# CYCLIC SIEVING, SKEW MACDONALD POLYNOMIALS AND SCHUR POSITIVITY 

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#### Abstract

When $\lambda$ is a partition, the specialized non-symmetric Macdonald polynomial $\mathrm{E}_{\lambda}(\mathbf{x} ; q ; 0)$ is symmetric and related to a modified Hall-Littlewood polynomial. We show that whenever all parts of the integer partition $\lambda$ is a multiple of $n$, the underlying set of fillings exhibit the cyclic sieving phenomenon (CSP) under a cyclic shift of the columns. The corresponding CSP polynomial is given by $\mathrm{E}_{\lambda}(\mathbf{x} ; q ; 0)$. In addition, we prove a refined cyclic sieving phenomenon where the content of the fillings is fixed. This refinement is closely related to an earlier result by B. Rhoades.

We also introduce a skew version of $\mathrm{E}_{\lambda}(\mathbf{x} ; q ; 0)$. We show that these are symmetric and Schur-positive via a variant of the Robinson-Schenstedt-Knuth correspondence and we also describe crystal raising- and lowering operators for the underlying fillings. Moreover, we show that the skew specialized non-symmetric Macdonald polynomials are in some cases vertical-strip LLT polynomials. As a consequence, we get a combinatorial Schur expansion of a new family of LLT polynomials.


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## 1. Introduction

The cyclic sieving phenomenon (CSP), introduced by V. Reiner, D. Stanton and D. White RSW04, is currently an active research topic. In this article, we
provide families of cyclic sieving on tableaux related to certain specializations of nonsymmetric Macdonald polynomials. This settles an earlier conjecture by the authors presented in Uhl19. The non-symmetric Macdonald polynomials are in our case closely related to the transformed Hall-Littlewood functions and Kostka-Foulkes polynomials, previously studied in the CSP context by B. Rhoades Rho10b. The family of polynomials we study is the specialization of the non-symmetric Macdonald polynomials $\mathrm{E}_{\lambda}\left(x_{1}, \ldots, x_{m} ; q, t\right)$ when $\lambda$ is an integer partition and $t=0$. They can be defined as a weighted sum over certain fillings of the Young diagram $\lambda$. We denote this set of fillings $\operatorname{COF}(\lambda, m)$, which is defined further down.
1.1. Main results. For an integer partition $\lambda$, we let $n \lambda:=\left(n \lambda_{1}, \ldots, n \lambda_{\ell}\right)$. We show that there is a natural action $\phi$ on the fillings $\operatorname{COF}(n \lambda, m)$ each block of $n$ consecutive columns is cyclically rotated one step. Consequently $\phi$ generates a $C_{n}$-action on $\operatorname{COF}(n \lambda, m)$. In Theorem 23 , we prove that for every integer partition $\lambda, m \in \mathbb{N}_{+}$and $n \in \mathbb{N}_{+}$, the triple

$$
\begin{equation*}
\left(\operatorname{COF}(n \lambda, m),\langle\phi\rangle, \mathrm{E}_{n \lambda}\left(1^{m}, q, 0\right)\right) \tag{1}
\end{equation*}
$$

exhibits the cyclic sieving phenomenon. Moreover, as $\lambda$ is held fixed and $n=$ $1,2,3, \ldots$, this family is a Lyndon-like family, a notion by P. Alexandersson, S. Linusson and S. Potka ALP19 (see also Gor19) meaning that fixed points under the group action are in natural bijection with smaller instances of the combinatorial objects. When $\lambda=(1)$, this phenomenon reduces to a classical cyclic sieving phenomenon on words of length $n$ in the alphabet [ $m$ ], see Example 13 below. A skew version of (1) is given in Theorem 37 .

We also prove a refined cyclic sieving phenomenon. Let $\operatorname{COF}(n \lambda, \nu)$ denote the set of coinversion-free fillings with shape $n \lambda$ and content $\nu$. In Theorem 28, we show that

$$
\begin{equation*}
\left(\operatorname{COF}(n \lambda, \nu),\langle\phi\rangle,\left[\mathrm{m}_{\nu}\right] \mathrm{E}_{n \lambda}(\mathrm{x}, q, 0)\right) \tag{2}
\end{equation*}
$$

exhibits the cyclic sieving phenomenon. When $\lambda=(n)$, we recover the cyclic sieving phenomenon on words of length $n$ with content $\nu$, and

$$
\left[\mathrm{m}_{\nu}\right] \mathrm{E}_{(n)}(\mathbf{x}, q, 0)=\left[\begin{array}{l}
n \\
\nu
\end{array}\right]_{q}
$$

a $q$-multinomial coefficient. We remark that if we take $\lambda=\left(1^{k}\right),\left[\mathrm{m}_{\nu}\right] \mathrm{E}_{n \lambda}(\mathrm{x}, q, 0)$ in (2) can be seen as a $q$-analogue of $n$-tuples of $k$-subsets of $[m]$ with content $\nu$. We expect that many properties of the usual major index extend to this generalization.

Finally in Section 5 we introduce a skew version of $\mathrm{E}_{\lambda}(\mathbf{x}, q, 0)$ and prove that these are symmetric and Schur positive. We provide an explicit Schur expansion using a generalization of charge in Theorem 34. As an application, in Theorem 45 we obtain a combinatorial Schur-expansion of a certain family of vertical-strip LLT polynomials, which has not been considered before. Combining Theorem 45 and Theorem 34 we have the following main result.

Theorem 1. Let $\lambda / \mu$ be a skew shape with $\ell$ rows, such that no column contains no more than two boxes. Let $\alpha$ be the composition defined as $\alpha_{i}=\lambda_{i}-\mu_{i}$ and let $\boldsymbol{\nu}$ be the tuple of skew shapes such that $\nu_{j}$ is the vertical strip $1^{\lambda_{j}} / 1^{\mu_{j}}$. Then

$$
\operatorname{LLT}_{\boldsymbol{\nu}}(\mathbf{x} ; q)=q^{\operatorname{mininv}(\boldsymbol{\nu})} \sum_{\rho \vdash\left|\lambda^{\prime} / \mu^{\prime}\right|} \mathrm{s}_{\rho^{\prime}}(\mathbf{x}) \sum_{T \in \operatorname{SSYT}(\rho, \alpha)} q^{\operatorname{charge}_{\mu}(T)}
$$

where charge ${ }_{\mu}$ is a natural generalization of the charge statistic, and $\operatorname{mininv}(\boldsymbol{\nu})$ is a simple statistic that only depend on the tuple $\boldsymbol{\nu}$.

The paper is structured as follows. In Section 2 we define the cyclic sieving phenomenon and give a brief overview of the relevant symmetric functions. In Section 3 we give a proof of the CSP in (1), and in Section 4 we prove (2). In Section 5 we introduce the skew specialized Macdonald polynomials and give the Schur expansion of these. We then prove a result in Section 7 which implies Theorem 1

We note that some of the results in this paper are based on earlier work done in the second author's master's thesis (Uhl19].

## 2. Preliminaries

### 2.1. Partitions and compositions.

Definition 2. Let $n$ and $r$ be natural numbers. A weak composition $\lambda$ of $n$ into $r$ parts is defined to be an $r$-tuple $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ of non-negative integers such that $\lambda_{1}+\cdots+\lambda_{r}=n$. We say that the numbers $\lambda_{1}, \ldots, \lambda_{r}$ are the parts of $\lambda$. If all parts of $\lambda$ are positive, we say that $\lambda$ is a composition, and we write $\lambda \vDash n$. If $\lambda$ has multiple parts of the same size, we may suppress them using exponents. As an example, $(7,7,0,1,1,1,4,4,4,4)$ may be expressed as $\left(7^{2}, 0,1^{3}, 4^{4}\right)$. We write $m_{j}(\lambda)$ for the number of parts of $\lambda$ equal to $j$ and $n \lambda:=\left(n \lambda_{1}, \ldots, n \lambda_{\ell}\right)$ for $n \in \mathbb{N}$. If further $\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{r}$, then $\lambda$ is a partition of $n$, and denote this by $\lambda \vdash n$. The parts of $\lambda$ are the positive entries of $\lambda$. The length of $\lambda$ is the number of parts and is denoted $\ell(\lambda)$. We identify partitions that only differ by trailing zeros, so $(4,2,2,1,0,0,0)=(4,2,2,1,0)=(4,2,2,1)$ as partitions. There is one unique partition of 0 , namely $\emptyset$ which is referred to as the empty partition.

Note that in some cases, the word parts is ambiguous. When $\lambda$ is a weak composition, a part can be zero whereas when $\lambda$ is a partition, a part must be a positive integer. This conflicting terminology is unfortunately very standard, see e.g. [Sta01].

### 2.2. Semistandard Young tableau.

Definition 3. Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right) \vdash n$. The Young diagram of $\lambda$ is defined as the set $\left\{(i, j) \in \mathbb{Z}^{2}: 1 \leq i \leq \lambda_{j}\right\}$. Geometrically, we think of this diagram as a set of $n$ boxes with $r$ left-justified rows and $i$ boxes in row $i$, counting from top to bottom, starting from row 1 and column 1 . The box in position $(i, j)$ is the box in the $i^{\text {th }}$ row and $j^{\text {th }}$ column. We use the notation $\lambda$ to both refer to the partition and to the Young diagram described by $\lambda$.

Define the conjugate of $\lambda$, denoted $\lambda^{\prime}$, to be the Young diagram obtained by transposing $\lambda$. Geometrically, the conjugate of $\lambda$ is obtained by reflecting $\lambda$ across the line $y=-x$. We write $\lambda^{\prime}=\left(\lambda_{1}^{\prime}, \ldots, \lambda_{l}^{\prime}\right)$. If $\lambda$ is a partition on the form $\lambda=\left(a^{b}\right)$, then $\lambda$ is a rectangular Young diagram.

Throughout this article, all the diagrams are displayed in English notation, using matrix coordinates, with a few exceptions in Section 7


Figure 1. To the left: A Young diagram of shape $\lambda=(5,5,2,1)$. To the right: A semistandard Young tableau of shape $\lambda$.

Definition 4. Let $\lambda$ be a Young diagram. A filling of $\lambda$ is a map $\lambda \rightarrow \mathbb{N}_{+}$and a semistandard Young tableau is a filling of $\lambda$ with positive integers such that in each row the entries are weakly increasing and in each row the entries are strictly increasing. The set of all semistandard Young tableaux of shape $\lambda$ is denoted $\operatorname{SSYT}(\lambda)$.

Let $T$ be a semistandard Young tableau. Define the reading word of $T$, denoted $\operatorname{rw}(T)$, as the word obtained by reading the entries $T$ from the bottom row to the top row and in each row from left to right. For example, the semistandard Young tableau in Figure 1 has reading word $\operatorname{rw}(T)=6452445511123$. We let $\mathbf{x}^{T}:=\prod_{j} x_{j}^{m_{j}(T)}$ where $m_{j}(T)$ is the number of entries in $T$ equal to $j$. The semistandard Young tableau $T$ in Figure 1 gives $\mathbf{x}^{T}=x_{1}^{3} x_{2}^{2} x_{3} x_{4}^{3} x_{5}^{2} x_{6}$.

There are several equivalent ways to define the Schur functions but the following is the most useful for our purposes. We let the Schur function indexed by the integer partition $\lambda$ be defined as

$$
\mathrm{s}_{\lambda}(\mathbf{x}):=\sum_{T \in \operatorname{SSYT}(\lambda)} \mathbf{x}^{T} .
$$

Example 5. Let $\lambda=(7,6,3,3,3,1)$ be a partition. Then $|\lambda|=23$, the conjugate partition, $\lambda^{\prime}$, is given by $(6,5,5,2,2,2,1)$ and $3 \lambda=(21,18,9,9,9,3)$. Furthermore, $m_{1}(\lambda)=1, m_{2}(\lambda)=0$ and $m_{3}(\lambda)=3$.
2.3. Burge words and RSK. The Robinson-Schenstedt-Knuth correspondence (RSK) is a famous combinatorial bijection with many different applications Sta01 Kra06. The version we use in this paper is a bijection between pairs of certain biwords and pairs of semistandard Young tableaux. We note that the biwords we consider are not lexigraphically ordered, which is otherwise typical.
Definition 6. A Burge word is a two-line array with positive integers

$$
W=\left(\begin{array}{llll}
i_{1} & i_{2} & \cdots & i_{m} \\
j_{1} & j_{2} & \cdots & j_{m}
\end{array}\right)
$$

sorted primarily increasingly in the first row and secondarily on the second row de-
 is a Burge word. A pair $\left(i_{c}, j_{c}\right)$ is called a biletter. The first row of $W$ is called the recording word and the second row of the biword is called the charge word - the reason for this terminology will be apparent in Proposition 33

We use the same row insertion bumping algorithm as the standard RSK on biwords, which we assume the readers are familiar with. Properties of our version of RSK is the third variant described by C. Krattenthaler Kra06].

| Inserted biletter | $\binom{1}{4}$ | $\binom{1}{1}$ | $\binom{2}{3}$ | $\binom{2}{2}$ | $\binom{4}{5}$ | $\binom{5}{4}$ | $\binom{5}{3}$ | $\binom{5}{1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| P | 4 | 1 <br> 4 | 1 3 <br> 4  | 1 2 <br> 3  <br> 4  | 1 2 5 <br> 3   <br> 4   | 1 2 4 <br> 3 5  <br> 4   | 1 2 3 <br> 3 4  <br> 4 5  | 1 1 3 <br> 2 4  <br> 3 5  <br> 4   |
| Q | 1 | 1 <br> 1 | 1 2 <br> 1  <br> 1  | 1 2 <br> 1  <br> 2  | 1 2 4 <br> 1   <br> 2   | 1 2 4 <br> 1 5  <br> 2   | 1 2 4 <br> 1 5  <br> 2 5  | 1 2 4 <br> 1 5  <br> 2 5  <br> 5   |

Table 1. Computing the image of a Burge word under RSK via a sequence of row insertions.

Proposition 7. The RSK-algorithm yields a bijection between Burge words and pairs of fillings $(P, Q)$ of the same shape such that the insertion tableau $P$ is semistandard and the recording tableau $Q$ has the property that $Q^{t}$ is semistandard.

As an example of Proposition 7, the procedure in Table 1 shows that we have the following correspondence.

$$
\left(\begin{array}{llllllll}
1 & 1 & 2 & 2 & 4 & 5 & 5 & 5 \\
4 & 1 & 3 & 2 & 5 & 4 & 3 & 1
\end{array}\right) \xrightarrow{R S K}\left(\begin{array}{|l|l|l}
\hline 1 & 1 & 3 \\
2 & 4 & \\
\hline 3 & 5 & \\
\hline 4 & &
\end{array}, \begin{array}{|l|l|l}
\hline 1 & 2 & 4 \\
\hline 1 & 5 & \\
\hline 2 & 5 \\
\hline 5 &
\end{array}\right) .
$$

2.4. $q$-analogues. A $q$-analogue of a certain expression is a rational function in the variable $q$ from which we can obtain the original expression by letting $q \rightarrow 1$.

Definition 8. Let $n \in \mathbb{N}$. Define the $q$-analogue of $n$ as $[n]_{q}:=1+q+\cdots+q^{n-1}$. Furthermore, define the $q$-factorial of $n$ as $[n]_{q}!:=[n]_{q}[n-1]_{q} \cdots[1]_{q}$. Lastly, the $q$-binomial coefficient are defined as

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q}:=\frac{[n]_{q}!}{[n-k]_{q}![k]_{q}!} \text { if } n \geq k \geq 0, \text { and } 0 \text { otherwise. }
$$

Theorem 9 ( $q$-Lucas theorem, see e.g Sag92]). Let $n, k \in \mathbb{N}$. Let $n_{1}, n_{0}, k_{1}, k_{0}$ be the unique natural numbers satisfying $0 \leq n_{0}, k_{0} \leq d-1$ and $n=n_{1} d+n_{0}$, $k=k_{1} d+k_{0}$. Then

$$
\left[\begin{array}{l}
n \\
k
\end{array}\right]_{q} \equiv\binom{n_{1}}{k_{1}}\left[\begin{array}{l}
n_{0} \\
k_{0}
\end{array}\right]_{q} \quad\left(\bmod \Phi_{d}(q)\right)
$$

where $\Phi_{d}(q)$ is the $d^{\text {th }}$ cyclotomic polynomial. In particular, we have

$$
\left[\begin{array}{c}
n  \tag{3}\\
k
\end{array}\right]_{\xi}=\binom{n_{1}}{k_{1}}\left[\begin{array}{l}
n_{0} \\
k_{0}
\end{array}\right]_{\xi} .
$$

if $\xi$ is a primitive $d^{\text {th }}$ root
Theorem 9 will be used in later sections.

### 2.5. Cyclic sieving.

Definition 10 (Cyclic sieving, see [RSW04]). Let $X$ be a set of combinatorial objects and $C_{n}$ be the cyclic group of order $n$ acting on $X$. Let $f(q) \in \mathbb{N}[q]$ be a polynomial with non-negative integer coefficients. We say that the triple $\left(X, C_{n}, f(q)\right)$ exhibits the cyclic sieving phenomenon, (CSP) if for all $d \in \mathbb{Z}$,

$$
\begin{equation*}
\#\left\{x \in X: g^{d} \cdot x=x\right\}=f\left(\xi^{d}\right) \tag{4}
\end{equation*}
$$

where $\xi$ is a primitive $n^{\text {th }}$ root of unity.
Note that it follows immediately from the definition that $\# X=f(1)$. It is almost always tacitly assumed that the group action of $C_{n}$ on $X$ and the polynomial $f(q)$ should be natural in some sense. The group action could be some form of rotation or cyclic shift of the elements of $X$. The polynomial usually has a closed form and is also typically the generating polynomial for some combinatorial statistic defined on $X$.

In Table 2 we summarize some of the most famous and relevant instances of cyclic sieving. For a more comprehensive list, see B. Sagan's article Sag11.

| Set | Group action | Polynomial | Reference |
| :--- | :--- | :--- | :--- |
| $k$-subsets of $[n]$ | Nearly free action | $\left[\begin{array}{l}n \\ k\end{array}\right]_{q}$ | RSW04 |
| Words with content $\alpha$ | Cyclic shift | $\left[\begin{array}{c}\alpha_{\alpha \mid} \\ \alpha_{1}, \ldots, \alpha_{\ell}\end{array}\right]_{q}$ | RSW04 |
| Non-cross. perf. matchings | Rotation | $\frac{1}{[n+1]_{q}}\left[\begin{array}{c}2 n \\ n\end{array}\right]_{q}$ | PPR08 |
| SYT $\left(n^{m}\right)$ | Promotion | $f^{\lambda}(q)$ | Rho10a |
| 01-matrices | Shift rows/columns | See Theorem 26 | Rho10b |

TABLE 2. A few known instances of cyclic sieving.

One of the main results of this paper, Theorem 23 , is a generalization of the first instance of cyclic sieving in Table 2 and it is also closely related to the last instance in the table.

Example 11 ( $k$-subset CSP, see RSW04]). Let $\binom{[n]}{k}$ be the set of $k$-subsets of [ $n$ ]. Suppose that $C_{N}$ is generated by a permutation $\sigma \in S_{n}$, where the cycles of $\sigma$ consists of $N$-cycles and one or zero singletons. Let $C_{N}$ act on $[n]$ in the natural way (this is referred to as $C_{N}$ acting nearly freely on $\left.[n]\right)$. Then $\left(\binom{[n]}{k}, C_{N},\left[\begin{array}{l}n \\ k\end{array}\right]_{q}\right)$ exhibits the cyclic sieving phenomenon.

The situation is even more interesting when different instances of the cyclic sieving phenomenon are related in a certain fashion.
Definition 12 (Lyndon-like CSP, ALP19]). Let $\left\{X_{n}\right\}_{n=1}^{\infty}$ be a family of combinatorial objects with a cyclic group action $C_{n}$ acting on $X_{n}$. Furthermore, let $\left\{f_{n}(q)\right\}_{n=1}^{\infty}$ be a sequence of polynomials in $\mathbb{N}[q]$, such that for each $n=1,2, \ldots$, the triple $\left(X_{n}, C_{n}, f_{n}(q)\right)$ exhibits the cyclic sieving phenomenon. We say that the family of triples $\left\{\left(X_{n}, C_{n}, f_{n}(q)\right)\right\}_{n=1}^{\infty}$ is Lyndon-like if $f_{n / d}(1)=f_{n}\left(e^{\frac{2 \pi i}{d}}\right)$ for all positive integers $d, n$ such that $d \mid n$.

Phrased in a different manner, the family is Lyndon-like if and only if the number of elements in $X_{n}$ fixed by $g^{d}$ is in bijection with $X_{d}$ where $g$ is an element of order $n$ in $C_{n}$. We note that the notion of Lyndon-like is also studied from a different perspective (called $q$-Gauß congruences) in Gor19.

Example 13. Let $W_{n k}$ be the set of words of length $n$ in the alphabet [ $k$ ]. Let $C_{n}$ act on $W_{n k}$ by cyclic rotation. Take $f_{n}(q)=\sum_{w \in W_{n k}} q^{\operatorname{maj}(w)}$. Then $\left(W_{n k}, C_{n}, f_{n}(q)\right)$ exhibits the cyclic sieving phenomenon. Furthermore, if we fix $k$, this family of CSP-triples is Lyndon-like.

One can show that the group action on a Lyndon-like family $X_{n}$ corresponds to rotation on some set of words of length $n$. When $X_{n}$ is the set of binary words of length $n$, the orbits of length $n$ are in bijection with Lyndon words, see A001037 in [Slo19]. Each Lyndon-like family of combinatorial objects then has an analogue of Lyndon words.
2.6. Symmetric functions and plethysm. We use standard notation (see e.g. Mac95, Sta01]) for symmetric functions. We have the elementary symmetric functions $\mathrm{e}_{\lambda}$, complete homogeneous symmetric functions $\mathrm{h}_{\lambda}$, the power-sum symmetric functions $p_{\lambda}$ and the Schur functions $s_{\lambda}$. Recall also the standard involution on symmetric functions $\omega$, with the defining properties that for $\lambda \vdash n$,

$$
\omega\left(\mathrm{h}_{\lambda}\right)=\mathrm{e}_{\lambda}, \quad \omega\left(\mathrm{s}_{\lambda}\right)=\mathrm{s}_{\lambda^{\prime}}, \quad \omega\left(\mathrm{p}_{\lambda}\right)=(-1)^{n-\ell(\lambda)} \mathrm{p}_{\lambda}
$$

We shall also require a few identities related to plethysm - for a comprehensive background on plethysm and the notation used, see J. Haglund's book Hag07. In this paper, we only need the following few properties. When $f$ is a symmetric function, we let the plethystic substitution $\mathrm{p}_{k}[f]$ for $k \in \mathbb{N}$ be defined as

$$
\begin{equation*}
\mathrm{p}_{k}[f]:=f\left(x_{1}^{k}, x_{2}^{k}, x_{3}^{k}, \ldots\right) . \tag{5}
\end{equation*}
$$

Note that in particular, $\mathrm{p}_{k}\left[\mathrm{p}_{m}\right]=\mathrm{p}_{k m}$. It is clear from the definition that for symmetric functions $f$ and $g$,

$$
\mathrm{p}_{k}[f+g]=\mathrm{p}_{k}[f]+\mathrm{p}_{k}[g] \text { and } \mathrm{p}_{k}[f \cdot g]=\mathrm{p}_{k}[f] \cdot \mathrm{p}_{k}[g] .
$$

Lemma 14. For any homogeneous symmetric function $f$ of degree $n$, we have that

$$
\mathrm{p}_{k}[\omega f]=(-1)^{(k+1) n} \omega\left(\mathrm{p}_{k}[f]\right)
$$

Proof. Since plethysm is linear, it suffices to prove the identity for $f=\mathrm{p}_{\lambda}$, where $\lambda \vdash n$. We have that $\mathrm{p}_{k}\left[\omega \mathrm{p}_{\lambda}\right]$ is equal to $\mathrm{p}_{k}\left[(-1)^{n-\ell(\lambda)} \mathrm{p}_{\lambda}\right]=(-1)^{n-\ell(\lambda)} \mathrm{p}_{k \lambda}=$ $(-1)^{(n-\ell(\lambda))+(k n-\ell(\lambda))} \omega\left(\mathrm{p}_{k \lambda}\right)$ which can be simplified to $(-1)^{(k+1) n} \omega\left(\mathrm{p}_{k}\left[\mathrm{p}_{\lambda}\right]\right)$.
2.7. Hall-Littlewood and non-symmetric Macdonald polynomials. The family of non-symmetric Macdonald polynomials, $\left\{\mathrm{E}_{\alpha}(\mathbf{x} ; q, t)\right\}_{\alpha}$ where $\alpha \in \mathbb{N}^{n}$ is a basis for $\mathbb{C}\left[x_{1}, \ldots, x_{n}\right]$. These were introduced by E. Opdam Opd95, Mac95, and further developed by I. Chrednik [Che95]. The first definition of non-symmetric Macdonald polynomials is quite cumbersome and indirect. J. Haglund, M. Haiman and N. Loehr HHL08 found a combinatorial formula for computing $\mathrm{E}_{\alpha}(\mathbf{x} ; q, t)$, using the notion of non-attacking fillings, thus generalizing F. Knop and S. Sahi's earlier formula for Jack polynomials KS97. In this paper, we shall only study a special case of the non-symmetric Macdonald polynomials, namely the case when
$\lambda$ is a partition and $t=0$. Here, we use the same notation as P . Alexandersson and M. Sawney [AS17, AS19], which differs slightly from Haglund et al. HHL08]. The notation $\mathrm{E}_{\alpha}(\mathbf{x} ; q, t)$ in this paper is equal to $E_{\operatorname{rev}(\alpha)}(\mathbf{x} ; q, t)$ in theirs where the composition has been reversed. Since we shall only study the specialization $\mathrm{E}_{\lambda}(\mathbf{x} ; q, 0)$, so we do not introduce the non-symmetric Macdonald polynomials in full generality.

Let $\lambda=\left(\lambda_{1}, \ldots, \lambda_{r}\right)$ be a Young diagram, $m \in \mathbb{N}$ with $m \geq r$. Let $F: \lambda \rightarrow[m]$ be a filling of $\lambda$. Three boxes $a, b c$ in $F$ form a triple if $a$ is just to the left of $b$ and $c$ is somewhere under $b$. The entries in a triple form an inversion-triple if they are ordered increasingly in a counter-clockwise orientation. If two entries in the triple are equal, then the entry with the largest subscript in (6) is considered to be the biggest.

A filling of shape $\lambda$ is called a coinversion-free filling if every triple is an inversiontriple and the first column is strictly decreasing from top to bottom. The set of such fillings where the entries are in $[m]$ is denoted $\operatorname{COF}(\lambda, m)$. Note that the conditions imply that every column in a coinversion-free filling must have distinct entries.

Remark 15. The definition of coinversion-free filling is essentially the same as used by P. Alexandersson and M. Sawhney [AS17] and by J. Uhlin [Uhl19] with the exception that the aforementioned texts also include basements. However, it is easy to see that these different definitions both yield $\mathrm{E}_{\lambda}(\mathbf{x} ; q, 0)$. Arguably, our definition makes the results in this article more natural. S. Assaf Ass18 and S. Assaf, N. Gonzáles AG18 study a generalized form of coinversion-free fillings, which also allows composition-shaped fillings. Therein, they are called semistandard key tabloids.

A descen $\prod^{\text {T }}$ of a filling $F$ is a box $(i, j)$ such that $F(i, j-1)<F(i, j)$. In particular, there are no descents in the first column. The set of descents is denoted $\operatorname{Des}(F)$. The leg of a box $b$ is the number of boxes that lie strictly to the right of $b$ in the diagram. In other words, if $b=(i, j)$, then $\operatorname{leg}(b)=\lambda_{i}-j$. The major index of $F$ is defined as

$$
\left.\operatorname{maj}(F)=\sum_{b \in \operatorname{Des}(F)}(\operatorname{leg}(b)+1)\right)
$$

Given a filling $F$ of shape $\lambda$, we let the weight of $F$, wt $(F)=\left(w_{1}, \ldots, w_{m}\right)$, be the vector such that $w_{i}$ count the number of occurrences of $i$ in $F$. Furthermore, we let $\mathbf{x}^{F}$ denote the monomial $\prod_{(i, j) \in \lambda} x_{F(i, j)}$, see Figure 2

The (specialized) non-symmetric Macdonald polynomial $\mathrm{E}_{\lambda}(\mathbf{x} ; q, 0)$ is then defined as

$$
\begin{equation*}
\mathrm{E}_{\lambda}\left(x_{1}, \ldots, x_{m} ; q, 0\right)=\sum_{F \in \operatorname{COF}(\lambda, m)} q^{\operatorname{maj}(F)} \mathbf{x}^{F} \tag{7}
\end{equation*}
$$

[^0]

Figure 2. A coinversion-free filling with descents marked and weight $(1,3,2,4,3,1,4)$. Its major index is $1+2+2+1=6$.

One can verify that this definition agrees with the one given in Ale15. Despite the name, the specialization $\mathrm{E}_{\lambda}(\mathbf{x} ; q, 0)$ is in fact a symmetric polynomia ${ }^{2}$ as we shall see below.

Example 16. As can be computed by summing all monomials in Figure 3, we have that $\mathrm{E}_{(2,1)}\left(x_{1}, x_{2}, x_{3} ; q, 0\right)$ is given by

$$
\begin{aligned}
& x_{1}^{2} x_{2}+x_{1} x_{2}^{2}+x_{1}^{2} x_{3}+x_{1} x_{3}^{2}+x_{2}^{2} x_{3}+x_{2} x_{3}^{2}+2 x_{1} x_{2} x_{3}+q x_{1} x_{2} x_{3} \\
& =(2+q) \mathrm{m}_{111}\left(x_{1}, x_{2}, x_{3}\right)+\mathrm{m}_{21}\left(x_{1}, x_{2}, x_{3}\right) .
\end{aligned}
$$

| 2 1 <br> 1  | 2 2 | 2 3 | 3 1 | 3 2 <br> 1  |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 1 | 1 | 1 |
| $x_{1}^{2} x_{2}$ | $x_{1} x_{2}^{2}$ | $q x_{1} x_{2} x_{3}$ | $x_{1}^{2} x_{3}$ | $x_{1} x_{2} x_{3}$ |
| 3 3 | 3 1 | 3 2 | 3 3 |  |
| 1 | 2 | 2 | 2 |  |
| $x_{1} x_{3}^{2}$ | $x_{1} x_{2} x_{3}$ | $x_{2}^{2} x_{3}$ | $x_{2} x_{3}^{2}$. |  |

Figure 3. All coinversion-free fillings of shape $\lambda=(2,1)$ in three variables with their respective contribution to Equation (7).

We shall also briefly make use of the modified Macdonald polynomials further down. Let $\lambda / \mu$ be a skew shape and let $F: \lambda / \mu \rightarrow[m]$ be a filling with no restrictions. The notion of inversion triples with the cases in (6) is extended to skew shapes, where the box $a$ may now lie outside the diagram.

Such boxes outside the diagram $\lambda / \mu$ are considered to have value $\infty$, in which case it is required $F\left(b_{1}\right)>F\left(c_{2}\right)$ in order for the triple to be an inversion triple. The notion of descent is extended to skew shapes, so that $(i, j)$ is a descent of $F$ if $F(i, j-1)<F(i, j)$, and we let $\operatorname{maj}(F)=\sum_{b \in \operatorname{Des}(F)}(\operatorname{leg}(b)+1)$ as before. Similarly,

[^1]the notion of weight is extended in the natural way. The (skew) modified Macdonald polynomial $\tilde{\mathrm{H}}_{\lambda / \mu}(\mathbf{x} ; q, t)$ is defined as
\[

$$
\begin{equation*}
\tilde{\mathrm{H}}_{\lambda / \mu}\left(x_{1}, \ldots, x_{m} ; q, t\right)=\sum_{F: \lambda / \mu \rightarrow[m]} q^{\operatorname{maj}(F)} t^{\operatorname{inv}(F)} \mathbf{x}^{F} \tag{9}
\end{equation*}
$$

\]

This stabilizes to a symmetric function as $m \rightarrow \infty$. Note that $\left[t^{*}\right] \tilde{\mathrm{H}}_{\lambda}(\mathbf{x} ; q, t)=$ $\mathrm{E}_{\lambda}(\mathbf{x} ; q, 0)$, that is, the coefficient of the highest power of $t$ that appears is given by a specialized Macdonald polynomial.

Proposition 17 (See AS17, AS19]). Let $\lambda$ be a partition with $\ell$ parts and let $S_{1}, S_{2}, \ldots, S_{\ell}$ be subsets of $[m]$ such that $\left|S_{j}\right|=\lambda_{j}$. Then there is a unique coinversion-free filling $F$ of shape $\lambda^{\prime}$ such that the entries in column $j$ of $F$ are given by $S_{j}$.

Proposition 17 implies that $\mathrm{E}_{\lambda^{\prime}}(\mathbf{x} ; 1 ; 0)=\mathrm{e}_{\lambda}(\mathbf{x})$. In fact, this identity generalizes to the non-symmetric setting, see AS19.

We shall also make use of the transformed Hall-Littlewood polynomials, $\mathrm{Q}_{\mu}^{\prime}(\mathbf{x} ; q)$. There are many different ways to define these, for example via the Kostka-Foulkes polynomials $K_{\lambda \mu}(q)$ :

$$
\begin{equation*}
\mathrm{Q}_{\mu}^{\prime}(\mathbf{x} ; q)=\sum_{\lambda} K_{\lambda \mu}(q) \mathrm{s}_{\lambda}(\mathbf{x}) \tag{10}
\end{equation*}
$$

We refer to Mac95, DLT94, TZ03 for more background and properties. For completeness, we provide a combinatorial method for computing $K_{\lambda \mu}(q)$ in Section 8 The transformed Hall-Littlewood polynomials are closely related to our specialization of the non-symmetric Macdonald polynomials.

Proposition 18 (See AS17, Thm. 14] or Hag07). For any partition $\lambda$

$$
\left.\mathrm{E}_{\lambda}(\mathbf{x} ; q ; 0)=\omega \overline{\mathrm{Q}_{\lambda^{\prime}}^{\prime}(\mathbf{x}} ; q\right)
$$

2.8. Evaluations at roots of unity. Our first goal is to evaluate $\mathrm{E}_{\lambda}(\mathbf{x} ; q ; 0)$ when $q$ is a root of unity. We need a few intermediate results.

Proposition 19 (See [LLT94, Thm. 2.1]). Let $\lambda$ be a partition of $n$. Let $d$ be a positive integer and write $m_{j}(\lambda)=d m_{j}^{\prime}+r_{j}$ with $0 \leq r_{j}<d$. Furthermore, let $\xi$ be a primitive $d^{\text {th }}$ root of unity. Then

$$
\begin{equation*}
\mathrm{Q}_{\lambda}^{\prime}(\mathbf{x} ; \xi)=\mathrm{Q}_{\tilde{\lambda}}^{\prime}(\mathbf{x} ; \xi) \prod_{j=1}^{n}\left(\mathrm{Q}_{\left(j^{d}\right)}^{\prime}(\mathbf{x} ; \xi)\right)^{m_{j}^{\prime}} \tag{11}
\end{equation*}
$$

where $\tilde{\lambda}$ is the partition $\left(1^{r_{1}}, 2^{r_{2}}, \ldots, n^{r_{n}}\right)$.
Proposition 20 (See LLT94, Thm. 2.2]). Let $\xi$ be a primitive $n^{\text {th }}$ root of unity. Then

$$
\begin{equation*}
\mathrm{Q}_{k^{n}}^{\prime}(\mathbf{x} ; \xi)=(-1)^{k(n-1)} \mathrm{p}_{n}\left[\mathrm{~h}_{k}(\mathbf{x})\right] \tag{12}
\end{equation*}
$$

Theorem 21. Let $\mu$ be an integer partition and $n \in \mathbb{N}$. Furthermore, let $\xi$ be $a$ primitive $n^{\text {th }}$ root of unity and $d$ a divisor of $n$. Then

$$
\begin{equation*}
\mathrm{E}_{n \mu}\left(1^{m} ; \xi^{d}, 0\right)=\prod_{j \geq 1}\binom{m}{j}^{d \cdot m_{j}\left(\mu^{\prime}\right)} \tag{13}
\end{equation*}
$$

Proof. We have that $\xi^{d}$ is a primitive $\left(\frac{n}{d}\right)^{\text {th }}$ root of unity. By Proposition 19

$$
\mathrm{Q}_{(n \mu)^{\prime}}^{\prime}\left(\mathbf{x} ; \xi^{d}\right)=\prod_{j \geq 1}\left(\mathrm{Q}_{j^{n / d}}^{\prime}\left(\mathbf{x} ; \xi^{d}\right)\right)^{d \cdot m_{j}\left(\mu^{\prime}\right)}
$$

Using Proposition 20, we then have

$$
\begin{equation*}
\mathrm{Q}_{(n \mu)^{\prime}}^{\prime}\left(\mathbf{x} ; \xi^{d}\right)=\prod_{j \geq 1}\left((-1)^{j(n / d-1)} \mathrm{p}_{n / d}\left[\mathrm{~h}_{j}(\mathbf{x})\right]\right)^{d \cdot m_{j}\left(\mu^{\prime}\right)} \tag{14}
\end{equation*}
$$

Applying the $\omega$-involution on both sides of 14 and using Proposition 18 and Lemma 14 gives

$$
\mathrm{E}_{n \mu}\left(\mathbf{x} ; \xi^{d}, 0\right)=\prod_{j \geq 1}\left(\mathrm{p}_{n / d}\left[\mathrm{e}_{j}(\mathbf{x})\right]\right)^{d \cdot m_{j}\left(\mu^{\prime}\right)}=\prod_{j \geq 1}\left(\mathrm{e}_{j}\left(x_{1}^{\frac{n}{d}}, x_{2}^{\frac{n}{d}}, \ldots\right)\right)^{d \cdot m_{j}\left(\mu^{\prime}\right)}
$$

Finally, we set $\left(x_{1}, \ldots, x_{m}\right)=\left(1, q, q^{2}, \ldots, q^{m-1}\right)$ and remaining variables are set to zero. Formulas for the principal specialization of elementary symmetric functions, see [Sta01, p. 303], give

$$
\mathrm{E}_{n \mu}\left(1, q, q^{2}, \ldots, q^{m-1} ; \xi^{d}, 0\right)=\prod_{j \geq 1}\left(q^{\frac{n}{d}\binom{j}{2}}\left[\begin{array}{c}
m  \tag{15}\\
j
\end{array}\right]_{q^{\frac{n}{d}}}\right)^{d \cdot m_{j}\left(\mu^{\prime}\right)}
$$

In particular, with $q=1$ we obtain 13 .
Example 22. Let $\mu=885322, n=12$ and $d=4$. Then $\mu^{\prime}=66433222$ and

$$
\mathrm{E}_{12 \mu}\left(1^{m} ; e^{\frac{2 \pi i 4}{12}}, 0\right)=\left(\binom{m}{2}^{3}\binom{m}{3}^{2}\binom{m}{4}\binom{m}{6}^{2}\right)^{4}
$$

## 3. CYCLIC SIEVING ON COINVERSION-FREE FILLINGS

Recall that $\operatorname{COF}(n \lambda, m)$ denotes the set of coinversion-free fillings of shape $n \lambda$ and entries in $[m]$. Let $\phi$ act on $\operatorname{COF}(n \lambda, m)$ by cyclically shifting the first $n$ columns one step to the right, the next $n$ columns one step to the right, and so on. Finally, the elements in each column are rearranged so that the result is again a coinversion-free filling in $\operatorname{COF}(n \lambda, m)$. This action is well-defined according to Proposition 17 Clearly, $\phi$ generates a cyclic group of order $n$ acting on $\operatorname{COF}(n \lambda, m)$, see Figure 4 for an example. We are now ready to prove one of the main results of

$$
\begin{array}{|l|l|l|l|l|l|}
\hline 6 & 4 & 2 & 1 & 1 & 6 \\
\hline 5 & 2 & 6 & 3 & 4 & 3 \\
\hline 2 & 1 & 4 & &
\end{array} \xrightarrow{\phi} \quad \begin{array}{|l|l|l|l|l|l|}
\hline 6 & 6 & 4 & 3 & 3 & 1 \\
\hline 4 & 2 & 2 & 6 & 1 & 4 \\
\hline 2 & 5 & 1 & & & \\
\hline
\end{array}
$$

Figure 4. The action of $\phi$ on a filling of shape $n \lambda=3(2,2,1)$.
the paper.
Theorem 23. For every integer partition $\lambda$ and $n \geq 1$, the triple

$$
\left(\operatorname{COF}(n \lambda, m),\langle\phi\rangle, \mathrm{E}_{n \lambda}\left(1^{m}, q, 0\right)\right)
$$

exhibits the cyclic sieving phenomenon. Moreover, the family

$$
\left\{\left(\operatorname{COF}(n \lambda, m),\langle\phi\rangle, \mathrm{E}_{n \lambda}\left(1^{m}, q, 0\right)\right)\right\}_{n=1}^{\infty}
$$

is a Lyndon-like family.

Proof. We first need to compute the number of elements fixed under $\phi^{d}$ whenever $d \mid n$. Since coinversion-free fillings are uniquely determined by their column sets, a coinversion-free filling in $\operatorname{COF}(n \lambda, m)$ fixed under $\phi^{d}$ is uniquely determined by the first $d$ columns in each consecutive block of $n$ columns. Hence,

$$
\begin{equation*}
\#\left\{F \in \operatorname{COF}(n \lambda, m): \phi^{d} \cdot F=F\right\}=\# \operatorname{COF}(d \lambda, m)=\prod_{j \geq 1}\binom{m}{k}^{d \cdot m_{j}\left(\lambda^{\prime}\right)} \tag{16}
\end{equation*}
$$

By using Theorem 21, the statements in the theorem now follows.

There is another natural group action on coinversion-free fillings. Pick $\sigma \in S_{m}$ and let $\sigma: \operatorname{COF}(\lambda, m) \rightarrow \operatorname{COF}(\lambda, m)$ act by letting $F^{\prime}=\sigma(F)$ be given by $F^{\prime}(i, j)=$ $\sigma F(i, j)$ for all $(i, j) \in \lambda$, followed by rearranging the elements in each column of $F^{\prime}$ to obtain a new coinversion-free filling. By Proposition 17, this action is well defined. Let $C_{M}=\langle\sigma\rangle$. Recall the notion of $C_{M}$ acting nearly freely on $[m]$ from Example 11. This induces a $C_{M}$-action on $\operatorname{COF}(\lambda, m)$.

Theorem 24. Let $n$ and $m$ be positive integers and let $\lambda$ be a partition. If $C_{n}$ acts nearly freely on $[m]$, then

$$
\left(\operatorname{COF}(\lambda, m), C_{n}, \mathrm{E}_{\lambda}\left(1, q, q^{2}, \ldots, q^{m-1} ; 1,0\right)\right)
$$

exhibits the cyclic sieving phenomenon.

Proof. The proof of the case when $\lambda$ is rectangular can be found in Uhl19, and extends to the general case without extra effort. Suppose $C_{n}$ is generated by $\sigma$. Write $\lambda^{\prime}=\left(\lambda_{1}^{\prime}, \ldots, \lambda_{c}^{\prime}\right)$ and consider the set of $c$-tuples $\mathbf{s}=\left(s_{1}, \ldots, s_{c}\right)$, so that $s_{i} \subseteq[m]$ and $\# s_{i}=\lambda_{i}^{\prime}$. We let $C_{n}$ act on such $c$-tuples by letting $\sigma \mathbf{s}=\left(\sigma s_{1}, \ldots, \sigma s_{c}\right)$. Proposition 17 implies that we can identify a coinversion-free filling with its columns sets. Hence, there is a bijection from $\operatorname{COF}(\lambda, m)$ to the set of $c$-tuples on the above form and it is clear that this bijection is equivariant with respect to $\sigma$, so it suffices to show that the set of $c$-tuples exhibits the cyclic sieving phenomenon.

But the set of $c$-tuples is a direct product of sets which we know exhbit the cyclic sieving phenomenon - this is just Example 11 Furthermore, the product of the CSP-polynomials in the example agree with (when $\xi=1$ ). It is straightforward to show that that CSP is preserved under taking direct products (see [Uhl19, Lem. $4.13(i)])$ so we are done.

Example 25. Note that the above theorem does in general not hold if $C_{M}$ does not act nearly freely on $[m]$. Suppose that $n=k=1$ and $C_{4}$ is generated by the permutation $\langle(1234)\rangle$. Then, the CSP-polynomial $\mathrm{E}_{n^{k}}\left(1, q, q^{2}, \ldots, q^{m-1} ; 1,0\right)=$ $1+q+q^{2}+q^{3}+q^{4}+q^{5}$, which evaluated at a primitive $4^{\text {th }}$ of unity $\xi=i$ yields $1+i$.

| 7 | 6 | 5 | 4 | 2 | 7 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 4 | 4 | 4 | 3 | 7 |  |
| 3 | 3 | 2 | 1 |  |  |
| 2 | 5 |  |  |  |  |

Figure 5. An example of the action of $\sigma=(1234567)$.

## 4. Refined CSP on $\operatorname{COF}(n \lambda)$

Let $\left[\mathrm{m}_{\nu}\right] f$ denote the coefficient of $\mathrm{m}_{\nu}$ in the symmetric function $f$. It follows from Proposition 18 and 10 that

$$
\begin{equation*}
\left[\mathrm{m}_{\nu}\right] \mathrm{E}_{\lambda}(\mathbf{x} ; q ; 0)=\sum_{\mu} K_{\mu \nu}(1) K_{\mu^{\prime} \lambda^{\prime}}(q) \tag{17}
\end{equation*}
$$

Theorem 26 (See Rho10b, Thm. 1.4]). Let $\mu$ and $\nu$ be compositions of $n$, with cyclic symmetries a and b, respectively. Let $X(\mu, \nu)$ be the set of $\ell(\lambda) \times \ell(\nu)$ binary matrices with row content $\mu$ and column-content $\nu$. Then the product $\mathbb{Z}_{\ell(\mu) / a} \times \mathbb{Z}_{\ell(\nu) / b}$ act on $X$ by a-fold row-rotation and b-fold column-rotation, respectively. Then

$$
\left(X(\mu, \nu), \mathbb{Z}_{\ell(\mu) / a} \times \mathbb{Z}_{\ell(\nu) / b}, \delta_{n}(q, t) \sum_{\lambda \vdash n} K_{\lambda \mu}(q) K_{\lambda^{\prime} \nu}(t)\right)
$$

exhibits the bi-cyclic sieving phenomenon. Here, $\delta_{n}(q, t)$ is a messy polynomial taking on values $\pm 1$ at relevant roots of unity. Furthermore, one can check that $\delta_{n}(q, t) \equiv 1$ in the case $q=1$.

By only considering the action on the columns in Theorem 26 we get the following corollary.

Corollary 27. Let $n \in \mathbb{N}, \lambda \vdash k$ where $\ell=\ell(\lambda), \nu \vDash n k$ with $m$ parts and let $X(\nu, n \lambda)$ be the set of $m \times n \lambda_{1}$ binary matrices with row-content $\nu$ and columncontent given by the conjugate of $n \lambda$. Let $\mathbb{Z}_{n}$ act on $X(\nu, n \lambda)$ by cyclic rotation of each block of $n$ consecutive columns. Then

$$
\left(X(\nu, n \lambda), \mathbb{Z}_{n}, \sum_{\mu \vdash n k} K_{\mu, \nu}(1) K_{\mu^{\prime},(n \lambda)^{\prime}}(q)\right)
$$

is a CSP-triple.

Proof. Note that there is an easy correspondence between $n$-fold rotation of columns and rotation of each block of $n$ consecutive columns.

Recall the definition of $\phi$ in Section 3, which cyclically shifts each block of $n$ consecutive columns.

Theorem 28. Let $n \in \mathbb{N}, \lambda \vdash k$ and let $\nu$ be a weak composition of $n k$ with $m$ parts. Then

$$
\left(\operatorname{COF}(n \lambda, \nu),\langle\phi\rangle,\left[\mathrm{m}_{\nu}\right] \mathrm{E}_{n \lambda}(\mathrm{x}, q, 0)\right)
$$

exhibits the cyclic sieving phenomenon.

Proof. Let $\varphi$ denote the one-step cyclic rotation of each block of $n$ consecutive columns in a matrix. By 17) it suffices to show that there is a bijection $A$ : $X(\nu, n \lambda) \rightarrow \operatorname{COF}(n \lambda, \nu)$ such that $\phi \circ A(M)=A \circ \varphi(M)$. We let $A(M)=F$ if and only if

$$
M(i, j)=1 \quad \Longleftrightarrow \quad \text { column } j \text { in } F \text { contains } i
$$

It is straightforward to verify that $\phi \circ A(M)=A \circ \varphi(M)$, so fixed-points under $\varphi^{d}$ are mapped to fixed-points under $\phi^{d}$ for all $d \in \mathbb{Z}$. This proves the theorem.

See Figure 6 for an illustration of Theorem 28 .


Figure 6. The orbits of $\operatorname{COF}(84,(8,2,2))$ under $\phi$. The CSPpolynomial is $f(q)=\left[\mathrm{m}_{(8,2,2)}\right] \mathrm{E}_{84}(\mathbf{x}, q, 0)=1+q+q^{2}+q^{3}+q^{4}+q^{6}$. One can easily check that for a primitive $4^{\text {th }}$ root of unity $\xi, f(\xi)=0$, $f\left(\xi^{2}\right)=2$ and $f\left(\xi^{4}\right)=6$ and thus agreeing with the definition of CSP in (4).

## 5. Skew specialized Macdonald polynomials

There is a natural generalization of $\mathrm{E}_{\lambda}(\mathbf{x} ; q, 0)$ to skew diagrams. In this section, we shall see that $\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0)$ is symmetric and Schur positive. Interestingly, the coefficients in the Schur expansion are not related to skew Kostka-Foulkes polynomials which at first glance is a natural guess.

A skew specialized Macdonald filling of shape $\lambda / \mu$ is a filling $F$ of the skew shape $\lambda / \mu$ such that each column of $F$ contains distinct entries, the first column is strictly decreasing, and every triple in $F$ is an inversion-triple as in 8. We let $\operatorname{COF}(\lambda / \mu)$ denote the set of all such fillings. It is not difficult to see that Proposition 17 can be generalized to the skew-case as well. In other words a skew specialized Macdonald filling is completely determined by the shape of the diagram and the ordered tuple of column sets.
Definition 29. Let $\lambda / \mu$ be a skew shape, and define the skew (specialized) nonsymmetric Macdonald polynomial as

$$
\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0)=\sum_{F \in \operatorname{COF}(\lambda / \mu)} \mathbf{x}^{F} q^{\operatorname{maj}(F)}
$$

One can quite easily see that these polynomials generalize the skew Schur functions:

$$
\begin{equation*}
\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; 0,0)=\mathrm{s}_{\lambda / \mu}(\mathbf{x}) \tag{18}
\end{equation*}
$$

As in the non-skew case, the functions $\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0)$ are actually symmetric, and we are justified to work in any number of variables. It is clear from the definition that

$$
\begin{equation*}
\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0):=\left[t^{*}\right] \tilde{\mathrm{H}}_{\lambda / \mu}(\mathbf{x} ; q, t) \tag{19}
\end{equation*}
$$

The fact that these are symmetric follows from Theorem 34 below. Symmetry was proved earlier in the non-skew case Uhl19 by using a variant of the LascouxSchützenberger involutions, see Definition 42 below.

Given a filling $F \in \operatorname{COF}(\lambda / \mu)$ and some large integer $M$, we define the extended filling $\hat{F}$ as the filling of shape $\lambda$ obtained from $F$ as

$$
\hat{F}(i, j)=\left\{\begin{array}{l}
M-i \text { if }(i, j) \in \mu  \tag{20}\\
F(i, j) \text { otherwise }
\end{array}\right.
$$

Note that $\hat{F}$ is a specialized Macdonald filling and that $\operatorname{maj}(F)=\operatorname{maj}(\hat{F})$. We shall make use of this definition in the next subsection.


Figure 7. Left: A skew specialized Macdonald filling of shape $\lambda / \mu$ with $\lambda=(6,5,4,3,3,2)$ and $\mu=(3,1,1)$ with descents marked. The weight of the filling is $(2,2,3,4,1,3,0,3)$. Right: $\hat{F}$ with $M=20$.
5.1. Charge, RSK and Schur expansion. We assume that the reader is familiar with the (row-insertion) Robinson-Schenstedt-Knuth correspondence, (RSK), described briefly in Section 2.3. See the appendix, Section 8 for the definition of charge on words and semistandard Young tableaux.
Definition 30. Let $\mu$ be a partition and let $w$ be a word with content $\beta$ (which can be a weak composition) such that $\mu+\beta$ is a partition. The postfix charge charge ${ }_{\mu}(w)$ is defined via the usual charge statistic as

$$
\operatorname{charge}_{\mu}(w):=\operatorname{charge}\left(w \cdot \ell^{\mu_{\ell}} \cdots 2^{\mu_{2}} 1^{\mu_{1}}\right)
$$

That is, we concatenate a postfix to $w$ with content $\mu$, where the letters appear in decreasing order. For example, charge ${ }_{21}(12231233)=$ charge $(12231233 \cdot 211)=2$.

Recall the definition of elementary Knuth transforms, stating that $y z x \sim_{K} y x z$ whenever $x<y \leq z$ and $x z y \sim_{K} z x y$ whenever $x \leq y<z$. Two words are Knuthequivalent, if one can obtain one from the other via a sequence of elementary Knuth transforms. If $w$ has partition-content, then its equivalence class contains a unique word which is the reading word of a semistandard Young tableau. Moreover, two words of partition content, $w$ and $w^{\prime}$ are Knuth-equivalent if and only if they insert to the same semistandard Young tableau under RSK.

Lemma 31. Let $\mu$ be a partition and suppose $u$ and $v$ are Knuth-equivalent. Then $u \cdot \ell^{\mu_{\ell}} \ldots 2^{\mu_{2}} 1^{\mu_{1}}$ and $v \cdot \ell^{\mu_{\ell}} \cdots 2^{\mu_{2}} 1^{\mu_{1}}$ are Knuth-equivalent and charge ${ }_{\mu}(u)=$ $\operatorname{charge}_{\mu}(v)$.

Proof. The first statement follows easily from the definition of Knuth-equivalence. Furthermore, if two words are Knuth-equivalent, they have the same charge, But94, Cor. 2.4.38].

Recall the notion of a Burge word from Definition 6 .
Definition 32. For each skew specialized Macdonald filling $F$ we associate a Burge word $W=W(F)$ as follows. For each entry $e=F(i, j)$, let $\binom{j}{e}$ be a biletter in $W$. Take $W$ to be the (unique) Burge word with all such biletters.

The non-skew case of Definition 32 was first given in AS17. Recall Proposition 17 and note that we can easily recover the column sets of $F$ from $W$. The map $W$ from fillings in $\operatorname{COF}(\lambda / \mu, m)$ to biwords where the top row is a weakly increasing sequence with $j$ entries equal to $\lambda_{j}^{\prime}-\mu_{j}^{\prime}$, and bottom row being elements in $[m$ ], strictly decreasing on each block of identical elements in the top row, is therefore a bijection. Note that $W$ is a bijection when we fix some shape $\lambda / \mu$. However, two skew specialized Macdonald fillings of different shapes may give the same biword. For example,

both have the same biword $\left(\begin{array}{llll}1 & 1 & 2 & 2 \\ 3 & 1 & 2 & 1\end{array}\right)$.
Note that the content of $F$ is equal to the content of $P$ and is also given by the bottom row of $W$ while the content of $Q$ is given by the top row of $W$.


Figure 8. A skew specialized Macdonald filling $F$, the corresponding biword, and the result $(P, Q)=(\operatorname{ins}(F), \operatorname{rec}(F))$ under RSK.

Proposition 33. Suppose $\lambda / \mu$ is a skew shape and $F \in \operatorname{COF}(\lambda / \mu)$. Then maj $(F)=$ charge $_{\mu^{\prime}}(\mathrm{cw}(F))$.

Proof. A short proof in the case $\mu=\emptyset$ was given in AS19. Let $F \in \operatorname{COF}(\lambda / \mu)$ and let $\hat{F}$ be the extended filling of $F$ as in 20 . Recall that $\operatorname{maj}(F)=\operatorname{maj}(\hat{F})$. It follows from the definition of the biword that $\operatorname{cw}(\hat{F})=\operatorname{cw}(F) \cdot \ell^{\mu_{\ell}^{\prime}} \cdots 2^{\mu_{2}^{\prime}} 1^{\mu_{1}^{\prime}}$. The case $\mu=\emptyset$ and Lemma $31 \operatorname{imply}$ that $\operatorname{maj}(F)=\operatorname{maj}(\hat{F})=\operatorname{charge}(\operatorname{cw}(\hat{F}))=$ charge $_{\mu^{\prime}}(\mathrm{cw}(F))$.

The Robinson-Schenstedt-Knuth correspondence has the essential property that if the word $w$ inserts to $P$ under RSK, then charge $(w)=$ charge $(P)$. From this property it follows that RSK provides a bijection

$$
\begin{equation*}
\operatorname{COF}(\lambda / \mu) \stackrel{R S K}{\longleftrightarrow} \bigcup_{\nu \vdash|\lambda / \mu|} \operatorname{SSYT}(\nu, \alpha) \times \operatorname{SSYT}\left(\nu^{\prime}\right), \quad \alpha_{i}:=\lambda_{i}^{\prime}-\mu_{i}^{\prime}, \tag{21}
\end{equation*}
$$

such that if $F \stackrel{R S K}{\longleftrightarrow}(P, Q)$ then maj $(F)=$ charge $_{\mu^{\prime}}(\operatorname{cw}(F))=$ charge $_{\mu^{\prime}}(P)$.
We now have the setup needed to prove the following theorem.
Theorem 34 (The Schur expansion of skew specialized Macdonald polynomials). Let $\lambda / \mu$ be a skew shape and let $\alpha$ be the weak composition given by $\alpha_{i}:=\lambda_{i}^{\prime}-\mu_{i}^{\prime}$. Then

$$
\begin{equation*}
\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0):=\sum_{\nu \vdash|\lambda / \mu|} \mathrm{s}_{\nu^{\prime}}(\mathbf{x}) \sum_{T \in \operatorname{SSYT}(\nu, \alpha)} q^{\text {charge }_{\mu^{\prime}}(T)} . \tag{22}
\end{equation*}
$$

Proof. By definition, $\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0)$ is equal to $\sum_{F \in \operatorname{COF}(\lambda / \mu)} \mathbf{x}^{F} q^{\operatorname{maj}(F)}$, which is equal to $\sum_{F \in \operatorname{COF}(\lambda / \mu)} \mathbf{x}^{F} q^{\text {charge }_{\mu^{\prime}}(\mathrm{cw}(F))}$ by using Proposition 33 Applying the RSK bijection in (21), we then have that

$$
\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0)=\sum_{\nu}\left(\sum_{P \in \operatorname{SSYT}\left(\nu^{\prime}, \alpha\right)} q^{\text {charge }_{\mu^{\prime}}(P)}\right) \mathrm{s}_{\nu^{\prime}}(\mathbf{x}),
$$

which is exactly the statement in (22).

| Shape $\lambda / \mu$ | Schur expansion of $\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0)$ |
| :--- | :--- |
| 1 | $\mathrm{~s}_{1}$ |
| 2 | $\mathrm{~s}_{2}+q \mathrm{~s}_{11}$ |
| 11 | $\mathrm{~s}_{11}$ |
| $21 / 1$ | $\mathrm{~s}_{2}+\mathrm{s}_{11}$ |
| 3 | $\mathrm{~s}_{3}+\left(q+q^{2}\right) \mathrm{s}_{21}+q^{3} \mathrm{~s}_{111}$ |
| 21 | $\mathrm{~s}_{21}+q \mathrm{~s}_{111}$ |
| 111 | $\mathrm{~s}_{111}$ |
| $22 / 1$ | $\mathrm{~s}_{21}+q \mathrm{~s}_{111}$ |
| $31 / 1$ | $\mathrm{~s}_{3}+(1+q) \mathrm{s}_{21}+q \mathrm{~s}_{111}$ |
| $211 / 1$ | $\mathrm{~s}_{21}+\mathrm{s}_{111}$ |
| $321 / 21$ | $\mathrm{~s}_{3}+2 \mathrm{~s}_{21}+\mathrm{s}_{111}$ |

Table 3. Here are the Schur expansions of $\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0)$ in the cases $|\lambda|-|\mu| \leq 3$.

Recall that the Littlewood-Richardson coefficients $c_{\mu \nu}^{\lambda}$ are defined via the relation $\mathrm{s}_{\mu} \mathrm{s}_{\nu}=\sum_{\lambda} c_{\mu \nu}^{\lambda} \mathrm{s}_{\lambda}$, and that the skew Schur functions expand as $\mathrm{s}_{\lambda / \mu}=\sum_{\nu} c_{\mu \nu}^{\lambda} \mathrm{s}_{\nu}$.
Corollary 35 (A Littlewood-Richardson rule). Let $\lambda / \mu$ be a skew shape and let $\alpha$ be the weak composition $\alpha_{i}:=\lambda_{i}^{\prime}-\mu_{i}^{\prime}$. Let

$$
\begin{equation*}
K_{\lambda / \mu}^{\nu}(q):=\sum_{T \in \operatorname{SSYT}(\nu, \alpha)} q^{\text {charge }_{\mu^{\prime}}(T)} . \tag{23}
\end{equation*}
$$

Then
(a) $K_{\lambda / \mu}^{\nu}(0)=c_{\mu \nu^{\prime}}^{\lambda}$, a Littlewood-Richardson coefficient,
(b) $K_{\lambda / \mu}^{\nu}(1)=K_{\nu \alpha}$, a Kostka coefficient,
(c) The product

$$
\mathrm{E}_{\lambda}(\mathbf{x} ; q, 0) \mathrm{E}_{\mu}(\mathbf{x} ; q, 0)=\sum_{\nu} K_{\kappa / c^{r}}^{\nu}(q) \mathrm{s}_{\nu^{\prime}}(\mathbf{x})
$$

where $c=\mu_{1}, r=\ell(\lambda)$, and $\kappa=\left(c+\lambda_{1}, \ldots, c+\lambda_{\ell(\lambda)}, \mu_{1}, \mu_{2}, \ldots, \mu_{\ell(\mu)}\right)$.
Proof. The first and second identity follows from Theorem 34 and 18 . The third identity follows from the observation that the product in the left hand side can be realized as a single skew specialized Macdonald polynomial, $\mathrm{E}_{(\lambda+c, \mu) / c^{r}}(\mathbf{x} ; q, 0)$, see the diagram in (24).


The coefficients $K_{\lambda / \mu}^{\nu}(q)$ might be related to the parabolic Kostka polynomials, whose constant terms are also Littlewood-Richardson coefficients, see [SW00, KSS01] for details.

We end this subsection with proving an additonal property of postfix-charge. Recall that the charge statistic is Mahonian, meaning that $\sum_{\sigma \in S_{n}} q^{\text {charge }(\sigma)}=[n]_{q}!$. There is a natural generalization of this identity for charge ${ }_{\mu}(\cdot)$.
Proposition 36. Let $\mu \vdash m$ and $n \geq 0$. Then

$$
\sum_{\sigma \in S_{n}} q^{\operatorname{charge}_{\mu}(\sigma)}=n!\prod_{i \geq 1} \frac{\left[\lambda_{i}-\mu_{i}\right]_{q}!}{\left(\lambda_{i}-\mu_{i}\right)!}
$$

where $\lambda^{\prime}=\left(\mu_{1}^{\prime}+1, \mu_{2}^{\prime}+1, \ldots, \mu_{n}^{\prime}+1, \mu_{n+1}, \ldots\right)$.
Proof. First note that the shape $\lambda / \mu$ has exactly one box in each of the first $n$ columns. For example, $\mu=53111$ and $n=8$ gives the following skew shape $\lambda / \mu$ :


We use Proposition 33 together with the fact that every permutation appear as charge word appears exactly one when summing over fillings with weight $1^{n}$. Hence,

$$
\left[x_{1} x_{2} \cdots x_{n}\right] \mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q)=\sum_{\sigma \in S_{n}} q^{\operatorname{charge}_{\mu}(\sigma)}
$$

But the left hand side is easy to compute directly since the rows of $\lambda / \mu$ occupy disjoint columns; there are $n!/\left(\left(\lambda_{1}-\mu_{1}\right) \cdots\left(\lambda_{\ell}-\mu_{\ell}\right)\right)$ ways to distribute $\{1,2, \ldots, n\}$ in the rows of $\lambda / \mu$. Furthermore, in row $i$, the major index gives the Mahonian
distribution $\left[\lambda_{i}-\mu_{i}\right]_{q}$ ! when summing over all permutations of the entries. This implies the formula.
5.2. CSP on skew specialized Macdonald polynomials. We can generalize Theorem 23 to the skew setting. We let $\phi$ act on $\operatorname{COF}(n \lambda / n \mu, m)$ as before, by cyclically shifting each block of $n$ consecutive columns one step to the right, followed by rearranging the entries in each column such that a (unique) specialized Macdonald filling is obtained. Again, $\langle\phi\rangle$ is a cyclic group of order $n$.

Theorem 37. For every skew shape $\lambda / \mu$ and $n \geq 1$, the triple

$$
\left(\operatorname{COF}(n \lambda / n \mu, m),\langle\phi\rangle, \mathrm{E}_{n \lambda / n \mu}\left(1^{m}, q, 0\right)\right)
$$

exhibits the cyclic sieving phenomenon. Furthermore, the family

$$
\left\{\left(\operatorname{COF}(n \lambda / n \mu, m),\langle\phi\rangle, \mathrm{E}_{n \lambda / n \mu}\left(1^{m}, q, 0\right)\right)\right\}_{n=1}^{\infty}
$$

is Lyndon-like, as described in Definition 12.
Proof. We first note that the number of descents between two adjacent columns of the same height only depends on the set of entries in each column, see Uhl19. Now consider the blocks of columns, where each block consists of $n$ consecutive columns (of the same height). Descents involving two entries from different blocks only contribute with a multiple of $n$ to the major index. Hence, in order to determine the major index mod $n$ of a filling, it suffices to examine the columns in each block separately. Let $\nu$ be the partition, such that the parts of $\nu^{\prime}$ is given by the multiset $\left\{\lambda_{i}^{\prime}-\mu_{i}^{\prime}: i=1,2, \ldots\right\}$. By the previous observations,

$$
\mathrm{E}_{n \lambda / n \mu}\left(1^{m}, q, 0\right) \equiv \mathrm{E}_{n \nu}\left(1^{m}, q, 0\right) \quad \bmod \left(q^{n}-1\right)
$$

It is then straightforward to use the same arguments as in the non-skew case, Theorem 23, to finish the proof.

Note that Theorem 28 can be generalized to the skew setting using a similar argument - we leave the details to the reader.

## 6. Crystal operators on words and SSYT

We now recall some minimal background on crystal operators on words and semistandard Young tableaux, see BS17, Shi] for more background.

The operators $\tilde{e}_{i}, \tilde{f}_{i}: \mathbb{N}^{k} \rightarrow \mathbb{N}^{k} \cup\{\emptyset\}$ are defined as follows. Given a word $w \in \mathbb{N}^{k}$ consider the subword $w_{i}$ consisting only of the letters $i$ and $i+1$. Replace each instance of $i$ with a right bracket and each $i+1$ with a left bracket. Remove all pairs of matching brackets and consider the remaining unmatched brackets, which now consist of $a$ right-brackets and $b$ left-brackets. These brackets correspond to a subword $w^{\prime}$ of the form $i^{a}(i+1)^{b}$ in $w$. The operator $\tilde{e}_{i}$ acting on $w$ turns the leftmost $i+1$ of $w^{\prime}$ into an $i$, if such an entry exists, otherwise, $\tilde{e}_{i}(w):=\emptyset$. Similarly, $\tilde{f}_{i}$ acting on $w^{\prime}$ turns the rightmost $i$ in $w^{\prime}$ into an $i+1$, if such an entry exists, otherwise, $\tilde{f}_{i}(w):=\emptyset$. For example,

$$
\tilde{e}_{1}(2,1,3,1,2,4,2,1,1,3,1, \underline{2}, 3,2,1,2,1)=2,1,3,1,2,4,2,1,1,3,1, \underline{1}, 3,2,1,2,1 \text {. }
$$

The operator $\tilde{e}_{i}$ is a crystal raising operator while $\tilde{f}_{i}$ is a crystal lowering operator. The operators also act on semistandard Young tableaux by acting on the reading word. We define a graph structure on words (or semistandard Young tableaux) by having a labeled directed edge $u \rightarrow v$ with label $i$ if $f_{i}(u)=v$. Examples on such components are given in Figure 9 .


Figure 9. A crystal graph on words and one on SSYT.
6.1. Crystal operators on COF and RSK. We shall now define crystal operators on the set $\operatorname{COF}(\lambda / \mu)$. These operators are considered in Uhl19, and the non-skew case was recently considered by S. Assaf and N. Gonzáles AG18, where the authors prove that they are indeed crystal operators. Here, we take a slightly different route and define the operators on biwords instead - it is straightforward to verify that our definitions are equivalent with theirs. These crystal biwords are closely related to the biwords we have seen previously.
Definition 38 (Crystal operators on fillings). Let $F \in \operatorname{COF}(\lambda / \mu)$ and define the crystal biword $\tilde{W}$, with entry $\binom{j}{c}$ appearing in $\tilde{W}$ if and only if there is a box with value $j$ in column $c$. The entries in $\tilde{W}$ are then sorted decreasingly, primarily on the bottom row entry, see Figure 10 Note that this is map to $\tilde{W}$ is invertible if $\lambda / \mu$ is fixed. We then define $\tilde{e}_{i}(F)$, and $\tilde{f}_{i}(F)$ as the result when applying $e_{i}$ and $f_{i}$, respectively, on the first row of $\tilde{W}$.

For the subset of coinversion-free fillings $F$ with $\operatorname{maj}(F)=0$, these operators are essentially a generalization of the raising- and lowering operators defined on semistandard Young tableaux.

Figure 10. A skew specialized Macdonald filling $F$, the corresponding crystal biword $\tilde{W}$ and the biword $W$ used for RSK. Note that the only difference between $\tilde{W}$ and $W$ is the ordering of the entries.

The following theorem was proven independently by S. Assaf and N. Gonzáles AG18 and the second author Uhl19].
Theorem 39 (See [Uhl19, AG18). The operators $\tilde{e}_{i}$ and $\tilde{f}_{i}$ preserve major index. That is, suppose that $F$ and $F^{\prime}$ are in $\operatorname{COF}(\lambda / \mu)$ and that $\tilde{e}_{i}(F), \tilde{f}_{i}\left(F^{\prime}\right) \neq \emptyset$. Then

$$
\operatorname{maj}\left(\tilde{e}_{i}(F)\right)=\operatorname{maj}(F) \text { and } \operatorname{maj}\left(\tilde{f}_{i}\left(F^{\prime}\right)\right)=\operatorname{maj}\left(F^{\prime}\right)
$$

Furthermore, $F, \tilde{e}_{i}(F)$ and $\tilde{f}_{i}(F)$ differ only at boxes with entries $i$ and $i+1$.
In Theorem 41 below we prove that the operators $\tilde{e}_{i}$ and $\tilde{f}_{i}$ define proper crystal graphs on the set $\operatorname{COF}(\lambda / \mu)$. See Figure 11 for examples of crystal graphs on coinversion-free fillings.

There is an important interaction between the crystal operators and the RSK correspondence, as we shall see in the following example.

Example 40. Suppose $F \in \operatorname{COF}(\lambda)$ with biword $W$

$$
W=\left(\begin{array}{llllllllll}
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\
4 & 3 & 1 & 3 & 2 & 1 & 5 & 2 & 1 & 2
\end{array}\right)
$$

The crystal biword is

$$
\tilde{W}=\left(\begin{array}{llllllllll}
3 & 1 & 2 & 1 & 4 & 3 & 2 & 3 & 2 & 1 \\
5 & 4 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1
\end{array}\right)
$$

Apply $e_{1}$ on the top row; We have the subword 121221 which turn into 121121 under $e_{1}$.

$$
\left(\begin{array}{llllllllll}
3 & 1 & 2 & 1 & 4 & 3 & 1 & 3 & 2 & 1 \\
5 & 4 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1
\end{array}\right)
$$

Sort columns again to obtain the biword $W^{\prime}$ that correspond to $\tilde{e}_{1}(F)$.

$$
\tilde{e}_{1}(F) \leftrightarrow\left(\begin{array}{llllllllll}
1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \\
4 & 3 & 2 & 1 & 3 & 1 & 5 & 2 & 1 & 2
\end{array}\right) .
$$

If we now perform RSK on $W$ and $W^{\prime}$ we see in Figure 12 that $\tilde{e}_{1}$ has a predictable effect on the corresponding insertion- and recording tableau, see Theorem 41

Theorem 41. Let $F \in \operatorname{COF}(\lambda / \mu)$ and suppose $\tilde{e}_{i}(F) \neq \emptyset$. Then $\operatorname{ins}(F)=\operatorname{ins}\left(\tilde{e}_{i}(F)\right)$ and $e_{i}(\operatorname{rec}(F))=\operatorname{rec}\left(\tilde{e}_{i}(F)\right)$. Stated equivalently on biwords: Let $W$ be a biword and let $\tilde{W}$ be its entries reordered such that it is a crystal biword. Suppose $\tilde{e}_{i}(\tilde{W}) \neq \emptyset$, and that $W^{\prime}$ is the biword corresponding to $\tilde{e}_{i}(\tilde{W})$. Then $\operatorname{ins}(W)=\operatorname{ins}\left(W^{\prime}\right)$ and $e_{i}(\operatorname{rec}(W))=\operatorname{rec}\left(W^{\prime}\right)$. The analogous statements for $\tilde{f}_{i}$ also hold.


Figure 11. The crystal structure on skew coinversion-free fillings of shape $\lambda / \mu$ with $\lambda=(3,2)$ and $\mu=(1)$ in 3 variables. Notice that the large component is isomorphic to the crystal graphs in Figure 9


Figure 12. An example of the raising operator at the level of RSK. The pairs of tableaux correspond to the two biwords in Example 40

Proof sketch: The fact that $\operatorname{ins}(F)=\operatorname{ins}\left(\tilde{e}_{i}(F)\right)$ follows from properties of the classical RSK algorithm.

The second property requires some more work, but every step is a routine transformation using known properties of various versions of RSK. First restate the property to the analogous statement about the dual RSK insertion algorithm (using column insertion), see [Kra06, Sec. 4.3]. One can then use that dual RSK and classical RSK are related in a simple manner (see e.g. [But94, Prop. 2.3.14]) and reduce the problem further to the case of the classical RSK. For classical RSK, the interaction with crystal operators is well-documented, see e.g. Shi Las03, BS17.

Using the bijection in Theorem 41, we see that the set $\operatorname{COF}(\lambda / \mu)$ is indeed a crystal graph under the raising- and lowering operators, since we have an equivariant bijection with crystals and crystal operators on semistandard tableaux. We remark that S. Assaf and N. Gonzáles AG18 proved the same result in the non-skew case by verifying the local characterization axioms introduced by Stembridge, see [Ste03].

We shall now briefly discuss an application of the crystal operators. Using the crystal operators $\tilde{e}_{i}$ and $\tilde{f}_{i}$, we can define involutions on $\operatorname{COF}(\lambda / \mu)$. For a coinversion-free filling $F$, denote $m_{i}=m_{i}(F)$ the number of $i$-entries of $F$.

Definition 42. For $i \in \mathbb{N}$, define the operators $\tilde{s}_{i}$ on $\operatorname{COF}(\lambda / \mu)$ by letting

$$
\tilde{s}_{i}(F):= \begin{cases}\left(\tilde{e}_{i}\right)^{m_{i+1}-m_{i}}(F) & \text { if } m<m_{i+1} \\ \left(\tilde{f}_{i}\right)^{m-m_{i+1}}(F) & \text { if } m_{i}>m_{i+1} \\ F & \text { if } m_{i}=m_{i+1}\end{cases}
$$

Restricted to the set of coinversion-free fillings with maj $=0$, the operators $\tilde{s}_{i}$ are essentially the famous Lascoux-Schützenberger involutions LS78. The difference being that the elements with maj $=0$ have weakly decreasing rows and strictly increasing columns as opposed to the weakly increasing in rows and strictly increasing in columns for in semistandard Young tableaux.

It is clear by Theorem 39 that the operators $\tilde{s}_{i}$ are maj-preserving involutions. Furthermore, if $F \in \operatorname{COF}(\lambda / \mu)$ and $\operatorname{wt}(F)=\left(w_{1}, \ldots, w_{r}\right)$, then $\tilde{s}_{i}(F)=$ $\left(w_{1}, \ldots, w_{i+1}, w_{i}, \ldots, w_{r}\right)$. This yields yet another proof that $\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0)$ is symmetric. In fact, it follows from general theory of crystals that the operators $\tilde{s}_{1}, \tilde{s}_{2}$, $\ldots, \tilde{s}_{r-1}$ generate an $S_{r}$-action on $\operatorname{COF}(\lambda / \mu, r)$.

## 7. SCHUR EXPANSION OF CERTAIN VERTICAL-STRIP LLT POLYNOMIALS

In this section, we briefly sketch that $\mathrm{E}_{\lambda / \mu}(\mathbf{x} ; q, 0)$ sometimes is a vertical strip LLT polynomial, up to a power of $q$. As a consequence, we therefore obtain an explicit formula for the Schur expansion of these particular LLT polynomials. Hence, we provide a new family of LLT polynomials with a combinatorial Schur expansion, not covered by previous results. We note that it is a major open problem in general to describe the LLT polynomials in the Schur basis.

Definition 43 (As in HHL05). Let $\boldsymbol{\nu}$ be a $k$-tuple of skew Young diagrams. Given such a tuple, we let $\operatorname{SSYT}(\boldsymbol{\nu})=\operatorname{SSYT}\left(\boldsymbol{\nu}^{1}\right) \times \operatorname{SSYT}\left(\boldsymbol{\nu}^{2}\right) \times \cdots \times \operatorname{SSYT}\left(\boldsymbol{\nu}^{k}\right)$ where $\operatorname{SSYT}(\lambda / \mu)$ is the set of skew semistandard Young tableaux of shape $\lambda / \mu$. Given $T=\left(T^{1}, T^{2}, \ldots, T^{k}\right) \in \operatorname{SSYT}(\boldsymbol{\nu})$, let $\mathbf{x}^{T}:=\mathbf{x}^{T^{1}} \cdots \mathbf{x}^{T^{k}}$ where $\mathbf{x}^{T^{i}}$ is the same monomial weight of $T^{i}$ as for Schur polynomials. Given a cell $u=(r, c)$ (row, column) in a skew diagram, the content is defined as $c(u):=c-r$. Entries $T^{i}(u)>T^{j}(v)$ in a tuple form an inversion if and only if

$$
i<j \text { and } c(u)=c(v), \quad \text { or } \quad i>j \text { and } c(u)=c(v)-1
$$

The LLT polynomial associated with the $k$-tuple $\boldsymbol{\nu}$ is given by

$$
\operatorname{LLT}_{\boldsymbol{\nu}}(\mathbf{x} ; q)=\sum_{T \in \operatorname{SSYT}(\boldsymbol{\nu})} q^{\operatorname{inv}(T)} \mathbf{x}^{T}
$$

where $\operatorname{inv}(T)$ is the total number of inversions appearing in $T$. One can show that $\operatorname{LLT}_{\boldsymbol{\nu}}(\mathbf{x} ; q)$ is a symmetric function, see HHL05] or AP18] for short proofs.

LLT polynomials such that each $\boldsymbol{\nu}^{j}$ is a skew shape of the form $1^{a} / 1^{b}$ with $a \geq b$ are called vertical strip LLT polynomials. Given a $k$-tuple $\boldsymbol{\nu}$, we let $\operatorname{mininv}(\boldsymbol{\nu})$ be the minimum number of inversions obtainable over all fillings. That is,

$$
\begin{equation*}
\operatorname{mininv}(\boldsymbol{\nu}):=\min _{T \in \operatorname{SSYT}(\boldsymbol{\nu})} \operatorname{inv}(T) \tag{25}
\end{equation*}
$$

Example 44. A $k$-tuple of skew shapes is traditionally illustrated using the French convention where box $(1,1)$ of each shape $\boldsymbol{\nu}^{i}$ is placed on the line $y=x$ with content 0 , and Cartesian coordinates are used. Below, we illustrate an element

$$
T \in \operatorname{SSYT}\left(1^{3} / \emptyset\right) \times \operatorname{SSYT}\left(1^{3} / 1^{1}\right) \times \operatorname{SSYT}\left(1^{2} / 1^{1}\right) \times \operatorname{SSYT}\left(1^{3} / \emptyset\right)
$$

which appears when computing the vertical-strip LLT polynomial $\operatorname{LLT}_{\boldsymbol{\nu}}(\mathbf{x} ; q)$ for $\boldsymbol{\nu}=(111 / \emptyset, 111 / 1,11 / 1,111 / \emptyset)$.


There are two inversions involving boxes $u$ and $v$ where $c(u)=c(v)$ and six inversions for which $c(u)=c(v)-1$. Hence, $T$ contributes with $q^{8} x_{1} x_{2}^{2} x_{3} x_{4}^{2} x_{5}^{2} x_{6}$. The full LLT polynomial $\operatorname{LLT}_{\boldsymbol{\nu}}(\mathbf{x} ; q)$ in the Schur basis is given by

$$
\begin{aligned}
& q^{8} \mathbf{S}_{333}+q^{7} \mathbf{s}_{432}+\left(q^{9}+q^{10}+q^{11}\right) \mathbf{s}_{3222}+\left(q^{8}+2 q^{9}+q^{10}\right) \mathrm{s}_{3321}+\left(q^{8}+q^{9}\right) \mathrm{s}_{4221} \\
& +q^{8} \mathrm{~S}_{4311}+\left(q^{10}+q^{11}+q^{12}+q^{13}\right) \mathrm{s}_{22221}+\left(q^{9}+3 q^{10}+2 q^{11}+q^{12}\right) \mathrm{s}_{32211} \\
& +\left(q^{9}+q^{10}+q^{11}\right) \mathrm{s}_{33111}+\left(q^{9}+q^{10}\right) \mathrm{s}_{42111}+\left(2 q^{11}+2 q^{12}+q^{13}+q^{14}\right) \mathrm{s}_{222111} \\
& +\left(q^{10}+2 q^{11}+2 q^{12}+q^{13}\right) \mathrm{s}_{321111}+q^{11} \mathrm{~s}_{411111}+\left(q^{12}+2 q^{13}+q^{14}+q^{15}\right) \mathrm{s}_{2211111} \\
& +\left(q^{12}+q^{13}+q^{14}\right) \mathrm{s}_{3111111}+\left(q^{14}+q^{15}+q^{16}\right) \mathrm{s}_{21111111}+q^{17} \mathrm{~s}_{111111111} .
\end{aligned}
$$

As $q^{7}$ is the lowest power of $q$ that appear in the expansion, we must have that $\operatorname{mininv}(\boldsymbol{\nu})=7$.

The current state-of-the-art regarding combinatorial proofs of Schur positivity of LLT polynomials is as follows.

- When all shapes in $\boldsymbol{\nu}$ are non-skew, the coefficients in the Schur basis are known to be certain parabolic Kazhdan-Lusztig polynomials, see [T00]. Hence, the coefficients are in $\mathbb{N}[q]$. In particular, this case contains the Hall-Littlewood symmetric functions.
- Whenever the $k$-tuple of shapes $\boldsymbol{\nu}$ consists of at most 3 shapes, all avoiding an arrangement of $2 \times 2$-boxes (that is, they are ribbons), Schur positivity is given by a combinatorial formula, see J. Blasiak Bla16.
- A few other cases when each shape in $\boldsymbol{\nu}$ is a single box is given in HNY18.

Theorem 45. Let $\lambda / \mu$ be a skew shape such that no column contains more than two boxes. Then

$$
\mathrm{E}_{\lambda^{\prime} / \mu^{\prime}}(\mathbf{x} ; q, 0)=q^{-\operatorname{mininv}(\boldsymbol{\nu})} \operatorname{LLT}_{\boldsymbol{\nu}}(\mathbf{x} ; q)
$$

where $\boldsymbol{\nu}_{j}$ is the vertical strip $\left(\lambda_{j}\right) /\left(\mu_{j}\right)$.

Proof sketch. For the modified Macdonald polynomials $\tilde{\mathrm{H}}_{\lambda}(\mathbf{x} ; q, t)$, we have the symmetry $\tilde{\mathrm{H}}_{\lambda}(\mathbf{x} ; q, t)=\tilde{\mathrm{H}}_{\lambda^{\prime}}(\mathbf{x} ; t, q)$. This interchanges the rôle of inversion triples and major index, see Hag07. This relationship extends to modified Macdonald polynomials indexed by skew shapes $\lambda / \mu$ as long as each column contains at most two boxes, see J. Brandlow, Bra07 Thm. 5].

There is a correspondence between inversion triples and inversions that appearing definition of LLT polynomials. In [HHL05, Eq. (23)], the authors provide (via a straightforward bijective argument) an expansion of the form

$$
\begin{equation*}
\tilde{\mathrm{H}}_{\lambda}(\mathbf{x} ; q, t)=\sum_{D} q^{\operatorname{maj}(D)} t^{-\operatorname{stat}(D)} \operatorname{LLT}_{\nu(D)}(\mathbf{x} ; t), \tag{26}
\end{equation*}
$$

where the sum runs over all subsets (possible descents) of boxes $(i, j)$ with $i>1$ of the diagram $\lambda$. In particular, the coefficient of the terms maximizing the major index a vertical-strip LLT polynomial. This expansion has a natural extension to skew shapes and one can check that stat $(\cdot)$ corresponds to mininv $(\cdot)$ for the highest-degree term. Combining all these observations we have

$$
\mathrm{E}_{\lambda^{\prime} / \mu^{\prime}}(\mathbf{x} ; q, 0)=\left[t^{*}\right] \tilde{\mathrm{H}}_{\lambda^{\prime} / \mu^{\prime}}(\mathbf{x} ; t, q)=\left[t^{*}\right] \tilde{\mathrm{H}}_{\lambda / \mu}(\mathbf{x} ; q, t)=q^{-\operatorname{mininv}(\boldsymbol{\nu})} \operatorname{LLT}_{\boldsymbol{\nu}}(\mathbf{x} ; q)
$$

The first identity is due to (19). The second identity is the tricky part and relies on Bra07]. The third identity is a simple generalization of (26).

Example 46. We illustrate Theorem 45 in the case $\lambda / \mu=4431 / 31$. The skew shape $\lambda / \mu$ is illustrated in (27) where we have labeled the boxes from right to left, top to bottom. The corresponding $k$-tuple of vertical strips is shown to the right. The labeling has the property that it maps inversion pairs in the filling to the right, to inversions in the LLT diagram, see HHL05 for details.


Notice that no column contains more than two boxes so the conditions in the theorem applies. The $k$-tuple $\boldsymbol{\nu}$ is $1111 / 111,1111 / 1,111 / \emptyset, 1 / \emptyset$, and it is easy (for
a computer) to check that

$$
\begin{aligned}
\mathrm{E}_{\lambda^{\prime} / \mu^{\prime}}(\mathbf{x} ; q, 0) & =\mathrm{s}_{332}+\mathrm{s}_{422}+\left(1+q^{2}\right) \mathrm{s}_{2222}+(2+2 q) \mathrm{s}_{3221}+\mathrm{s}_{3311}+\mathrm{s}_{4211} \\
& +\left(3 q+q^{2}\right) \mathrm{s}_{22211}+4 q \mathrm{~s}_{32111}+q \mathrm{~s}_{41111}+4 q^{2} \mathrm{~s}_{221111}+3 q^{2} \mathrm{~s}_{311111} \\
& +3 q^{3} \mathrm{~s}_{2111111}+q^{4} \mathrm{~s}_{11111111}
\end{aligned}
$$

and that this is also equal to $q^{-1} \operatorname{LLT}_{\boldsymbol{\nu}}(\mathbf{x} ; q)$.
As a final check, we verify one of the coefficients with the combinatorial formula. Using the notation in Theorem $34 \alpha=1331$. The term $(2+2 q) \mathrm{s}_{3221}$ then arises from the four semistandard tableaux

| 1 2 2 2 | 1 2 2 3 | 1 2 2 3 | 1 2 2 4 |
| :---: | :---: | :---: | :---: |
| 3 3  | 233 | 234 | 233 |
| 4 | 4 | 3 | 3 |

where the value of $\operatorname{charge}_{31}(w)=\operatorname{charge}(w \cdot 2111)$ is shown under each tableau.
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## 8. Appendix: How to compute Kostka-Foulkes polynomials

We shall briefly describe how to compute the coefficients $K_{\lambda \mu}(q)$ appearing in Equation 10). This combinatorial model was first described by A. Lascoux and M. Schützenberger in LS78]. For a permutation $\sigma \in S_{k}$, let $\operatorname{Des}(\sigma):=\{i \in[k-1]$ : $\left.\sigma_{i+1}<\sigma_{i}\right\}$, and let $\operatorname{rev}(\sigma)$ be the reverse, $\sigma_{n}, \sigma_{k-1}, \ldots, \sigma_{1}$. We can now introduce the notion of charge of a permutation.

$$
\begin{equation*}
\operatorname{charge}(\sigma):=\operatorname{maj}\left(\operatorname{rev}\left(\sigma^{-1}\right)\right)=\sum_{i \notin \operatorname{Des}\left(\sigma^{-1}\right)}(k-i) \tag{28}
\end{equation*}
$$

For example,

$$
\operatorname{charge}(198423765)=\operatorname{maj}(\operatorname{rev}(156498732))=\operatorname{maj}(237894651)=20
$$

Given a word $w$ with content $\mu \vdash n$, we partition its entries into standard subwords as follows. Start from the right of $w$ and mark the first occurrence of 1. Proceed to the left, and mark the first occurrence of 2 , then 3 and so on, wrapping around the end if nessecary, until $\mu_{1}^{\prime}$ entries have been marked. This subword is the first standard subword of $w$. Remove this subword, and repeat the process to find the second standard subword, of length $\mu_{2}^{\prime}$.

For example, the first standard subword in $w=21123543411223$ has been circled.

$$
2,1,1,(2), 3,(5), 4,3,(4), 1,(1), 2,2,(3) .
$$

In total, we have four standard subwords in $w$, with corresponding charge values

$$
\operatorname{charge}(25413)=3, \quad \operatorname{charge}(2431)=2 \quad \text { charge }(132)=2 \quad \operatorname{charge}(12)=1,
$$

and we define charge $(w)$ as the sum of the charge values of the standard subwords. In the example above, charge $(w)=8$.

Recall the definition of the reading word $\mathrm{rw}(T)$ of a semistandard Young tableau from Definition 4 We then define charge $(T):=\operatorname{charge}(\operatorname{rw}(T))$ and the KostkaFoulkes polynomial $K_{\lambda \mu}(q)$ may be computed as

$$
\begin{equation*}
K_{\lambda \mu}(q)=\sum_{T \in \operatorname{SSYT}(\lambda, \mu)} q^{\operatorname{charge}(T)} \tag{29}
\end{equation*}
$$

Example 47 (Computing a Kostka-Foulkes polynomial). Consider the case $\lambda=421$, $\mu=3211$. There are four tableaux in $\operatorname{SSYT}(\lambda, \mu)$. Below, these are displayed, each with the list of standard subwords and corresponding charge values.

| 1 1 1 4 | 1 1 1 3 | 1 1 1 2 | 1 1 1 2 |
| :---: | :---: | :---: | :---: |
| 22 | 22 | 24 | 23 |
| 3 | 4 | 3 | 4 |
| 3214, 21, | 4213, 21, 1 | 3241 | 4231 |
| , | ${ }_{2+0+0}$ | S |  |

Hence, $K_{\lambda \mu}(q)=q+2 q^{2}+q^{3}$.
We remark that there is also the notion of skew Kostka-Foulkes polynomials, $K_{\lambda / \mu, \nu}(q)$ see But94. They are defined as in 29, where the sum now ranges over the elements in $\operatorname{SSYT}(\lambda / \mu, \nu)$. To our knowledge, there is no obvious relationship between these skew Kostka-Foulkes polynomials and the polynomials

$$
\sum_{T \in \operatorname{SSYT}\left(\nu^{\prime}, \alpha\right)} q^{\text {charge }_{\mu^{\prime}}(T)}
$$

appearing in 22).

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[^0]:    ${ }^{1}$ Note that this seems non-standard compared to descents in words. This terminology is due to the usage of skyline diagrams used when describing the non-symmetric Macdonald polynomials HHL08. We use English notation rather than skyline diagrams.

[^1]:    ${ }^{2}$ There is an extension of the notion of inversion triples to diagrams indexed by weak compositions and then one has that $\mathrm{E}_{\lambda}(\mathbf{x} ; q, 0)$ is independent of the order of entries in $\lambda$, see Ale15 Prop. 17].

