$ (n,n)-clans as a subset of type $AIII$ (n,n)-clans whose elements satisfy extra conditions, and this inclusion is reflected in the containment of the respective orbits. Then for each type $DIII$ (n,n)-clan γ, to obtain a representative flag for the L-orbit Q_γ, it suffices to produce an isotropic flag $F_\bullet(\gamma)$ satisfying

$$\dim(F_n(\gamma) \cap E_n) \equiv n \bmod 2$$

which can also be produced by [21], Theorem 2.2.14, as this theorem provides flags for type $AIII$ clans. This will give us a full set of representative flags on which we can perform the sect analysis.

Theorem 6.4. *Given an (n,n)-clan $\gamma = c_1 \cdots c_{2n}$ of type $DIII$ with default permutation σ, define a flag $F_\bullet(\gamma) = \langle \mathbf{v}_1, \ldots, \mathbf{v}_{2n} \rangle$ by making the following assignments.*

- *If $c_i = \pm$, set*

$$v_i = \mathbf{e}_{\sigma(i)}.$$

- *If $c_i = c_j \in \mathbb{N}$ where $i < j$, so that $c_{2n+1-i} = c_{2n+1-j}$ as well, with $i < 2n+1-j$, then set*

$$\begin{aligned}
\mathbf{v}_i &= \frac{1}{\sqrt{2}}(\mathbf{e}_{\sigma(i)} + \mathbf{e}_{\sigma(j)}), \\
\mathbf{v}_j &= \frac{1}{\sqrt{2}}(\mathbf{e}_{\sigma(i)} - \mathbf{e}_{\sigma(j)}), \\
\mathbf{v}_{2n+1-i} &= \frac{1}{\sqrt{2}}(\mathbf{e}_{\sigma(2n+1-i)} + \mathbf{e}_{\sigma(2n+1-j)}), \\
\mathbf{v}_{2n+1-j} &= \frac{1}{\sqrt{2}}(\mathbf{e}_{\sigma(2n+1-i)} - \mathbf{e}_{\sigma(2n+1-j)}).
\end{aligned}$$

Then $F_\bullet(\gamma)$ is a representative flag for the L-orbit Q_γ. Furthermore, if g_γ is the matrix defined by letting $\mathbf{v}_i$ be its i^{th} column, then $Q_\gamma = Lg_\gamma B/B$ in G/B. Matrices/flags obtained in this way from type $DIII$ (n,n)-clans constitute a full set of representative flags for L-orbits in G/B.

Proof. The verification that $g_\gamma \in SO_{2n}$ is routine (if tedious) linear algebra. From this it follows that $F_\bullet(\gamma)$ is isotropic with respect to J_{2n}.

Next we argue that

$$\dim(F_n(\gamma) \cap E_n) \equiv n \bmod 2.$$

If a $-$ appears at c_i, then $\mathbf{v}_i = \mathbf{e}_r$ for some $r > n$. Thus, for each $-$ among the first n symbols, $\dim(F_n(\gamma) \cap E_n)$ is reduced by one (compared to when $F_n = E_n$). For each $c_i = c_j \in \mathbb{N}$, the vector subspace spanned by $\mathbf{v}_i$ and $\mathbf{v}_j$ is equal to $\text{span}(\mathbf{e}_q, \mathbf{e}_r)$ for some $q \leq n$ and some $r > n$. Then, each pair of matching natural numbers among $c_1, \ldots, c_n$ reduces $\dim(F_n(\gamma) \cap E_n)$ by one as well. Since there are an even number of $-$'s and pairs of matching natural numbers among the first n symbols,

$$\dim(F_n(\gamma) \cap E_n) = n - 2k \equiv n \bmod 2,$$

for some natural number k.

Finally, we mention how to obtain this flag from [21], Theorem 2.2.14. Let a *family* in γ mean a collection of symbols $c_i, c_j, c_{2n+1-j}, c_{2n+1-i}$ with $c_i = c_j \in \mathbb{N}$, $i < j$, and $i \leq 2n+1-j$. For each family, we modify the default signed clan by flipping the signatures of $\tilde{c}_i$ and $\tilde{c}_j$ so that they are $+$ and $-$, respectively. Denote the signed clan so obtained by $\tilde{\gamma}'$. To reflect this adjustment, we also modify the default permutation σ by swapping $\sigma(i)$ and $\sigma(j)$ for all such families. Denote the permutation so obtained by σ'.

Then σ' satisfies the conditions of [21], Theorem 2.2.14, and we claim the flag produced by that theorem using $\tilde{\gamma}'$ and σ', and denoted by $F'_\bullet(\gamma) = \langle \mathbf{v}'_1, \ldots, \mathbf{v}'_{2n} \rangle$, is the same flag as $F_\bullet(\gamma)$ constructed above. Indeed, applying that theorem, one finds $\mathbf{v}'_i = \mathbf{v}_i = \mathbf{e}_{\sigma(i)}$ whenever $c_i = \pm$, and for any family $c_i, c_j, c_{2n+1-j}, c_{2n+1-i}$, we get

$$\mathbf{v}_i = \mathbf{v}'_i \quad \text{and} \quad \mathbf{v}_j = \mathbf{v}'_j \quad \text{and} \quad \mathbf{v}_{2n+1-j} = -\mathbf{v}'_{2n+1-j} \quad \text{and} \quad \mathbf{v}_{2n+1-i} = \mathbf{v}'_{2n+1-i}.$$

Clearly, these generate the same flags. Thus, the theorem is proved. $\square$

For example, the matrix representative for the clan $\gamma = +1212-$ from Theorem 6.4 is

$$g_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 6.5. Given an (n, n)-clan $\gamma = c_1 \cdots c_{2n}$, one obtains the *base clan* associated to γ by replacing each symbol $\tilde{c}_i$ of the default signed clan $\tilde{\gamma}$ by its signature.

Continuing with our running example, the base clan for $+1212-$ is $+--++-$. Now we can use the flags produced by Theorem 6.4 to form the sects.

Proposition 6.6. *Let Q_γ and Q_τ be L-orbits in G/B corresponding to (n, n)-clans γ and τ of type DIII. Then Q_γ and Q_τ lie in the same P-orbit of G/B if and only if γ and τ have the same base clan.*

Proof. Assume that γ has base clan τ, where $\gamma = c_1 \cdots c_{2n}$ and $\tau = t_1 \cdots t_{2n}$, and let $F_\bullet(\gamma) = \langle \mathbf{v}_1, \ldots, \mathbf{v}_{2n} \rangle$ and $F_\bullet(\tau) = \langle \mathbf{u}_1, \ldots, \mathbf{u}_{2n} \rangle$ be the corresponding flags constructed by Theorem 6.4. As each clan has the same signature at symbols of the same index, they have the same default permutation. Then, we have two kinds of cases to examine.

Suppose we have a family, $c_i, c_j, c_{2n+1-j}, c_{2n+1-i}$. Then, Theorem 6.4 will yield flag $F_\bullet(\gamma)$ with

$$(\mathbf{v}_i, \mathbf{v}_j) = (\frac{1}{\sqrt{2}}(\mathbf{e}_r + \mathbf{e}_{2n+1-s}), \frac{1}{\sqrt{2}}(\mathbf{e}_r - \mathbf{e}_{2n+1-s}))$$

and

$$(\mathbf{v}_{2n+1-j}, \mathbf{v}_{2n+1-i}) = (\frac{1}{\sqrt{2}}(-\mathbf{e}_s + \mathbf{e}_{2n+1-r}), \frac{1}{\sqrt{2}}(\mathbf{e}_s + \mathbf{e}_{2n+1-r})),$$

where $n < r = \sigma(i)$ and $n < s = \sigma(2n + 1 - j)$. We also obtain the flag $F_\bullet(\tau)$ with

$$(\mathbf{u}_i, \mathbf{u}_j) = (\mathbf{e}_r, \mathbf{e}_{2n+1-s})$$

and

$$(\mathbf{u}_{2n+1-j}, \mathbf{u}_{2n+1-i}) = (\mathbf{e}_s, \mathbf{e}_{2n+1-r}),$$

Then define a linear map by

$$\begin{aligned} p^{r,s} : \mathbf{e}_r &\longmapsto \mathbf{e}_r + \mathbf{e}_{2n+1-s}, \\ \mathbf{e}_s &\longmapsto \mathbf{e}_s - \mathbf{e}_{2n+1-r}, \\ \mathbf{e}_i &\longmapsto \mathbf{e}_i \qquad\qquad \text{for } i \neq r, s. \end{aligned}$$

It is again routine to check that this map defines an element of P, so $p^{r,s} \cdot F_\bullet(\tau)$ is a flag in the same P-orbit. Also, this map takes $\mathbf{u}_i$ to $\mathbf{v}_i$, and $\mathbf{u}_{2n+1-j}$ to the span of $\mathbf{v}_{2n+1-j}$, yielding pairs with the same span

$$(p^{r,s} \cdot \mathbf{u}_i, p^{r,s} \cdot \mathbf{u}_j) \quad \text{and} \quad (\mathbf{v}_i, \mathbf{v}_j),$$

and

$$(p^{r,s} \cdot \mathbf{u}_{2n+1-j}, p^{r,s} \cdot \mathbf{u}_{2n+1-i}) \quad \text{and} \quad (\mathbf{v}_{2n+1-j}, \mathbf{v}_{2n+1-i}).$$

Now, after we act on the flag $F_\bullet(\tau)$ by the appropriate element of the form $p^{r,s}$ for each family,

$$\{c_i = c_j, c_{2n+1-j} = c_{2n+1-i} \mid j \neq 2n+1-i\},$$

then we obtain a flag which is an equivalent presentation of $F_\bullet(\gamma)$. Thus Q_γ is in the same P-orbit as Q_τ.

The proof of the converse is exactly as in type CI case, which can be found in [2].

□

By flipping the $L\backslash G/B$ double cosets around and applying the map π, we obtain the following corollary. See also [2].

Corollary 6.7. *Let R_γ and R_τ be B-orbits in G/L corresponding to clans γ and τ, and let $\pi : G/L \to G/P$ denote the canonical projection. Then $\pi(R_\gamma) = \pi(R_\tau)$ if and only if γ and τ have the same base clan.*

Schubert cells of SO_{2n}/P are in bijection with subsets $I \subset [2n]$ such that $|I| = n$ and if $i \in I$, then $2n+1-i \notin I$ ([5], pp. 34). P stabilizes the maximal isotropic subspace E_n, and in fact each B-orbit of G/P, denoted C_I, is represented by the isotropic subspace which is spanned by $\{\mathbf{e}_i \mid i \in I\}$. The subset I can be associated to a base clan γ_I by assigning

$$c_i = \begin{cases} + & \text{if } i \in I \\ - & \text{if } i \notin I, \end{cases} \tag{6.3}$$

and just as in [2], Bg_{γ_I}P$= C_I$. Then we have the following analog of a result from [2], whose proof is identical to the one given there, except for the fact that in this case $g_{\gamma_I}^{-1} = g_{\gamma_I}$, since it is the matrix of an even involution.

Theorem 6.8. *Let C_I be the Schubert cell corresponding to $I \subset [2n]$, and $\pi : G/L \to G/P$ the natural projection. Associate to I a base clan γ_I as in equation (6.3), and denote the set of clans with base clan γ_I by Σ_I. If R_γ denotes the B-orbit of G/L associated to the clan γ, then*

$$\pi^{-1}(C_I) = \bigsqcup_{\gamma \in \Sigma_I} R_\gamma. \tag{6.4}$$

Consequently, each sect has a base clan which corresponds to a closed orbit.

7 The Big Sect

In this section, we are going to investigate the number of (n,n)-clans lying in the largest sect; we denote this number by ϵ_n. These are the clans whose corresponding B-orbits comprise the preimage of the dense Schubert cell under the map $\pi : G/L \to G/P$. Since this sect must include the dense B-orbit corresponding to the clan γ_0 of (5.3), we see that this sect has base clan

$$\underbrace{- - \cdots - -}_{\text{first } n\text{-spots}} + + \cdots + + \quad \text{or} \quad \underbrace{- - \cdots - +}_{\text{first } n\text{-spots}} - + \cdots + +$$

depending on whether n is even or odd, respectively.

Recall that the number of $-$ symbols plus the number of pairs of matching natural numbers among the first n symbols of $\gamma \in \Delta(n)$ is always even. Then a clan lies in the largest sect only if

(a) it has natural number pairs only in Π_0 when n is even,

(b) or it has at most two Π_1 pairs at (c_i, c_n) and (c_{2n+1-i}, c_{n+1}) when n is odd.

Let $2r$ be the total number of pairs of matching natural numbers in a type $DIII$ clan $\gamma = c_1 \cdots c_{2n}$ which lies in the largest sect. These will occupy $2r$ of the symbols among $c_1 \cdots c_n$; obviously, this can be done in $\binom{n}{2r}$ many different ways. Then, the $2r$ positions form r pairs[7], which can be done in $\frac{(2r)!}{2^r r!}$ different ways. Summing over possible values for r, we have

$$\epsilon_n = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2r)!2^r r!}, \tag{7.1}$$

which happens to be the number of involutions on n letters ([15], A000085). This coincidence reveals the following.

Proposition 7.1. *Taking $\epsilon_0 = 1$, the number of clans in the largest sect satisfies the recurrence relation*

$$\epsilon_n = \epsilon_{n-1} + (n-1)\epsilon_{n-2}, \tag{7.2}$$

and has exponential generating function

$$\sum_{n=0}^{\infty} \epsilon_n \frac{x^n}{n!} = e^{x + \frac{x^2}{2}}. \tag{7.3}$$

Recall that a *partial permutation* is a map $x : \{1, \dots, m\} \longrightarrow \{0, \dots, n\}$ satisfying the following rule:

- if $x(i) = x(j)$ and $x(i) \neq 0$, then $i = j$ for each $1 \leq i, j \leq m$.

[7] These paired positions (i,j) determine first symbols of opposing Π_0 pairs unless n is odd and $j = n$ in which case the pairing forces $(c_i, c_j) \in \Pi_1$.

A partial permutation x can be represented by an $m \times n$ matrix (x_{ij}), where x_{ij} is 1 if and only if $x(i) = j$ and is 0 otherwise. Note that under this convention we view our matrices as acting on vectors from the right. Partial permutations are sometimes called *rook placements*[8], and in case $m = n$, they form a monoid under matrix multiplication called the *rook monoid* and denoted $\mathcal{R}_n$. We refer to [1] and Chapter 15 of [7] for background theory.

Definition 7.2. A *partial involution* on n elements is a partial permutation which is represented by a symmetric $n \times n$ matrix. A partial involution with no fixed points is called a *partial fixed-point-free involution*, and the set of such partial involutions is denoted $\mathcal{PF}_n$.

There is a bijection between $\mathcal{PF}_n$ and the set of invertible involutions $\mathcal{I}_n$ as follows: a partial involution matrix can be completed to the matrix of an involution by placing a 1 on the diagonal of any row/column without a 1. However, we will prove that $\epsilon_n = |\mathcal{PF}_n|$ by exhibiting an explicit bijection between the clans in the largest sect and the partial fixed-point-free involutions.

Let $\gamma = c_1 \cdots c_{2n} \in \Delta(n)$ lie in the largest sect. We construct the associated $x \in \mathcal{PF}_n$ as follows.

(i) If $c_i = \pm$, then take $x(i) = 0$ for all $1 \le i \le n$.

(ii) If $(c_i, c_j) \in \Pi_0$, then take $x(i) = 2n + 1 - j$ and $x(2n + 1 - j) = i$ for all $1 \le i \le n < j \le 2n$.

(iii) If $(c_i, c_n) \in \Pi_1$, then take $x(i) = n$ and $x(n) = i$.

It is easy to show that this map is invertible. Let us start with a partial fixed-point-free involution $x \in \mathcal{PF}_n$ and determine its associated (n, n)-clan by assigning the first n symbols and then completing using skew-symmetry.

(i) If $x(i) = 0$, then take $c_i = -$ unless $i = n$ and n is odd, in which case $c_i = +$.

(ii) If $x(i) = j$ and $x(j) = i$ with $i < j$, then we take $(c_i, c_{2n+1-j}) \in \Pi_0$ unless $j = n$ and n is odd in which case we take $(c_i, c_n) \in \Pi_1$.

The algorithm and its reverse are clearly injective, completing proof of the following.

Theorem 7.3. *Partial fixed-point-free involutions on n letters and (n, n)-clans of type DIII in the largest sect are in bijection.*

The elements of $\mathcal{PF}_n$ also parameterize the congruence orbits of the invertible upper triangular $n \times n$ matrices on the skew-symmetric $n \times n$ matrices (with complex entries). This endows them with a poset structure which is the containment order of the corresponding orbit closures, studied in [8] and [4]. Let this poset be denoted $(\mathcal{PF}_n, \le_{con})$. The order relation $\le_{con}$ admits a simple combinatorial description in terms of rank-control matrices.

[8] The rook placements of Section 3.2 are a particular instance, viewing the boards as matrices.

From [17], we know that the full closure order (or *Bruhat order*) on clans can be deduced from the weak order using a simple recursive rule. However, for type $DIII$ clans, a general combinatorial description that covers arbitrary n is lacking. Moreover, in [20] it is pointed out that the Bruhat order on type $DIII$ (n, n)-clans fails in general to be the restriction of the Bruhat order on all (n, n)-clans, though this is conjectured to be true in type CI, for instance. In particular, Wyser points out that $DIII$ clans $1 + -12 + -2$ and 12341234 are not related in the Bruhat order in type $DIII$ (as can be observed in Figure 5.1), though they are related as type $AIII$ clans.

Despite these obstacles and having obtained similar results in types $AIII$ and CI in [3] and [2], we find it appropriate to close this paper by conjecturing that the Bruhat order on type $DIII$ (n, n)-clans, restricted to the big sect, is isomorphic as a poset to $(\mathcal{PF}_n, \leq_{con})$.

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