Bioperational Multisets in Various Semi-rings

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July 2019

Abstract. One can find lists of whole numbers having equal sum and product. We call such a creature a *bioperational multiset*. No one seems to have seriously studied them in areas outside whole numbers such as the rationals, Gaussian integers, or semi-rings. We enumerate all possible sum-products for a bioperational multiset over whole numbers and six additional domains.

1 Introduction

The numbers $1, 2$, and 3 have the property that their sum is also their product. That is, $1 + 2 + 3 = 1 \cdot 2 \cdot 3 = 6$. As Matt Parker [1] has pointed out, this gives us as a strange sort of byproduct:

$$\log(1 + 2 + 3) = \log 1 + \log 2 + \log 3,$$

due to the identity

$$\log(ab) = \log a + \log b.$$

We coin the term *bioperational* here to refer to any such multiset $\{a_i\}_{i=1}^n$ having an equal sum and product. That is to say,

$$\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i.$$

A *multiset*, as can be guessed, is a set in which a number can occur multiple times (instead of just once or not at all).

There are some open conjectures about the number of bioperational multisets over $\mathbb{N}$ of length n as n gets bigger and bigger [2]. There is also a smattering of analysis available on math.stackexchange including uniqueness of solutions and connections with trigonometry [3][4][5][6]. One can also find a surprisingly complicated solution algorithm [7]. But it seems no one has formally categorized bioperational multisets over $\mathbb{N}$ (unless we count Matt's "pseudoproof" and the passing comments of others).

In addition to $\mathbb{N}$, it is interesting to explore bioperational multisets in other environments. They are well-defined anywhere addition and multiplication are well-defined (hence in any semi-ring). We would suspect therefore to find these creatures lurking in $\mathbb{Z}, \mathbb{C}, \mathbb{F}_p, \mathrm{GL}_n(\mathbb{R})$, and many other places (even in non-Abelian rings!). In this paper, we limit ourselves to analyzing bioperational multisets over

- non-negative integers ($\mathbb{N}$) in Section 3,
- integers ($\mathbb{Z}$) in Section 4,
- general fields ($\mathbb{Q}, \mathbb{F}_p$, etc) in Section 5,
- lunar integers ($\mathbb{L}$) in Section 6,
- Gaussian integers ($\mathbb{Z}[i]$) in Section 7,
- Eisenstein integers ($\mathbb{Z}[\omega]$) in Section 8,
- and integers with $\sqrt{2}$ appended ($\mathbb{Z}[\sqrt{2}]$) in Section 9.

2 Some Definitions

Firstly, we have to blow some dead leaves out of the way to see clearly. To do so requires the leafblower of *vocabulary.* Suppose we have a bioperational multiset $S = \{a_i\}_{i=1}^n$. We say

- S is *trivial* if it contains only one element,
- S *vanishes* if the sum-product is zero, and
- S is *minimal* if no proper subset of its terms forms a bioperational set of the same sum-product.

Note that we are using 'trivial' differently than in [2].

All the examples we are considering are 1) integral domains and 2) Abelian. That means 1) if any two numbers have a product of zero then one or both of them must also be zero ($ab = 0$ implies $a = 0$ or $b = 0$) And 2) the order we multiply stuff doesn't matter (so $ab = ba$). This has some consequences on our analysis.

1) In an integral domain, any vanishing bioperational multisets must contain zero – which is rather boring. An analysis of vanishing bioperational multisets over non-integral domains might be interesting. In fact, the first example $(1, 2, 3)$ vanishes if we place it in $\mathbb{Z}/6\mathbb{Z}$. But the adventure of non-integral domains in general will be neglected here.

2) Since the order of multiplication doesn't matter in our examples, we call our subjects bioperational *multisets.* However, the author greatly hopes that bioperational multisets will be explored in non-Abelian rings (in which case they would be bioperational *sequences*). The author would have loved to explore these objects in the quaternions ($\mathbb{H}$) themselves but was too ignorant for the attempt.

For two multisets A and B we let "$A + B$" denote their *multiset sum* which is best explained with an example:

$$\{2,7,2,2,3\} + \{1,2,7,7\} = \{1,2,2,2,2,3,7,7,7\}.$$

In technical terms, we are summing the multiplicities of all elements involved. Similarly "$A - B$" will denote subtracting multiplicites:

$$\{2,7,2,2,3\} - \{2,2,3\} = \{2,7\}.$$

We also use coefficients of multisets to denote scaling multiplicities. Or in other words

$$3\{2,5,5\} = \{2,2,2,5,5,5,5,5,5\}.$$

Lastly, to keep our equations less messy, for a multiset S we will write $\sigma(S)$ for the sum of its elements and $\pi(S)$ for the product. They look nicer than $\sum_{i=1}^{n} a_i$ and $\prod_{i=1}^{n} a_i$.

3 Non-negative integers ($\mathbb{N}$)

We begin with

Theorem 3.1. *There is exactly one non-vanishing bioperational multiset over $\mathbb{N}$ of length n for $n = 2, 3, 4$ with constructions*

$$2 + 2 = 2 \cdot 2 = 4,$$

$$1 + 2 + 3 = 1 \cdot 2 \cdot 3 = 6,$$

$$\textit{and} \quad 1 + 1 + 2 + 4 = 1 \cdot 1 \cdot 2 \cdot 4 = 8.$$

This confirms Matt's conjecture for $n = 2$ and is stated without proof in [2].

Proof. We first take $n = 2$. Suppose we have $ab = a + b$. Rearrangement yields $ab - a - b + 1 = (a-1)(b-1) = 1$. Since the only way to factor 1 as two positive integers is $1 = 1 \cdot 1$ it follows that $a - 1 = b - 1 = 1$ or equivalently, that $a = b = 2$.

A phenomenal proof of $n = 3$ was given by Mark Bennet in [3]. We repeat it here. Suppose we have $a + b + c = abc$. At least one term must be 1 since if otherwise $a \geq b \geq c \geq 2$ and we would have

$$3a \geq a + b + c = abc \geq 4a$$

which is true only if $a \leq 0$. But that is a contractiction since we are assuming $a \geq 2$..

Okay, so let $c = 1$. Then we have $a + b + 1 = ab$. Rearranging yields $ab - a - b + 1 = (a-1)(b-1) = 2$. It follows that $\{a-1, b-1\} = \{1,2\}$ or equivalently, that $\{a,b\} = \{2,3\}$.

Similarly for $n = 4$, suppose $a + b + c + d = abcd$. Similar to the case $n = 3$, we know at least one term must be 1 since $a \geq b \geq c \geq d \geq 2$ would imply

$$4a \geq a + b + c = abc \geq 8a.$$

So let $d = 1$. But we can play the same trick again. If we have $a \geq b \geq c \geq 2$ then

$$3a + 1 \geq a + b + c + 1 = abc \geq 4a$$

from which it follows that $a \leq 1$ which again contradicts our assumption $a \geq 2$. Let $c = 1$ also.

We have $a + b + 2 = ab$. Rearranging yields $(a - 1)(b - 1) = 3$ from which it follows $\{a, b\} = \{4, 2\}$. □

One may be tempted to generalize this proof technique and keep tackling larger and larger n (In fact, we wrote a program to do exactly this [7]. See [8] for another such solution algorithm). For example, $n = 5$ yields

Theorem 3.2. *There are* 3 *non-vanishing bioperational multisets over* $\mathbb{N}$ *of length* $n = 5$ *with constructions*

$$1 + 1 + 2 + 2 + 2 = 1 \cdot 1 \cdot 2 \cdot 2 \cdot 2 = 8,$$

$$1 + 1 + 1 + 3 + 3 = 1 \cdot 1 \cdot 1 \cdot 3 \cdot 3 = 9,$$

$$\text{and} \quad 1 + 1 + 1 + 2 + 5 = 1 \cdot 1 \cdot 1 \cdot 2 \cdot 5 = 10.$$

Proof. From computation. □

But there turns out to be an easier way to catalog all bioperational multisets. We first need a lower foothold (or "lemma" as they're called).

Lemma 3.3. *The product of one or more real numbers, all greater than or equal to* 2*, is greater than or equal to their sum. That is, if* $a_i \geq 2$*,* $i = 1, .., n$ *then*

$$\prod_{i=1}^{n} a_i \geq \sum_{i=1}^{n} a_i.$$

Proof. Induction will be used on n. The base case, $n = 1$, of a single number is clearly true since every number is equal to itself – and therefore is greater than or equal to itself ($a_1 \geq a_1$).

Next suppose we have some multiset $S = \{a_i\}_{i=1}^{n}$ for which the theorem statement is true. So $\pi(S) \geq \sigma(S)$. We have to show the theorem true for a new multiset S' formed by appending a new element $a_{n+1} \geq 2$ since any multiset can be built up one element at a time.

Let k be the largest integer such that $a_{n+1} > \pi(S)^k$. From this we grab two crimps,

$$a_{n+1} - 1 \geq \pi(S)^k \quad \text{and} \quad a_{n+1} < \pi(S)^{k+1},$$

with which the last bit of the proof can be shown easily.

$$\pi(S') = a_{n+1}\pi(S) = (a_{n+1}-1)\pi(S) + \pi(S) \geq \pi(S)^k\pi(S) + \sigma(S)$$
$$= \pi(S)^{k+1} + \sigma(S) > a_{n+1} + \sigma(S) = \sigma(S')$$

Technically this is a bit overkill since we've shown $\pi(S') > \sigma(S')$ when all we needed was $\pi(S') \geq \sigma(S')$. Oh well. □

With that Lemma we can catalog all bioperational multisets over $\mathbb{N}$ by their sum-product.

Theorem 3.4. *For every composite integer $m \in \mathbb{N}$ there exists a non-trivial bioperational multiset over $\mathbb{N}$ with a sum-product of m.*

Proof. Suppose a composite integer $m = a_1a_2...a_k$ with $k > 1$ and $a_i \geq 2$ for $i = 1, ..., k$. Let $S = \{a_i\}_{i=1}^k$. By Lemma 3.3, we know $\pi(S) \geq \sigma(S)$. So let the non-negative integer $d = \pi(S) - \sigma(S)$ be their difference. The multiset

$$S' = \{a_1, a_2, ..., a_k, \overbrace{1, ..., 1}^{d \text{ times}}\}$$

is bioperational with sum-product m since

$$\sigma(S') = a_1 + ... + a_k + \overbrace{1 + ... + 1}^{d \text{ times}} = \sigma(S) + (\pi(S) - \sigma(S)) = \pi(S) = \pi(S') = m.$$

□

From the proof of the previous theorem we can also make a statement about the lengths of bioperational multisets.

Corollary 3.4.1. *For every factorization of a composite integer $m = a_1a_2...a_k$ there exists a non-vanishing bioperational multiset over $\mathbb{N}$ of length $m+k-\sum a_i$.*

Proof. Let S and S' denote the same multisets as in the proof of Theorem 3.4. S' is bioperational and contains $k + d = k + (\pi(S) - \sigma(S)) = m + k - \sum a_i$ elements. □

Starting at $n = 2$ the number of non-vanishing bioperational multisets over $\mathbb{N}$ of length n is

$$1, 1, 1, 3, 1, 2, 2, 2, 2, 3, 2, 4, 2, ...$$

(sequence A033178 in OEIS [8]). The positions of record in this list occur at

$$n = 2, 5, 13, 25, 37, 41, 61, 85, 113, 181, 361, 421, 433, ...$$

(sequence A309230 in OEIS). The terms all appear to have fewer prime factors than their neighbors.

4 Integers ($\mathbb{Z}$)

An interesting thing happens once negatives are on the playing field. A multiset can be extended without changing either sum or product. Consider

$$S = \{1, 2, 3, -1, -1, 1, 1\}$$

which is bioperational with sum-product 6. This is the first example of a non-minimal bioperational multiset. In $\mathbb{N}$ every non-vanishing bioperational multiset is also minimal. Not so in $\mathbb{Z}$!

Accordingly, we now use *bioperation* as a verb as well. We say a multiset has been *bioperated* if it has been made bioperational by means of changing its sum with appendages. For example, we may bioperate $S = \{3, -5\}$ which has a sum $\sigma(S) = -2$ and product $\pi(S) = -15$. Since appending $T = \{-1, -1, 1\}$ decrements $\sigma(S)$ and fixes $\pi(S)$, bioperation is accomplished by just repeatedly appending T. In particular,

$$S' = S + 13T$$

is bioperational. Note however S' is not minimal. We can trim it down to minimality by shaving off groups of $\{-1, -1, 1, 1\}$ which have no effect on the sum-product obtaining

$$S'' = S + 13T - 6\{-1, -1, 1, 1, 1\} = \{3, -5, \overbrace{-1, ..., -1}^{14 \text{ times}}, 1\}$$

which is, in fact, minimal.

There are three important appendages in $\mathbb{Z}$ which fix the product.

label	appendage	$\Delta\sigma(S)$
T_1	$\{1\}$	$+1$
T_0	$\{1, 1, -1, -1\}$	0
T_{-1}	$\{1, -1, -1\}$	-1

We now give the parallel of Theorem 3.4 for $\mathbb{Z}$.

Theorem 4.1. *For every composite integer $m \in \mathbb{Z}$ there exists a non-trivial minimal bioperational multiset over $\mathbb{Z}$ with a sum-product of m.*

Proof. Choose a factorization $m = a_1...a_n$ with $n \geq 2$ where each a_i may be positive or negative and $|a_i| \geq 2$ for $i = 1, ..., n$. The multiset $S = \{a_i\}_{i=1}^n$ has the desired product. Bioperate S producing S' such that $\sigma(S') = \pi(S') = \pi(S)$. This is done by appending $T_{\pm 1}$ as needed. To be

Finally, if S' is not minimal we may take a minimal bioperational multiset, S'', from it. S'' must include the non-units $a_1, ..., a_n$ (a "unit" by the way is a fancy name for a number with an inverse in its same ring; in this case 1 and -1). Since $n \geq 2$ we are assured that S'' is non-trivial. □

5 Fields

Bioperational multisets turn out disappointingly abundant in fields.

Lemma 5.1. *Given any multiset* $S = \{a_i\}_{i=1}^n$ *whose elements are in a field* F *and such that* $\pi(S) \neq 1$*, one can bioperate* S *into a unique multiset* S' *by appending a single element,*

$$a_{n+1} = \frac{\sigma(S)}{\pi(S) - 1}.$$

This was stated for $F = \mathbb{Q}$ *and* $n = 4$ *by Robert Israel in [6].*

Proof. Any element $a_{n+1} \in F$ which might bioperate S must satisfy $\sigma(S) + a_{n+1} = \pi(S)a_{n+1}$. Rearranging yields $a_{n+1} = \frac{\sigma(S)}{\pi(S)-1}$ which exists if $\pi(S) \neq 1$. □

The lemma turns out to be an exhaustive description.

Theorem 5.2. *In any field, all non-trivial bioperational multisets can be produced with Lemma 5.1.*

Proof. Suppose we have some bioperational multiset $S = \{a_i\}_{i=1}^n$ which cannot be produced by the lemma. Let S_i' be the multiset formed by removing a_i. It follows from the lemma $\pi(S_i') = \frac{\pi(S)}{a_i} = 1$ for all $i = 1, ..., n$. This in turn implies $\pi(S) = a_i$ for $i = 1, ..., n$ and we see that all a_i are equal. We therefore have a solution to

$$a_1^n = na_1.$$

But dividing out an a_1 from both sides gives us $n = a_1^{n-1} = \pi(S_1') = 1$ showing S is trivial. □

Before leaving this territory, we note that there are solutions to $a^{n-1} = n$ leading to bioperational sets of a single value. Take for instance $\{2, 2, 2, 2, 2\}$ which is bioperational in $\mathbb{F}_{11}$.

6 Lunar Integers ($\mathbb{L}$)

The Lunar Integers are the only strictly *semi*-ring to be considered. Their arithmetic is well analyzed in [10] (there called "Dismal" Arithmetic) and Neil Sloane gives a wonderful introduction in a Numberphile interview [11].

We neglect to explain the arithmetic here ourselves. We need only note some properties of the number of digits. If we let $D(a)$ denote the number of digits of a lunar integer $a \in \mathbb{L}$, then

$$D(ab) = D(a) + D(b) - 1 \quad \text{and} \quad D(a + b) = \max\{D(a), D(b)\}.$$

These give us

Lemma 6.1. *In any Lunar Bioperational Set, there is at most one element with 2 or more digits.*

Proof. Suppose $S = \{a_i\}_{i=1}^n \subset \mathbb{L}$ is bioperational and that $D(a_1) \geq D(a_i)$ for $i = 2, ..., n$. From the aforementioned identities

$$D(\sigma(S)) = \max\{D(a_i)\}_{i=1}^n = D(a_1)$$

and

$$D(\pi(S)) = 1 + \sum_{i=1}^n (D(a_i) - 1) = D(a_1) + \sum_{i=2}^n (D(a_i) - 1).$$

Since $D(\pi(S)) = D(\sigma(S))$, it follows that $\sum_{i=2}^n (D(a_i) - 1) = 0$ and hence that $D(a_i) = 1$ for $i = 2, ..., n$. □

Apparently bioperational multisets can't breathe well on the moon:

Theorem 6.2. *Every minimal bioperational multiset of Lunar integers is trivial.*

Proof. We prove the contrapositive. Suppose $S = \{a_i\}_{i=1}^n \subset \mathbb{L}$ bioperational and non-trivial. From Lemma 6.1 we may assume $D(a_i) = 1$ for $i = 2, ..., n$. For $a \in \mathbb{L}$, let $F(a) \in \mathbb{L}$ denote be the last digit of a. From the definitions of addition and multiplication over $\mathbb{L}$

$$\max\{F(a_1), a_2, ..., a_n\} = F(\sigma(S)) = F(\pi(S)) = \min\{F(a_i), a_2, ..., a_n\}.$$

But this implies $F(a_1) = a_2 = ... = a_n$. In which case the multiset $S' = \{a_1\}$ is trivially bioperational with the same sum-product as S and hence S is not minimal. □

So there are bioperational multisets in $\mathbb{L}$, like $\{17, 7\}$ and $\{2, 2, 2\}$, but they aren't very interesting.

7 Gaussian Integers ($\mathbb{Z}[i]$)

Gaussian integers are numbers of the form $a + bi$ where a and b are integers and $i^2 = -1$ (so like $2 + 3i$ or $-1 - 19i$ for example). In addition to the appendages T_{-1}, T_0, and T_1 given in Section 4, two more appear in $\mathbb{Z}[i]$,

$$T_{\pm 2i} = \{\pm i, \pm i, -1, 1\},$$

which perturb the sum by $\pm 2i$ and fix the product. So sometimes bioperate the imaginary part of a multiset sum.

Take, for instance, $S = \{1 + 2i, 2 + 3i\}$. We have

$$\sigma(S) = 3 + 5i \quad \text{and} \quad \pi(S) = -4 + 7i.$$

The difference is $\pi(S) - \sigma(S) = -7 + 2i$. We bioperate by 1) appending T_{-1} seven times, 2) appending T_{2i} once, and 3) shaving off T_0 until minimality is reached. The result is

$$S' = \{1 + 2i, 2 + 3i, i, i, -1, -1, -1, -1, -1, -1, -1\}$$

which is bioperational with $\pi(S') = \sigma(S') = -4 + 7i$.

We need a couple lemmas before the result analogous to Theorem 3.4.

Lemma 7.1. *A Gaussian integer $a + bi$ is a multiple of $1 + i$ if and only if a and b have the same parity (that is, are both odd or both even).*

Proof. Firstly, suppose $a + bi = (1 + i)(c + di)$ is a multiple of $1 + i$. Then $a = c - d$ and $b = c + d$. a and b therefore have the same parity since $b = a + 2d$.

Conversely, suppose a and b have the same parity. If both even, then we may write

$$a + bi = 2\Big(\frac{a}{2} + \frac{b}{2}i\Big) = (1 + i)(1 - i)\Big(\frac{a}{2} + \frac{b}{2}i\Big)$$

and are done. If both odd, then we may write

$$a + bi = (1 + i)\Big(\frac{a + b}{2} + \frac{b - a}{2}i\Big).$$

□

Lemma 7.2. *For any Gaussian integers $\alpha_1, ..., \alpha_n \in \mathbb{Z}[i]$ such that $1 + i$ does not divide any α_i,*

$$Im\Big(\prod \alpha_i\Big) \equiv Im\Big(\sum \alpha_i\Big) \mod 2.$$

Proof. Let $\varphi(a + bi) = \overline{b \,\%\, 2} \in \mathbb{F}_2$. It follows that $\varphi(\alpha + \beta) = \varphi(\alpha) + \varphi(\beta)$. But more interestingly, it turns out that when neither of α nor β are multiples of $1 + i$ we have also $\varphi(\alpha\beta) = \varphi(\alpha) + \varphi(\beta)$. From lemma 7.1 it follows that the residues of α and β in $\mathbb{Z}[i]/(2) \cong \mathbb{F}_2[i]$ are in $\{1, i\}$. It is enought to check that φ has the desired property on $\{1, i\}$:

$$0 = \varphi(1) = \varphi(1 \cdot 1) = \varphi(1) + \varphi(1) = 0 + 0 = 0$$

$$1 = \varphi(i) = \varphi(1 \cdot i) = \varphi(1) + \varphi(i) = 0 + 1 = 1$$

$$0 = \varphi(-1) = \varphi(i \cdot i) = \varphi(i) + \varphi(i) = 1 + 1 = 0$$

The lemma follows noting

$$\overline{\mathrm{Im}\Big(\prod \alpha_i\Big) \,\%\, 2} = \varphi\Big(\prod \alpha_i\Big) = \sum \varphi(\alpha_i) = \varphi\Big(\sum \alpha_i\Big) = \overline{\mathrm{Im}\Big(\sum \alpha_i\Big) \,\%\, 2}.$$

□

The enzymes of $\mathbb{Z}[i]$ have been assembled. We are ready to digest the theorem.

Theorem 7.3. *For every $\mu \in \mathbb{Z}[i]$ which factors into non-units (i.e. $\mu = \alpha\beta$ with $\alpha, \beta \notin \{1, -1, i, -i\}$) there exists a non-trivial minimal bioperationl multiset over $\mathbb{Z}[i]$ with a sum-product of μ.*

Proof. Pick some factorization $\mu = a_1 ... a_n$ and let $S = \{a_i\}_{i=1}^n$ with at least two a_i non-units. We break into two cases.

Case 1) if $\mathrm{Im}(\pi(S))$ and $\mathrm{Im}(\sigma(S))$ have the same parity, we may bioperate S by appending $T_{\pm 1}$ and $T_{\pm 2i}$ as needed. The result is S'; bioperational with sum-product μ. If S' is not minimal, we may take a minimal subset S''. And we are assured S'' is non-trivial since otherwise $S'' = \{\mu\}$ which implies $a_i = \mu$ for some. And $a_i = \mu$ implies all other α_j for $j \neq i$ are units since $\alpha_1 ... \alpha_{i-1}\alpha_{i+1} ... \alpha_n = 1$ (note we are using in this last step the fact that $\mathbb{Z}[i]$ is an integral domain).

Case 2) if $\mathrm{Im}(\pi(S))$ and $\mathrm{Im}(\sigma(S))$ have different parities, we may suppose from Lemma 7.2 that some α_j is divisible by $1 + i$. We create a new multiset by removing α_j from S and appending $\{i\alpha_j, i, -1\}$. In notation,

$$S' = \{\alpha_i\}_{i \neq j} + \{i\alpha_j, i, -1\}.$$

The product remains unchanged since

$$\pi(S') = \frac{\pi(S)}{\alpha_j}(i^2\alpha_j)(-1) = \pi(S).$$

More importantly, it is claimed that $\mathrm{Im}(\sigma(S'))$ and $\mathrm{Im}(\sigma(S))$ have different parity. Their difference is

$$\mathrm{Im}(\sigma(S')) - \mathrm{Im}(\sigma(S)) = \mathrm{Im}(\sigma(S') - \sigma(S)) = \mathrm{Im}(i\alpha_j + i - 1 - \alpha_j).$$

Let $\alpha_j = a + bi$ for some integers a and b. Substitution gives

$$\mathrm{Im}(\sigma(S')) - \mathrm{Im}(\sigma(S)) = \mathrm{Im}(ai - b + i - 1 - a - bi) = a - b + 1.$$

From Lemma 7.1 we may suppose that a and b have the same parity since $1 + i | \alpha_j$. It follows that $a - b + 1$ is odd, that $\mathrm{Im}(\sigma(S'))$ and $\mathrm{Im}(\sigma(S))$ have different parity, and therefore that $\mathrm{Im}(\sigma(S'))$ and $\mathrm{Im}(\pi(S)) = \mathrm{Im}(\pi(S'))$ have the same parity. And so we return to the first case to bioperate S'. ☐

8 Eisenstein Integers ($\mathbb{Z}[\omega]$)

Eisenstein integers are similar to the Gaussians in that they are all of the form $a + b\omega$ where a and b are integers and ω is a strictly complex number such that $\omega^3 = 1$. Right away this gives us our first appendage,

$$T_{3\omega} = \{\omega, \omega, \omega\}.$$

One can show further show that $\omega^2 = -1 - \omega$ from which we get

$$T_{-2\omega} = \{-\omega, -1 - \omega, -1, 1, 1\}.$$

Thus we have $T_\omega = T_{3\omega} + T_{-2\omega}$ and $T_{-\omega} = T_{3\omega} + 2T_{-2\omega}$ at our disposal. Suprisingly, we can therefore bioperate any multiset over $\mathbb{Z}[\omega]$. Our main theorem in this section will therefore run almost identically to its analog over $\mathbb{Z}$.

Theorem 8.1. *For $\mu \in \mathbb{Z}[\omega]$ which factors into non-units there exists a non-trivial minimal bioperational multiset over $\mathbb{Z}[\omega]$ with a sum-product of μ.*

Proof. Choose a factorization $\mu = \alpha_1 ... \alpha_n$ with 2 non-units and let $S = \{\alpha_i\}_{i=1}^n$. Bioperate S with $T_{\pm 1}$ and $T_{\pm\omega}$. The resulting bioperational multiset S' can be shaved down to minimality without becoming trivial since the non-units cannot be trimmed off. □

9 Integers $\sqrt{2}$ Appended ($\mathbb{Z}[\sqrt{2}]$)

Lastly, we consider the integer ring of a real quadratic number field. Numbers in $\mathbb{Z}[\sqrt{2}]$ are of the form $a + b\sqrt{2}$ where a and b are integers (again, very similar to $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$). We use the fact that

$$(1 + \sqrt{2})(-1 + \sqrt{2}) = 1$$

to create

$$T_{\pm 2\sqrt{2}} = \{\pm 1 \pm \sqrt{2}. \mp 1 \pm \sqrt{2}\}.$$

There's good reason to think that this is the best we can do (I.e. that $T_{\pm\sqrt{2}}$ doesn't exist over $\mathbb{Z}[\sqrt{2}]$). But the best proof the author could come up with for such a fact uses difficult results about quadratic number fields and complicated induction. Instead, we take a route similar to that taken through $\mathbb{Z}[i]$.

Lemma 9.1. *The number $a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ is a multiple of $\sqrt{2}$ if and only if a is even.*

Proof. Note $(c + d\sqrt{2})\sqrt{2} = 2d + c\sqrt{2}$. □

Lemma 9.2. *For numbers $\alpha_1, ..., \alpha_n \in \mathbb{Z}[\sqrt{2}]$ let*

$$a + b\sqrt{2} = \sum \alpha_i \quad \textit{and} \quad c + d\sqrt{2} = \prod \alpha_i.$$

If no α_i is divisible by $\sqrt{2}$ then $b \equiv d \mod 2$.

Proof. We again create a strange homomorphism. Let $\varphi(a+b\sqrt{2}) = \overline{b \ \% \ 2} \in \mathbb{F}_2$. It follows that $\varphi(\alpha + \beta) = \varphi(\alpha) + \varphi(\beta)$. We claim if $\sqrt{2}$ divides neither α nor β then $\varphi(\alpha\beta) = \varphi(\alpha) + \varphi(\beta)$ as well. From the previous lemma, we see that the residues of such α and β with coefficients in $\mathbb{F}_2$ are in $\{1, 1 + \sqrt{2}\}$. We check φ by hand:

$$0 = \varphi(1) = \varphi(1 \cdot 1) = \varphi(1) + \varphi(1) = 0 + 0 = 0$$

$$1 = \varphi(1 + \sqrt{2}) = \varphi(1 \cdot (1 + \sqrt{2})) = \varphi(1) + \varphi(1 + \sqrt{2}) = 0 + 1 = 1$$

$$0 = \varphi(1) = \varphi((1 + \sqrt{2}) \cdot (1 + \sqrt{2})) = \varphi(1 + \sqrt{2}) + \varphi(1 + \sqrt{2}) = 1 + 1 = 0$$

We end noting

$$\overline{d \% 2} = \varphi\Big(\prod \alpha_i\Big) = \sum \varphi(\alpha_i) = \varphi\Big(\sum \alpha_i\Big) = \overline{b \% 2}.$$

□

It's probable that if the author knew more about ring isomorphisms, the results of this section and those of Section 7 could have been demonstrated simultaneously.

Theorem 9.3. *For every $\mu \in \mathbb{Z}[\sqrt{2}]$ which factors into non-units there exists a non-trivial minimal bioperationl multiset over $\mathbb{Z}[\sqrt{2}]$ with a sum-product of μ.*

Proof. Pick a factorization $\mu = \alpha_1 ... \alpha_n$ and let $S = \{\alpha_i\}_{i=1}^n$, $a + b\sqrt{2} = \sigma(S)$, and $c + d\sqrt{2} = \pi(S)$. If b and d have the same parity, S can be bioperated into the desired result. If not, we may pick some α_j a multiple of $\sqrt{2}$. Letting

$$S' = \{\alpha_i\}_{i \neq j} + \{(1 + \sqrt{2})\alpha_j, -1 + \sqrt{2}\}$$

Letting $\alpha_j = x + y\sqrt{2}$, the change $\sigma(S)$ is

$$\sigma(S') - \sigma(S) = (x + y\sqrt{2})(1 + \sqrt{2}) + (-1 + \sqrt{2}) - (x + y\sqrt{2}) = (2y - 1) + (x + 1)\sqrt{2}.$$

But from Lemma 9.1, we may suppose that x is even and that therefore $\sigma(S')$ and $\sigma(S)$ have $\sqrt{2}$ coefficients of different parity. It follows that $\sigma(S')$ and $\pi(S') = \pi(S)$ have $\sqrt{2}$ coefficients of the same parity and that S' can therefore be bioperated into the desired result. □

10 Generalization and Open Problems

Let's start this section by bundling up our main theorems into a single statement

Theorem 10.1. *If R is one of $\mathbb{N}, \mathbb{Z}, \mathbb{Z}[i], \mathbb{Z}[\omega]$, or $\mathbb{Z}[\sqrt{2}]$ then for every $\mu \in R$ which factors into non-units, there exists a non-trivial minimal bioperational multiset over R with a sum-product of μ.*

Proof. Theorems 3.4, 4.1, 7.3, 8.1, 9.3. □

Some open problems of interest:

- Does Theorem 10.1 hold over the quaternions?
  Order of multiplication now matters. We at least have $T_{\pm 2i}, T_{\pm 2j}$, and $T_{\pm 2k}$ at our disposal since

  $$T_{\pm 2v} = (v, v, -1, 1)$$

  has a product of 1 for $v \in \{i, j, k\}$.

- Does Theorem 10.1 hold for all integer rings of real quadratic number fields?
  There are families of such rings that admit easy attack. Take for instance $\mathbb{Z}[\sqrt{t^2 \pm 1}]$. From
  $$(t + \sqrt{t^2 \pm 1})(t - \sqrt{t^2 \pm 1}) = \mp 1$$
  we can construct appendages $T_{\pm 2\sqrt{t^2 \pm 1}}$ which gives us pretty good flexibility for bioperation. And in general, for $d = t(b^2 t \pm 2)$ we can construct appendages $T_{\pm 2b\sqrt{d}}$. The first values not covered by these parametrizations are
  $$13, 19, 21, 22, 28, 29, 31, 33, 39, 41, 43, 44, 45, 46, 52, 53, 54, 55, 57, 58, 59, 61, 67, 69, ...$$
  Perhaps $\mathbb{Z}[\sqrt{13}]$, which has a relatively large fundamental unit, is our first example for which Theorem 10.1 fails. One would think it easy to construct a counter-example ring to the theorem. However the handful of examples the author toyed with proved dead ends.

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