ON THE AUTOMATICITY OF SEQUENCES DEFINED BY CONTINUED FRACTIONS

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ABSTRACT. Continued fraction expansions of automatic sequences have been extensively studied during the last decades. The research interests are, on one hand, in the degree or automaticity of the partial quotients following the seminal paper of Baum and Sweet in 1976, and on the other hand, in calculating the Hankel determinants and irrationality exponent, as one can find in the works of Allouche-Peyrière-Wen-Wen, Bugeaud, and the first author. The present paper is motivated by the converse problem: to study continued fractions whose coefficients form an automatic sequence. We consider two such continued fractions defined by the Thue-Morse and period-doubling sequences respectively, and prove that they are congruent to algebraic series in $\mathbb{Z}[[x]]$ modulo 4. Consequently, the sequences of the coefficients of the power series expansions of the two continued fractions modulo 4 are 2-automatic. Our approach is to first guess the explicit formulas of certain subsequences of $(P_n(x))$ and $(Q_n(x))$, where $P_n(x)/Q_n(x)$ is the canonical representation of the truncated continued fractions, then prove these formulas by an intricate induction involving eight subsequences while exploiting the relations between these subsequences.

1. INTRODUCTION

Continued fraction expansions of automatic sequences have been extensively studied during the last decades. By the well-known Theorem of Christol [14], an element in $\mathbb{F}_q((1/x))$ is algebraic over $\mathbb{F}_q(x)$ if and only if its coefficients form a q-automatic sequence. In 1976, Baum and Sweet [7] proved that the continued fraction expansion of the unique solution $\varphi(x)$ in $\mathbb{F}_2((1/x))$ of the equation

$$xf^3 + f + x = 0,$$

has partial quotients of bounded degree. They also gave examples of algebraic elements of degree greater than 2 in $\mathbb{F}_q((1/x))$ whose continued fraction expansion have partial quotients of unbounded degree. In contrast, we do not know any algebraic real number of degree greater than 2 that has bounded or unbounded partial quotients. In [8], for every even degree d, Baum and Sweet gave examples of elements in $\mathbb{F}_2((1/x))$, algebraic of degree d, with bounded partial quotients. In [29], Mills and Robbins gave explicitly the continued fraction expansion of $\varphi(x)$. They also gave examples, for each odd prime p, of algebraic elements of degree greater than 2 in $\mathbb{F}_p((1/x))$ whose partial quotients are linear; this was largely generalized by Lasjaunias and Yao in [26]. Allouche proved in [2] that the sequences of partial

quotients for the examples in [29] are automatic. In contrast, Mkaouar proved that the sequence parital quotients of $\varphi(x)$, while being morphic, is not automatic. In [27] and [28], Lasjaunias and Yao considered the sequence of leading coefficients of the partial quotients instead of the partial quotients themselves and described several families of hyperquadratic series for which the leading coefficients of the partial quotients form automatic sequences.

On the other hand, continued fraction expansions of automatic sequences have been studied for calculating the Hankel determinants and irrationality exponents [22, 23, 19, 6]. In 1998, Allouche, Peyrière, Wen and Wen proved that all the Hankel determinants of the Thue-Morse sequence are nonzero [1]. This property allowed Bugeaud to prove that the irrationality exponents of the Thue-Morse-Mahler numbers are exactly 2 [10]. Since then, the Hankel determinants for several other automatic sequences, in particular, the paperfolding sequence, the Stern sequence and the period-doubling sequence, are studied by Coons, Vrbik, Guo, Wu, Wen, Bugeaud and Fu [15, 21, 20, 12]. Using Jacobi continued fractions, the first author found a simple proof of APWW's result [22]. Finally, The Euler-Lagrange theorem says that the continued fraction expansion of a quadratic irrational number is ultimately periodic. The first author obtained similar result for quadratic power series on finite fields [23].

The present paper is motivated by the converse problem: to study continued fractions whose coefficients form an automatic sequence.

We now give a brief introduction to automatic sequences. We refer the readers to [5, p. 185] for more details. Automatic sequences appear naturally in the study of various domains of mathematics and theoretical computer science. One of the equivalent definitions of automatic sequences is the following: for an integer $k \geq 2$, a sequence $(u_n)_{n \geq 0}$ is said to be k-*automatic* if its k-*kernel*, defined as

$$\{(u(k^d n + j))_{n \geq 0} \mid d \in \mathbb{N},\, 0 \leq j \leq k^d - 1\},$$

is finite. Thus, if we denote by Λ_j the Cartier operators [5, p. 376] that maps $\sum_{n=0}^{\infty} a_n x^n$ to $\sum_{n=0}^{\infty} a_{kn+j} x^n$, then the k-kernel of $(u_n)_{n \geq 0}$ is in bijection with the smallest set containing the series $\sum_{n=0}^{\infty} u_n x^n$ that is stable under the operations of Λ_j $(j = 0, 1, \ldots, k-1)$. We use a double list L to encode the structure of the kernel, by $L[i][j] = i'$ we mean that the i-th element of the kernel is mapped to the i'-th by Λ_j, with the sequence itself denoted by the 0-th element.

In this article we will consider the *Thue-Morse sequence* $\mathbf{t} = (t_n)$ defined by the recurrence relations (see [33], [4])

$$
\begin{aligned}
t_0 &= 1; \\
t_{2n} &= t_n; & (n \geq 1) \\
t_{2n+1} &= -t_n, & (n \geq 0)
\end{aligned}
$$

and the *period-doubling sequence* $\mathbf{s} = (s_n)$ defined by the recurrence relations [31, 19]

$$
\begin{aligned}
s_{2n} &= 1; & (n \geq 0) \\
s_{2n+1} &= -s_n. & (n \geq 0)
\end{aligned}
$$

We see from the definition that the 2-kernel of the Thue-Morse sequence is

$$\{(t_n)_n, (t_{2n+1})_n\},$$

and the 2-kernel of the period-doubling sequence is

$$\{(s_n)_n, (s_{2n})_n, (s_{2n+1})_n, (s_{4n+1})_n\}.$$

Therefore they are both 2-automatic. The structures of the above two 2-kernels are represented by $[[0,1],[1,0]]$ and $[[1,2],[1,1],[3,0],[3,3]]$ respectively.

Basic definition and properties of continued fractions will be recalled in Section 2. We consider the continued fractions defined by the Thue-Morse and the period-doubling sequence:

$$(1.1) \qquad C(x) := \sum_{n\geq 0} c_n x^n := \cfrac{t_0}{1+\cfrac{t_1 x}{1+\cfrac{t_2 x}{1+\cfrac{t_3 x}{1+\cfrac{t_4 x}{\ddots}}}}} = \cfrac{1}{1-\cfrac{x}{1-\cfrac{x}{1+\cfrac{x}{1-\cfrac{x}{\ddots}}}}}$$

and

$$(1.2) \qquad D(x) := \sum_{n\geq 0} d_n x^n := \cfrac{s_0}{1+\cfrac{s_1 x}{1+\cfrac{s_2 x}{1+\cfrac{s_3 x}{1+\cfrac{s_4 x}{\ddots}}}}} = \cfrac{1}{1-\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{1+\cfrac{x}{\ddots}}}}}$$

The above two continued fractions will be called *Thue-Morse* continued fraction and *Period-doubling* continued fraction respectively. Write $\bar{c}_n = \pi(c_n)$, $\bar{C}(x) = \sum_{n\geq 0} \bar{c}_n x^n$, and $\bar{d}_n = \pi(d_n)$, $\bar{D}(x) = \sum_{n\geq 0} \bar{d}_n x^n$, where π is the canonical surjection of $\mathbb{Z}$ onto $\mathbb{Z}/4\mathbb{Z}$. Note that we could also have defined $\bar{C}(x)$ and $\bar{D}(x)$ by (1.1) and (1.2) while viewing ± 1 as elements in $\mathbb{Z}/4\mathbb{Z}$. The first terms of these sequences are listed below.

$$
\begin{aligned}
(t_n) &= (1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, \ldots), \\
(c_n) &= (1, 1, 2, 3, 4, 6, 8, 11, 14, 18, 20, 22, 16, 4, -32, -93, -220, \ldots), \\
(\bar{c}_n) &= (1, 1, 2, 3, 0, 2, 0, 3, 2, 2, 0, 2, 0, 0, 0, 3, 0, 2, 0, 2, 0, 0, 0, \ldots), \\
(s_n) &= (1, -1, 1, 1, 1, -1, 1, -1, 1, -1, 1, 1, 1, -1, 1, 1, \ldots), \\
(d_n) &= (1, 1, 0, 1, -2, 4, -8, 17, -36, 74, -152, 316, -656, 1352, \ldots), \\
(\bar{d}_n) &= (1, 1, 0, 1, 2, 0, 0, 1, 0, 2, 0, 0, 0, 0, 0, 1, 2, 0, 0, 2, 0, 0, 0, 0, \ldots).
\end{aligned}
$$

Notice that the sequences $(c_n), (\bar{c}_n), (d_n), (\bar{d}_n)$ are not in the OEIS.

In the present paper we study the above two continued fractions and obtain the following properties of the sequences $(\bar{c}_n)$ and $(\bar{d}_n)$.

Theorem 1.1. *We have the following congruence:*

$$(1.3) \qquad C(x) \equiv \frac{\sqrt{1-4x}-1}{2x} + 1 + \sqrt{2\sqrt{1-4x}-1} \pmod 4.$$

Theorem 1.2. *We have the following congruence:*

$$(1.4) \qquad D(x) \equiv \frac{(1 + \sqrt{1 + 4x})\sqrt{2\sqrt{1 - 4x^2} - 1} - 2}{2x} \pmod 4.$$

The following Theorem from [16] then allows us to conclude that $(\bar{c}_n)_n$ and $(\bar{d}_n)_n$ are 2-automatic.

Theorem 1.3 (Denef-Lipschitz). *Suppose that the power series $f(x_1, \ldots, x_k) \in \mathbb{Z}_p[[x_1, \ldots, x_k]]$ is algebraic over $\mathbb{Z}_p[x_1, \ldots, x_k]$. Then for each α, the coefficient sequence of $f \pmod{p^\alpha}$ is p-automatic.*

The automaticity of $(\bar{c}_n)_n$ and $(\bar{d}_n)_n$ can also be proved by a direct calculation of their 2-kernels.

Theorem 1.4. *The sequence $(\bar{c}_n)$ is 2-automatic; the structure of its 2-kernel is represented by $[[1, 2], [3, 4], [5, 6], [1, 7], [4, 7], [5, 4], [8, 6], [7, 7], [8, 4]]$.*

Theorem 1.5. *The sequence $(\bar{d}_n)$ is 2-automatic; the structure of its 2-kernel is represented by $[[1, 0], [2, 3], [1, 4], [3, 3], [4, 3]]$*

The right hand side of congruence (1.3) and (1.4) are respectively of degree 4 and 8 over $\mathbb{Z}(x)$. This raises the question of what the minimal degree of polynomial equations that $\bar{C}$ and $\bar{D}$ satisfy is. Concerning this, we have the following result.

Theorem 1.6. *Let $S(x, y) = (xy^2 + y + 1)^2 \in \mathbb{Z}/4\mathbb{Z}[x, y]$, then for both series $\bar{C}(x)$ and $\bar{D}(x)$ in $\mathbb{Z}/4\mathbb{Z}$, we have $S(x, \bar{C}(x)) = S(x, \bar{D}(x)) = 0$. Furthermore, there is no polynomial in $\mathbb{Z}/4\mathbb{Z}[x, y]$ that, seen as a polynomial in y, has degree less than 4, and, whose leading coefficient is invertible in the ring of Laurent series $\mathbb{Z}/4\mathbb{Z}((x))$, that annihilates either $\bar{C}(x)$ or $\bar{D}(x)$.*

Informally put, Theorem 1.6 says that $\bar{C}(x)$ and $\bar{D}(x)$ are of degree 4, while their continued fraction expansion are 2-automatic. It may be interesting to compare Theorem 1.6 to the following result concerning automatic sequences and real continued fractions [11].

Theorem 1.7 (Bugeaud 2013). *The continued fraction expansion of an algebraic number of degree at least three cannot be generated by a finite automaton.*

The *Hankel determinant* of order n of the formal power series $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ (or of the sequence $(a_0, a_1, a_2, \cdots)$ is defined by

$$H_n(f(x)) = H_n(a_0, a_1, a_2, \ldots) := \det(a_{i+j})_{0 \le i,j \le n-1}$$

for $n \ge 1$, and $H_0(f(x)) = H_0(a_0, a_1, a_2, \ldots) = 1$ if $n = 0$.

Concerning the Hankel determinants of $C(x)$ and $D(x)$, we have the following result.

Theorem 1.8. *The sequences of Hankel determinants $(H_n(C(x)))$ and $(H_n(D(x)))$ are 2-automatic.*

Based on our results, we put forward the following conjecture.

Conjecture 1.9. *The sequences $c_n \pmod{2^m}$ and $d_n \pmod{2^m}$ are 2-automatic for all $m \ge 1$.*

Theorem 1.4 and 1.5 says that Conjecture 1.9 is true for $m = 2$. Note that if the conjecture is true for $m = k$, then it is also true for all positive integers $m < k$. For $m = 1$, we can also see directly that

$$C(x) \equiv D(x) \equiv \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{\ddots}}}} \pmod 2.$$

The right hand side of the congruence is the generating function for the Catalan numbers [9]. Being quadratic, it is 2-automatic modulo 2.

When $m = 3$, experiments suggest that $c_n \pmod{2^3}$ and $d_n \pmod{2^3}$ are 2-automatic with the following kernel structure for $c_n \pmod{2^3}$

$$[[1, 2], [3, 4], [5, 6], [7, 8], [9, 10], [11, 12], [13, 6], [3, 14], [8, 10], [4, 8],$$
$$[10, 10], [11, 15], [12, 8], [16, 12], [17, 10], [15, 8], [16, 15], [14, 8]];$$

and for $d_n \pmod{2^3}$

$$[[1, 2], [3, 4], [5, 2], [6, 7], [4, 4], [8, 9], [3, 9], [10, 4], [11, 12],$$
$$[9, 4], [7, 9], [8, 4], [13, 4], [12, 9]].$$

This article is structured as follows: in Section 2, we give the definitions and properties of Stieltjes and Jacobi continued fractions. In Section 3, we exploit the structure of the Thue-Morse sequence and obtain the relations between certain subsequences of $P_n(x)$ and $Q_n(x)$, with $P_n(x)/Q_n(x)$ being the canonical representation of the n-th convergent of the continued fraction $C(x)$. Then we prove by induction the explicit expression of eight subsequences. We only use two of them but we need all eight for the induction hypotheses. Taking the limit, we obtain the explicit expression of the Thue-Morse continued fraction $\bar{C}(x)$ as a power series and prove that it is equal to an algebraic series with integer coefficients modulo 4. In consequence, its coefficients form a 2-automatic sequence. In Section 4 we obtain similar results for the period-doubling continued fraction $\bar{D}(x)$ using what we have proved for $\bar{C}(x)$ and the relation between the Thue-Morse and the period-doubling sequences. In Section 5 we prove Theorem 1.6. Finally in Section 6 we prove that the sequences of Hankel determinants $H_n(C(x))$ and $H_n(D(x))$ are 2-automatic.

2. Stieltjes and Jacobi continued fractions

Stieltjes and Jacobi continued fraction are studied in enumerative combinatorics for their link with the orthogonal polynomials and the weighted Motzkin paths (see [25, p.386, p.389], [34], [17]). For a sequence $\mathbf{a} = (a_n)_n$ taking values in a field $\mathbb{K}$,

and for each positive integer n, we define the rational fraction:

$$(2.1) \qquad \mathrm{Stiel}_n(\mathbf{a}) := \cfrac{a_0}{1 + \cfrac{a_1 x}{1 + \cfrac{a_2 x}{\ddots \atop 1 + \cfrac{a_{n-1}x}{1 + a_n x}}}},$$

which we also denote by $[\![a_0, a_1, \ldots, a_n]\!]$ for short.

We define two sequence of polynomials $P_n(x)$ and $Q_n(x)$ by the initial conditions $P_0(x) = a_0$, $Q_0(x) = 1$, $P_1(x) = a_0$ and $Q_1(x) = 1 + a_1 x$, and for $n \geq 2$

$$(2.2) \qquad \begin{pmatrix} 1 & a_n x \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_2 x \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P_1(x) & Q_1(x) \\ P_0(x) & Q_0(x) \end{pmatrix} = \begin{pmatrix} P_n(x) & Q_n(x) \\ P_{n-1}(x) & Q_{n-1}(x) \end{pmatrix}.$$

We have $\mathrm{Stiel}_n(\mathbf{a}) = P_n(x)/Q_n(x)$ for all n. A proof of the following theorem can be found in [18, p. 257].

Theorem 2.1. *The sequence of formal power series $P_n(x)/Q_n(x)$ is convergent.*

The infinite Stieltjes continued fraction $\mathrm{Stiel}(\mathbf{a})$ is defined to be

$$\lim_{n \to \infty} P_n(x)/Q_n(x),$$

the rational fraction $P_n(x)/Q_n(x)$ is called the *n-th convergent* of $\mathrm{Stiel}(\mathbf{a})$ and the unsimplified fraction $P_n(x)/Q_n(x)$ the *canonical representation* of $\mathrm{Stiel}_n(\mathbf{a})$.

For $0 \leq k < n$, if $P(x)/Q(x)$ is the canonical representation of the Stieltjes continued fraction $[\![a_k, \ldots, a_n]\!]$, then it can be easily shown from (2.2) that

$$(2.3) \qquad P_n(x) = Q(x)P_{k-1}(x) + xP(x)P_{k-2}(x),$$

$$(2.4) \qquad Q_n(x) = Q(x)Q_{k-1}(x) + xP(x)Q_{k-2}(x).$$

We define the Jacobi continued fractions in a similar way. For two sequences $\mathbf{u} = (u_n)_n$ and $\mathbf{v} = (v_n)_n$ with $v_i \neq 0$ for all $i \in \mathbb{N}$, $\mathrm{Jac}(\mathbf{u}, \mathbf{v})$ is defined to be the infinite continued fraction

$$(2.5) \qquad \mathrm{Jac}(\mathbf{u}, \mathbf{v}) = \cfrac{v_0}{1 + u_1 x - \cfrac{v_1 x^2}{1 + u_2 x - \cfrac{v_2 x^2}{1 + u_3 x - \cfrac{v_3 x^2}{\ddots}}}}.$$

The basic properties on Stieltjes and Jacobi continued fractions can be found in [17, 35, 32, 24]. We emphasize the fact that the Hankel determinants can be calculated from the Stieltjes and Jacobi continued fractions by means of the following fundamental relation, first stated by Heilermann in 1846 [24]:

Theorem 2.2. *The nth-order Hankel determinants of the Stieltjes* (2.1) *and Jacobi* (2.5) *continued fractions are given by*

$$H_n(\mathrm{Stiel}(\mathbf{a})) = a_0^n (a_1 a_2)^{n-1} (a_3 a_4)^{n-2} \cdots (a_{2n-3} a_{2n-2}),$$

$$H_n(\mathrm{Jac}(\mathbf{u}, \mathbf{v})) = v_0^n v_1^{n-1} v_2^{n-2} \cdots v_{n-2}^2 v_{n-1}.$$

The following contraction theorem establishes a link between the Stieltjes and Jacobi continued fractions [35, 30, 32].

Theorem 2.3. *[Contraction Theorem] The Stieltjes continued fraction* $\mathrm{Stiel}(\mathbf{a})$ *and Jacobi continued fraction* $Jac(\mathbf{u}, \mathbf{v})$ *are equal, if*

$$u_1 = a_1;$$
$$u_k = a_{2k-2} + a_{2k-1}; \qquad\qquad (k \geq 2)$$
$$v_0 = a_0;$$
$$v_k = a_{2k-1}a_{2k}. \qquad\qquad (k \geq 1)$$

Using the above notation, the two power series $C(x)$ and $D(x)$ defined in Section 1 can be written as $C(x) = \mathrm{Stiel}(\mathbf{t})$ and $D(x) = \mathrm{Stiel}(\mathbf{s})$.

3. THUE-MORSE CONTINUED FRACTION

First we consider the n-th convergent $P_n(x)/Q_n(x)$ of the Thue-Morse continued fraction $C(x)$. Making use of the structure of the Thue-Morse sequence, we establish the following recurrence relations of P_n and Q_n.

Lemma 3.1. *Let* $P_n(x)/Q_n(x)$ *be the canonical representation of* $\mathrm{Stiel}_n(a)$. *The two sequences* $P_n(x)$ *and* $Q_n(x)$ *are characterized by the initial conditions*

$$P_0(x) = P_1(x) = Q_0(x) = 1, \; Q_1(x) = 1 - x$$

and the following recurrence relations for $m \geq 1$ *and* $1 \leq \epsilon \leq 2^m$:

$$U_{2^{m+1}-\epsilon}(x) = Q_{2^m-\epsilon}(-x)U_{2^m-1}(x) - xP_{2^m-\epsilon}(-x)U_{2^m-2}(x),$$

where U is either of the sequences P or Q.

Proof. For a fixed $1 \leq \epsilon \leq 2^m$, let $P(x)/Q(x)$ be the canonical representation of the Stieltjes continued fraction $[\![t_{2^m}, t_{2^m+1}, \ldots, t_{2^{m+1}-\epsilon}]\!]$. From the definition of the Thue-Morse sequence, we see that $t_n = 1$ if the number of 1's in the binary expansion of n is even, and $t_n = -1$ otherwise, and therefore $t_{2^m+j} = -t_j$ for all $m \geq 0$ and $0 \leq j \leq 2^m - 1$. Hence $P(x)/Q(x)$ is in fact the canonical representation of $[\![-t_0, -t_1, \ldots, -t_{2^m-\epsilon}]\!]$. By (2.2), $P(x) = -P_{2^m-\epsilon}(-x)$ and $Q(x) = Q_{2^m-\epsilon}(-x)$. Using formula (2.3) and (2.4) we get the desired result. □

From the above recurrence relations of $P_n(x)$ and $Q_n(x)$, we are able to derive by induction the explicit expression of $P_{2^{2m}-2}(x)$ and $Q_{2^{2m}-2}(x)$, which we will then use to calculate $C(x) = \lim P_n(x)/Q_n(x)$.

To simplify notations, we define, for $m \geq 0$,

$$S_m(x) = \sum_{j=0}^{m-1} x^{2^j}, \quad S_m^e(x) = \sum_{j=0}^{m-1} x^{2^{2j}}, \quad S_m^o(x) = \sum_{j=0}^{m-1} x^{2^{2j+1}},$$

and

$$S_\infty(x) = \sum_{j=0}^{\infty} x^{2^j}, \quad S_\infty^e(x) = \sum_{j=0}^{\infty} x^{2^{2j}}, \quad S_\infty^o(x) = \sum_{j=0}^{\infty} x^{2^{2j+1}}.$$

If the parameter is x, we write without the parameter as $S_m := S_m(x)$, etc. Recall that the Kronecker delta symbol $\delta_{i,j}$ is 1 if $i = j$, and 0 otherwise.

We are only interested in 3) and 7) from the following proposition, but we need the others for the proof by induction.

Proposition 3.2. *We have the following explicit values for the polynomials $P_n(x)$ and $Q_n(x)$ for $n = 2^k - 1$ and $n = 2^k - 2$.*

1) $P_{2^{2m}-1}(x) \equiv 1 + 2S_{m-1}^o(x) \pmod 4; (m \geq 1)$

2) $P_{2^{2m+1}-1}(x) \equiv 1 + 2x(1 - \delta_{m,0}) + 2S_m^e(x) \pmod 4; (m \geq 0)$

3) $P_{2^{2m}-2}(x) \equiv 1 + x^{-1}S_{2m-1}(x)^2 - 2S_m^e(x) \pmod 4; (m \geq 1)$

4) $P_{2^{2m+1}-2}(x) \equiv 1 + x^{-1}S_{2m}(x)^2 - 2S_m^o(x) \pmod 4; (m \geq 0)$

5) $Q_{2^{2m}-1}(x) \equiv 1 - x + 2x^{2^{2m-1}} - S_{2m-1}(x)^2 + 2xS_m^e(x) \pmod 4; (m \geq 1)$

6) $Q_{2^{2m+1}-1}(x) \equiv 1 - x + 2x^{2^{2m}}(1 - \delta_{m,0}) - S_{2m}(x)^2 + 2xS_m^o(x) \pmod 4; (m \geq 0)$

7) $Q_{2^{2m}-2}(x) \equiv 1 + 2S_{2m-1}(x) \pmod 4; (m \geq 1)$

8) $Q_{2^{2m+1}-2}(x) \equiv 1 + 2x(1 - \delta_{m,0}) + 2S_{2m}(x) \pmod 4. (m \geq 0)$

Proof. We prove this result by induction on

$$n \in \{2^k - 1 \mid k\} \cup \{2^k - 2 \mid k\}.$$

When we compute $P_n(x)$ or $Q_n(x)$, the induction hypothesis is that the expressions for $P_\ell(x)$ and $Q_\ell(x)$ are true for $\ell < n$ and $\ell \in \{2^k - 1 \mid k\} \cup \{2^k - 2 \mid k\}$. Relations 1) - 8) are true for $m \equiv 0$ or $m \equiv 1$. In the sequel let $m \geq 2$.

1) Using the induction hypothesis, we have

$$
\begin{aligned}
Q_{2^{2m-1}-1}(-x) &\equiv 1 + x + 2x^{2^{2m-2}} - S_{2m-2}(x)^2 - 2xS_{m-1}^o(x); \\
P_{2^{2m-1}-1}(x) &\equiv 1 + 2x + 2S_{m-1}^e(x); \\
P_{2^{2m-1}-1}(-x) &\equiv 1 - 2x + 2S_{m-1}^e(x); \\
P_{2^{2m-1}-2}(x) &\equiv 1 + x^{-1}S_{2m-2}(x)^2 - 2S_{m-1}^o(x).
\end{aligned}
$$

By Lemma 3.1, we obtain

$$
\begin{aligned}
P_{2^{2m}-1} &\equiv Q_{2^{2m-1}-1}(-x)P_{2^{2m-1}-1}(x) - xP_{2^{2m-1}-1}(-x)P_{2^{2m-1}-2}(x) \\
&\equiv (1 + 2x + 2S_{m-1}^e(x)) \\
&\quad \times ((1 + x + 2x^{2^{2m-2}} - S_{2m-2}(x)^2 - 2xS_{m-1}^o(x)) \\
&\quad - x(1 + x^{-1}S_{2m-2}(x)^2 - 2S_{m-1}^o(x))) \\
&\equiv (1 + 2x + 2S_{m-1}^e(x))(1 + 2x^{2^{2m-2}} - 2S_{2m-2}(x)^2) \\
&\equiv 1 + 2x + 2S_{m-1}^e(x) + 2x^{2^{2m-2}} - 2S_{2m-2}(x)^2 \\
&\equiv 1 + 2S_{m-1}^e(x) + 2x^{2^{2m-2}} - 2S_{2m-1}(x) \\
&\equiv 1 + 2S_{m-1}^o(x).
\end{aligned}
$$

2) Using the induction hypothesis, we have

$$
\begin{aligned}
Q_{2^{2m}-1}(-x) &\equiv 1 + x + 2x^{2^{2m-1}} - S_{2m-1}(x)^2 + 2xS_m^e(x); \\
P_{2^{2m}-1}(x) &\equiv 1 + 2S_{m-1}^o(x);
\end{aligned}
$$

$$P_{2^{2m}-1}(-x) \equiv 1 + 2S^o_{m-1}(x);$$
$$P_{2^{2m}-2}(x) \equiv 1 + x^{-1}S_{2m-1}(x)^2 - 2S^e_m(x).$$

By Lemma 3.1, we obtain

$$P_{2^{2m+1}-1} \equiv Q_{2^{2m}-1}(-x)P_{2^{2m}-1}(x) - xP_{2^{2m}-1}(-x)P_{2^{2m}-2}(x)$$
$$\equiv (1 + 2S^o_{m-1}(x))$$
$$\times \left((1 + x + 2x^{2^{2m-1}} - S_{2m-1}(x)^2 + 2xS^e_m(x)) \right.$$
$$\left. - x(1 + x^{-1}S_{2m-1}(x)^2 - 2S^e_m(x)) \right)$$
$$\equiv (1 + 2S^o_{m-1}(x))(1 + 2x^{2^{2m-1}} - 2S_{2m-1}(x)^2)$$
$$\equiv 1 + 2S^o_{m-1}(x) + 2x^{2^{2m-1}} - 2S_{2m-1}(x)^2$$
$$\equiv 1 + 2S^o_{m-1}(x) + 2x^{2^{2m-1}} - 2S_{2m}(x) + 2x$$
$$\equiv 1 + 2x + 2S^e_m(x).$$

3) Using the induction hypothesis, we have

$$Q_{2^{2m-1}-2}(x) \equiv 1 + 2x + 2S_{2m-2};$$
$$P_{2^{2m-1}-1}(x) \equiv 1 + 2x + 2S^e_{m-1};$$
$$P_{2^{2m-1}-2}(x) \equiv 1 + x^{-1}S^2_{2m-2} - 2S^o_{m-1};$$
$$P_{2^{2m-1}-2}(-x) \equiv 1 - x^{-1}S^2_{2m-2} - 2S^o_{m-1}.$$

By Lemma 3.1, we obtain

$$P_{2^{2m}-2}(x) \equiv Q_{2^{2m-1}-2}(-x)P_{2^{2m-1}-1}(x) - xP_{2^{2m-1}-2}(-x)P_{2^{2m-1}-2}(x)$$
$$\equiv (1 + 2x + 2S_{2m-2})(1 + 2x + 2S^e_{m-1})$$
$$- x(1 + x^{-1}S^2_{2m-2} - 2S^o_{m-1})(1 - x^{-1}S^2_{2m-2} - 2S^o_{m-1})$$
$$\equiv 1 + 2S_{2m-2} + 2S^e_{m-1} - x(1 - 2S^o_{m-1})^2 + x(x^{-1}S^2_{2m-2})^2$$
$$\equiv 1 + 2S_{2m-2} + 2S^e_{m-1} - x + x^{-1}(S_{2m-1} - x)^2$$
$$\equiv 1 + x^{-1}S^2_{2m-1} + 2S^e_m.$$

4) Using the induction hypothesis, we have

$$P_{2^{2m}-1}(x) \equiv 1 + 2S^o_{m-1}(x);$$
$$P_{2^{2m}-2}(x) \equiv 1 + x^{-1}S_{2m-1}(x)^2 - 2S^e_m(x);$$
$$Q_{2^{2m}-2}(x) \equiv 1 + 2S_{2m-1}(x).$$

By Lemma 3.1, we obtain

$$P_{2^{2m+1}-2}(x) \equiv Q_{2^{2m}-2}(-x)P_{2^{2m}-1}(x) - xP_{2^{2m}-2}(-x)P_{2^{2m}-2}(x)$$
$$\equiv (1 + 2S_{2m-1}(-x))(1 + 2S^o_{m-1}(x))$$
$$- x(1 - x^{-1}S_{2m-1}(-x)^2 - 2S^e_m(-x))(1 + x^{-1}S_{2m-1}(x)^2 - 2S^e_m(x))$$
$$\equiv 1 + 2S_{2m-1}(x) + 2S^o_{m-1}(x) - x + x^{-1}S_{2m-1}(x)^4$$
$$\equiv 1 + 2S_{2m-1}(x) + 2S^o_{m-1}(x) - x + x^{-1}(S_{2m}(x)^2 - 2xS_{2m}(x) + x^2)$$
$$\equiv 1 + x^{-1}S_{2m}(x)^2 - 2S^o_m(x).$$

5) Using the induction hypothesis, we have

$$P_{2^{2m-1}-1}(x) \equiv 1 + 2x + 2S_{m-1}^e(x);$$

$$Q_{2^{2m-1}-1}(x) \equiv 1 - x + 2x^{2^{2m-2}} - S_{2m-2}(x)^2 + 2xS_{m-1}^o(x);$$

$$Q_{2^{2m-1}-2}(x) \equiv 1 + 2x + 2S_{2m-2}(x).$$

By Lemma 3.1, we obtain

$$Q_{2^{2m}-1}(x) \equiv Q_{2^{2m-1}-1}(-x)Q_{2^{2m-1}-1}(x) - xP_{2^{2m-1}-1}(-x)Q_{2^{2m-1}-2}(x)$$

$$\equiv (1 + x + 2x^{2^{2m-2}} - S_{2m-2}^2 + 2xS_{m-1}^o)$$

$$(1 - x + 2x^{2^{2m-2}} - S_{2m-2}^2 + 2xS_{m-1}^o)$$

$$- x(1 + 2x + 2S_{m-1}^e)(1 + 2x + 2S_{2m-2})$$

$$\equiv (1 - S_{2m-2}^2)^2 - x^2 - (x + 2xS_{m-1}^e + 2xS_{2m-2})$$

$$\equiv 1 + 2S_{2m-2}^2 + S_{2m-1}^2 + 2xS_{2m-1} - (x + 2xS_{m-1}^e + 2xS_{2m-2})$$

$$\equiv 1 - x + 2x^{2^{2m-1}} - S_{2m-1}^2 + 2xS_m^e.$$

6) Using the induction hypothesis, we have

$$P_{2^{2m}-1}(x) \equiv 1 + 2S_{m-1}^o(x);$$

$$Q_{2^{2m}-1}(x) \equiv 1 - x + 2x^{2^{2m-1}} - S_{2m-1}(x)^2 + 2xS_m^e(x);$$

$$Q_{2^{2m}-2}(x) \equiv 1 + 2S_{2m-1}(x).$$

By Lemma 3.1, we obtain

$$Q_{2^{2m+1}-1}(x) \equiv Q_{2^{2m}-1}(-x)Q_{2^{2m}-1}(x) - xP_{2^{2m}-1}(-x)Q_{2^{2m}-2}(x)$$

$$\equiv (1 + x + 2x^{2^{2m-1}} - S_{2m-1}^2 + 2xS_m^e)$$

$$\equiv (1 - x + 2x^{2^{2m-1}} - S_{2m-1}^2 + 2xS_m^e)$$

$$- x(1 + 2S_{m-1}^o)(1 + 2S_{2m+1})$$

$$\equiv 1 + 3S_{2m}^2 + 2xS_{2m} + 2x^{2^{2m}} - x + 2xS_{m-1}^o + 2xS_{2m-1}$$

$$\equiv 1 - x + 2x^{2^{2m}} - S_{2m}^2 + 2xS_m^o.$$

7) Using the induction hypothesis, we have

$$P_{2^{2m-1}-2}(x) \equiv 1 + x^{-1}S_{2m-2}(x)^2 - 2S_{m-1}^o(x);$$

$$Q_{2^{2m-1}-1}(x) \equiv 1 - x + 2x^{2^{2m-2}} - S_{2m-2}(x)^2 + 2xS_{m-1}^o(x);$$

$$Q_{2^{2m-1}-2}(x) \equiv 1 + 2x + 2S_{2m-2}(x).$$

By Lemma 3.1, we obtain

$$Q_{2^{2m}-2}(x)$$

$$\equiv Q_{2^{2m-1}-2}(-x)Q_{2^{2m-1}-1}(x) - xP_{2^{2m-1}-2}(-x)Q_{2^{2m-1}-2}(x)$$

$$\equiv (1 + 2x + 2S_{2m-2})(1 - x + 2x^{2^{2m-2}} - S_{2m-2}^2 + 2xS_{m-1}^o)$$

$$- x(1 - x^{-1}S_{2m-2}^2 + 2S_{m-1}^o)(1 + 2x + 2S_{2m-2})$$

$$\equiv 1 + x + 2x2^{2m-2} - S_{2m-2}^2 + 2xS_{m-1}^o + 2x^2 + 2xS_{2m-2}^2 + 2S_{2m-2} + 2xS_{2m-2}$$

$$+ 2S_{2m-2}^3 - x(1 + 2x + 2S_{2m-2} - x^{-1}S_{2m-2}^2 + 2S_{2m-2}^2 + 2x^{-1}S_{2m-2}^3 + 2S_{m-1}^o)$$

$$\equiv 1 + 2S_{2m-1}.$$

8) Using the induction hypothesis, we have

$$P_{2^{2m}-2}(x) \equiv 1 + x^{-1}S_{2m-1}(x)^2 - 2S_m^e(x);$$
$$Q_{2^{2m}-1}(x) \equiv 1 - x + 2x^{2^{2m-1}} - S_{2m-1}(x)^2 + 2xS_m^e(x);$$
$$Q_{2^{2m}-2}(x) \equiv 1 + 2S_{2m-1}(x);$$

By Lemma 3.1, we obtain

$$Q_{2^{2m+1}-2}(x) \equiv Q_{2^{2m}-2}(-x)Q_{2^{2m}-1}(x) - xP_{2^{2m}-2}(-x)Q_{2^{2m}-2}(x)$$
$$\equiv (1 + 2S_{2m-1})(1 - x + 2x^{2^{2m-1}} - S_{2m-1}^2 + 2xS_m^e)$$
$$\quad - x(1 - x^{-1}S_{2m-1}^2 + 2S_m^e)(1 + 2S_{2m-1})$$
$$\equiv 1 - x + 2x^{2^{2m-1}} - S_{2m-1}^2 + 2xS_m^e + 2S_{2m-1} + 2xS_{2m-1} + 2S_{2m-1}^3$$
$$\quad - (x - S_{2m-1}^2 + 2xS_m^e + 2xS_{2m-1} + 2S_{2m-1}^3)$$
$$\equiv 1 + 2x + 2S_{2m-1} + 2x^{2^{2m-1}}$$
$$\equiv 1 + 2x + 2S_{2m}. \qquad \square$$

The explicit expressions of $P_{2^{2m}-2}(x)$ and $P_{2^{2m}-2}(x)$ give the explicit expression for $C(x)$.

Proposition 3.3.

(3.1) $$C(x) \equiv 1 - \sum_{i,j=0}^{\infty} x^{2^i + 2^j - 1} + 2\sum_{k=0}^{\infty} x^{2^{2k}} \pmod{4}.$$

Proof. By Theorem 2.1,

$$C(x) = \lim_{m\to\infty} P_{2^{2m}-2}(x)/Q_{2^{2m}-2}(x).$$

The constant term of $Q_{2^{2m}+2}(x)$ being 1, $1/Q_{2^{2m}+2}(x)$ belongs to $\mathbb{Z}[[x]]$. By 3) and 7) of Proposition 3.2,

$$C(x) \equiv \lim_{m\to\infty} \frac{1 + x^{-1}S_{2m-1}(x)^2 - 2S_m^e(x)}{1 + 2S_{2m-1}(x)}$$
$$\equiv \lim_{m\to\infty} (1 + x^{-1}S_{2m-1}(x)^2 - 2S_m^e(x))(1 + 2S_{2m-1}(x))$$
$$\equiv \lim_{m\to\infty} 1 + x^{-1}S_{2m-1}(x)^2 - 2S_m^e(x) + 2S_{2m-1}(x) + 2x^{-1}S_{2m-1}(x)^3$$
$$\equiv 1 + x^{-1}\left(\sum_{j=0}^{\infty} x^{2^j}\right)^2 + 2\sum_{j=0}^{\infty} x^{2^{2j}} + 2\sum_{j=0}^{\infty} x^{2^j} + 2x^{-1}\left(\sum_{j=0}^{\infty} x^{2^j}\right)^3$$
$$\equiv 1 + x^{-1}\left(\sum_{j=0}^{\infty} x^{2^j}\right)^2 + 2\sum_{j=0}^{\infty} x^{2^{2j}} + 2\sum_{j=0}^{\infty} x^{2^j} + 2x^{-1}\left(\sum_{j=1}^{\infty} x^{2^j}\right)\left(\sum_{j=0}^{\infty} x^{2^j}\right)$$
$$\equiv 1 + x^{-1}\left(\sum_{j=0}^{\infty} x^{2^j}\right)^2 + 2\sum_{j=0}^{\infty} x^{2^{2j}} + 2\sum_{j=0}^{\infty} x^{2^j} + 2x^{-1}\left(\sum_{j=0}^{\infty} x^{2^j}\right)^2 + 2\sum_{j=0}^{\infty} x^{2^j}$$
$$\equiv 1 - x^{-1}\left(\sum_{j=0}^{\infty} x^{2^j}\right)^2 + 2\sum_{j=0}^{\infty} x^{2^{2j}}$$

$$\equiv 1 - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1} + 2 \sum_{k=0}^{\infty} x^{2^{2k}} \pmod 4. \qquad \square$$

Now we prove Theorem 1.4 by repeated applying the Cartier operators to the right hand side of (3.1).

Proof of Theorem 1.4. We recall that $\bar{C}(x)$ denotes the series in $\mathbb{Z}/4\mathbb{Z}[[x]]$ that is the reduction modulo 4 of $C(x)$. We prove that by applying Λ_0 and Λ_1 repeatedly to $\bar{C}(x)$, we can only obtain a finite number of series. Indeed, we have

$$\bar{C}(x) =: f_0,$$

$$\Lambda_0 f_0 = \Lambda_0 \bar{C}(x) = 1 + 2\sum_{j=0}^{\infty} x^{2^j} + 2\sum_{k=0}^{\infty} x^{2^{2k+1}} = 1 + 2\sum_{j=0}^{\infty} x^{2^{2j}} =: f_1,$$

$$\Lambda_1 f_0 = \Lambda_1 \bar{C}(x) = -1 - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1} + 2 = 1 - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1} =: f_2,$$

$$\Lambda_0 f_1 = \Lambda_0 \Big(1 + 2\sum_{j=0}^{\infty} x^{2^{2j}}\Big) = 1 + 2\sum_{j=0}^{\infty} x^{2^{2j+1}} =: f_3,$$

$$\Lambda_1 f_1 = \Lambda_1 \Big(1 + 2\sum_{j=0}^{\infty} x^{2^{2j}}\Big) = 2 =: f_4,$$

$$\Lambda_0 f_2 = \Lambda_0 \Big(1 - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1}\Big) = 1 + 2\sum_{j=0}^{\infty} x^{2^j} =: f_5,$$

$$\Lambda_1 f_2 = \Lambda_1 \Big(1 - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1}\Big) = - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1} =: f_6,$$

$$\Lambda_0 f_3 = \Lambda_0 \Big(1 + 2\sum_{j=0}^{\infty} x^{2^{2j+1}}\Big) = 1 + 2\sum_{j=0}^{\infty} x^{2^{2j}} = f_1,$$

$$\Lambda_1 f_3 = \Lambda_1 \Big(1 + 2\sum_{j=0}^{\infty} x^{2^{2j+1}}\Big) = 0 =: f_7,$$

$$\Lambda_0 f_4 = \Lambda_0 2 = 2 = f_4,$$
$$\Lambda_1 f_4 = \Lambda_1 2 = 0 = f_7,$$

$$\Lambda_0 f_5 = \Lambda_0 \Big(1 + 2\sum_{j=0}^{\infty} x^{2^j}\Big) = f_5,$$

$$\Lambda_1 f_5 = \Lambda_1 \Big(1 + 2\sum_{j=0}^{\infty} x^{2^j}\Big) = 2 = f_4,$$

$$\Lambda_0 f_6 = \Lambda_0 \Big(- \sum_{i,j=0}^{\infty} x^{2^i+2^j-1}\Big) = -1 + 2\sum_{j=0}^{\infty} x^{2^j} =: f_8,$$

$$\Lambda_1 f_6 = \Lambda_1 \Big(- \sum_{i,j=0}^{\infty} x^{2^i+2^j-1}\Big) = - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1} = f_6,$$

$$\Lambda_0 f_8 = \Lambda_0 \Big(-1 + 2\sum_{j=0}^{\infty} x^{2^j}\Big) = f_8,$$

$$\Lambda_1 f_8 = \Lambda_0\left(-1 + 2\sum_{j=0}^{\infty} x^{2^j}\right) = 2 = f_4.$$

We see from the computation above that the 2-kernel of $\bar{C}(x)$ consists of 9 elements, f_0 through f_8. The structure of the 2-kernel is

$$[[1,2],[3,4],[5,6],[1,7],[4,7],[5,4],[8,6],[7,7],[8,4]]. \quad \square$$

The following lemma is used in the proof of Theorem 1.1 (see, for example, [22]).

Lemma 3.4.
$$\sqrt{1-4x} \equiv 1 + 2\sum_{k=1}^{\infty} x^{2^k} \quad (\text{mod } 4).$$

Proof of Theorem 1.1. From the proof of Proposition 3.3 we know that

$$C(x) \equiv 1 + x^{-1}S_\infty(x)^2 + 2S_\infty^e(x) + 2S_\infty(x) + 2x^{-1}S_\infty(x)^3 \quad (\text{mod } 4),$$

therefore, we only need to find $S_\infty(x)$ (mod 2), $(S_\infty(x))^2$ (mod 4) and $S_\infty^e(x)$ (mod 2). By Lemma 3.4,

$$(3.2) \qquad\qquad S_\infty(x) \equiv \frac{1 - \sqrt{1-4x}}{2} \quad (\text{mod } 2),$$

so that

$$(3.3) \qquad S_\infty(x)^2 \equiv \left(\frac{1 - \sqrt{1-4x}}{2}\right)^2 \equiv \frac{1 - 2x - \sqrt{1-4x}}{2} \quad (\text{mod } 4).$$

To calculate $S_\infty^e(x)$ (mod 2), we notice that

$$S_\infty^e(x)^2 + S_\infty^e(x) = S_\infty^e(x^2) + S_\infty^e(x) + 2x\psi(x)$$
$$= S_\infty(x) + 2x\psi(x)$$
$$= \frac{1 - \sqrt{1-4x}}{2} + 2x\xi(x) + 2x\psi(x),$$

where

$$\psi(x) = \frac{1}{2x}\left(S_\infty^e(x)^2 - S_\infty^e(x^2)\right) \quad \text{and} \quad \xi(x) = \frac{1}{2x}\left(S_\infty(x) - \frac{1 - \sqrt{1-4x}}{2}\right)$$

are in $\mathbb{Z}[[x]]$. We remark that by Lemma 3.4, if

$$f(x), g(x) \in \mathbb{Z}[[x]] \quad \text{and} \quad f(x) \equiv g(x) \quad (\text{mod } 2),$$

then

$$\sqrt{1 + 4xf(x)} \equiv \sqrt{1 + 4xg(x)} \quad (\text{mod } 4).$$

Therefore

$$S_\infty^e(x) = \frac{-1 + \sqrt{1 - (2 - 2\sqrt{1-4x}) + 2x\xi(x) + 2x\psi(x)}}{2}$$

$$(3.4) \qquad\qquad \equiv \frac{-1 + \sqrt{2\sqrt{1-4x} - 1}}{2} \quad (\text{mod } 2).$$

Finally

$$C(x) \equiv 1 + x^{-1}S_\infty(x)^2 + 2S_\infty^e(x) + 2S_\infty(x) + 2x^{-1}S_\infty(x)^3$$
$$\equiv 1 + \frac{1 - 2x - \sqrt{1-4x}}{2x} + \left(-1 + \sqrt{2\sqrt{1-4x} - 1}\right)$$

$$+ \left(1 - \sqrt{1 - 4x}\right) + 2x^{-1} \left(\frac{1 - \sqrt{1 - 4x}}{2}\right)^3$$

$$\equiv \frac{\sqrt{1 - 4x} - 1}{2x} + 1 + \sqrt{2\sqrt{1 - 4x} - 1} \pmod 4. \qquad \square$$

4. Period-doubling continued fraction

In this section we prove Theorem 1.2 using Theorem 1.1 and Theorem 2.3. As a corollary, we get the explicit expression of $\bar{D}(x)$ as a power series and from this we calculate the 2-kernel of the sequence $(\bar{d}_n)$.

Proof of Theorem 1.2. In Theorem 2.3, if we let

$$u_1 = t_1 = -1;$$
$$u_n = t_{2n-2} + t_{2n-1} = 0; \qquad (n \geq 2)$$
$$v_0 = t_0 = 1;$$
$$v_n = t_{2n-1}t_{2n} = -t_{n-1}t_n = s_{n-1}, \qquad (n \geq 1)$$

we get

$$(4.1) \quad C(x) = \cfrac{t_0}{1 + \cfrac{t_1 x}{1 + \cfrac{t_2 x}{1 + \cfrac{t_3 x}{1 + \cfrac{t_4 x}{\ddots}}}}} = \cfrac{1}{1 - x - \cfrac{s_0 x^2}{1 - \cfrac{s_1 x^2}{1 - \cfrac{s_2 x^2}{\ddots}}}} = \frac{1}{1 - x - x^2 D(-x^2)}.$$

We define

$$H_1(x) = \frac{\sqrt{1 - 4x} - 1}{2x} + 1 + \sqrt{2\sqrt{1 - 4x} - 1} = 1 - 3x + \cdots$$

$$H_2(x) = \frac{1 + \sqrt{1 + 4x}}{2} = 1 + x + \cdots$$

$$H_3(x) = \sqrt{2\sqrt{1 - 4x^2} - 1} = 1 - 2x^2 + \cdots$$

Then our goal (1.4) can be written as

$$D(x) \equiv \frac{H_1(x)H_3(x) - 1}{x} \pmod 4.$$

Since $C(x) \equiv H(x) \pmod 4$ and the constant term of $C(x)$ and $H_1(x)$ is 1, by (4.1) we know that

$$-x^2 D(-x^2) = \frac{1}{C(x)} - (1 - x) \equiv \frac{1}{H_1(x)} - 1 + x \pmod 4.$$

We only need to show that

$$\frac{1}{H_1(x)} - 1 + x \equiv -x^2 \times \frac{H_2(-x^2)H_3(-x^2) - 1}{-x^2} \pmod 4,$$

that is,

$$\frac{1}{H_1(x)} + x \equiv H_2(-x^2)H_3(-x^2) \pmod 4.$$

Since the constant term of $H_1(x)$ is 1, this is equivalent to

$$H_1(x)(H_2(-x^2)H_3(-x^2) - x) \equiv 1 \pmod 4.$$

By (3.3), (3.4) and (3.2),

$$H_1(x) \equiv -\frac{S_\infty(x)^2}{x} + 2S_\infty^e + 1 \pmod 4,$$

$$H_2(-x^2) = \frac{1 + \sqrt{1 - 4x^2}}{2}$$

$$\equiv 1 - x^2 - S_\infty(x^2)^2$$

$$\equiv 1 - x^2 - (S_\infty(x) - x)^2$$

$$\equiv 1 + (2x - 1)S_\infty(x)^2 \pmod 4,$$

$$H_3(-x^2) = \sqrt{2\sqrt{1 - 4x^4} - 1}$$

$$\equiv 1 + 2S_\infty^e(x^4)$$

$$\equiv 1 - 2x + 2S_\infty^e(x) \pmod 4.$$

Taking account of the above congruence relations and after rearranging the terms, we get

(4.2)
$$xH_1(x)(H_2(-x^2)H_3(-x^2) - x)$$

$$\equiv x - xS_\infty(x)^2 + x^2 - S_\infty(x)^2(1 - S_\infty(x)^2 + x)$$

$$+ S_\infty^e(x)(2x^2 + 2S_\infty(x)^2(1 - S_\infty(x)^2)) \pmod 4.$$

Since by (3.2) and (3.3) we have

$$2S_\infty(x)^2(1 - S_\infty(x)^2) \equiv 2(S_\infty(x) - x)(1 - S_\infty(x) + x)$$

$$\equiv 2(S_\infty(x) - S_\infty(x)^2 - x - x^2)$$

$$\equiv 2x^2 \pmod 4,$$

and

$$S_\infty(x)^4 \equiv (S_\infty(x) - x)^2$$

$$\equiv S_\infty(x)^2 + x^2 - 2xS_\infty(x) \pmod 4,$$

congruence (4.2) becomes

$$xH_1(x)(H_2(-x^2)H_3(-x^2) - x)$$

$$\equiv x - xS_\infty(x)^2 + x^2 - S_\infty(x)^2(1 - S_\infty(x)^2 + x)$$

$$\equiv x + 2x^2 + 2xS_\infty(x)^2 + 2xS_\infty(x)$$

$$\equiv x \pmod 4. \qquad \square$$

From Theorem 1.2 we obtain the following explicit expression for $D(x) \pmod 4$.

Corollary 4.1.

(4.3)
$$D(x) \equiv 1 - \sum_{i,j=0}^{\infty} x^{2^i + 2^j - 1} + 2\sum_{k=0}^{\infty} x^{2^{2k+1} - 1}\left(1 + \sum_{j=0}^{\infty} x^{2^j}\right) \pmod 4.$$

Proof of Corollary 4.1. We obtained the following congruence in Theorem 1.2,

$$D(x) \equiv \frac{(1 + \sqrt{1 + 4x})\sqrt{2\sqrt{1 - 4x^2} - 1} - 2}{2x} \quad (\mathrm{mod}\ 4).$$

From (3.3) we know

$$\frac{1 + \sqrt{1 + 4x}}{2} \equiv 1 + x - S_\infty(-x)^2 \quad (\mathrm{mod}\ 4)$$

$$\equiv 1 + x - S_\infty(x)^2 \quad (\mathrm{mod}\ 4)$$

$$\equiv 1 + x - \left(\sum_{j=0}^\infty x^{2^j}\right)^2 \quad (\mathrm{mod}\ 4)$$

$$\equiv 1 + x - \sum_{i,j=0}^\infty x^{2^i + 2^j} \quad (\mathrm{mod}\ 4).$$

By (3.4)

$$\sqrt{2\sqrt{1 - 4x^2} - 1} \equiv 1 + 2\sum_{k=0}^\infty x^{2^{2k+1}} \quad (\mathrm{mod}\ 4).$$

Therefore

$$D(x) \equiv \frac{1}{x}\left(\left(1 + x - \sum_{i,j=0}^\infty x^{2^i + 2^j}\right)\left(1 + 2\sum_{k=0}^\infty x^{2^{2k+1}}\right) - 1\right)$$

$$\equiv 1 - \sum_{i,j=0}^\infty x^{2^i + 2^j - 1} + 2\sum_{k=0}^\infty x^{2^{2k+1} - 1}\left(1 + x - \sum_{i,j=0}^\infty x^{2^i + 2^j}\right)$$

$$\equiv 1 - \sum_{i,j=0}^\infty x^{2^i + 2^j - 1} + 2\sum_{k=0}^\infty x^{2^{2k+1} - 1}\left(1 + x - \sum_{j=0}^\infty x^{2^{j+1}}\right)$$

$$\equiv 1 - \sum_{i,j=0}^\infty x^{2^i + 2^j - 1} + 2\sum_{k=0}^\infty x^{2^{2k+1} - 1}\left(1 + \sum_{j=0}^\infty x^{2^j}\right). \qquad \square$$

Proof of Theorem 1.5. By Theorem 1.2 and Theorem 1.3 we know that the sequence $(\bar{d}_n)$ is 2-automatic. Using Corollary 4.1, we calculate the 2-kernel of $(\bar{d}_n)$. First we compute $\Lambda_0(\bar{D}(x))$ and $\Lambda_1(\bar{D}(x))$. We define three power series in $\mathbb{Z}/4\mathbb{Z}[[x]]$:

$$A := -\sum_{i,j=0}^\infty x^{2^i + 2^j - 1}, \quad B := 2\sum_{k=0}^\infty x^{2^{2k+1} - 1}, \quad C := 2\left(\sum_{k=0}^\infty x^{2^{2k+1} - 1}\right)\left(\sum_{j=0}^\infty x^{2^j}\right),$$

so that $\bar{D}(x) = 1 + A + B + C$. We have

$$\Lambda_0(A) = \Lambda_0(2\sum_{j=1}^\infty x^{2^j}) = \Lambda_0(2\sum_{j=0}^\infty (x^2)^{2^j}) = 2\sum_{j=0}^\infty x^{2^j},$$

$$\Lambda_1(A) = \Lambda_1\left(-x - \sum_{i,j=1}^\infty x^{2^i + 2^j - 1}\right)$$

$$= \Lambda_1\left(x\left(-1 - \sum_{i,j=1}^\infty (x^2)^{2^{i-1} + 2^{j-1} - 1}\right)\right)$$

$$= -1 - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1},$$

$$\Lambda_0(B) = 0,$$

$$\Lambda_1(B) = \Lambda_1\Big(2x \sum_{k=0}^{\infty} x^{2^{2k+1}-2}\Big) = \Lambda_1\Big(2x \sum_{k=0}^{\infty} (x^2)^{2^{2k}-1}\Big) = 2 \sum_{k=0}^{\infty} x^{2^{2k}-1},$$

$$\Lambda_0(C) = \Lambda_0\Big(2 \sum_{k=0}^{\infty} x^{2^{2k+1}}\Big) = \Lambda_0\Big(2 \sum_{k=0}^{\infty} (x^2)^{2^{2k}}\Big) = 2 \sum_{k=0}^{\infty} x^{2^{2k}},$$

$$\Lambda_1(C) = \Lambda_1\Big(2x \sum_{k=0,j=1}^{\infty} x^{2^{2k+1}+2^j-2}\Big)$$

$$= \Lambda_1\Big(2x \sum_{k=0,j=1}^{\infty} (x^2)^{2^{2k}+2^{j-1}-1}\Big)$$

$$= 2 \sum_{k,j=0}^{\infty} x^{2^{2k}+2^j-1}.$$

Thus, if we let f_0 denote $\bar{D}(x)$, then

$$\Lambda_0(f_0) = \Lambda_0(\bar{D}(x)) = 1 + \Lambda_0(A) + \Lambda_0(B) + \Lambda_0(C)$$

$$= 1 + 2 \sum_{j=0}^{\infty} x^{2^j} + 2 \sum_{k=0}^{\infty} x^{2^{2k}}$$

$$= 1 + 2 \sum_{j=0}^{\infty} x^{2^{2j+1}}$$

$$=: f_1,$$

and

$$\Lambda_1(f_0) = \Lambda_1(\bar{D}(x)) = \Lambda_1(A) + \Lambda_1(B) + \Lambda_1(C)$$

$$= -1 - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1} + 2 \sum_{k=0}^{\infty} x^{2^{2k}-1} + 2 \sum_{k,j=0}^{\infty} x^{2^{2k}+2^j-1}$$

$$= \bar{D}(x) = f_0.$$

The last equality holds because

$$\bar{D}(x) - \Big(-1 - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1} + 2 \sum_{k=0}^{\infty} x^{2^{2k}-1} + 2 \sum_{k,j=0}^{\infty} x^{2^{2k}+2^j-1}\Big)$$

$$= 2 + 2 \sum_{k=0}^{\infty} x^{2^{2k+1}-1}\Big(1 + \sum_{j=1}^{\infty} x^{2^j}\Big) + 2 \sum_{k=0}^{\infty} x^{2^{2k}-1}\Big(1 + \sum_{j=1}^{\infty} x^{2^j}\Big)$$

$$= 2 + 2 \sum_{k=0}^{\infty} x^{2^k-1}\Big(1 + \sum_{j=1}^{\infty} x^{2^j}\Big)$$

$$= 2 + 2 \sum_{k=0}^{\infty} x^{2^k-1} + 2 \sum_{j,k=0}^{\infty} x^{2^k+2^j-1}$$

$$= 2 + 2\sum_{k=0}^{\infty} x^{2^k-1} + 2\sum_{j=0}^{\infty} x^{2^j+2^j-1}$$

$$= 2 + 2\sum_{k=0}^{\infty} x^{2^k-1} + 2\sum_{j=0}^{\infty} x^{2^{j+1}-1}$$

$$= 2 + 2x^{2^0-1}$$

$$= 0.$$

Then we calculate $\Lambda_0(f_1)$ and $\Lambda_1(f_1)$:

$$\Lambda_0(f_1) = \Lambda_0\Big(1 + 2\sum_{j=0}^{\infty} x^{2^{2j+1}}\Big) = \Lambda_0\Big(1 + 2\sum_{j=0}^{\infty} (x^2)^{2^{2j}}\Big) = 1 + 2\sum_{j=0}^{\infty} x^{2^{2j}} =: f_2,$$

$$\Lambda_1(f_1) = \Lambda_1\Big(1 + 2\sum_{j=0}^{\infty} x^{2^{2j+1}}\Big) = 0 =: f_3.$$

Finally we calculate $\Lambda_0(f_2)$ and $\Lambda_1(f_2)$:

$$\Lambda_0(f_2) = \Lambda_0\Big(1 + 2\sum_{j=1}^{\infty} x^{2^{2j}}\Big)$$

$$= \Lambda_0\Big(1 + 2\sum_{j=0}^{\infty} (x^2)^{2^{2j+1}}\Big)$$

$$= 1 + 2\sum_{j=0}^{\infty} x^{2^{2j+1}}$$

$$= \Lambda_0(\bar{D}(x)) = f_1,$$

$$\Lambda_1(f_2) = \Lambda_1(2x) = 2 =: f_4. \quad \square$$

We see that the structure of the 2-kernel of $(\bar{d}_n)$ is $[[1,0],[2,3],[1,4],[3,3],[4,3]]$.

5. Proof of Theorem 1.6

In this section we prove Theorem 1.6. First we recall that from Theorem 1.1 and 1.2 that

$$C(x) \equiv \varphi(x) \pmod{4},$$
$$D(x) \equiv \psi(x) \pmod{4},$$

where

$$\varphi(x) = \frac{\sqrt{1-4x}-1}{2x} + 1 + \sqrt{2\sqrt{1-4x}-1} \in \mathbb{Z}[[x]],$$

$$\psi(x) = \frac{(1+\sqrt{1+4x})\sqrt{2\sqrt{1-4x^2}-1}-2}{2x} \in \mathbb{Z}[[x]].$$

By rearranging the terms and squaring both sides of the equalities, we obtain annihilating polynomials $P(x,y)$ and $Q(x,y)$ of $\varphi(x)$ and $\psi(x)$ respectively:

$$P(x,y) = y^4x^2 - 4\,y^3x^2 + 2\,y^3x^1 + 8\,y^2x^2 - 4\,y^2x^1 + 8\,yx^2$$
$$+ 16\,x^3 + y^2 - 16\,x^2 + 8\,x^1 - 1,$$

$$Q(x,y) = y^8x^7 + 8\,y^7x^6 + 4\,y^6x^6 + 30\,y^6x^5 + 32\,y^4x^7 + 24\,y^5x^5$$
$$+ 64\,y^4x^6 + 68\,y^5x^4 + 14\,y^4x^5 + 128\,y^3x^6 + 48\,y^4x^4$$

$$+ 256\, y^3 x^5 + 64\, y^2 x^6 + 97\, y^4 x^3 + 56\, y^3 x^4 + 224\, y^2 x^5$$
$$+ 256\, x^7 + 32\, y^3 x^3 + 372\, y^2 x^4 + 128\, yx^5 + 84\, y^3 x^2 + 78\, y^2 x^3$$
$$+ 192\, yx^4 - 96\, x^5 - 12\, y^2 x^2 + 232\, yx^3 + 64\, x^4 + 40\, y^2 x$$
$$+ 44\, yx^2 + 73\, x^3 - 24\, yx + 52\, x^2 + 8\, y + 8\, x - 8.$$

For $n \in \mathbb{N}^*$, we let π_n denote the canonical projection of $\mathbb{Z}$ onto $\mathbb{Z}/n\mathbb{Z}$, and by abuse of notation, the canonical projection of $\mathbb{Z}[[x]]$ onto $\mathbb{Z}/n\mathbb{Z}[[x]]$, of $\mathbb{Z}[x,y]$ onto $\mathbb{Z}/n\mathbb{Z}[x,y]$, etc.

Since $P(x, \varphi(x)) = 0$, $Q(x, \psi(x)) = 0$, and

$$\pi_2(P(x,y)) = x^2 y^4 + y^2 + 1 = (xy^2 + y + 1)^2,$$
$$\pi_2(Q(x,y)) = x^7 y^8 + x^3 y^4 + x^3 = x^3 (xy^2 + y + 1)^4,$$

we have

$$(5.1) \qquad\qquad x\varphi(x)^2 + \varphi(x) + 1 \equiv 0 \pmod 2,$$

$$(5.2) \qquad\qquad x\psi(x)^2 + \psi(x) + 1 \equiv 0 \pmod 2,$$

and therefore

$$(x\varphi(x)^2 + \varphi(x) + 1)^2 \equiv 0 \pmod 4,$$
$$(x\psi(x)^2 + \psi(x) + 1)^2 \equiv 0 \pmod 4.$$

In other words, the polynomial $S(x,y) = (xy^2 + y + 1)^2 \in \mathbb{Z}/4\mathbb{Z}[x,y]$ is an annihilating polynomial for both $\bar{C} = \pi_4(\varphi)$ and $\bar{D} = \pi_4(\psi)$.

Now we prove that there is no polynomial in $\mathbb{Z}/4\mathbb{Z}[x,y]$ that, seen as a polynomial in y, has degree less than 4, and, whose leading coefficient is invertible in the ring of Laurent series $\mathbb{Z}/4\mathbb{Z}((x))$, that annihilates either $\bar{C}(x)$ or $\bar{D}(x)$. By absurdity, suppose that $Q(x,y) = Q_n(x)y^n + \cdots + Q_1(x)y + Q_0(x)$ is such a polynomial of minimal degree on y. By assumption, n is less than 4, $Q_n(x)$ is invertible in $\mathbb{Z}/4\mathbb{Z}((x))$ and $Q(x,y)$ annihilates either $\bar{C}(x)$ or $\bar{D}(x)$. Since $Q_n(x)$ is invertible in $\mathbb{Z}/4\mathbb{Z}((x))$, we can effectuate Euclidean division of $P(x,y)$ by $Q(x,y)$, and by minimality of n, we obtain

$$Q_n(x)P(x,y) = Q(x,y)R(x,y)$$

for some $R(x,y) \in \mathbb{Z}/4\mathbb{Z}[x,y]$.

Reducing modulo 2 (where we use π_2 by abuse of notation), we get

$$\pi_2(Q(x,y))\pi_2(R(x,y)) = \pi_2(Q_n(x)P(x,y)) = \pi_2(Q_n(x))(xy^2 + y + 1)^2.$$

Since $Q_n(x)$ is invertible in $\mathbb{Z}/4\mathbb{Z}((x))$, $\pi_2(Q(x))$ is non-zero. As factorization into irreducible factors of $\pi_2(Q(x,y)R(x,y))$ in $\mathbb{F}_2(x)[y]$ is unique up to multiplication by elements in $\mathbb{F}_2(x)$, and $1 \le n \le 3$, we know that there exists $\alpha(x) \in \mathbb{Z}[x]$ taking coefficients in $\{0, 1\}$, such that $\pi_2(\alpha(x))$ is a factor of $\pi_2(Q_n(x))$ and

$$\pi_2(Q(x,y)) = \pi_2(\alpha(x)) \cdot (xy^2 + y + 1).$$

Therefore there exist polynomials $\beta_0(x), \beta_1(x), \beta_2(x)$ in $\mathbb{Z}[x]$ taking coefficients in $\{0, 1\}$, such that

$$Q(x,y) = \pi_4(\alpha(x)) \cdot (xy^2 + y + 1) + 2x\pi_4(\beta_2(x))y^2 + 2\pi_4(\beta_1(x))y + 2\pi_4(\beta_0(x)).$$

Since, by assumption, $Q(x, \pi_4(f(x))) = 0$, where f stands for one of C and D, we have

$$\alpha(x)(xf(x)^2 + f(x) + 1) \equiv 2x\beta_2(x)f(x)^2 + 2\beta_1(x)f(x) + 2\beta_0(x) \pmod 4.$$

We let $g(x)$ denote the series $(xf(x)^2 + f(x) + 1)/2$, by (5.1) and (5.2) we know that $g(x)$ has integer coefficients. We rewrite the above congruence as

$$\alpha(x)g(x) \equiv x\beta_2(x)f(x)^2 + \beta_1(x)f(x) + \beta_0(x) \pmod 2,$$

in other words,

(5.3) $\pi_2(\alpha(x))\pi_2(g(x)) = x\pi_2(\beta(x))\pi_2(f(x))^2 + \pi_2(\beta_1(x))\pi_2(f(x)) + \pi_2(\beta_0(x)).$

In light of (5.1) and (5.2), $\pi_2(f(x))$ is of degree 2 over $\mathbb{F}_2(x)$, so that the right hand side of (5.3) lives in a quadratic extension of $\mathbb{F}_2(x)$. We will prove that the left hand side of (5.3) is of degree 4 over $\mathbb{F}_2(x)$, which will lead to a contradiction. Also, $\pi_2(\alpha(x))$ being a non-zero element in $\mathbb{F}_2(x)$, we only need to prove that the degree of $\pi_2(g(x))$ over $\mathbb{F}_2(x)$ is 4.

In case $f(x) = C(x)$, since $C(x) \equiv \varphi(x) \pmod 4$, we have

$$xC(x)^2 + C(x) + 1 \equiv x\varphi(x)^2 + \varphi(x) + 1 \pmod 4,$$

and therefore

$$g(x) \equiv (x\varphi(x)^2 + \varphi(x) + 1)/2 \pmod 2.$$

From Theorem 1.1, we find that $(x\varphi(x)^2 + \varphi(x) + 1)/2$ is equal to

$$x\sqrt{-4\,x+1} + x\sqrt{2\sqrt{-4\,x+1}-1} + \frac{1}{2}\sqrt{-4\,x+1}\sqrt{2\sqrt{-4\,x+1}-1} + \frac{1}{2}\sqrt{-4\,x+1}$$

Its annihilating polynomial is

$$\begin{aligned}
T(x,y) =\,& 16\,x^6 + 8\,y^2x^3 + 32\,yx^4 - 32\,x^5 + y^4 + 24\,y^2x^2 - 40\,yx^3 + 8\,x^4 \\
& - 6\,y^2x - 8\,yx^2 + 16\,x^3 + 8\,yx - 8\,x^2 - y + x.
\end{aligned}$$

Therefore $\pi_2(T(x,y)) = y^4 + y + x$ is an annihilating polynomial of $\pi_2(g(x))$. Let us verify that it is irreducible in $\mathbb{F}_2[x][y]$. If $y^4 + y + x$ factorizes into a cubic and a linear factor, then the linear factor must be $(y + x)$ or $(y + 1)$. However, $y^4 + y + x$ is divisible by neither. If it factorizes into two quadratic factors, then it must be of the form $(y^2 + \xi(x)y + 1)(y^2 + \eta(x)y + x)$, where $\xi(x)$ and $\eta(x)$ are in $\mathbb{F}_2[x]$. When we expand and compare the coefficients, we see that $\xi(x)$ and $\eta(x)$ must satisfy simultaneously $\xi(x) + \eta(x) = 0$ and $\xi(x)\eta(x) + x + 1 = 0$, which is impossible.

In case $f(x) = D(x)$, we find that

$$g(x) \equiv (x\psi(x)^2 + \psi(x) + 1)/2 \pmod 2.$$

We could have computed an annihilating polynomial for $\pi_2(g(x))$ the same way that we did in the case $f(x) = C(x)$, but we would have to deal with too many terms in the calculation involving $D(x)$. So we choose to work directly in $\mathbb{F}_2[[x]]$, by using Corollary 4.1 to find the 2-kernel of $\pi_2(g(x))$, from which we will obtain the minimal polynomial of $\pi_2(g(x))$ following the method in the proof of Theorem 1 from [14].

We prove now that the structure of the 2-kernel of $\pi_2(g(x))$ is

$$[[1,0],[2,3],[1,4],[3,4],[4,4]].$$

By Corollary 4.1,

(5.4) $$D(x) \equiv 1 - \sum_{i,j=0}^{\infty} x^{2^i+2^j-1} + 2\sum_{k=0}^{\infty} x^{2^{2k+1}-1}\left(1 + \sum_{j=0}^{\infty} x^{2^j}\right) \pmod 4.$$

Therefore

$$D(x) \equiv 1 + \sum_{i,j=0}^{\infty} x^{2^i+2^j-1} \equiv 1 + \sum_{j=1}^{\infty} x^{2^j-1} \equiv \sum_{j=0}^{\infty} x^{2^j-1} \pmod{2},$$

$$xD(x)^2 \equiv \sum_{i,j=0}^{\infty} x^{2^i+2^j-1} \pmod{4},$$

and

$$xD(x)^2 + D(x) + 1 \equiv 2 + 2\sum_{k=0}^{\infty} x^{2^{2k+1}-1}\left(1 + \sum_{j=0}^{\infty} x^{2^j}\right) \pmod{4}.$$

We now have an explicit expression for $\pi_2(g(x))$

$$\pi_2(g(x)) = 1 + 1\sum_{k=0}^{\infty} x^{2^{2k+1}-1}\left(1 + \sum_{j=0}^{\infty} x^{2^j}\right) =: g_0(x).$$

To compute the 2-kernel of $g_0(x) = \pi_2(g(x))$, we apply the operators Λ_0 and Λ_1:

$$\Lambda_0 g_0(x) = 1 + \sum_{k=0}^{\infty} x^{2^{2k}} =: g_1(x),$$

$$\Lambda_1 g_0(x) = g_0(x),$$

$$\Lambda_0(g_1(x)) = 1 + \sum_{k=0}^{\infty} x^{2^{2k+1}} =: g_2(x),$$

$$\Lambda_1 g_1(x) = 1 =: g_3(x),$$

$$\Lambda_0 g_2(x) = g_1(x),$$

$$\Lambda_1 g_2(x) = 0 =: g_4(x).$$

Therefore the 2-kernel of g is

$$[[1,0],[2,3],[1,4],[3,4],[4,4]].$$

The following identities are just another way of writing out 2-kernel.

$$g_0(x) = g_1(x)^2 + xg_0(x)^2,$$

$$g_1(x) = g_2(x)^2 + xg_3(x)^2 = g_2(x)^2 + x,$$

$$g_2(x) = g_1(x)^2 + xg_4(x)^2 = g_1(x)^2.$$

From these, we deduce that $\pi_2(g(x)) = g_0(x)$ is a root of the polynomial

$$x^4y^8 + y^4 + xy^2 + y + x^2$$

in $\mathbb{F}_2[x,y]$, which factorizes as

$$(xy^4 + y^3 + 1)(x^3y^4 + x^2y^3 + xy^2 + y + x^2).$$

By computing the first few terms of $\pi_2(g(x))$ we find that the second factor is not an annihilating polynomial for $\pi_2(g(x))$, and therefore $xy^4 + y^3 + 1$ is. As

$$xy^4 + y^3 + 1 = y^4\left(\left(\frac{1}{y}\right)^4 + \frac{1}{y} + x\right),$$

and we have just shown in the case $f = C(x)$ that $y^4 + y + x$ is irreducible in $\mathbb{F}_2[x,y]$, $xy^4 + y^3 + 1$ is also irreducible. This shows that $\pi_2(g(x))$ has degree 4 over $\mathbb{F}_2(x)$ and completes our proof.

6. Hankel determinants

The Hankel determinants of $C(x)$ and $D(x)$ can be calculated by Heilermann's theorem. To prove their automacity, we need the following theorem.

Theorem 6.1 (see [3]). *Let X be an alphabet on which is defined an associative operation $*$. Let $x = (x_n)$ be a q-automatic sequence on the alphabet X. The sequence $y = (y_n)$ defined by*

$$y_1 = x_0$$
$$y_2 = x_1 * x_0$$
$$\vdots$$
$$y_n = x_{n-1} * x_{n-2} * \cdots * x_0$$

is q-automatic.

From the definition of the Thue-Morse and the period-doubling sequence, it is easy to see that

$$t_{2k+1}t_{2k+2} = s_k, \quad \text{and} \quad s_{2k+1}s_{2k+2} = -s_k.$$

By Theorem 2.2, we have

$$H_n(C(x)) = t_0^n(t_1t_2)^{n-1}(t_3t_4)^{n-2}\cdots(t_{2n-3}t_{2n-2})^1$$
$$= s_0^{n-1}s_1^{n-2}\cdots s_{n-2}^1.$$

and

$$H_n(D(x)) = s_0^n(s_1s_2)^{n-1}(s_3s_4)^{n-2}\cdots(s_{2n-3}s_{2n-2})^1$$
$$= (-1)^{n(n-1)/2}s_0^{n-1}s_1^{n-2}\cdots s_{n-2}^1$$
$$= (-1)^{n(n-1)/2}H_n(C(x)).$$

Define $u_n := s_0s_1\cdots s_{n-1}$. By Theorem 6.1, (u_n) is 2-automatic, and consequently $H_n(C(x)) = u_0u_1\cdots u_{n-1}$ is 2-automatic. Since $(-1)^{n(n-1)/2}$ is periodic, $H_n(D(x))$ is also 2-automatic. Finally $H_n(\bar{C}(x))$ and $H_n(\bar{C}(x))$ are 2-automatic as the reduction modulo 4 of $H_n(C(x))$ and $H_n(D(x))$.

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