Composition polynomials of the RNA matrix and B-composition polynomials of the Riordan pseudo-involution

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Abstract

Let $(a(x), xa(x))$ is the Riordan matrix from the Bell subgroup. We denote $(a(x), xa(x))^{\varphi} = \left(a^{(\varphi)}(x), xa^{(\varphi)}(x)\right)$, where power of the matrix is defined in the standard way. Polynomials $c_n(x)$ such that $a^{(\varphi)}(x) = \sum_{n=0}^{\infty} c_n(\varphi) x^n$ will be called composition polynomials. We consider composition polynomials of the RNA matrix. Construction associated with these polynomials allows the following generalization. If the matrix $(a(x), xa(x))$ is a pseudo-involution, then there exists numerical sequence (B-sequence) with the generating function $B(x)$ such that $a(x) = 1 + xa(x) B\left(x^2 a(x)\right)$. Matrix, whose B-sequence has the generating function $\varphi B(x)$, will be denoted by $\left(a^{[\varphi]}(x), xa^{[\varphi]}(x)\right)$. Polynomials $u_n(x)$ such that $a^{[\varphi]}(x) = \sum_{n=0}^{\infty} u_n(\varphi) x^n$ will be called B-composition polynomials. Coefficients of these polynomials are expressed in terms of the B-sequence. We show that the matrices whose rows correspond to the B-composition polynomials are connected with the exponential Riordan matrices in a certain way.

1 Introduction

Matrices that we will consider correspond to operators in the space of formal power series. We will associate rows and columns of matrices with the generating functions of their elements, i.e. with the formal power series. Thus, the expression $Aa(x) = b(x)$ means that the column vector multiplied by the matrix A has the generating function $a(x)$, resultant column vector has the generating function $b(x)$. nth coefficient of the series $a(x)$ denote $[x^n] a(x)$; (n, m)th element, nth row, nth descending diagonal, nth ascending diagonal and nth column of the matrix A will be denoted respectively by

$$(A)_{n,m}, \qquad [n, \rightarrow] A, \qquad [n, \searrow] A, \qquad [n, \nearrow] A, \qquad Ax^n.$$

Infinite lower triangular matrix $(f(x), g(x))$, nth column of which has the generating function $f(x) g^n(x)$, $g_0 = 0$, is called Riordan matrix (Riordan array). It is the product of two matrices that correspond to the operators of multiplication and composition of series:

$$(f(x), g(x)) = (f(x), x)(1, g(x)),$$

$$(f(x), x) a(x) = f(x) a(x), \qquad (1, g(x)) a(x) = a(g(x)),$$

$$(f(x), g(x))(b(x), a(x)) = (f(x) b(g(x)), a(g(x))).$$

Elements of the matrix $(1, g(x))$ are expressed through coefficients of the series $g(x)$ by the formula

$$((1, g(x)))_{n,m} = \sum_{n,m} \frac{m!}{m_1! m_2! \dots m_n!} g_1^{m_1} g_2^{m_2} \dots g_n^{m_n},$$

where the summation is over all monomials $g_1^{m_1} g_2^{m_2} ... \ g_n^{m_n}$ for which $n = \sum_{i=1}^{n} i m_i$, $m = \sum_{i=1}^{n} m_i$.

If $f_0 \neq 0$, $g_1 \neq 0$, matrix $(f(x), g(x))$ is called proper. Proper Riordan matrices form a group called the Riordan group. Matrices of the form $(f(x), x)$ form a subgroup called the Appell subgroup; matrices of the form $(f(x), xf(x))$ form a subgroup called the Bell subgroup.

Matrices

$$|e^x|^{-1}(f(x), g(x))|e^x| = (f(x), g(x))_E,$$

where $|e^x|$ is the diagonal matrix, $|e^x| x^n = x^n/n!$, are called exponential Riordan matrices. Denote $[n, \to](f(x), g(x))_E = s_n(x)$. Then

$$\sum_{n=0}^{\infty} \frac{s_n(\varphi)}{n!} x^n = f(x) \exp(\varphi g(x)).$$

If $g(x) = x$, then the sequence of polynomials $s_n(x)$ is called Appel sequence. Matrix, power of which is defined by the identity

$$P^\varphi = \left(\frac{1}{1-\varphi x}, \frac{x}{1-\varphi x}\right) = (e^{\varphi x}, x)_E, \qquad [n, \to] P^\varphi = (\varphi + x)^n,$$

is called Pascal matrix.

Riordan matrix $(f(x), xg(x))$, $g_0 = \pm 1$, having property

$$(f(x), xg(x))^{-1} = (1, -x)(f(x), xg(x))(1, -x) = (f(-x), xg(-x)),$$

is called pseudo-involution in the Riordan group [1] – [8]. Example of pseudo-involution is the power of the Pascal matrix. For each pseudo-involution $(f(x), xg(x))$, $g_0 = 1$, there exists numerical sequence $B = (b_0, b_1, b_2, ...)$, with the generating function $B(x)$, such that

$$g(x) = 1 + xg(x) B\left(x^2 g(x)\right).$$

Sequence B is called B-sequence of the matrix $(f(x), xg(x))$ ([4],[5]; in [4] this sequence is called Δ-sequence). Generating function of this sequence will be called B-function of the matrix $(f(x), xg(x))$.

Consider the following construction for the Bell subgroup matrices $(a(x), xa(x))$, $a_0 = 1$. Denote

$$(a(x), xa(x))^\varphi = \sum_{n=0}^{\infty} \binom{\varphi}{n} ((a(x), xa(x)) - I)^n,$$

$$\log(a(x), xa(x)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} ((a(x), xa(x)) - I)^n.$$

where $I = (1, x)$. Then

$$(a(x), xa(x))^\varphi = \sum_{n=0}^{\infty} \frac{\varphi}{n!} (\log(a(x), xa(x)))^n.$$

Build the matrix $L(a(x))$ by the rule $L(a(x)) x^n = (\log(a(x), xa(x)))^n x^0$. Denote $(a(x), xa(x))^\varphi = \left(a^{(\varphi)}(x), xa^{(\varphi)}(x)\right)$, $[n, \to] L(a(x)) |e^x| = c_n(x)$. Then $a^{(\varphi)}(x) = \sum_{n=0}^{\infty} c_n(\varphi) x^n$. Polynomials $c_n(x)$ will be called composition polynomials. (If in this construction we replace the Bell subgroup matrices with the Appell subgroup matrices, then we get $a^{(\varphi)}(x) = a^\varphi(x)$, $L(a(x)) = (1, \log a(x))$; in this case, the polynomials $c_n(x)$ are called convolution polynomials [9]).

Note that if the matrix $(a(x), xa(x))$ is pseudo-involution, i.e. $a^{(-1)}(x) = a(-x)$, then the polynomial $c_{2n}(x)$ is even function, the polynomial $c_{2n+1}(x)$ is odd function.
Example 1.

$$a(x) = (1-x)^{-1}, \qquad (a(x), xa(x)) = (e^x, x)_E,$$

$$\log(a(x), xa(x)) = (x, x)_E, \qquad L(a(x)) = |e^x|^{-1}, \qquad c_n(x) = x^n.$$

In Section 2, we consider composition polynomials of the RNA matrix. Construction associated with these polynomials allows a generalization, which we introduce in Section 3. Matrix, whose B-sequence has the generating function $\varphi B(x)$, will be denoted by $(a(x), xa(x))^{[\varphi]} = \left(a^{[\varphi]}(x), xa^{[\varphi]}(x)\right)$, $a^{[1]}(x) = a(x)$. Polynomials $u_n(x)$ such that $a^{[\varphi]}(x) = \sum_{n=0}^{\infty} u_n(\varphi) x^n$ will be called B-composition polynomials. Coefficients of these polynomials are expressed in terms of the B-sequences of the matrix $(a(x), xa(x))$ by a certain formula. Using this formula, we can build the matrix, rows of which correspond to the B-composition polynomials. We call such matrix B-composition matrix. In Section 4, Section 5, we build B-composition matrices for the cases $B = 1 + x$, $B = C(x)$, where $C(x)$ is the Catalan series. In Section 6, we prove a simple but unexpected theorem on the connection of B-composition matrices with exponential Riordan matrices. Using this connection, in Section 7 we introduce the B-composition-convolution polynomials such that $\left(a^{[\varphi]}(x)\right)^{\beta} = \sum_{n=0}^{\infty} u_n(\beta, \varphi) x^n$.

2 Composition polynomials of the RNA matrix

Let $(R(x), xR(x))$ is the RNA matrix:

$$(R(x), xR(x)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 2 & 3 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 4 & 6 & 6 & 4 & 1 & 0 & 0 & \cdots \\ 8 & 13 & 13 & 10 & 5 & 1 & 0 & \cdots \\ 17 & 28 & 30 & 24 & 15 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$(R(x), xR(x))^{\varphi} = \left(C\left(x^2\right), xC\left(x^2\right)\right)^{-1} P^{\varphi} \left(C\left(x^2\right), xC\left(x^2\right)\right) =$$

$$= \left(\frac{1}{1+x^2}, \frac{x}{1+x^2}\right) \left(\frac{1}{1-\varphi x}, \frac{x}{1-\varphi x}\right) \left(\frac{1-\sqrt{1-4x^2}}{2x^2}, \frac{1-\sqrt{1-4x^2}}{2x}\right),$$

$$R^{(\varphi)}(x) = \frac{1 - \varphi x + x^2 - \sqrt{\left(1 - \varphi x + x^2\right)^2 - 4x^2}}{2x^2}.$$

Matrix $(R(x), xR(x))^{\varphi}$ is the pseudo-evolution. In [6] it is shown that if $B(x)$ is the B-function of the matrix $(a(x), xa(x))$, then the coefficients of the series $a^v(x)$ are expressed through coefficients of the series $B(x)$ by the formula

$$[x^n]\, a^v(x) = \sum_n \frac{v(v+k-1)_{q-1}}{m_0! m_1! ... m_p!} b_0^{m_0} b_1^{m_1} ... b_p^{m_p},$$

$$(v+k-1)_{q-1} = (v+k-1)(v+k-2)...(v+k-q+1),$$

$$p = \left\lfloor \frac{n-1}{2} \right\rfloor, \qquad k = \sum_{i=0}^{p} (i+1) m_i, \qquad q = \sum_{i=0}^{p} m_i,$$

where the summation is over all monomials $b_0^{m_0} b_1^{m_1} ... b_p^{m_p}$ for which $n = \sum_{i=0}^{p} m_i (2i+1)$. This formula is called B-expansion. Series $R^{(\varphi)}(x)$ is solution to the equation

$$a(x) = 1 + x a(x) \left(\frac{\varphi}{1 - x^2 a(x)} \right),$$

so that B-function of the matrix $(R(x), xR(x))^{\varphi}$ is the series $\varphi(1-x)^{-1}$. Hence, composition polynomials of the RNA matrix (we denote them $r_n(x)$) have the form

$$r_0(x) = 1, \qquad r_n(x) = \sum_{m=0}^{n} \left(\sum_{n,m} \frac{\left(\frac{n+m}{2}\right)_{m-1}}{m_0! m_1! ... m_p!} \right) x^m,$$

where the summation of the coefficient of x^m is over all partitions $n = \sum_{i=0}^{p} m_i (2i+1)$, $\sum_{i=0}^{p} m_i = m$. Using this formula, we will begin to build the matrix $R = L(R(x)) |e^x|$, $[n, \to] R = r_n(x)$:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 6 & 0 & 10 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 20 & 0 & 15 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 10 & 0 & 50 & 0 & 21 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 50 & 0 & 105 & 0 & 28 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 15 & 0 & 175 & 0 & 196 & 0 & 36 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Form of this matrix leads to the assumption that $[2n, \nearrow] R = N_n(x)$, where $N_n(x)$ are the Narayana polynomials:

$$N_0(x) = 1, \qquad N_n(x) = \frac{1}{n} \sum_{m=0}^{n} \binom{n}{m-1} \binom{n}{m} x^m.$$

Let's turn to the matrix N (A090181), $[n, \to] N = N_n(x)$:

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 3 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 6 & 6 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 10 & 20 & 10 & 1 & 0 & \cdots \\ 0 & 1 & 15 & 50 & 50 & 15 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 1.

$$N x^{n+1} = \frac{x^n N_n(x)}{(1-x)^{2n+1}}, \qquad n > 0.$$

Proof. Generating function of the sequence of Narayana polynomials is

$$N(t, x) = \sum_{n=0}^{\infty} N_n(t) x^n = \frac{1 + x(1-t) - \sqrt{1 - 2x(1+t) + x^2(1-t)^2}}{2x}.$$

Then

$$N(x,t)=\sum_{n=0}^{\infty}N_n(x)\,t^n=\frac{1+t(1-x)-\sqrt{1-2t(1+x)+t^2(1-x)^2}}{2t},$$

$$N\frac{1}{1-tx}=1-t+t\sum_{n=0}^{\infty}\frac{N_n(x)\,x^nt^n}{(1-x)^{2n+1}}=1-t+\frac{t}{1-x}N\left(x,\frac{xt}{(1-x)^2}\right)=$$

$$=\frac{1+x(1-t)-\sqrt{1-2x(1+t)+x^2(1-t)^2}}{2x}=\sum_{n=0}^{\infty}N_n(t)\,x^n.$$

Theorem 2.

$$[2n,\nearrow]\,R=N_n(x).$$

Proof. Denote $\tilde{N}_0(x)=1,\ \tilde{N}_n(x)=(1/x\ )\,N_n(x)$. Then

$$\tilde{N}(x,t)=\sum_{n=0}^{\infty}\tilde{N}_n(x)\,t^n=\frac{1-t(1-x)-\sqrt{1-2t(1+x)+t^2(1-x)^2}}{2xt}.$$

By Theorem 1, if $[2n,\nearrow]\,R=N_n(x)$, then

$$Rx^{n+1}=\frac{x^{n+1}\tilde{N}_n(x^2)}{(1-x^2)^{2n+1}}.$$

Then

$$R\frac{1}{1-tx}=1+xt\sum_{n=0}^{\infty}\frac{\tilde{N}_n(x^2)\,x^nt^n}{(1-x^2)^{2n+1}}=1+\frac{xt}{1-x^2}\tilde{N}\left(x^2,\frac{xt}{(1-x^2)^2}\right)=$$

$$=\frac{1-tx+x^2-\sqrt{(1-tx+x^2)^2-4x^2}}{2x^2}=R^{(t)}(x).$$

Thus,

$$r_{2n}(x)=\sum_{m=0}^{n}N_{n+m,2m}x^{2m},\qquad r_{2n+1}(x)=\sum_{m=0}^{n}N_{n+m+1,2m+1}x^{2m+1},$$

where $N_n(x)=\sum_{m=0}^{n}N_{n,m}x^m$, or

$$r_{2n}(x)=\sum_{m=0}^{n}\frac{1}{n+m}\binom{n+m}{2m-1}\binom{n+m}{2m}x^{2m},$$

$$r_{2n+1}(x)=\sum_{m=0}^{n}\frac{1}{n+m+1}\binom{n+m+1}{2m}\binom{n+m+1}{2m+1}x^{2m+1}.$$

Generalization of the RNA matrix is the matrix$(R(\beta,x)\,,xR(\beta,x))$:

$$(R(\beta,x)\,,xR(\beta,x))^{\varphi}=\left(C\left(\beta x^2\right),xC\left(\beta x^2\right)\right)^{-1}P^{\varphi}\left(C\left(\beta x^2\right),xC\left(\beta x^2\right)\right)=$$

$$=\left(\frac{1}{1+\beta x^2},\frac{x}{1+\beta x^2}\right)\left(\frac{1}{1-\varphi x},\frac{x}{1-\varphi x}\right)\left(\frac{1-\sqrt{1-4\beta x^2}}{2\beta x^2},\frac{1-\sqrt{1-4\beta x^2}}{2\beta x}\right),$$

$$R^{(\varphi)}(\beta,x)=\frac{1-\varphi x+\beta x^2-\sqrt{(1-\varphi x+\beta x^2)^2-4\beta x^2}}{2\beta x^2}.$$

Series $R^{(\varphi)}(\beta, x)$ is solution to the equation

$$a(x) = 1 + xa(x)\left(\frac{\varphi}{1 - \beta x^2 a(x)}\right),$$

so that B-function of the matrix $(R(\beta, x), xR(\beta, x))^{\varphi}$ is the series $\varphi(1 - \beta x)^{-1}$. Hence, composition polynomials of the matrix $(R(\beta, x), xR(\beta, x))$ have the form

$$\sum_{m=0}^{n}\left(\sum_{n,m}\frac{\binom{\frac{n+m}{2}}{m-1}}{m_0!m_1!...m_p!}\right)\beta^{\frac{n-m}{2}}x^m = \left(\sqrt{\beta}\right)^n r_n\left(x/\sqrt{\beta}\right).$$

3 B-composition polynomials

Matrix, whose B-sequence has the generating function $\varphi B(x)$, will be denoted by $(a(x), xa(x))^{[\varphi]} = \left(a^{[\varphi]}(x), xa^{[\varphi]}(x)\right)$. Polynomials $u_n(x)$ such that $a^{[\varphi]}(x) = \sum_{n=0}^{\infty} u_n(\varphi)x^n$ will be called B-composition polynomials.

Theorem 3. *Let $B = (b_0, b_1, b_2, ...)$ and $u_n(x)$ are the B-sequence and B-composition polynomials of the matrix $(a(x), xa(x))$. Then*

$$[x^m]\,u_n(x) = \binom{\frac{n+m}{2}}{m-1}\sum_{n,m}\frac{b_0^{m_0}b_1^{m_1}...b_p^{m_p}}{m_0!m_1!...m_p!},$$

where summation is over all partitions $n = \sum_{i=0}^{p} m_i(2i+1)$, $\sum_{i=0}^{p} m_i = m$.

Proof. This is obvious property of the B-expansion.

Properties of the B-expansion also imply that if the B-function $B(x)$ is associated with the polynomials $u_n(x)$, then B-function $B(\beta x)$ is associated with the polynomials $\left(\sqrt{\beta}\right)^n u_n\left(x/\sqrt{\beta}\right)$.

B-expansion when $v = 1$ we call B_1-expansion. Initial terms of the B_1-expansion are:

$$a_0 = 1, \qquad a_1 = b_0, \qquad a_2 = b_0^2, \qquad a_3 = b_0^3 + b_1, \qquad a_4 = b_0^4 + 3b_0b_1,$$

$$a_5 = b_0^5 + 6b_0^2b_1 + b_2, \qquad a_6 = b_0^5 + 10b_0^3b_1 + 4b_0b_2 + 2b_1^2,$$

$$a_7 = b_0^7 + 15b_0^4b_1 + 10b_0^2b_2 + 10b_0b_1^2 + b_3,$$

$$a_8 = b_0^8 + 21b_0^5b_1 + 20b_0^3b_2 + 30b_0^2b_1^2 + 5b_0b_3 + 5b_1b_2,$$

$$a_9 = b_0^9 + 28b_0^6b_1 + 35b_0^4b_2 + 70b_0^3b_1^2 + 15b_0^2b_3 + 30b_0b_1b_2 + 5b_1^3 + b_4,$$

$$a_{10} = b_0^{10} + 36b_0^7b_1 + 56b_0^5b_2 + 140b_0^4b_1^2 + 35b_0^3b_3 + 35b_0b_1^3 + 105b_0^2b_1b_2 + \\ + 6b_0b_4 + 6b_1b_3 + 3b_2^2.$$

Using Theorem 3, we can build the matrix, rows of which correspond to the B-compositions polynomials. We call such matrix a B-composition matrix. Note that the first column of such matrix has the generating function $xB(x^2)$.

4 Case $B = 1 + x$

Series

$${}_{(1)}R^{[\varphi]}(x) = \frac{1 - \varphi x - \sqrt{(1 - \varphi x)^2 - 4\varphi x^3}}{2\varphi x^3}$$

is solution to the equation $a(x) = 1 + xa(x)\varphi(1 + x^2a(x))$, so that B-function of the matrix $\left({}_{(1)}R(x), x{}_{(1)}R(x)\right)^{[\varphi]}$ is the series $\varphi(1+x)$. B-composition matrix has the form

$$
{}_{(1)}R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 6 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 2 & 0 & 10 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 10 & 0 & 15 & 0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 30 & 0 & 21 & 0 & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 5 & 0 & 70 & 0 & 28 & 0 & 1 & 0 & \cdots \\
0 & 0 & 0 & 0 & 35 & 0 & 140 & 0 & 36 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

Coefficient of monomial $b_0^p b_1^v$ in the B_1-expansion is equal to

$$
\frac{(p+2v)_{p+v-1}}{p!v!} = \frac{(p+2v)!}{(1+v)!p!v!} = \frac{1}{1+v}\binom{p+2v}{p}\binom{2v}{v} = C_v\binom{2v+p}{p},
$$

$C_v = [x^v]\,C(x)$. Monomial $b_0^p b_1^v$ corresponds to partition of the number $n = p + 3v$ into $m = p + v$ parts. Hence,

$$
\left({}_{(1)}R\right)_{n,m} = C_{(n-m)/2}\binom{(n+m)/2}{(3m-n)/2},
$$

where $C_{(n-m)/2} = 0$, if $n-m$ is odd,

$$
[2n, \searrow]\,{}_{(1)}R = \sum_{m=0}^{\infty} C_n\binom{n+m}{m-n}x^m = x^n C_n\left(\frac{1}{1-x}\right)^{2n+1},
$$

$$
[2n, \nearrow]\,{}_{(1)}R = \sum_{m=0}^{n} C_{n-m}\binom{n}{2m-n}x^m,
$$

$$
{}_{(1)}Rx^{2n} = x^{2n}\sum_{m=0}^{n} C_m\binom{2n+m}{2n-m}x^{2m}, \quad {}_{(1)}Rx^{2n+1} = x^{2n+1}\sum_{m=0}^{n} C_m\binom{2n+1+m}{2n+1-m}x^{2m},
$$

$$
[2n, \rightarrow]\,{}_{(1)}R = {}_{(1)}r_{2n}(x) = \sum_{m=0}^{n} C_{n-m}\binom{n+m}{3m-n}x^{2m},
$$

$$
[2n+1, \rightarrow]\,{}_{(1)}R = {}_{(1)}r_{2n+1}(x) = \sum_{m=0}^{n} C_{n-m}\binom{n+1+m}{3m+1-n}x^{2m+1}.
$$

Let's turn to the polynomials $P_n(x)$ (A033282):

$$
P_n(x) = \frac{1}{n+1}\sum_{m=0}^{n}\binom{n+1}{m+1}\binom{n+m+2}{m}x^m = (1+x)^n\tilde{N}_{n+1}\left(\frac{x}{1+x}\right).
$$

Since

$$
\sum_{n=0}^{\infty}\tilde{N}_{n+1}(x)\,t^n = \frac{1 - t(1+x) - \sqrt{1 - 2t(1+x) + t^2(1-x)^2}}{2xt^2} = \bar{N}(x,t),
$$

then

$$P(x,t)=\sum_{n=0}^{\infty}P_n(x)\,t^n=\bar{N}\left(\frac{x}{1+x},(1+x)\,t\right)=\frac{1-t(1+2x)-\sqrt{1-2t(1+2x)+t^2}}{2xt^2(1+x)}.$$

Theorem 4.

$${}_{(1)}Rx^{n+1}=x^{n+1}P_n\left(x^2\right)\left(1+x^2\right).$$

Proof.

$${}_{(1)}R\frac{1}{1-tx}=1+tx\left(1+x^2\right)\sum_{n=0}^{\infty}P_n\left(x^2\right)x^nt^n=1+tx\left(1+x^2\right)P\left(x^2,xt\right)=$$

$$=\frac{1-tx-\sqrt{(1-tx)^2-4tx^3}}{2tx^3}={}_{(1)}R^{[t]}(x).$$

5 Case $B=C(x)$

Series

$${}_{(2)}R^{[\varphi]}(x)=\frac{1+((2/\varphi)-\varphi)\,x-\sqrt{1-2\varphi x+(\varphi^2-4)\,x^2}}{2x\,(1/\varphi)}$$

is solution to the equation

$$a(x)=1+xa(x)\,\varphi\left(\frac{1-\sqrt{1-4x^2a(x)}}{2x^2a(x)}\right),$$

so that B-function of the matrix $\left({}_{(2)}R(x),x_{(2)}R(x)\right)^{[\varphi]}$ is the series $\varphi C(x)$. B-composition matrix has the form

$${}_{(2)}R=\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&\cdots\\
0&1&0&0&0&0&0&0&0&0&0&\cdots\\
0&0&1&0&0&0&0&0&0&0&0&\cdots\\
0&1&0&1&0&0&0&0&0&0&0&\cdots\\
0&0&3&0&1&0&0&0&0&0&0&\cdots\\
0&2&0&6&0&1&0&0&0&0&0&\cdots\\
0&0&10&0&10&0&1&0&0&0&0&\cdots\\
0&5&0&30&0&15&0&1&0&0&0&\cdots\\
0&0&35&0&70&0&21&0&1&0&0&\cdots\\
0&14&0&140&0&140&0&28&0&1&0&\cdots\\
0&0&126&0&420&0&252&0&36&0&1&\cdots\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots
\end{pmatrix}.$$

We assume that $[2n,\searrow]\,{}_{(2)}R=(1/x^{n-1})\,[2n,\searrow]\,{}_{(1)}R$, $n>0$. Let's turn to the matrix ${}_{(1,2)}R$ (A107131), ${}_{(1,2)}Rx^{n+1}=x^{n+1}P_n(x)\,(1+x)$:

$${}_{(1,2)}R=\begin{pmatrix}
1&0&0&0&0&0&0&0&\cdots\\
0&1&0&0&0&0&0&0&\cdots\\
0&1&1&0&0&0&0&0&\cdots\\
0&0&3&1&0&0&0&0&\cdots\\
0&0&2&6&1&0&0&0&\cdots\\
0&0&0&10&10&1&0&0&\cdots\\
0&0&0&5&30&15&1&0&\cdots\\
0&0&0&0&35&70&21&1&\cdots\\
\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots
\end{pmatrix}.$$

Denote $[n,\to]\,_{(1,2)}R = F_n(x)$.
Theorem 5.
$$[n+1,\to]\,_{(2)}R = \frac{1}{x^{n-1}} F_n\left(x^2\right).$$
Proof.
$$_{(1,2)}R\frac{1}{1-tx} = F(t,x) = \sum_{n=0}^{\infty} F_n(t)\,x^n = 1 + xt(1+x)\sum_{n=0}^{\infty} P_n(x)\,x^n t^n =$$
$$= \frac{1 - xt - \sqrt{1 - 2xt(1+2x) + x^2t^2}}{2x^2t};$$
$$_{(2)}R\frac{1}{1-tx} = 1 + xt\sum_{n=0}^{\infty} F_n\left(t^2\right)\frac{x^n}{t^n} = 1 + xtF\left(t^2, x/t\right) =$$
$$= \frac{1 + ((2/t) - t)\,x - \sqrt{1 - 2tx + (t^2-4)\,x^2}}{2x\,(1/t)} = {}_{(2)}R^{[t]}(x).$$
Thus,
$$\left({}_{(2)}R\right)_{n,m} = C_{(n-m)/2}\binom{n-1}{m-1},$$
$$[2n,\to]\,_{(2)}R = {}_{(2)}r_{2n}(x) = \sum_{m=0}^{n} C_{n-m}\binom{2n-1}{2m-1}x^{2m},$$
$$[2n+1,\to]\,_{(2)}R = {}_{(2)}r_{2n+1}(x) = \sum_{m=0}^{n} C_{n-m}\binom{2n}{2m}x^{2m+1}.$$
Denote $(1/x)\,_{(2)}r_{n+1}(x) = {}_{(2)}\bar{r}_n(x)$. Since
$$(x,x)^T\,_{(2)}R\,(x,x) = \left(\bar{C}(x), x\right)_E, \quad \bar{C}(x) = \sum_{n=0}^{\infty} C_n\frac{x^{2n}}{(2n)!},$$
then sequence of polynomials $_{(2)}\bar{r}_n(x)$ is Appel sequence:
$$\sum_{n=0}^{\infty}\frac{_{(2)}\bar{r}_n(\varphi)}{n!}x^n = \bar{C}(x)\,e^{\varphi x}.$$
Thus,
$$[x^n]\,_{(2)}R^{[\varphi]}(x) = \varphi\,(n-1)!\left[x^{n-1}\right]\bar{C}(x)\,e^{\varphi x}.$$
B-composition matrix will be denoted by $\langle B(x)\rangle$. If
$$(x,x)^T\langle B(x)\rangle(x,x) = \left(\bar{B}(x), x\right)_E, \qquad \bar{B}(x) = \sum_{n=0}^{\infty} b_n\frac{x^{2n}}{(2n)!},$$
matrix $\langle B(x)\rangle$ we call the Appell type matrix.
Theorem 6. *If the matrix $\langle B(x)\rangle$ is Appell type matrixя, then $b_n = C_n b_1^n$.*
Proof. If the matrix $\langle B(x)\rangle$ is Appell type matrixя, then identity takes place
$$\sum_{m=0}^{n} b_{n-m}\binom{2n+1}{2m+1} = \sum_{2(n+1)}\frac{(k)_{q-1}}{m_0!m_1!...m_n!}b_0^{m_0}b_1^{m_1}...b_n^{m_n},$$
$$k = \sum_{i=0}^{n}(i+1)\,m_i, \qquad q = \sum_{i=0}^{n} m_i,$$

where in the right part the summation is over all monomials $b_0^{m_0}b_1^{m_1}...b_n^{m_n}$ for which $2(n+1) = \sum_{i=0}^{n} m_i (2i+1)$. We will consider this identity as equation with unknowns b_1, b_2 ... b_n (it's obvious that $b_0 = 1$). Since monomial $b_0 b_n$ corresponds to partition of the number $2(n+1)$ into two parts, equal to $2n+1$ and 1, the equation can be represented as

$$(2n+1)\,b_n + f(b_1, b_2, ..., b_{n-1}) = (n+2)\,b_n + g(b_1, b_2, ..., b_{n-1}),$$

where $f(b_1, b_2, ..., b_{n-1})$, $g(b_1, b_2, ..., b_{n-1})$ are independent of b_n. Thus, the nth term of the B-sequence, starting from the second, is uniquely expressed through the previous terms. Result is known: $b_n = C_n b_1^n$.

6 Connection theorem

Generating function of the nth descending diagonal of the exponential Riordan matrix has the form $h_n(x)/(1-x)^{2n+1}$, where $h_n(x)$ is the polynomial of degree $\le n$ ([10] - [13]). In particular,

$$[n,\searrow]\left(1, \frac{x}{1-x}\right)_E = \frac{(n+1)!N_n(x)}{(1-x)^{2n+1}},$$

$$[n,\searrow](1, x(1+x))_E = \frac{((2n)!/n!\,)\,x^n}{(1-x)^{2n+1}}, \quad [n,\searrow](1, xC(x))_E = \frac{((2n)!/n!\,)\,x}{(1-x)^{2n+1}},$$

(in the latter case $n > 0$). Thus,

$$[2n,\searrow]\left\langle\frac{1}{1-x}\right\rangle = \frac{1}{(n+1)!}[n,\searrow]\left(1, \frac{x}{1-x}\right)_E,$$

$$[2n,\searrow]\langle 1+x\rangle = \frac{1}{(n+1)!}[n,\searrow](1, x(1+x))_E,$$

$$[2n,\searrow]\langle C(x)\rangle = \frac{1}{(n+1)!}[n,\searrow](1, xC(x))_E.$$

This observation leads to the following theorem.
Theorem 7.

$$[2n,\searrow]\langle B(x)\rangle = \frac{1}{(n+1)!}[n,\searrow](1, xB(x))_E.$$

Proof.

$$((1, xB(x))_E)_{n,m} = n!\sum_{n,m}\frac{b_0^{m_0}b_1^{m_1}...b_{n-1}^{m_{n-1}}}{m_0!m_1!...m_{n-1}!},$$

where the summation is over all monomials $b_0^{m_0}b_1^{m_1}...b_{n-1}^{m_{n-1}}$ for which $n = \sum_{i=0}^{n-1} m_i(i+1)$, $m = \sum_{i=0}^{n-1} m_i$;

$$(\langle B(x)\rangle)_{n,m} = \left(\frac{n+m}{2}\right)_{m-1}\sum_{n,m}\frac{b_0^{m_0}b_1^{m_1}...b_p^{m_p}}{m_0!m_1!...m_p!},$$

where the summation is over all monomials $b_0^{m_0}b_1^{m_1}...b_p^{m_p}$ for which $n = \sum_{i=0}^{p} m_i(2i+1)$, $m = \sum_{i=0}^{p} m_i$. We must prove that

$$(\langle B(x)\rangle)_{2n-m,m} = \frac{1}{(n-m+1)!}((1, xB(x))_E)_{n,m}.$$

This comes down to the proof that

$$\sum_{n,m}\frac{b_0^{m_0}b_1^{m_1}...b_{n-1}^{m_{n-1}}}{m_0!m_1!...m_{n-1}!} = \sum_{2n-m,m}\frac{b_0^{m_0}b_1^{m_1}...b_p^{m_p}}{m_0!m_1!...m_p!},$$

where on the left the summation is carried by the rule $n=\sum_{i=0}^{n-1} m_i\,(i+1)$, $m=\sum_{i=0}^{n-1} m_i$, on the right – by the rule $2n-m=\sum_{i=0}^{p} m_i\,(2i+1)$, $m=\sum_{i=0}^{p} m_i$. Isomorphism between the set of partitions of the number n into m parts and the set of partitions of the number $2n-m$ into m odd parts (each partition $n=\sum_{i=0}^{n-m} m_i\,(i+1)$ corresponds to the partition $2n-m=\sum_{i=0}^{n-m} m_i\,(2i+1)$, and vice versa) is proof.

7 B-composition-convolution polynomials

Let $s_n\,(x)$ is the convolution polynomials of the series $B\,(x)$: $B^m\,(x)=\sum_{n=0}^{\infty} s_n\,(m)\,x^n$. Then

$$((1,xB\,(x))_E)_{n,m}=\frac{n!s_{n-m}\,(m)}{m!},$$

$$[n,\to]\,\langle B\,(x)\rangle=u_n\,(x)=\sum_{m=0}^{n}\frac{\left(\frac{n+m}{2}\right)_{m-1}s_{\frac{n-m}{2}}\,(m)}{m!}x^m,$$

$$u_{2n}\,(x)=\sum_{m=0}^{n}\binom{n+m}{2m}\frac{s_{n-m}\,(2m)}{n-m+1}x^{2m},$$

$$u_{2n+1}\,(x)=\sum_{m=0}^{n}\binom{n+m+1}{2m+1}\frac{s_{n-m}\,(2m+1)}{n-m+1}x^{2m+1}.$$

Example 2.

$$B\,(x)=e^x,\qquad u_{2n}\,(x)=\sum_{m=0}^{n}\binom{n+m}{2m}\frac{(2m)^{n-m}}{(n-m+1)!}x^{2m},$$

$$u_{2n+1}\,(x)=\sum_{m=0}^{n}\binom{n+m+1}{2m+1}\frac{(2m+1)^{n-m}}{(n-m+1)!}x^{2m+1}.$$

We use all possibilities of the B-expansion. Denote

$$u_n\,(x)=u_n^{(1)}\,(x)=\sum_{m=0}^{n}u_mx^m,\qquad u_n^{(v)}\,(x)=\sum_{m=0}^{n}\frac{v\left(v+\frac{n+m}{2}-1\right)_{v-1}}{\left(v+\frac{n-m}{2}\right)_{v-1}}u_mx^m,$$

Then

$$\left(a^{[\varphi]}\,(x)\right)^v=\sum_{n=0}^{\infty}u_n^{(v)}\,(\varphi)\,x^n.$$

Example 3.

$$\left[x^{2n}\right]R^v\,(x)=\sum_{m=0}^{n}\frac{v(v+n+m-1)_{v-1}}{(v+n-m)_{v-1}}N_{n+m,2m},$$

$$\left[x^{2n+1}\right]R^v\,(x)=\sum_{m=0}^{n}\frac{v(v+n+m)_{v-1}}{(v+n-m)_{v-1}}N_{n+m+1,2m+1}.$$

Theorem 8. *If $s_n\,(x)$ is the convolution polynomials of the B-function of the matrix $(a\,(x)\,,xa\,(x))$, $g_n\,(x)$ is the convolution polynomials of the series $a\,(x)$, then*

$$g_0\,(x)=1,\qquad g_{2n}\,(x)=\sum_{m=0}^{n}x(x+n+m-1)_{2m-1}\frac{s_{n-m}\,(2m)}{(2m)!},$$

$$g_{2n+1}\,(x)=\sum_{m=0}^{n}x(x+n+m)_{2m}\frac{s_{n-m}\,(2m+1)}{(2m+1)!}.$$

Proof. From the definition of the B-expansion it follows that

$$g_0(x) = 1, \quad g_n(x) = \sum_{m=0}^{n} x\left(x + \frac{n+m}{2} - 1\right)_{m-1} \sum_{n,m} \frac{b_0^{m_0} b_1^{m_1} \dots b_p^{m_p}}{m_0! m_1! \dots m_p!},$$

where the summation of the coefficient of $x\left(x + \frac{n+m}{2} - 1\right)_{m-1}$ is over all partitions $n = \sum_{i=0}^{p} m_i(2i+1)$, $\sum_{i=0}^{p} m_i = m$. By Theorem 7

$$\sum_{n,m} \frac{b_0^{m_0} b_1^{m_1} \dots b_p^{m_p}}{m_0! m_1! \dots m_p!} = \frac{s_{(n-m)/2}(m)}{m!}.$$

Example 4.

$$\left[x^{2n}\right] R^{\beta}(x) = \sum_{m=0}^{n} \frac{\beta(\beta + n + m - 1)_{2m-1}}{(2m)!} \binom{n+m-1}{n-m},$$

$$\left[x^{2n+1}\right] R^{\beta}(x) = \sum_{m=0}^{n} \frac{\beta(\beta + n + m)_{2m}}{(2m+1)!} \binom{n+m}{n-m}.$$

Denote

$$u_n(\beta, x) = \sum_{m=0}^{n} \beta\left(\beta + \frac{n+m}{2} - 1\right)_{m-1} \frac{s_{(n-m)/2}(m)}{m!} x^m.$$

Then

$$\left(a^{[\varphi]}(x)\right)^{\beta} = \sum_{n=0}^{\infty} u_n(\beta, \varphi) x^n.$$

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